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OPTIMAL PROPERTIES FOR DEFICIENT QUARTIC SPLINES OF MARSDEN TYPE

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Abstract. In this work, we obtain an improved error estimate in the interpolation with the Hermite C^2-smooth deficient complete quartic spline that has the distribution of nodes following the Marsden type scheme and investigate the possibilities to compute the derivatives on the knots such that the obtained spline $S \in C^1[a,b]$ has minimal curvature and minimal L^2-norm of S' and S'''. In each case, the interpolation error estimate is performed in terms of the modulus of continuity.

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1. INTRODUCTION

Motivated by the nice properties of complete cubic splines, and trying to increase the degree of C^2-smooth complete spline interpolant, but preserving the bandwidth of the diagonally dominant system, Howell introduced in [9] the deficient complete quartic spline that match the interpolated function $f \in C[a,b]$ on the interpolation nodes and on midpoints of a grid following the Marsden scheme (see [17] and [25]). More precisely, considering a grid

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$$

and the midpoints $z_i = x_{i-1/2} = \frac{x_{i-1}+x_i}{2}$, $i = \overline{1,n}$, Howell proved the existence and uniqueness of the deficient quartic spline $S \in C^2[a,b]$ taking prescribed values on x_i, $i = \overline{0,n}$ and on $x_{i-1/2}$, $i = \overline{1,n}$, with the endpoint interpolation

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conditions $S'(a) = f'(a)$, $S'(b) = f'(b)$. Moreover, by the continuity condition $S \in C^2[a,b]$ the spline must satisfy the tridiagonal system of equations

(1)
$$-\frac{1}{h_i}\cdot m_{i-1} + \left(\frac{4}{h_i} + \frac{4}{h_{i+1}}\right)\cdot m_i - \frac{1}{h_{i+1}}\cdot m_{i+1} =$$
$$= \frac{5}{h_i^2} f(x_{i-1}) - \frac{5}{h_{i+1}^2} f(x_{i+1}) + \left(\frac{11}{h_i^2} - \frac{11}{h_{i+1}^2}\right) f(x_i) + \frac{16}{h_{i+1}^2} f(x_{i+1/2}) - \frac{16}{h_i^2} f(x_{i-1/2})$$
$$= d_i, \quad i = \overline{1, n-1}$$

with known $m_0 = f'(a)$, $m_n = f'(b)$, where $h_i = x_i - x_{i-1}$, $i = \overline{1,n}$, and $m_i = S'(x_i)$, $i = \overline{0,n}$. With the notation $y_i = f(x_i)$, $i = \overline{0,n}$, and $t = \frac{x - x_{i-1}}{h_i} \in [0,1]$ for $x \in [x_{i-1}, x_i]$, this quartic spline has the following expression on each interval $[x_{i-1}, x_i]$, $i = \overline{1,n}$:

(2)
$$S(x) = (1-t)^2(1-2t)(4t+1)\cdot y_{i-1} + 16t^2(1-t)^2\cdot y_{i-1/2} +$$
$$+ t^2(2t-1)(5-4t)\, y_i + h_i t(1-t)^2(1-2t)\, m_{i-1} + h_i t^2(1-t)(1-2t)\, m_i$$
$$= A_i(x)\cdot y_{i-1} + B_i(x)\cdot y_{i-1/2} + C_i(x)\cdot y_i + D_i(x)\cdot m_{i-1} + E_i(x)\cdot m_i.$$

Solving this system, the local derivatives m_i, $i = \overline{0,n}$ are uniquely determined. Concerning the interpolation error estimate, in [10] was established the following result.

Theorem 1. [10, Th.2] *Let $f \in C^5[0,1]$. Then we have:*

$$|f(x) - S(x)| \le \frac{C_0 h^5}{5!}\cdot \max_{x\in[0,1]} \left|f^V(x)\right|, \quad x \in [0,1]$$

where

(3)
$$C_0 = \left(\frac{1}{30} + \frac{\sqrt{30}}{3}\right)\cdot\sqrt{\left(\frac{1}{4} - \frac{1}{\sqrt{30}}\right)} = \max_{x\in[0,1]} |c(t)|$$
$$c(t) = \frac{3t^2(1-2t)(1-t)^2 + t(1-t)(1-2t)}{6}.$$

Also we have

(4)
$$\left|f'(x_i) - S'(x_i)\right| \le \frac{h^4}{6!}\cdot \max_{x\in[0,1]}\left|f^V(x)\right|, \quad i = \overline{1, n-1}.$$

Furthermore, the constant C_0 in (3) cannot be improved for an equally spaced partition. Inequality (4) is also best possible. Also we have

(5)
$$\left|f'(x) - S'(x)\right| \le C_1 \frac{h^4}{6!}\cdot \max_{x\in[0,1]}\left|f^V(x)\right|.$$

In [9] Howell conjectured that $C_1 = 1$. Volkov [24] proved that this conjecture holds true. For much less smooth class of functions $f \in C[a,b]$ and in the case of uniform partition, considering a simplified endpoint condition $S'(a) = S'(b) = 0$, the corresponding error estimate is obtained in terms of the modulus of continuity as follows:

THEOREM 2 ([9], [10]). *Let* $f \in C[a,b]$. *If* $\{x_i\}_{i=0}^n$ *is the partition of equally spaced knots, then for* $x_{i-1} \le x \le z_i = \frac{x_{i-1}+x_i}{2}$ *and* $t = \frac{x-x_{i-1}}{h_i}$, $i = \overline{1,n}$, *we have*

$$|f(x) - S(x)| \le c(t)\,\omega(f,h) \le c_2\omega(f,h), \quad t \in [0, \tfrac{1}{2}] \tag{6}$$

and for $x_i \le x \le z_{i+1}$, *or* $\frac{1}{2} \le t \le 1$

$$|f(x) - S(x)| \le c(1-t)\,\omega(f,h) \tag{7}$$

where $c(t) = 1 + \frac{13}{3}t - 3t^2 - \frac{58}{3}t^3 + 16t^4$ *and* $c_2 = \max\limits_{t\in[0,\frac{1}{2}]} |c(t)| \cong 1.6572$.

In [9] the author had extended this result for nonuniform partitions considering the constant $\beta = \max\{h_i : i = \overline{1,n}\}/\min\{h_i : i = \overline{1,n}\}$ and $h = \max\{h_i : i = \overline{1,n}\}$, obtaining:

THEOREM 3 ([9, Th. 4.1.3]). *Let* $f \in C[a,b]$ *and let* $S \in C^2[a,b]$ *be the quartic spline* (2). *Then for* $x_{i-1} \le x \le z_i = \frac{x_{i-1}+x_i}{2}$ *(i.e. for* $0 \le t \le \frac{1}{2}$ *with* $t = \frac{x-x_{i-1}}{h_i}$*)*

$$|f(x) - S(x)| \le c_1(t)\,\omega(f,h) \tag{8}$$

and for $x_i \le x \le z_{i+1}$ *(i. e. for* $\frac{1}{2} \le t \le 1$*)*

$$|f(x) - S(x)| \le c_1(1-t)\,\omega(f,h)$$

where $c_1(t) = 1 + 10t^2 - 28t^3 + 16t^4 + \frac{8}{3}(\beta^2+\beta)\,t(1-t)(1-2t)$.

Error estimates for C^3-smooth quartic splines and for quartic splines that on midpoints matches with the first derivative were established in [4], [5], [12], [18], [20] and [21]. In [20], Theorem 1 is generalized by replacing the position of midpoints to a general type of interior points of the form $z_i = x_{i-1} + \theta h_i$, $i = \overline{1,n}$ with $\theta \in [\frac{1}{4}, \frac{3}{4}]$. In this paper we improve the error estimates (6)–(7) obtaining a smaller constant in terms of the modulus of continuity in the case $f \in C[a,b]$, and investigate the corresponding error estimates for the situations when the endpoint condition $S'(a) = S'(b) = 0$ is replaced by other classical ones such as $S''(a) = S''(b) = 0$, $S'(a) = f'(a)$ and $S'(b) = f'(b)$, or $S''(a) = f''(a)$ and $S''(b) = f''(b)$.

On the other hand, in the present work we investigate the posibilities to determine the local derivatives m_i, $i = \overline{0,n}$, on the nodes of the deficient quartic spline $S \in C^1[a,b]$ given in (2), such that the mean curvature

$$\sqrt{\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} (S''(x))^2\,dx}$$

and the functionals

$$J_k(S) = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \left(S^{(k)}(x)\right)^2 dx$$

for $k \in \{1, 2, 3\}$, are minimized.

Shape preserving properties for quartic splines were investigated in [7], [22] and [26]. Optimal properties for quartic splines were obtained in [6], [14], [15], and [18]. In [14] and [15] the properties of C^3-smooth quartic splines having interpolation points t_i, $i = \overline{1,n}$ with $x_{i-1} < t_i < x_i$, different by the grid nodes, were investigated. Natural, complete and periodic C^3-smooth quartic splines in connection with the minimization of the curvature are considered in [14], while the parameters of spline are determined in [15] under the minimization of the functionals $J_k(S)$, $k \in \{0, 1, 2, 3\}$, by using quadratic programming and the technique of pseudoinverse solution of linear systems in a similar way as was performed in [13] for cubic splines.

The paper is organized as follows. In Section 2 we obtain some improvements of the error estimates presented in [10] and in [9], regarding the interpolation of a continuous function $f \in C[a, b]$ by the deficient complete quartic spline $S \in C^2[a, b]$ given in (2). Section 3 is devoted to the optimal properties of the deficient quartic spline $S \in C^1[a, b]$ in connection with the minimization of the functionals $J_k(S)$ above presented, of a special interest being the minimal curvature and the minimal slope of the graph of S. In order to illustrate the obtained theoretical results, a numerical example is presented in Section 4 and some concluding remarks are pointed out in the last section.

2. ERROR ESTIMATES FOR DEFICIENT QUARTIC SPLINES

Consider the quartic spline $S \in C^2[a, b]$ given in (2) under the endpoint condition $S'(a) = S'(b) = 0$ interpolating a continuous function $f \in C[a, b]$ on a uniform parition $\{x_i\}_{i=0}^n$ and matching f on the midpoints $x_{i-1/2}$, $i = \overline{1,n}$. In that follows, we get an improvement of the estimates (6)–(7) considering both the cases $f \in C[a, b]$ and $f \in \mathrm{Lip}[a, b]$, where $\mathrm{Lip}[a, b] = \{f \in C[a, b] : \exists L > 0$ with $|f(x) - f(y)| \leq L|x - y|, \forall x, y \in [a, b]\}$.

Theorem 4. *If the quartic spline S presented in (2) interpolates $f \in C[a, b]$ on the uniform partition $\{x_i\}_{i=0}^n$ with the endpoint condition $S'(a) = S'(b) = 0$, then*

$$(9)\quad |S(x) - f(x)| \leq$$
$$\leq \begin{cases} \max\limits_{t\in[0,\frac{1}{2}]} |P(t)|\,\omega\left(f, \frac{h}{2}\right) + \frac{1125}{8192}\omega(f, h), & x \in [x_{i-1}, x_{i-1/2}] \\ \max\limits_{t\in[\frac{1}{2},1]} |P(1-t)|\,\omega\left(f, \frac{h}{2}\right) + \frac{1125}{8192}\omega(f, h), & x \in [x_{i-1/2}, x_i] \end{cases} \quad i = \overline{1,n}$$

where $P(t) = 8t^4 - \frac{10}{3}t^3 - 11t^2 + \frac{16}{3}t + 1$, and $h = \frac{b-a}{n}$. If $f \in \mathrm{Lip}[a, b]$ with the Lipschitz constant L, then the error estimate becomes

$$(10)\qquad |S(x) - f(x)| \leq 0.95084 \cdot Lh, \quad \forall x \in [a, b].$$

Proof. Firstly, we observe that $A_i(x) \geq 0$, $B_i(x) \geq 0$, $C_i(x) \leq 0$, $D_i(x) \geq 0$, $E_i(x) \geq 0$ for $x \in [x_{i-1}, x_{i-1/2}]$ and $A_i(x) \leq 0$, $B_i(x) \geq 0$, $C_i(x) \geq 0$,

$D_i(x) \leq 0$, $E_i(x) \leq 0$ for $x \in [x_{i-1/2}, x_i]$. Since $A_i(x) + B_i(x) + C_i(x) = 1$, $\forall x \in [x_{i-1}, x_i]$ we infer that on the interval $[x_{i-1}, x_{i-1/2}]$ we have

$$\begin{aligned} |S(x) - f(x)| \leq & |A_i(x) + B_i(x)| \cdot \max\{|y_{i-1} - f(x)|, |y_{i-1/2} - f(x)|\} \\ & + |C_i(x)| \cdot |y_i - f(x)| + |D_i(x) + E_i(x)| \cdot \max\{|m_{i-1}|, |m_i|\} \end{aligned} \tag{11}$$

and on $[x_{i-1/2}, x_i]$ we get

$$\begin{aligned} |S(x) - f(x)| \leq & |A_i(x)| \cdot |y_{i-1} - f(x)| \\ & + |B_i(x) + C_i(x)| \cdot \max\{|y_i - f(x)|, |y_{i-1/2} - f(x)|\} \\ & + |D_i(x) + E_i(x)| \cdot \max\{|m_{i-1}|, |m_i|\} \end{aligned} \tag{12}$$

with $|D_i(x) + E_i(x)| = t(1-t)|1-2t| \cdot h$, where $t = \frac{x - x_{i-1}}{h} \in [0,1]$. For estimating $\max\{|m_i| : i = \overline{0,n}\}$ we see that the tridiagonal system (1) becomes in the case of equally spaced knots:

$$-\tfrac{1}{8} \cdot m_{i-1} + m_i - \tfrac{1}{8} \cdot m_{i+1} = \tfrac{5(y_{i-1} - y_{i+1})}{8h} + \tfrac{2\left(y_{i+1/2} - y_{i-1/2}\right)}{h} = d_i$$

for $i = \overline{1, n-1}$, with $m_0 = m_n = 0$. Intending to estimate $|d_i|$, $i = \overline{1, n-1}$, we get

$$\begin{aligned} |d_i| \leq & \tfrac{5\left|y_{i-1} - y_{i-1/2}\right|}{8h} + \tfrac{11\left|y_i - y_{i-1/2}\right|}{8h} + \tfrac{11\left|y_{i+1/2} - y_i\right|}{8h} + \tfrac{5\left|y_{i+1/2} - y_{i+1}\right|}{8h} \\ \leq & \tfrac{4}{h} \cdot \omega\left(f, \tfrac{h}{2}\right), \quad \forall i = \overline{1, n-1} \end{aligned}$$

and since the matrix A of the tridiagonal system is diagonally dominant, we infer that $\|A^{-1}\| \leq \frac{4}{3}$ and then,

$$\|m\|_\infty = \max\{|m_i| : i = \overline{0,n}\} \leq \|A^{-1}\| \cdot \max\{|d_i| : i = \overline{1, n-1}\} \leq \tfrac{16\omega\left(f, \frac{h}{2}\right)}{3h}.$$

Consequently, for the last term in (11) and (12) we get

$$|D_i(x) + E_i(x)| \cdot \max\{|m_{i-1}|, |m_i|\} \leq t(1-t)|1-2t| \cdot \tfrac{16}{3} \cdot \omega\left(f, \tfrac{h}{2}\right)$$

for all $x \in [x_{i-1}, x_i]$, $i = \overline{1,n}$. Now, by (11) we obtain

$$\begin{aligned} & |S(x) - f(x)| \leq \\ & \leq \left((1-t)^2(1-2t)(4t+1) + 16t^2(1-t)^2 + \tfrac{16}{3}t(1-t)(1-2t)\right) \cdot \omega\left(f, \tfrac{h}{2}\right) \\ & \quad + \left|t^2(2t-1)(5-4t)\right| \omega(f,h) \\ & \leq \max_{t \in [0, \frac{1}{2}]} \left|8t^4 - \tfrac{10}{3}t^3 - 11t^2 + \tfrac{16}{3}t + 1\right| \cdot \omega\left(f, \tfrac{h}{2}\right) + \max_{t \in [0, \frac{1}{2}]} |C_i(x)| \cdot \omega(f,h) \\ & \leq \max_{t \in [0, \frac{1}{2}]} |P(t)| \cdot \omega\left(f, \tfrac{h}{2}\right) + \tfrac{1125}{8192} \omega(f,h) \end{aligned}$$

for $x \in [x_{i-1}, x_{i-1/2}]$. Analogously, by (12) it obtains

$$|s(x) - f(x)| \leq \max_{t\in[\frac{1}{2},1]} \left|8t^4 - \tfrac{86}{3}t^3 + 27t^2 - \tfrac{16}{3}t\right| \cdot \omega\left(f, \tfrac{h}{2}\right) + \max_{t\in[\frac{1}{2},1]} |A_i(x)| \cdot \omega(f,h)$$
$$\leq \max_{t\in[\frac{1}{2},1]} |P(1-t)| \cdot \omega\left(f, \tfrac{h}{2}\right) + \tfrac{1125}{8192}\omega(f,h)$$

when $x \in [x_{i-1/2}, x_i]$, and the estimate (9) follows. After elementary calculus, we see that

$$\max_{t\in[0,\frac{1}{2}]} |P(t)| = \max_{t\in[\frac{1}{2},1]} |P(1-t)| \simeq 1.627,$$

and when $f \in \mathrm{Lip}[a,b]$, the estimate (9) becomes

$$|S(x) - f(x)| \leq 1.627 \cdot \tfrac{Lh}{2} + \tfrac{1125}{8192} \cdot Lh \simeq 0.95084 \cdot Lh, \quad \forall x \in [x_{i-1}, x_i], i = \overline{1,n}$$

that is the estimate (10). □

Remark 5. By Theorem 2, the estimate (6) is $|s(x) - f(x)| \leq 1.6572 \cdot \omega(f,h)$, $\forall x \in [a,b]$ and when $f \in \mathrm{Lip}[a,b]$ it becomes $|s(x) - f(x)| \leq 1.6572 \cdot Lh$. So, the estimate (10) is better because gives a smaller constant in the case of Lischitzian functions. We can assert that the estimate (9) is better than (6) because in (9) it appears $\omega(f, \frac{h}{2})$ near $\max_{t\in[0,\frac{1}{2}]} |P(t)| \simeq 1.627$ and the factor $\omega(f, \frac{h}{2})$ considerable reduces the error in comparison with $\omega(f,h)$. Therefore the estimates obtained in Theorem 4 represent an improvement of Theorem 2, especially in the case of Lipschitzian functions. □

Remark 6. An interesting property of the deficient quartic spline $S \in C^2[a,b]$ given in (2) can be observed in the case of uniform partition when integrate this spline over the interval $[a,b]$. It obtains the corrected Simpson composite quadrature formula which is exact for polynomials of degree 5 or less. This corrected quadrature formula is usually obtained by applying the Richardson extrapolation and Grüss type inequalities (see [16]), or by using a finite difference technique (see [23]). □

Concerning the estimate from Theorem 3, in the case of nonuniform partition, we can state the following.

Corollary 7. *If the quartic spline S presented in (2) interpolates $f \in C[a,b]$ on a nonuniform partition $\{x_i\}_{i=0}^n$ with the endpoint condition $S'(a) = S'(b) = 0$ and $\beta = \max\{h_i : i = \overline{1,n}\} / \min\{h_i : i = \overline{1,n}\}$, $h = \max\{h_i : i = \overline{1,n}\}$, $\underline{h} = \min\{h_i : i = \overline{1,n}\}$, then*

$$|S(x) - f(x)| \leq \left(\tfrac{9317}{8192} + \tfrac{4\sqrt{3}(\beta^2+1)}{27}\right) \cdot \omega\left(f, \tfrac{h}{2}\right) + \tfrac{1125}{8192} \cdot \omega(f,h) \tag{13}$$

for all $x \in [a,b]$.

Proof. The matrix A of the tridiagonal, diagonally dominant system (1) has $\|A^{-1}\| \leq \frac{4}{3}$ and when estimate $|d_i|$, $i = \overline{1, n-1}$, we get

$$\begin{aligned}|d_i| \leq & \tfrac{h_{i+1}}{4h_i(h_i+h_{i+1})} \cdot (5|y_{i-1} - y_{i-1/2}| + 11|y_i - y_{i-1/2}|) + \\ & + \tfrac{h_i}{4h_{i+1}(h_i+h_{i+1})} \left(5|y_{i+1/2} - y_{i+1}| + 11|y_{i+1/2} - y_i|\right) \\ \leq & \tfrac{4}{(h_i+h_{i+1})} \cdot [\tfrac{h_{i+1}}{h_i} \cdot \omega\left(f, \tfrac{h_i}{2}\right) + \tfrac{h_i}{h_{i+1}} \omega\left(f, \tfrac{h_{i+1}}{2}\right)] \\ \leq & \tfrac{4\omega\left(f, \frac{h}{2}\right)}{(h_i+h_{i+1})} \left(\beta + \tfrac{1}{\beta}\right) \leq \tfrac{2\left(\beta^2+1\right)}{\beta \underline{h}} \omega\left(f, \tfrac{h}{2}\right)\end{aligned}$$

for all $i = \overline{1, n-1}$. Then, $\max\{|m_{i-1}|, |m_i|\} \leq \frac{8}{3\underline{h}} \left(\beta + \frac{1}{\beta}\right) \cdot \omega\left(f, \frac{h}{2}\right)$ and since

$$\max_{t\in[0,\frac{1}{2}]} |A_i(x) + B_i(x)| = \max_{t\in[\frac{1}{2},1]} |B_i(x) + C_i(x)| = \tfrac{9317}{8192}$$

and $\max_{t\in[0,\frac{1}{2}]} |C_i(x)| = \max_{t\in[\frac{1}{2},1]} |A_i(x)| = \frac{1125}{8192}$, the maximum values being attained in $t = \frac{5}{16}$ and $t = \frac{11}{16}$, respectively, by (11) and (12) we obtain,

$$\begin{aligned}& |S(x) - f(x)| \leq \\ & \leq \tfrac{9317}{8192} \cdot \omega\left(f, \tfrac{h}{2}\right) + \tfrac{1125}{8192} \cdot \omega(f, h) + \max_{t\in[0,1]} |D_i(x) + E_i(x)| \tfrac{8\left(\beta^2+1\right)}{3\beta\underline{h}} \omega\left(f, \tfrac{h}{2}\right) \\ & = \left(\tfrac{9317}{8192} + \tfrac{h\sqrt{3}}{18} \cdot \tfrac{8\left(\beta^2+1\right)}{3\beta\underline{h}}\right) \cdot \omega\left(f, \tfrac{h}{2}\right) + \tfrac{1125}{8192} \omega(f, h) \\ & = \left(\tfrac{9317}{8192} + \tfrac{4\sqrt{3}\left(\beta^2+1\right)}{27}\right) \omega\left(f, \tfrac{h}{2}\right) + \tfrac{1125}{8192} \omega(f, h)\end{aligned}$$

for all $x \in [a, b]$. □

If we compare the estimate (13) with the result from Theorem 4.1.3 in [9], since $\max_{t\in[0,\frac{1}{2}]} |t(1-2t)(1-t)| = \frac{\sqrt{3}}{18}$ and $\max_{t\in[0,\frac{1}{2}]} |16t^4 - 28t^3 + 10t^2 + 1| = \frac{9317}{8192} + \frac{1125}{8192} = 1.2747$, we see that the presence in (13) of the factor $\omega(f, \frac{h}{2})$ instead of $\omega(f, h)$ represents an improvement of the result from Theorem 4.1.3 in [9].

The endpoint condition $S'(a) = S'(b) = 0$ imposed in [10] in order to simplify the study of the error estimate in the case $f \in C[a, b]$ can be replaced by other classical ones. For instance, in that follows we investigate the modification of the error estimate when other two supplementary endpoint conditions are included. In the first case, mentioned in [10], we can consider $S'(a) = f'(a)$, $S'(b) = f'(b)$ and the linear system (1) has the central lines

$$\begin{aligned}-\tfrac{h_{i+1}\cdot m_{i-1} + h_i \cdot m_{i+1}}{4(h_i+h_{i+1})} + m_i = & \tfrac{h_{i+1}\cdot\left(5y_{i-1} - 16y_{i-1/2} + 11y_i\right)}{4h_i(h_i+h_{i+1})} - \tfrac{h_i\cdot\left(5y_{i+1} - 16y_{i+1/2} + 11y_i\right)}{4h_{i+1}(h_i+h_{i+1})} \\ = & d_i, \quad i = \overline{2, n-2}\end{aligned}$$

and the first and the last equations becomes

$$m_1 - \frac{h_1 \cdot m_2}{4(h_1+h_2)} = \frac{h_2^2\left(5y_0-16y_{1/2}+11y_1-f'(a)\right)}{4h_1h_2(h_1+h_2)} - \frac{h_1^2\left(5y_2-16y_{2-1/2}+11y_1\right)}{4h_1h_2(h_1+h_2)}$$

and

$$-\frac{h_n \cdot m_{n-2}}{4(h_{n-1}+h_n)} + m_{n-1} =$$

$$= \frac{h_n \cdot \left(5y_{n-2}-16y_{n-1-1/2}+11y_{n-1}\right)-h_n \cdot f'(b)}{4h_{n-1}(h_{n-1}+h_n)} - \frac{h_{n-1} \cdot \left(5y_n-16y_{n-1/2}+11y_{n-1}\right)}{4h_{i+1}(h_i+h_{i+1})} = d_{n-1}$$

obtaining the same inequality $|d_i| \le \frac{4h}{\underline{h}^2} \cdot \omega\left(f, \frac{h}{2}\right)$, $i = \overline{2, n-2}$ and

$$|d_1| \le \frac{4h}{\underline{h}^2} \cdot \omega\left(f, \frac{h}{2}\right) + \frac{h}{8\underline{h}^2}\left|f'(a)\right|, \quad |d_{n-1}| \le \frac{4h}{\underline{h}^2} \cdot \omega\left(f, \frac{h}{2}\right) + \frac{h}{8\underline{h}^2}\left|f'(b)\right|.$$

Then, $\|m\|_\infty = \max\{|m_i| : i = \overline{0,n}\} \le \frac{4}{3} \cdot \left(\frac{4h}{\underline{h}^2} \cdot \omega\left(f, \frac{h}{2}\right) + \frac{h \cdot m'}{8\underline{h}^2}\right)$ and the error estimate becomes

$$|S(x) - f(x)| \le \left(\frac{9317}{8192} + \left(\frac{4\sqrt{3}}{27} + \frac{\sqrt{3} \cdot m'}{108}\right)\left(1+\beta^2\right)\right) \cdot \omega\left(f, \frac{h}{2}\right) + \frac{1125}{8192} \cdot \omega(f, h)$$

for all $x \in [a, b]$, where $m' = \max\{|f'(a)|, |f'(b)|\}$.

REMARK 8. If $f \in C^1[a, b]$, then in the case of equally spaced knots this error estimate is,

$$(14) \quad |S(x) - f(x)| \le \left(\frac{11\,567}{16\,384} + \frac{11\sqrt{3}}{72}\right) \cdot M'h \simeq 0.97061 \cdot M'h, \quad \forall x \in [a, b]$$

where $M' = \max\{|f'(x)| : x \in [a, b]\}$. □

Taking the natural type endpoint condition $S''(a) = S''(b) = 0$ the linear system (1) receives two supplementary equations

$$(15) \quad \begin{cases} m_0 - \frac{1}{4}m_1 = \frac{-11y_0+16y_{1-1/2}-5y_1}{4h_1} = d_0 \\ -\frac{1}{4}m_{n-1} + m_n = \frac{5y_{n-1}-16y_{n-1/2}+11y_n}{4h_n} = d_n \end{cases}$$

obtaining the estimates $|d_0| \le \frac{4}{\underline{h}} \cdot \omega\left(f, \frac{h}{2}\right)$, $|d_n| \le \frac{4}{\underline{h}} \cdot \omega\left(f, \frac{h}{2}\right)$ and the interpolation error estimate is the same as in (13).

We can consider now, the second type of complete endpoint conditions $S''(a) = f''(a)$, $S''(b) = f''(b)$ when the values $f''(a)$ and $f''(b)$ are given, the supplementary equations becoming

$$\begin{cases} m_0 - \frac{1}{4}m_1 = \frac{-11y_0+16y_{1-1/2}-5y_1}{4h_1} + \frac{h_1}{8}f''(a) = d_0 \\ -\frac{1}{4}m_{n-1} + m_n = \frac{5y_{n-1}-16y_{n-1/2}+11y_n}{4h_n} + \frac{h_n}{8}f''(b) = d_n \end{cases}$$

with $|d_0| \le \frac{4}{\underline{h}} \cdot \omega\left(f, \frac{h}{2}\right) + \frac{h}{8}\left|f''(a)\right|$, and $|d_n| \le \frac{4}{\underline{h}} \cdot \omega\left(f, \frac{h}{2}\right) + \frac{h}{8}\left|f''(b)\right|$. In this case, the error estimate is

$$|S(x) - f(x)| \le \left(\frac{9317}{8192} + \frac{4\sqrt{3}\left(1+\beta^2\right)}{27}\right) \cdot \omega\left(f, \frac{h}{2}\right) + \frac{1125}{8192} \cdot \omega(f, h) + \frac{m''\sqrt{3}h^2}{108}$$

where $m'' = \max\{|f''(a)|, |f''(b)|\}$.

Remark 9. When the values $f''(a)$ and $f''(b)$ are not available, in order to preserve $\mathcal{O}(h^5)$ accuracy, we can consider the endpoint conditions $S''(x_0) = p_0''(x_0)$, $S''(x_n) = p_n''(x_n)$, where p_0 is the quartic Lagrange polynomial interpolating the points (x_0, y_0), $(x_{1-1/2}, y_{1-1/2})$, (x_1, y_1), $(x_{2-1/2}, y_{2-1/2})$, (x_2, y_2) and p_n is the quartic Lagrange polynomial interpolating the points (x_{n-2}, y_{n-2}), $(x_{n-1-1/2}, y_{n-1-1/2})$, (x_{n-1}, y_{n-1}), $(x_{n-1/2}, y_{n-1/2})$, $(x_n . y_n)$. Based on [11], we have $f''(x_0) = p_0''(x_0) + \frac{u_0''(x_0)}{5!} \cdot f^{(5)}(\xi_1)$ and $f''(x_n) = p_n''(x_n) + \frac{u_n''(x_n)}{5!} \cdot f^{(5)}(\xi_n)$, where $\xi_1 \in (x_0, x_2)$, $\xi_n \in (x_{n-2}, x_n)$, and

$$u_0(x) = (x - x_0)(x - x_{1-1/2})(x - x_1)(x - x_{2-1/2})(x - x_2)$$
$$u_n(x) = (x - x_{n-2})(x - x_{n-1-1/2})(x - x_{n-1})(x - x_{n-1/2})(x - x_n).$$

Similar treatment at endpoints can be realized when the values $f'(a)$ and $f'(b)$ are not available, by considering the conditions $S'(x_0) = p_0'(x_0)$ and $S'(x_n) = p_n'(x_n)$. □

3. OPTIMAL PROPERTIES FOR DEFICIENT QUARTIC SPLINES

3.1. Minimal mean curvature. In this section we consider the deficient quartic spline (2) $S \in C^1[a,b]$ and in this case the spline local derivatives m_i, $i = \overline{0,n}$ remain free. We will determine m_i, $i = \overline{0,n}$, in order to minimize the functionals

$$J_k(S) = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \left(S^{(k)}(x)\right)^2 dx, \quad k \in \{1,2,3\}.$$

According to [2], $\sqrt{J_1(S)}$ is an average of the slope of the graph of S and $\sqrt{J_2(S)}$ represents the mean curvature of the graph of S, while $\sqrt{J_3(S)}$ is related to the mean curvature of the graph of S'. Since $S^{(4)}$ is piecewise constant discontinuous function, the minimization of $J_4(S)$ is without of interest.

By (2) we see that in each interval $[x_{i-1}, x_i]$, $i = \overline{1,n}$ we have

$$S''(x) = A_i''(x)\, y_{i-1} + B_i''(x)\, y_{i-1/2} + C_i''(x)\, y_i + D_i''(x)\, m_{i-1} + E_i''(x)\, m_i$$

and therefore

$$J_2(S)(m_0, m_1, \ldots, m_n) =$$
$$= \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} (S''(x))^2 dx$$
$$= \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} [A_i''(x)\, y_{i-1} + B_i''(x)\, y_{i-1/2} + C_i''(x)\, y_i + D_i''(x)\, m_{i-1} + E_i''(x)\, m_i]^2 dx.$$

In order to minimize $J_2(S)$ the system of normal equations $\frac{\partial J_2}{\partial m_i} = 0$, $i = \overline{0,n}$, is

$$m_0 \int_{x_0}^{x_1} (D_1''(x))^2 dx + m_1 \int_{x_0}^{x_1} D_1''(x) E_1''(x) dx =$$
$$= -\int_{x_0}^{x_1} A_1''(x) D_1''(x) dx \cdot y_0 - \int_{x_0}^{x_1} B_1''(x) D_1''(x) dx \cdot y_{1-1/2} - \int_{x_0}^{x_1} C_1''(x) D_1''(x) dx \cdot y_1,$$

$$m_{i-1} \int_{x_{i-1}}^{x_i} E_i''(x) D_i''(x) dx + m_i \left(\int_{x_{i-1}}^{x_i} (E_i''(x))^2 dx + \int_{x_i}^{x_{i+1}} (D_{i+1}''(x))^2 dx \right) +$$
$$+ m_{i+1} \int_{x_i}^{x_{i+1}} D_{i+1}''(x) E_{i+1}''(x) dx =$$
$$= - \int_{x_{i-1}}^{x_i} E_i''(x) [A_i''(x) y_{i-1} + B_i''(x) y_{i-1/2} + C_i''(x) \cdot y_i] dx$$
$$- \int_{x_i}^{x_{i+1}} D_{i+1}''(x) [A_{i+1}''(x) y_i + B_{i+1}''(x) y_{i+1/2} + C_{i+1}''(x) y_{i+1}] dx, \quad i = \overline{1, n-1}$$

$$m_{n-1} \int_{x_{n-1}}^{x_n} D_n''(x) E_n''(x) dx + m_n \int_{x_{n-1}}^{x_n} (E_n''(x))^2 dx =$$
$$= - \int_{x_{n-1}}^{x_n} E_n''(x) [A_n''(x) \cdot y_{n-1} + B_n''(x) \cdot y_{n-1/2} + C_n''(x) \cdot y_n] dx$$

and after elementary calculus becomes

$$(16) \quad \begin{cases} m_0 - \frac{1}{6} m_1 = \frac{-47y_0 + 64y_{1-1/2} - 17y_1}{18h_1} = d_0'' \\ -\frac{h_{i+1} \cdot m_{i-1}}{6(h_i + h_{i+1})} + m_i - \frac{h_i \cdot m_{i+1}}{6(h_i + h_{i+1})} = \\ = \frac{h_{i+1} \cdot \left(17y_{i-1} - 64y_{i-1/2}\right)}{18h_i(h_i + h_{i+1})} + \frac{47y_i}{18} \cdot \left(\frac{1}{h_i} - \frac{1}{h_{i+1}}\right) + \frac{h_i \cdot \left(64y_{i+1/2} - 17y_{i+1}\right)}{18h_{i+1}(h_i + h_{i+1})} \\ = d_i'', \quad i = \overline{1, n-1} \\ -\frac{1}{6} m_{n-1} + m_n = \frac{17y_{n-1} - 64y_{n-1/2} + 47y_n}{18h_n} = d_n''. \end{cases}$$

We see that the matrix A'' of this system is strictly diagonally dominant with the index of diagonally dominance $1/6$, better than for the system (1), and $\left\| (A'')^{-1} \right\| \leq \frac{6}{5}$(see [19]). Moreover, since $a_{ii}'' > 0$ and $a_{ij}'' < 0$ for $i \neq j$, $i, j = \overline{0, n}$, we infer that all elements of the matrix $(A'')^{-1}$ are nonnegative. The strictly diagonally dominance ensures the existence and uniqueness of the solution of system (16) and the numerical stability of the LU factorization method for solving this system. In this way we obtain the following result.

THEOREM 10. *There is a unique solution* $(m_0, m_1, \ldots, m_n)$ *which minimize the functional* $J_2(S)$ *and the mean curvature of the graph of* S*, too. The interpolation error estimate of the obtained quartic spline* $S \in C^1[a,b]$ *interpolating a function* $f \in C[a,b]$ *is,*

$$|S(x) - f(x)| \leq \left(\frac{9317}{8192} + \frac{32\sqrt{3}\beta^2}{135}\right) \cdot \omega\left(f, \frac{h}{2}\right) + \frac{1125}{8192} \cdot \omega(f,h) \tag{17}$$

for all $x \in [a,b]$.

Proof. As above, the system of normal equations associated to the functional $J_2(S)$ has unique solution due to the strictly diagonally dominance of its matrix and this solution can be obtained by applying the iterative algorithm presented in [1, pp. 14–15], for tridiagonal linear systems. Moreover, by the strictly diagonally dominance of the matrix A'' we infer that all the diagonal minors of the Hessian matrix $\left(\frac{\partial^2 J_2}{\partial m_i \partial m_j}\right)_{i,j=\overline{0,n}}$ are strictly positive and thus $(m_0, m_1, \ldots, m_n)$ is a real minimum point of $J_2(S)$. For obtaining the estimate (17) we observe by (16) that

$$|d_0''| \leq \frac{47|y_{1-1/2}-y_0|+17|y_{1-1/2}-y_1|}{18h_1} \leq \frac{32}{9\underline{h}} \cdot \omega\left(f, \frac{h}{2}\right),$$

$$|d_n''| \leq \frac{17|y_{n-1}-y_{n-1/2}|+47|y_n-y_{n-1/2}|}{18h_n} \leq \frac{32}{9\underline{h}} \cdot \omega\left(f, \frac{h}{2}\right)$$

and

$$\begin{aligned} |d_i''| \leq {} & \frac{h_{i+1}(17|y_{i-1}-y_{i-1/2}|+47|y_i-y_{i-1/2}|)}{18h_i(h_i+h_{i+1})} + \frac{47h_i|y_{i+1/2}-y_i|}{18h_{i+1}(h_i+h_{i+1})} + \\ & + \frac{17h_i|y_{i+1/2}-y_{i+1}|}{18h_{i+1}(h_i+h_{i+1})} \leq \frac{32h_{i+1}\cdot\omega\left(f,\frac{h}{2}\right)}{9h_i(h_i+h_{i+1})} + \frac{32h_i\cdot\omega\left(f,\frac{h}{2}\right)}{9h_{i+1}(h_i+h_{i+1})} \leq \frac{32h}{9\underline{h}^2} \cdot \omega\left(f, \frac{h}{2}\right) \end{aligned}$$

for all $i = \overline{1, n-1}$, and consequently, $\|m\|_\infty = \max\{|m_i| : i = \overline{0,n}\} \leq \frac{64h}{15\underline{h}^2} \cdot \omega\left(f, \frac{h}{2}\right)$. Similarly as in the proof of Corollary 7 we obtain

$$\begin{aligned} |S(x) - f(x)| & \leq \frac{9317}{8192} \cdot \omega\left(f, \frac{h}{2}\right) + \frac{1125}{8192} \cdot \omega(f,h) + \frac{h\sqrt{3}}{18} \cdot \frac{64h}{15\underline{h}^2}\omega\left(f, \frac{h}{2}\right) \\ & \leq \left(\frac{9317}{8192} + \frac{32\sqrt{3}\beta^2}{135}\right) \cdot \omega\left(f, \frac{h}{2}\right) + \frac{1125}{8192} \cdot \omega(f,h), \quad \forall x \in [a,b]. \end{aligned}$$

Since the solution of (16) minimize $J_2(S)$ we infer that it minimize the mean curvature $\sqrt{J_2(S)}$ of the graph of S, too. □

REMARK 11. The solution of (16) provides a less smooth quartic spline $S \in C^1[a,b]$ with increased deficiency, and for the Marsden's type quartic spline $S \in C^2[a,b]$ given in (2) we cannot obtain neither a natural kind spline, nor the continuity property $S \in C^3[a,b]$. Altough, natural quartic spline can be obtained with high degree of continuity $S \in C^3[a,b]$, and a way that leads to natural quartic spline were obtained in [3], for the class of interpolating-derivative splines with minimal $J_3(S)$. For quartic splines $S \in C^3[a,b]$ with different type interpolation points t_i, $i = \overline{1,n}$ with $x_{i-1} < t_i < x_i$, the property of minimal curvature (minimal J_2) was obtained in [14] in connection with the endpoint type conditions $S''(a) = S''(b) = S'''(a) = S'''(b) = 0$ (natural quartic spline) and $S^{(j)}(a) = f^{(j)}(a)$, $S^{(j)}(b) = f^{(j)}(b)$, $j \in \{0,1\}$ (complete quartic spline). □

Concerning the minimal mean curvature $\sqrt{J_3(S)}$ of the graph of S', since

$$J_3(S)(m_0, m_1, \ldots, m_n) = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} [S'''(x)]^2 dx$$

the minimization of $\sqrt{J_3(S)}$ can be obtained by solving the system of normal equations $\frac{\partial J_3}{\partial m_i}$, $i = \overline{0,n}$. After elementary calculus, this system becomes

(18)
$$\begin{cases} m_0 - \frac{13}{19} m_1 = \frac{-70y_0 + 157y_{1-1/2} - 87y_1}{19h_1} = d_0''' \\ -\frac{13h_{i+1}^3}{19\left(h_i^3 + h_{i+1}^3\right)} \cdot m_{i-1} + m_i - \frac{13h_i^3}{19\left(h_i^3 + h_{i+1}^3\right)} \cdot m_{i+1} = d_i''' = \\ = \frac{h_{i+1}^3 \cdot \left(87y_{i-1} - 157y_{i-1/2} + 70y_i\right)}{19h_i\left(h_i^3 + h_{i+1}^3\right)} + \frac{h_i^3 \cdot \left(-70y_i + 157y_{i+1/2} - 87y_{i+1}\right)}{19h_{i+1}\left(h_i^3 + h_{i+1}^3\right)}, \quad i = \overline{1, n-1} \\ -\frac{13}{19} m_{n-1} + m_n = \frac{87y_{n-1} - 157y_{n-1/2} + 70y_n}{19h_n} = d_n''' \end{cases}$$

being diagonally dominant and thus has unique solution that minimizes $J_3(S)$. Since

$$\begin{aligned} \left|d_0'''\right| &\leq \frac{70\left|y_0 - y_{1-1/2}\right| + 87\left|y_{1-1/2} - y_1\right|}{19h_1} \leq \frac{157}{19\underline{h}} \cdot \omega\left(f, \frac{h}{2}\right), \\ \left|d_n'''\right| &\leq \frac{157}{19\underline{h}} \cdot \omega\left(f, \frac{h}{2}\right) \\ \left|d_i'''\right| &\leq \frac{157h_{i+1}^3}{19h_i\left(h_i^3 + h_{i+1}^3\right)} \omega\left(f, \frac{h_i}{2}\right) + \frac{157h_i^3}{19h_{i+1}\left(h_i^3 + h_{i+1}^3\right)} \omega\left(f, \frac{h_{i+1}}{2}\right) \leq \frac{157h^3}{19\underline{h}^4} \cdot \omega\left(f, \frac{h}{2}\right), \end{aligned}$$

$\forall i = \overline{1, n-1}$ and since the matrix A''' of the linear system (18) has the inverse with $\| (A''')^{-1} \| \leq \frac{19}{6}$, for the solution $m = (m_0, m_1, \ldots, m_n)$ of (18) we obtain the estimate

$$\|m\|_\infty = \max\{|m_i| : i = \overline{0,n}\} \leq \frac{157h^3}{6\underline{h}^4} \cdot \omega\left(f, \frac{h}{2}\right).$$

In this way it obtains the following result.

Corollary 12. *The functional $J_3(S)$ has unique minimum point which minimize the mean curvature of the graph of the derivative S'. The interpolation error estimate of the obtained deficient quartic spline with minimal curvature of the graph of S' is,*

(19) $$|S(x) - f(x)| \leq \left(\frac{9317}{8192} + \frac{157\sqrt{3}\beta^4}{108}\right) \cdot \omega\left(f, \frac{h}{2}\right) + \frac{1125}{8192} \cdot \omega(f, h), \quad \forall x \in [a,b].$$

3.2. Minimal average slope of the graph. In order to establish the parameters m_i, $i=\overline{0,n}$, of the quartic spline with minimal average slope of the graph $\sqrt{J_1(S)}$, we minimize the functional $J_1(S)$ which has the expression

$$J_1(S)(m_0,m_1,\dots,m_n)=\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}[S'(x)]^2dx$$

The corresponding system of normal equations $\frac{\partial J_1}{\partial m_i}$, $i=\overline{0,n}$, will be after some computation,

$$(20)\qquad \begin{cases} m_0+\frac{5}{16}m_1=\frac{-29y_0+16y_{1-1/2}+13y_1}{16h_1}=d'_0\\ \frac{5h_i}{16(h_i+h_{i+1})}\cdot m_{i-1}+m_i+\frac{5h_{i+1}}{16(h_i+h_{i+1})}\cdot m_{i+1}=\\ =\frac{-13y_{i-1}-16y_{i-1/2}+16y_{i+1/2}+13y_{i+1}}{16(h_i+h_{i+1})}=d'_i,\quad i=\overline{1,n-1}\\ \frac{5}{16}m_{n-1}+m_n=\frac{-13y_{n-1}-16y_{n-1/2}+29y_n}{16h_n}=d'_n \end{cases}$$

and since the matrix of this system is diagonally dominant, the system (20) has unique solution. In this way we obtain the following result.

THEOREM 13. *The functional $J_1(S)$ has unique minimum point $(m_0,m_1,\dots,m_n)$ and for the corresponding quartic spline $S\in C^1[a,b]$ interpolating a function $f\in C[a,b]$, we have the following error estimate:*

$$(21)\quad |S(x)-f(x)|\le\left(\frac{9317}{8192}+\frac{7\sqrt{3}\beta}{33}\right)\cdot\omega\left(f,\frac{h}{2}\right)+\frac{1125}{8192}\cdot\omega(f,h),\qquad\forall x\in[a,b].$$

Proof. Based on the strictly diagonal dominance of the matrix A' of the system (20) we infer that $\|(A')^{-1}\|_\infty\le\frac{16}{11}$, this system has unique solution and all the diagonal minors of the Hessian matrix $\left(\frac{\partial^2J_1}{\partial m_i\partial m_j}\right)_{i,j=\overline{0,n}}$ are strictly positive. Then, this solution is the unique extremal point of the functional $J_1(S)$ and it is a minimum point. In order to obtain the error estimate, firstly we see that

$$\begin{aligned} |d'_0|&\le\frac{29|y_{1-1/2}-y_0|+13|y_1-y_{1-1/2}|}{16h_1}\le\frac{21}{8\underline{h}}\cdot\omega\left(f,\frac{h}{2}\right),\\ |d'_n|&\le\frac{21}{8\underline{h}}\cdot\omega\left(f,\frac{h}{2}\right),\\ d'_i&=\frac{-13y_{i-1}-16y_{i-1/2}}{16(h_i+h_{i+1})}+\frac{29y_i}{16(h_i+h_{i+1})}+\frac{16y_{i+1/2}+13y_{i+1}}{16(h_i+h_{i+1})}-\frac{29y_i}{16(h_i+h_{i+1})}\\ &=\frac{13(y_{i-1/2}-y_{i-1})}{16(h_i+h_{i+1})}+\frac{29(y_i-y_{i-1/2})}{16(h_i+h_{i+1})}+\frac{29(y_{i+1/2}-y_i)}{16(h_i+h_{i+1})}+\frac{13(y_{i+1}-y_{i+1/2})}{16(h_i+h_{i+1})} \end{aligned}$$

and consequently,

$$|d'_i|\le\frac{42\cdot\omega(f,\frac{h_i}{2})}{16(h_i+h_{i+1})}+\frac{42\cdot\omega(f,\frac{h_{i+1}}{2})}{16(h_i+h_{i+1})}\le\frac{21}{8\underline{h}}\cdot\omega(f,\frac{h}{2}),\quad i=\overline{1,n-1}.$$

It follows that $\|m\|_\infty=\max\{|m_i|:i=\overline{0,n}\}\le\frac{42}{11\underline{h}}\cdot\omega(f,\frac{h}{2})$ obtaining,

$$|S(x)-f(x)|\le\frac{9317}{8192}\cdot\omega\left(f,\frac{h}{2}\right)+\frac{1125}{8192}\cdot\omega(f,h)+\frac{h\sqrt{3}}{18}\cdot\frac{42}{11\underline{h}}\omega\left(f,\frac{h}{2}\right)\le$$

$$\leq \left(\frac{9317}{8192} + \frac{7\sqrt{3}\beta}{33}\right) \cdot \omega\left(f, \frac{h}{2}\right) + \frac{1125}{8192} \cdot \omega\left(f, h\right), \quad \forall x \in [a, b].$$

□

Remark 14. Considering the index of the diagonally dominant property introduced in [8] for a matrix $A = (a_{ij})_{i,j=\overline{0,n}}$, be the constant

$$D\left(A\right) = \max_{i=\overline{0,n}} \left(\frac{1}{|a_{ii}|} \cdot \sum_{j=0, j\neq i}^{n} |a_{ij}| \right)$$

and denoting by D, D'', D''', D', the index of the diagonally dominant property of the matrices of the linear systems (1), (16), (18), (20), respectively, we see that $D = \frac{1}{4}$, $D'' = \frac{1}{6}$, $D''' = \frac{13}{19}$, $D' = \frac{5}{16}$, and $D'' < D < D' < D'''$. So, the matrix of the system of normal equations associated to the minimal curvature of the graph has stronger diagonally dominant property than the others. Investigating the error estimates obtained in (13), (17), (19), and (21), and considering even the case of equally spaced knots, when $\beta = 1$, we see that

$$\frac{7\sqrt{3}}{33} \simeq 0.3674 < \frac{32\sqrt{3}}{135} \simeq 0.41056 < \frac{8\sqrt{3}}{27} \simeq 0.5132 < \frac{157\sqrt{3}}{108} \simeq 2.5179.$$

Consequently, the quartic spline with minimal average slope of the graph has the best error estimate both for uniform and nonuniform partitions. □

4. NUMERICAL EXPERIMENT

In order to illustrate the theoretical results consider $n = 5$ and the following data presented in Section 4.

i	0	1	2	3	4	5
x_i	0	2	4	6	8	10
y_i	16	20	28	21	24	28
$y_{i-1/2}$		12	23	32	18	30

Table 1. The input data.

According to (1), (15) and (16), let $SC \in C^2[0, 10]$ be the deficient quartic spline with the endpoint conditions as in (15) and $SMC \in C^1[0, 10]$ be the deficient quartic spline with minimal mean curvature of the graph $\sqrt{J_2}$ obtained according to Theorem 10. With SD we denote the deficient quartic spline with minimal average slope of the graph $\sqrt{J_1}$ obtained in Theorem 13 after solving the linear system (18), and with SDC we denote the quartic spline with minimal mean curvature of the graph of the first derivative $\sqrt{J_3}$, according to (20). By solving the linear systems (1)+(15), (16), (18) and (20) we obtain the corresponding local derivatives m_i, $i = \overline{0, 5}$, for each of the above mentioned quartic splines, and the results are presented in Section 4.

These quartic splines are represented in Fig. 4.1 and Fig. 4.2 illustrating their interpolation properties. In Fig. 4.1 we represent with solid line the

splines	m_0	m_1	m_2	m_3	m_4	m_5
SC	−8.7018	7.1929	8.2452	−10.731	7.9057	−4.5236
SMC	−7.8476	6.9145	7.488	−10.225	7.8167	−4.1417
SD	−1.9689	5.1006	2.5249	−5.5601	5.4596	−1.0811
SDC	−21.343	6.3453	13.621	−17.530	6.2924	−13.116

Table 2. The local derivatives m_i, $i = \overline{0,5}$.

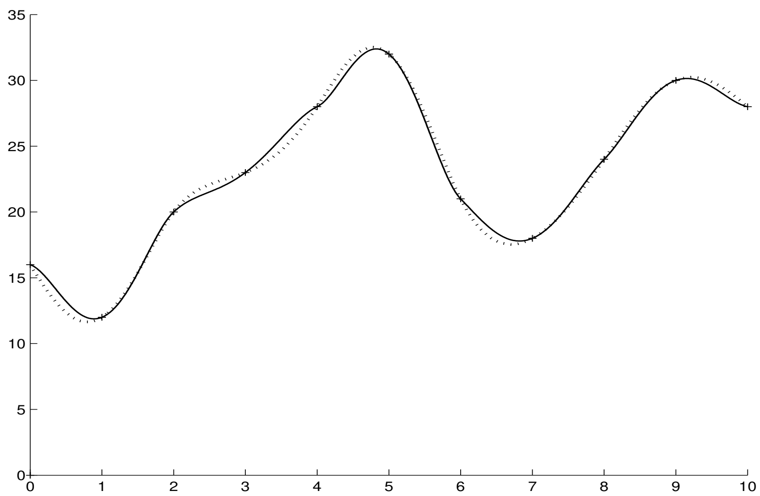

Fig. 4.1. The graphs of the C^2-smooth quartic spline SC (....), and of SD (—) with minimal $\sqrt{J_1(S)}$.

quartic spline SD having minimal average slope of the graph and in dots is plotted the quartic spline $SC \in C^2[0,10]$ obtained by solving the system (1)+(15). The quartic spline SMC with minimal mean curvature of the graph is represented under solid line in Fig. 4.2, while the quartic spline SDC with minimal mean curvature of the graph of the first derivative is plotted in dots. Investigating Figs. 4.1 and 4.2 we see that smaller oscillation can be observed at the quartic spline SD with minimal average slope of the graph and at the quartic spline SMC with minimal mean curvature of the graph, respectively. In order to illustrate this geometric property observed in Figs. 4.1 and 4.2 we compute the length $L(S)$ of the graph for this four quartic splines and the results are sumarized in Section 4. As was expected, the deficient quartic spline SD with minimal average slope of the graph has the smallest graph length.

The graphs and figures were obtained by using the Matlab application.

splines:	SC	SD	SMC	SDC
$L(S)$:	68.676	63.15	68.237	72.735

Table 3. The length of graph.

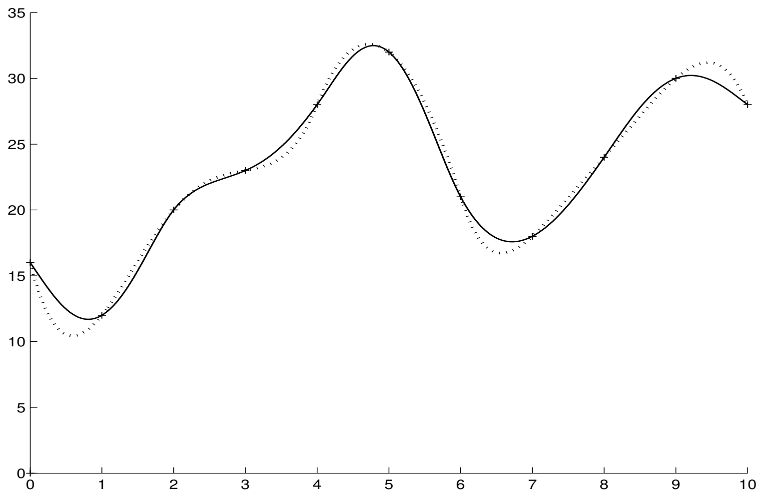

Fig. 4.2. Graphs of SMC (—) with minimal $\sqrt{J_2(S)}$ and SDC (....) with minimal $\sqrt{J_3(S)}$.

5. CONCLUSIONS

Observing the possibility to express the error estimate of the deficient quartic spline interpolant $S \in C^2[a,b]$ in terms of both $\omega\left(f, \frac{h}{2}\right)$ and $\omega(f,h)$, in the case of interpolated functions $f \in C[a,b]$, in this work we improve the error estimates from [10] and [9], the results being obtained in Theorem 4 and Corollary 7. This fact is revealed in Remark 5 by observing a smaller constant for equally spaced knots in the case of Lipschitzian interpolated functions. The possibility to determine the local derivatives m_i, $i = \overline{0,n}$, for obtaining certain optimal properties of the deficient quartic spline $S \in C^1[a,b]$, is investigated. In this context, the deficient quartic spline with minimal mean curvature of the graph of S and S', respectively, are obtained in Theorem 10 and Corollary 12. Related to the error estimate in terms of the modulus of continuity, a better bound is observed at the deficient quartic spline with minimal average slope of the graph, which is obtained in Theorem 13.

The numerical example shows the quality of the interpolation properties of the above presented four quartic splines. The technique of minimizing the mean curvature and the average slope of the graph, presented in this work, can be extended to parametric quartic spline curves, too, and this is the subject of a future work, investigating the use of both the chordal and centripetal parametrization.

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