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TRIGONOMETRIC APPROXIMATION ON THE HEXAGON

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Dedicated to the memory of Dr. Figen Kiraz

Abstract. The degree of trigonometric approximation of continuous functions, which are periodic with respect to the hexagon lattice, is estimated in uniform and Hölder norms. Approximating trigonometric polynomials are matrix means of hexagonal Fourier series.

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1. INTRODUCTION

Approximation problems of functions of several variables defined on cubes of the Euclidean space are usually studied by assuming that the functions are periodic in each of their variables (see, for example [20, §§ 5.3, 6.3], [23, vol. II, ch. XVII], [22, part 2], [15], [16] and [17]). But, we need other definitions of periodicity to study approximation problems on non-tensor product domains, for example on hexagonal domains of $\mathbb{R}^2$. The periodicity defined by lattices is the most useful one.

In the Euclidean plane $\mathbb{R}^2$, besides the standard lattice $\mathbb{Z}^2$ and the rectangular domain $[-\frac{1}{2},\frac{1}{2})^2$, the simplest lattice is the hexagon lattice and the simplest spectral set is the regular hexagon. The hexagon lattice has importance, since it offers the densest packing of the plane with unit circles. In this section we give basic information about hexagonal lattice and hexagonal Fourier series. More detailed information can be found in [11] and [21].

The generator matrix and the spectral set of the hexagonal lattice $H\mathbb{Z}^2$ are given by

$$H = \begin{pmatrix} \sqrt{3} & 0 \\ -1 & 2 \end{pmatrix}$$

and

$$\Omega_H = \left\{(x_1,x_2) \in \mathbb{R}^2 : -1 \le x_2, \tfrac{\sqrt{3}}{2}x_1 \pm \tfrac{1}{2}x_2 < 1\right\}.$$

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It is more convenient to use the homogeneous coordinates (t_1, t_2, t_3) that satisfies $t_1+t_2+t_3 = 0$. As it is pointed out in [21], using homogeneous coordinates reveals symmetry in various formulas. If we set

$$t_1 := -\tfrac{x_2}{2} + \tfrac{\sqrt{3}x_1}{2},\ t_2 := x_2,\ t_3 := -\tfrac{x_2}{2} - \tfrac{\sqrt{3}x_1}{2},$$

the hexagon Ω_H becomes

$$\Omega = \left\{(t_1, t_2, t_3) \in \mathbb{R}^3 : -1 \leq t_1, t_2, -t_3 < 1,\ t_1 + t_2 + t_3 = 0\right\},$$

which is the intersection of the plane $t_1 + t_2 + t_3 = 0$ with the cube $[-1, 1]^3$.

We use bold letters $\mathbf{t}$ for homogeneous coordinates and we set

$$\mathbb{R}^3_H = \left\{\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}^3 : t_1 + t_2 + t_3 = 0\right\},\ \mathbb{Z}^3_H := \mathbb{Z}^3 \cap \mathbb{R}^3_H.$$

A function $f : \mathbb{R}^2 \to \mathbb{C}$ is called H-periodic if

$$f(x + Hk) = f(x)$$

for all $k \in \mathbb{Z}^2$ and $x \in \mathbb{R}^2$. If we define $\mathbf{t} \equiv \mathbf{s} \pmod 3$ as

$$t_1 - s_1 \equiv t_2 - s_2 \equiv t_3 - s_3 \pmod 3$$

for $\mathbf{t} = (t_1, t_2, t_3)$, $\mathbf{s} = (s_1, s_2, s_3) \in \mathbb{R}^3_H$, it follows that the function f is H-periodic if and only if $f(\mathbf{t}) = f(\mathbf{t} + \mathbf{s})$ whenever $\mathbf{s} \equiv \mathbf{0} \pmod 3$, and

$$\int_{\Omega} f(\mathbf{t} + \mathbf{s})\, d\mathbf{t} = \int_{\Omega} f(\mathbf{t})\, d\mathbf{t} \quad \left(\mathbf{s} \in \mathbb{R}^3_H\right)$$

for H-periodic integrable function f [21].

$L^2(\Omega)$ becomes a Hilbert space with respect to the inner product

$$\langle f, g\rangle_H := \tfrac{1}{|\Omega|}\int_{\Omega} f(\mathbf{t})\, \overline{g(\mathbf{t})} d\mathbf{t},$$

where $|\Omega|$ denotes the area of Ω. The functions

$$\phi_{\mathbf{j}}(\mathbf{t}) := e^{\frac{2\pi i}{3}\langle \mathbf{j}, \mathbf{t}\rangle} \qquad (\mathbf{t} \in \mathbb{R}^3_H),$$

where $\langle \mathbf{j}, \mathbf{t}\rangle$ is the usual Euclidean inner product of $\mathbf{j}$ and $\mathbf{t}$, are H-periodic, and by a theorem of B. Fuglede the set

$$\left\{\phi_{\mathbf{j}} : \mathbf{j} \in \mathbb{Z}^3_H\right\}$$

becomes an orthonormal basis of $L^2(\Omega)$ [3] (see also [11]).

For every natural number n, we define a subset of $\mathbb{Z}^3_H$ by

$$\mathbb{H}_n := \left\{\mathbf{j} = (j_1, j_2, j_3) \in \mathbb{Z}^3_H : -n \leq j_1, j_2, j_3 \leq n\right\}.$$

The subspace

$$\mathcal{H}_n := \operatorname{span}\{\phi_{\mathbf{j}} : \mathbf{j} \in \mathbb{H}_n\} \qquad (n \in \mathbb{N})$$

has dimension $\#\mathbb{H}_n = 3n^2 + 3n + 1$, and its members are called hexagonal trigonometric polynomials of degree n.

The hexagonal Fourier series of an H-periodic function $f \in L^1(\Omega)$ is

$$f(\mathbf{t}) \sim \sum_{\mathbf{j}\in\mathbb{Z}_H^3} \widehat{f}_{\mathbf{j}}\phi_{\mathbf{j}}(\mathbf{t}), \tag{1}$$

where

$$\widehat{f}_{\mathbf{j}} = \frac{1}{|\Omega|}\int_{\Omega} f(\mathbf{t})\,\overline{\phi_{\mathbf{j}}(\mathbf{t})}d\mathbf{t} \qquad (\mathbf{j}\in\mathbb{Z}_H^3).$$

The nth hexagonal partial sum of the series (1) is defined by

$$S_n(f)(\mathbf{t}) := \sum_{\mathbf{j}\in\mathbb{H}_n} \widehat{f}_{\mathbf{j}}\phi_{\mathbf{j}}(\mathbf{t}) \qquad (n\in\mathbb{N}).$$

It is clear that

$$S_n(f)(\mathbf{t}) = \frac{1}{|\Omega|}\int_{\Omega} f(\mathbf{t}-\mathbf{u})\,D_n(\mathbf{u})\,d\mathbf{u},$$

where D_n is the Dirichlet kernel, defined by

$$D_n(\mathbf{t}) := \sum_{\mathbf{j}\in\mathbb{H}_n} \phi_{\mathbf{j}}(\mathbf{t}).$$

It is known that the Dirichlet kernel can be expressed as

$$D_n(\mathbf{t}) = \Theta_n(\mathbf{t}) - \Theta_{n-1}(\mathbf{t}) \qquad (n\geq 1), \tag{2}$$

where

$$\Theta_n(\mathbf{t}) := \frac{\sin\frac{(n+1)(t_1-t_2)\pi}{3}\sin\frac{(n+1)(t_2-t_3)\pi}{3}\sin\frac{(n+1)(t_3-t_1)\pi}{3}}{\sin\frac{(t_1-t_2)\pi}{3}\sin\frac{(t_2-t_3)\pi}{3}\sin\frac{(t_3-t_1)\pi}{3}} \tag{3}$$

for $\mathbf{t} = (t_1,t_2,t_3)\in\mathbb{R}_H^3$ [11].

The degree of approximation of H-periodic continuous functions by Cesàro, Riesz and Nörlund means of their hexagonal Fourier series was investigated by us in [4], [5], [6], [7] and [8]. In this paper, we studied the degree of approximation by matrix means of hexagonal Fourier and we obtained generalizations of previous results.

2. MAIN RESULTS

Let $C_H(\overline{\Omega})$ be the Banach space of complex valued H-periodic continuous functions defined on $\mathbb{R}_H^3$, whose norm is the uniform norm:

$$\|f\|_{C_H(\overline{\Omega})} := \sup\left\{|f(\mathbf{t})| : \mathbf{t}\in\overline{\Omega}\right\}.$$

The modulus of continuity of the function $f\in C_H(\overline{\Omega})$ is defined by

$$\omega_H(f,\delta) := \sup_{0<\|\mathbf{t}\|\leq\delta} \|f - f(\cdot+\mathbf{t})\|_{C_H(\overline{\Omega})},$$

where

$$\|\mathbf{t}\| := \max\{|t_1|,|t_2|,|t_3|\}$$

for $\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}_H^3$. $\omega_H(f, \cdot)$ is a nonnegative and nondecreasing function, and satisfies

$$\omega_H(f, \lambda\delta) \le (1+\lambda)\,\omega_H(f, \delta) \tag{4}$$

for $\lambda > 0$ [21].

A function $f \in C_H(\overline{\Omega})$ is said to belong to the Hölder space $H^\alpha(\overline{\Omega})$ $(0 < \alpha \le 1)$ if

$$\Lambda^\alpha(f) := \sup_{\mathbf{t} \neq \mathbf{s}} \frac{|f(\mathbf{t}) - f(\mathbf{s})|}{\|\mathbf{t} - \mathbf{s}\|^\alpha} < \infty.$$

$H^\alpha(\overline{\Omega})$ becomes a Banach space with respect to the Hölder norm

$$\|f\|_{H^\alpha(\overline{\Omega})} := \|f\|_{C_H(\overline{\Omega})} + \Lambda^\alpha(f).$$

Let $A = (a_{n,k})$ $(n, k = 0, 1, \ldots)$ be a lower triangular infinite matrix of real numbers. The A-transform of the sequence $(S_n(f))$ of partial sums the series (1) is defined by

$$T_n^{(A)}(f)(\mathbf{t}) := \sum_{k=0}^{n} a_{n,k} S_k(f)(\mathbf{t}) \qquad (n \in \mathbb{N}).$$

We shall assume that the lower triangular matrix $A = (a_{n,k})$ satisfies the conditions

$$a_{n,k} \ge 0 \qquad (n = 0, 1, \ldots, 0 \le k \le n), \tag{5}$$

$$a_{n,k+1} \ge a_{n,k} \qquad (n = 0, 1, \ldots, 0 \le k \le n-1), \tag{6}$$

and

$$\sum_{k=0}^{n} a_{n,k} = 1 \qquad (n = 0, 1, \ldots). \tag{7}$$

Also we use the notations

$$A_{n,k}^* := \sum_{\nu=k}^{n} a_{n,\nu} \;\; (0 \le k \le n), \;\; A_n^*(u) := A_{n,n-[u]}^*\,, \; a_n^*(u) := a_{n,n-[u]} \;\; (u > 0),$$

where $[u]$ is the integer part of u.

Hereafter, the relation $x \lesssim y$ will mean that there exists an absolute constant $c > 0$ such that $x \le cy$ holds for quantities x and y.

Theorem 1. *Let $f \in C_H(\overline{\Omega})$ and let $A = (a_{n,k})$ $(n, k = 0, 1, \ldots)$ be a lower triangular infinite matrix of real numbers which satisfies (5), (6) and (7). Then the estimate*

$$\left\| f - T_n^{(A)}(f) \right\|_{C_H(\overline{\Omega})} \lesssim \log(n+1) \left\{ \omega_H(f, a_{n,n}) + \sum_{k=1}^{n} \frac{\omega_H\left(f, \frac{1}{k}\right)}{k} A_{n,n-k}^* \right\} \tag{8}$$

holds.

Corollary 2. *If $f \in H^{\alpha}(\overline{\Omega})$ $(0 < \alpha \leq 1)$ and $A = (a_{n,k})$ $(n, k = 0, 1, \ldots)$ as in Theorem 1, then*

$$\left\| f - T_n^{(A)}(f) \right\|_{C_H(\overline{\Omega})} \lesssim \log(n+1) \left\{ a_{n,n}^{\alpha} + \sum_{k=1}^{n} \frac{A_{n,n-k}^{*}}{k^{1+\alpha}} \right\}. \tag{9}$$

Theorem 3. *Let $0 \leq \beta < \alpha \leq 1$, $f \in H^{\alpha}(\overline{\Omega})$ and let $A = (a_{n,k})$ $(n, k = 0, 1, \ldots)$ be a lower triangular infinite matrix of real numbers which satisfies (5), (6) and (7). Then*

(10)

$$\left\| f - T_n^{(A)}(f) \right\|_{H^{\beta}(\overline{\Omega})} \lesssim \log(n+1) \left(\sum_{k=1}^{n} \frac{A_{n,n-k}^{*}}{k} \right)^{\frac{\beta}{\alpha}} \left\{ a_{n,n}^{\alpha-\beta} + \left(\sum_{k=1}^{n} \frac{A_{n,n-k}^{*}}{k^{1+\alpha}} \right)^{1-\frac{\beta}{\alpha}} \right\}.$$

Analogues of these results were obtained in [10], [13], [14] and [2] for matrix means of trigonometric Fourier series of 2π-periodic continuous functions.

3. PROOFS OF MAIN RESULTS

Proof of Theorem 1. It is clear that

$$\begin{aligned} \left| f(\mathbf{t}) - T_n^{(A)}(f)(\mathbf{t}) \right| &\leq \frac{1}{|\Omega|} \int_{\Omega} |f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u})| \left| \sum_{k=0}^{n} a_{n,k} D_k(\mathbf{u}) \right| d\mathbf{u} \\ &\lesssim \frac{1}{|\Omega|} \int_{\Omega} \omega_H(f, \|\mathbf{u}\|) \left| \sum_{k=0}^{n} a_{n,k} D_k(\mathbf{u}) \right| d\mathbf{u}. \end{aligned}$$

If we set $\Theta_{-1}(\mathbf{u}) := 0$, by (2) we get

$$\begin{aligned} &\int_{\Omega} \omega_H(f, \|\mathbf{u}\|) \left| \sum_{k=0}^{n} a_{n,k} D_k(\mathbf{u}) \right| d\mathbf{u} = \\ &= \int_{\Omega} \omega_H(f, \|\mathbf{u}\|) \left| \sum_{k=0}^{n} a_{n,k} \Big(\Theta_k(\mathbf{u}) - \Theta_{k-1}(\mathbf{u}) \Big) \right| d\mathbf{u}. \end{aligned}$$

The function

$$\mathbf{t} \to \omega_H(f, \|\mathbf{t}\|) \left| \sum_{k=0}^{n} a_{n,k} \left(\Theta_k(\mathbf{t}) - \Theta_{k-1}(\mathbf{t}) \right) \right|$$

is symmetric with respect to variables t_1, t_2 and t_3, where $\mathbf{t} = (t_1, t_2, t_3) \in \Omega$. Hence it is sufficient to estimate the integral over the triangle

$$\begin{aligned} \Delta &:= \left\{ \mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}_H^3 : 0 \leq t_1, t_2, -t_3 \leq 1 \right\} \\ &= \left\{ (t_1, t_2) : t_1 \geq 0,\ t_2 \geq 0,\ t_1 + t_2 \leq 1 \right\}, \end{aligned}$$

which is one of the six equilateral triangles in $\overline{\Omega}$. By considering the formula (3), we obtain

$$\int_{\Delta}\omega_H\left(f,\|\mathbf{t}\|\right)\left|\sum_{k=0}^{n}a_{n,k}\left(\Theta_k\left(\mathbf{t}\right)-\Theta_{k-1}\left(\mathbf{t}\right)\right)\right|d\mathbf{t} =$$

$$= \int_{\Delta}\omega_H\left(f,t_1+t_2\right)\left|\sum_{k=0}^{n}a_{n,k}\left(\frac{\sin\frac{(k+1)(t_1-t_2)\pi}{3}\sin\frac{(k+1)(t_2-t_3)\pi}{3}\sin\frac{(k+1)(t_3-t_1)\pi}{3}}{\sin\frac{(t_1-t_2)\pi}{3}\sin\frac{(t_2-t_3)\pi}{3}\sin\frac{(t_3-t_1)\pi}{3}}\right.\right.$$
$$\left.\left.-\frac{\sin\frac{k(t_1-t_2)\pi}{3}\sin\frac{k(t_2-t_3)\pi}{3}\sin\frac{k(t_3-t_1)\pi}{3}}{\sin\frac{(t_1-t_2)\pi}{3}\sin\frac{(t_2-t_3)\pi}{3}\sin\frac{(t_3-t_1)\pi}{3}}\right)\right|d\mathbf{t}.$$

If we use the change of variables

(11) $$s_1 := \tfrac{t_1-t_3}{3} = \tfrac{2t_1+t_2}{3},\ s_2 := \tfrac{t_2-t_3}{3} = \tfrac{t_1+2t_2}{3},$$

the integral becomes

$$3\int_{\widetilde{\Delta}}\omega_H\left(f,s_1+s_2\right)\left|\sum_{k=0}^{n}a_{n,k}\left(\frac{\sin((k+1)(s_1-s_2)\pi)\sin((k+1)s_2\pi)\sin((k+1)(-s_1\pi))}{\sin((s_1-s_2)\pi)\sin(s_2\pi)\sin(-s_1\pi)}\right.\right.$$
$$\left.\left.-\frac{\sin(k(s_1-s_2)\pi)\sin(ks_2\pi)\sin(k(-s_1\pi))}{\sin((s_1-s_2)\pi)\sin(s_2\pi)\sin(-s_1\pi)}\right)\right|ds_1ds_2,$$

where $\widetilde{\Delta}$ is the image of Δ in the plane, that is

$$\widetilde{\Delta} := \left\{(s_1,s_2) : 0\le s_1\le 2s_2,\ 0\le s_2\le 2s_1,\ s_1+s_2\le 1\right\}.$$

Since the integrated function is symmetric with respect to s_1 and s_2, estimating the integral over the triangle

$$\Delta^* := \left\{(s_1,s_2)\in\widetilde{\Delta} : s_1\le s_2\right\} = \left\{(s_1,s_2) : s_1\le s_2\le 2s_1,\ s_1+s_2\le 1\right\},$$

which is the half of $\widetilde{\Delta}$, will be sufficient. The change of variables

(12) $$s_1 := \tfrac{u_1-u_2}{2},\ s_2 := \tfrac{u_1+u_2}{2}$$

transforms the triangle Δ^* to the triangle

$$\Gamma := \left\{(u_1,u_2) : 0\le u_2\le \tfrac{u_1}{3},\ 0\le u_1\le 1\right\}.$$

Thus we have to estimate the integral

$$I_n := \int_{\Gamma}\omega_H\left(f,u_1\right)\left|\sum_{k=0}^{n}a_{n,k}D_k^*\left(u_1,u_2\right)\right|du_1du_2,$$

where

$$\begin{aligned} D_k^*\left(u_1,u_2\right) \quad = \quad & \frac{\sin((k+1)(u_2)\pi)\sin\left((k+1)\frac{u_1+u_2}{2}\pi\right)\sin\left((k+1)\left(\frac{u_1-u_2}{2}\pi\right)\right)}{\sin((u_2)\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)} \\ & -\frac{\sin(k(u_2)\pi)\sin\left(k\frac{u_1+u_2}{2}\pi\right)\sin\left(k\left(\frac{u_1-u_2}{2}\pi\right)\right)}{\sin((u_2)\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)}. \end{aligned}$$

By elementary trigonometric identities, we obtain

$$D_k^*(u_1,u_2) = D_{k,1}^*(u_1,u_2) + D_{k,2}^*(u_1,u_2) + D_{k,3}^*(u_1,u_2), \tag{13}$$

where

$$D_{k,1}^*(u_1,u_2) := 2\cos\left((k+\tfrac{1}{2})u_2\pi\right)\frac{\sin\left(\frac{1}{2}u_2\pi\right)\sin\left((k+1)\frac{u_1+u_2}{2}\pi\right)\sin\left((k+1)\frac{u_1-u_2}{2}\pi\right)}{\sin(u_2\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)},$$

$$D_{k,2}^*(u_1,u_2) := 2\cos\left((k+\tfrac{1}{2})\tfrac{u_1+u_2}{2}\pi\right)\frac{\sin(ku_2\pi)\sin\left(\frac{1}{2}\frac{u_1+u_2}{2}\pi\right)\sin\left((k+1)\frac{u_1-u_2}{2}\pi\right)}{\sin(u_2\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)}$$

and

$$D_{k,3}^*(u_1,u_2) := 2\cos\left((k+\tfrac{1}{2})\tfrac{u_1-u_2}{2}\pi\right)\frac{\sin(ku_2\pi)\sin\left(k\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{1}{2}\frac{u_1-u_2}{2}\pi\right)}{\sin(u_2\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)}.$$

We partition the triangle Γ as $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where

$$\begin{aligned}
\Gamma_1 &:= \left\{(u_1,u_2)\in\Gamma : u_1 \le a_{n,n}\right\},\\
\Gamma_2 &:= \left\{(u_1,u_2)\in\Gamma : u_1 \ge a_{n,n},\ u_2 \le \tfrac{a_{n,n}}{3}\right\},\\
\Gamma_3 &:= \left\{(u_1,u_2)\in\Gamma : u_1 \ge a_{n,n},\ u_2 \ge \tfrac{a_{n,n}}{3}\right\}.
\end{aligned}$$

Hence $I_n = I_{n,1} + I_{n,2} + I_{n,3}$, where

$$I_{n,j} := \int_{\Gamma_j} \omega_H(f,u_1)\left|\sum_{k=0}^{n} a_{n,k} D_k^*(u_1,u_2)\right| du_1 du_2 \qquad (j=1,2,3).$$

We need the inequalities

$$\left|\frac{\sin nt}{\sin t}\right| \le n, \qquad (n\in\mathbb{N}), \tag{14}$$

and

$$\sin t \ge \tfrac{2}{\pi}t, \qquad \left(0\le t\le\tfrac{\pi}{2}\right) \tag{15}$$

to estimate integrals $I_{n,1}, I_{n,2}$ and $I_{n,3}$.

We divide Γ_1 into three parts to estimate $I_{n,1}$ as follows:

$$\begin{aligned}
\Gamma_1' &:= \left\{(u_1,u_2)\in\Gamma_1 : u_1 \le \tfrac{1}{n+1}\right\},\\
\Gamma_1'' &:= \left\{(u_1,u_2)\in\Gamma_1 : u_1 \ge \tfrac{1}{n+1}, u_2 \le \tfrac{1}{3(n+1)}\right\},\\
\Gamma_1''' &:= \left\{(u_1,u_2)\in\Gamma_1 : u_1 \ge \tfrac{1}{n+1}, u_2 \ge \tfrac{1}{3(n+1)}\right\}.
\end{aligned}$$

Hence we have

$$I_{n,1} = \left(\int_{\Gamma_1'} + \int_{\Gamma_1''} + \int_{\Gamma_1'''}\right)\omega_H(f,u_1)\left|\sum_{k=0}^{n} a_{n,k} D_k^*(u_1,u_2)\right| du_1 du_2.$$

By (14),

$$\int\limits_{\Gamma_1'} \omega_H(f,u_1)\left|\sum_{k=0}^{n} a_{n,k} D_k^*(u_1,u_2)\right| du_1 du_2 \lesssim$$

$$\lesssim \int\limits_{\Gamma_1'} \omega_H(f,u_1)\left(\sum_{k=0}^{n}(k+1)^2 a_{n,k}\right) du_1 du_2$$

$$\leq (n+1)^2 \int\limits_{0}^{\frac{1}{3(n+1)}} \int\limits_{3u_2}^{\frac{1}{n+1}} \omega_H(f,u_1)\, du_1 du_2$$

$$\leq \omega_H\left(f,\tfrac{1}{n+1}\right) \leq \omega_H(f,a_{n,n}).$$

By (15),

$$\int\limits_{\Gamma_1''} \omega_H(f,u_1)\left|\sum_{k=0}^{n} a_{n,k} D_{k,1}^*(u_1,u_2)\right| du_1 du_2 \lesssim$$

$$\lesssim \int\limits_{0}^{\frac{1}{3(n+1)}} \int\limits_{\frac{1}{n+1}}^{a_{n,n}} \frac{\omega_H(f,u_1)}{u_1^2} du_1 du_2 \leq \omega_H(f,a_{n,n}) \int\limits_{0}^{\frac{1}{3(n+1)}} \int\limits_{\frac{1}{n+1}}^{a_{n,n}} \frac{1}{u_1^2} du_1 du_2 \leq \omega_H(f,a_{n,n}).$$

(14) and (15) gives for $j=1,2$,

$$\int\limits_{\Gamma_1''} \omega_H(f,u_1)\left|\sum_{k=0}^{n} a_{n,k} D_{k,j}^*(u_1,u_2)\right| du_1 du_2 \lesssim$$

$$\lesssim \quad (n+1) \int\limits_{0}^{\frac{1}{3(n+1)}} \int\limits_{\frac{1}{n+1}}^{a_{n,n}} \frac{\omega_H(f,u_1)}{u_1} du_1 du_2 \leq (n+1)\,\omega_H(f,a_{n,n}) \int\limits_{0}^{\frac{1}{3(n+1)}} \int\limits_{\frac{1}{n+1}}^{a_{n,n}} \frac{1}{u_1} du_1 du_2$$

$$\leq \quad \log(n+1)\,\omega_H(f,a_{n,n}).$$

Since

$$\sin 2x + \sin 2y + \sin 2z = -4 \sin x \sin y \sin z$$

for $x+y+z=0$, we also get the expression

(16) $$D_k^*(u_1,u_2) = H_{k,1}(u_1,u_2) + H_{k,2}(u_1,u_2) + H_{k,3}(u_1,u_2),$$

where

$$H_{k,1}(u_1,u_2) := \frac{1}{2}\frac{\cos((2k+1)u_2\pi)}{\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)},$$

$$H_{k,2}(u_1,u_2) := -\frac{1}{2}\frac{\cos\left((2k+1)\frac{u_1+u_2}{2}\pi\right)}{\sin(u_2\pi)\sin\left(\frac{u_1-u_2}{2}\pi\right)},$$

$$H_{k,3}(u_1,u_2) := \frac{1}{2}\frac{\cos\left((2k+1)\frac{u_1-u_2}{2}\pi\right)}{\sin(u_2\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)}.$$

By the method used in [12, p. 179] we get

$$\left|\sum_{k=0}^{n} a_{n,k}\cos(2k+1)t\right| \lesssim A_n^*\left(\frac{1}{t}\right) + \frac{1}{\sin t}a_n^*\left(\frac{1}{t}\right) \qquad (0<t<\pi), \tag{17}$$

and

$$\left|\sum_{k=0}^{n} a_{n,k}\cos(2k+1)t\right| \lesssim A_n^*\left(\frac{1}{t}\right) \qquad \left(0<t\leq\frac{\pi}{2}\right). \tag{18}$$

By aim of (18) and (15) we obtain

$$\left|\sum_{k=0}^{n} a_{n,k}H_{k,1}(u_1,u_2)\right| \lesssim \frac{1}{u_1^2}A_n^*\left(\frac{1}{\pi u_2}\right) \tag{19}$$

and

$$\left|\sum_{k=0}^{n} a_{n,k}H_{k,3}(u_1,u_2)\right| \lesssim \frac{1}{u_1u_2}A_n^*\left(\frac{3}{\pi u_1}\right), \tag{20}$$

where both of for u_1 and u_2 are away from the origin. Also, for such u_1 and u_2, it follows from (17), (15) and from the fact

$$\sin\left(\frac{u_1\pi}{2}\right) \lesssim \sin\left(\frac{(u_1+u_2)\pi}{2}\right)$$

that

$$\left|\sum_{k=0}^{n} a_{n,k}H_{k,2}(u_1,u_2)\right| \lesssim \frac{1}{u_1u_2}A_n^*\left(\frac{3}{\pi u_1}\right). \tag{21}$$

Using (19) and the inequality

$$\frac{\omega_H(f,\delta_2)}{\delta_2} \leq 2\frac{\omega_H(f,\delta_1)}{\delta_1} \qquad (\delta_1<\delta_2), \tag{22}$$

which is obtained from (4), and considering that the function A_n^* is nondecreasing yield

$$\int_{\Gamma_1'''} \omega_H(f,u_1)\left|\sum_{k=0}^{n} a_{n,k}H_{k,1}(u_1,u_2)\right| du_1du_2 \lesssim$$

$$\lesssim \int_{\frac{1}{3(n+1)}}^{\frac{a_{n,n}}{3}} \int_{3u_2}^{a_{n,n}} \frac{\omega_H(f,u_1)}{u_1^2} A_n^*\left(\frac{1}{\pi u_2}\right) du_1 du_2$$

$$\leq 2 \int_{\frac{1}{3(n+1)}}^{\frac{a_{n,n}}{3}} \int_{3u_2}^{a_{n,n}} \frac{\omega_H(f,3u_2)}{3u_1 u_2} A_n^*\left(\frac{1}{\pi u_2}\right) du_1 du_2$$

$$= \frac{2}{3} \int_{\frac{1}{3(n+1)}}^{\frac{a_{n,n}}{3}} \frac{\omega_H(f,3u_2)}{u_2} \log\left(\frac{a_{n,n}}{3u_2}\right) A_n^*\left(\frac{1}{\pi u_2}\right) du_2$$

$$\leq \log\left((n+1)\, a_{n,n}\right) \int_{\frac{1}{3(n+1)}}^{\frac{a_{n,n}}{3}} \frac{\omega_H(f,3u_2)}{u_2} A_n^*\left(\frac{1}{\pi u_2}\right) du_2$$

$$\leq \log(n+1) \int_{\frac{1}{3(n+1)}}^{\frac{a_{n,n}}{3}} \frac{\omega_H(f,3u_2)}{u_2} A_n^*\left(\frac{1}{\pi u_2}\right) du_2 = \log(n+1) \int_{\frac{3}{\pi}}^{\frac{3}{\pi}(n+1)} \frac{\omega_H\left(f,\frac{3}{\pi t}\right)}{t} A_n^*(t)\, dt$$

$$= \log(n+1) \sum_{k=1}^{n} \left(\int_{\frac{3}{\pi}k}^{\frac{3}{\pi}(k+1)} \frac{\omega_H\left(f,\frac{3}{\pi t}\right)}{t} A_n^*(t)\, dt \right)$$

$$\leq \log(n+1) \sum_{k=1}^{n} \frac{\omega_H\left(f,\frac{1}{k}\right)}{k} A_n^*(k+1).$$

For $j = 2, 3$, by (20) and (21)

$$\int_{\Gamma_1'''} \omega_H(f,u_1) \left| \sum_{k=0}^{n} a_{n,k} H_{k,j}(u_1,u_2) \right| du_1 du_2 \lesssim$$

$$\lesssim \int_{\frac{1}{n+1}}^{a_{n,n}} \int_{\frac{1}{3(n+1)}}^{\frac{u_1}{3}} \frac{\omega_H(f,u_1)}{u_1 u_2} A_n^*\left(\frac{3}{\pi u_1}\right) du_2 du_1$$

$$= \int_{\frac{1}{n+1}}^{a_{n,n}} \frac{\omega_H(f,u_1)}{u_1} \log\left((n+1)\, u_1\right) A_n^*\left(\frac{3}{\pi u_1}\right) du_1$$

$$\leq \log\left((n+1)\, a_{n,n}\right) \int_{\frac{1}{n+1}}^{a_{n,n}} \frac{\omega_H(f,u_1)}{u_1} A_n^*\left(\frac{3}{\pi u_1}\right) du_1 =$$

$$= \log\left((n+1)\,a_{n,n}\right) \int\limits_{\frac{3}{\pi}}^{\frac{3}{\pi}(n+1)} \frac{\omega_H\left(f, \frac{3}{\pi t}\right)}{t} A_n^*(t)\,dt$$

$$\leq \log(n+1) \sum_{k=1}^{n} \frac{\omega_H\left(f, \frac{1}{k}\right)}{k} A_n^*(k+1)\,.$$

Thus we get the estimate

$$I_{n,1} \lesssim \log(n+1)\left\{\omega_H(f, a_{n,n}) + \sum_{k=1}^{n} \frac{\omega_H\left(f, \frac{1}{k}\right)}{k} A_n^*(k+1)\right\}. \tag{23}$$

We divide Γ_2 into two domains as

$$\Gamma_2' := \left\{(u_1, u_2) \in \Gamma_2 : u_2 \leq \frac{a_{n,n}}{3(n+1)}\right\},$$
$$\Gamma_2'' := \left\{(u_1, u_2) \in \Gamma_2 : u_2 \geq \frac{a_{n,n}}{3(n+1)}\right\}$$

to estimate $I_{n,2}$. By (15) and (22),

$$\int\limits_{\Gamma_2'} \omega_H(f, u_1) \left| \sum_{k=0}^{n} a_{n,k} D_{k,1}^*(u_1, u_2) \right| du_1 du_2 \lesssim$$

$$\lesssim \int\limits_{0}^{\frac{a_{n,n}}{3(n+1)}} \int\limits_{a_{n,n}}^{1} \frac{\omega_H(f, u_1)}{u_1^2} du_1 du_2 \leq 2\frac{\omega_H(f, a_{n,n})}{a_{n,n}} \int\limits_{0}^{\frac{a_{n,n}}{3(n+1)}} \int\limits_{a_{n,n}}^{1} \frac{1}{u_1} du_1 du_2$$

$$= 2\frac{\omega_H(f, a_{n,n})}{a_{n,n}} \log\left(\frac{1}{a_{n,n}}\right) \frac{a_{n,n}}{3(n+1)} \leq \log(n+1)\,\omega_H(f, a_{n,n})\,.$$

(14), (15) and (22) yield

$$\int\limits_{\Gamma_2'} \omega_H(f, u_1) \left| \sum_{k=0}^{n} a_{n,k} D_{k,j}^*(u_1, u_2) \right| du_1 du_2 \lesssim$$

$$\lesssim (n+1) \int\limits_{0}^{\frac{a_{n,n}}{3(n+1)}} \int\limits_{a_{n,n}}^{1} \frac{\omega_H(f, u_1)}{u_1} du_1 du_2 \leq 2(n+1)\frac{\omega_H(f, a_{n,n})}{a_{n,n}} \int\limits_{0}^{\frac{a_{n,n}}{3(n+1)}} \int\limits_{a_{n,n}}^{1} du_1 du_2$$

$$\leq \omega_H(f, a_{n,n})$$

for $j = 2, 3$. Since (15) implies $\left|H_{k,1}(u_1, u_2)\right| \lesssim \frac{1}{u_1^2}$ for $(u_1, u_2) \in \Gamma_2''$, we get

$$\int\limits_{\Gamma_2''} \omega_H(f, u_1) \left| \sum_{k=0}^{n} a_{n,k} H_{k,1}(u_1, u_2) \right| du_1 du_2 \lesssim$$

$$\lesssim \int_{\frac{a_{n,n}}{3(n+1)}}^{\frac{a_{n,n}}{3}} \int_{a_{n,n}}^{1} \frac{\omega_H(f,u_1)}{u_1^2} du_1 du_2 \leq 2\frac{\omega_H(f,a_{n,n})}{a_{n,n}} \int_{\frac{a_{n,n}}{3(n+1)}}^{\frac{a_{n,n}}{3}} \int_{a_{n,n}}^{1} \frac{1}{u_1} du_1 du_2$$

$$= 2\frac{\omega_H(f,a_{n,n})}{a_{n,n}} \log\left(\frac{1}{a_{n,n}}\right) \frac{a_{n,n}}{3}\left(1-\frac{1}{n+1}\right) \leq \log(n+1)\,\omega_H(f,a_{n,n}).$$

By considering (20) and (21) we obtain

$$\int_{\Gamma_2''} \omega_H(f,u_1) \left|\sum_{k=0}^{n} a_{n,k} H_{k,j}(u_1,u_2)\right| du_1 du_2 \lesssim$$

$$\lesssim \int_{\frac{a_{n,n}}{3(n+1)}}^{\frac{a_{n,n}}{3}} \int_{a_{n,n}}^{1} \frac{\omega_H(f,u_1)}{u_1 u_2} A_n^*\left(\frac{3}{\pi u_1}\right) du_1 du_2 = \log(n+1) \int_{a_{n,n}}^{1} \frac{\omega_H(f,u_1)}{u_1} A_n^*\left(\frac{3}{\pi u_1}\right) du_1$$

$$\leq \log(n+1) \int_{\frac{1}{n+1}}^{1} \frac{\omega_H(f,u_1)}{u_1} A_n^*\left(\frac{3}{\pi u_1}\right) du_1 \leq \log(n+1) \sum_{k=1}^{n} \frac{\omega_H\left(f,\frac{1}{k}\right)}{k} A_n^*(k+1)$$

for $j=2,3$. Thus we obtain

$$I_{n,2} \lesssim \log(n+1)\left\{\omega_H(f,a_{n,n}) + \sum_{k=1}^{n} \frac{\omega_H\left(f,\frac{1}{k}\right)}{k} A_n^*(k+1)\right\}. \tag{24}$$

If we use (19) and (22),

$$\int_{\Gamma_3} \omega_H(f,u_1) \left|\sum_{k=0}^{n} a_{n,k} H_{k,1}(u_1,u_2)\right| du_1 du_2 \lesssim$$

$$\lesssim \int_{\frac{a_{n,n}}{3}}^{\frac{1}{3}} \int_{3u_2}^{1} \frac{\omega_H(f,u_1)}{u_1^2} A_n^*\left(\frac{1}{\pi u_2}\right) du_1 du_2$$

$$\leq \frac{2}{3} \int_{\frac{a_{n,n}}{3}}^{\frac{1}{3}} \int_{3u_2}^{1} \frac{\omega_H(f,3u_2)}{u_1 u_2} A_n^*\left(\frac{1}{\pi u_2}\right) du_1 du_2$$

$$= \frac{2}{3} \int_{\frac{a_{n,n}}{3}}^{\frac{1}{3}} \frac{\omega_H(f,3u_2)}{u_2} \log\left(\frac{1}{3u_2}\right) A_n^*\left(\frac{1}{\pi u_2}\right) du_2$$

$$\leq \frac{2}{3} \log\left(\frac{1}{a_{n,n}}\right) \int_{\frac{a_{n,n}}{3}}^{\frac{1}{3}} \frac{\omega_H(f,3u_2)}{u_2} A_n^*\left(\frac{1}{\pi u_2}\right) du_2 \leq$$

$$\leq \log(n+1) \int_{\frac{1}{3(n+1)}}^{\frac{1}{3}} \frac{\omega_H(f,3u_2)}{u_2} A_n^*\left(\frac{1}{\pi u_2}\right) du_2$$

$$= \log(n+1) \int_{\frac{3}{\pi}}^{\frac{3}{\pi}(n+1)} \frac{\omega_H\left(f,\frac{3}{\pi t}\right)}{t} A_n^*(t)\, dt$$

$$\leq \log(n+1) \sum_{k=1}^{n} \frac{\omega_H\left(f,\frac{1}{k}\right)}{k} A_n^*(k+1).$$

By (20) and (21) we get

$$\int_{\Gamma_3} \omega_H(f,u_1) \left| \sum_{k=0}^{n} a_{n,k} H_{k,j}(u_1,u_2) \right| du_1 du_2 \lesssim$$

$$\lesssim \int_{a_{n,n}}^{1} \int_{\frac{a_{n,n}}{3}}^{\frac{u_1}{3}} \frac{\omega_H(f,u_1)}{u_1 u_2} A_n^*\left(\frac{3}{\pi u_1}\right) du_2 du_1 = \int_{a_{n,n}}^{1} \frac{\omega_H(f,u_1)}{u_1} \log\left(\frac{u_1}{a_{n,n}}\right) A_n^*\left(\frac{3}{\pi u_1}\right) du_1$$

$$\leq \log\left(\frac{1}{a_{n,n}}\right) \int_{\frac{1}{n+1}}^{1} \frac{\omega_H(f,u_1)}{u_1} A_n^*\left(\frac{3}{\pi u_1}\right) du_1 \leq \log(n+1) \sum_{k=1}^{n} \frac{\omega_H\left(f,\frac{1}{k}\right)}{k} A_n^*(k+1)$$

for $j = 2, 3$. Thus, we have

$$I_{n,3} \lesssim \log(n+1) \sum_{k=1}^{n} \frac{\omega_H\left(f,\frac{1}{k}\right)}{k} A_n^*(k+1). \tag{25}$$

Inequalities (23), (24), (25) and the fact $A_n^*(k+1) \lesssim A_{n,n-k}^*$ imply (8). □

Proof of Theorem 3. The method used in proof of Theorem 1 gives

$$\int_{\Omega} \left| \sum_{k=0}^{n} a_{n,k} D_k(\mathbf{u}) \right| d\mathbf{u} \lesssim \log(n+1) \sum_{k=1}^{n} \frac{A_{n,n-k}^*}{k} \tag{26}$$

and

$$\int_{\Omega} \|\mathbf{u}\|^{\alpha} \left| \sum_{k=0}^{n} a_{n,k} D_k(\mathbf{u}) \right| d\mathbf{u} \lesssim \log(n+1) \left\{ a_{n,n}^{\alpha} + \sum_{k=1}^{n} \frac{A_{n,k}}{k^{1+\alpha}} \right\} \qquad (0 < \alpha \leq 1). \tag{27}$$

If we set $e_n(\mathbf{t}) := f(\mathbf{t}) - T_n^{(A)}(f)(\mathbf{t})$, we have

$$\left\| f - T_n^{(A)}(f) \right\|_{H^{\beta}(\overline{\Omega})} = \left\| f - T_n^{(A)}(f) \right\|_{C_H(\overline{\Omega})} + \Lambda^{\beta}(e_n). \tag{28}$$

Since

$$|e_n(\mathbf{t}) - e_n(\mathbf{s})| \le \frac{1}{|\Omega|}\int_\Omega \Big| f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u}) \Big| \left|\sum_{k=0}^{n} a_{n,k} D_k(\mathbf{u})\right| d\mathbf{u},$$

we have to estimate the integral

$$J_n := \int_\Omega \Big| f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u}) \Big| \left|\sum_{k=0}^{n} a_{n,k} D_k(\mathbf{u})\right| d\mathbf{u}.$$

Since $f \in H^\alpha(\overline{\Omega})$ we have

$$\Big| f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u}) \Big| \lesssim \|\mathbf{t}-\mathbf{s}\|^\alpha,$$

hence by (26) we get

$$\begin{aligned}
(J_n)^{\frac{\beta}{\alpha}} &= \left(\int_\Omega \Big| f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u}) \Big| \left|\sum_{k=0}^{n} a_{n,k} D_k(\mathbf{u})\right| d\mathbf{u}\right)^{\frac{\beta}{\alpha}} \\
&\lesssim \|\mathbf{t}-\mathbf{s}\|^\beta \left(\int_\Omega \left|\sum_{k=0}^{n} a_{n,k} D_k(\mathbf{u})\right| d\mathbf{u}\right)^{\frac{\beta}{\alpha}} \\
&\lesssim \|\mathbf{t}-\mathbf{s}\|^\beta \left(\log(n+1) \sum_{k=1}^{n} \frac{A^*_{n,n-k}}{k}\right)^{\frac{\beta}{\alpha}}.
\end{aligned}$$

We also have

$$|f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u})| \lesssim \|\mathbf{u}\|^\alpha$$

for $f \in H^\alpha(\overline{\Omega})$. Thus by (27) we obtain

$$\begin{aligned}
(J_n)^{1-\frac{\beta}{\alpha}} &\lesssim \left(\int_\Omega \|\mathbf{u}\|^\alpha \left|\sum_{k=0}^{n} a_{n,k} D_k(\mathbf{u})\right| d\mathbf{u}\right)^{1-\frac{\beta}{\alpha}} \\
&\lesssim \left(\log(n+1)\left\{a^\alpha_{n,n} + \sum_{k=1}^{n} \frac{A^*_{n,n-k}}{k^{1+\alpha}}\right\}\right)^{1-\frac{\beta}{\alpha}}.
\end{aligned}$$

Since

$$\begin{aligned}
&|e_n(\mathbf{t}) - e_n(\mathbf{s})| \le \\
&\le J_n = (J_n)^{\frac{\beta}{\alpha}} (J_n)^{1-\frac{\beta}{\alpha}} \\
&\lesssim \|\mathbf{t}-\mathbf{s}\|^\beta \log(n+1) \left(\sum_{k=1}^{n} \frac{A^*_{n,n-k}}{k}\right)^{\frac{\beta}{\alpha}} \left\{a^{\alpha-\beta}_{n,n} + \left(\sum_{k=1}^{n} \frac{A^*_{n,n-k}}{k^{1+\alpha}}\right)^{1-\frac{\beta}{\alpha}}\right\},
\end{aligned}$$

we get

$$\frac{|e_n(\mathbf{t})-e_n(\mathbf{s})|}{\|\mathbf{t}-\mathbf{s}\|^{\beta}} \lesssim \log(n+1)\left(\sum_{k=1}^{n}\frac{A^*_{n,n-k}}{k}\right)^{\frac{\beta}{\alpha}}\left\{a^{\alpha-\beta}_{n,n}+\left(\sum_{k=1}^{n}\frac{A^*_{n,n-k}}{k^{1+\alpha}}\right)^{1-\frac{\beta}{\alpha}}\right\}\quad(\mathbf{t}\neq\mathbf{s}),$$

which implies

$$\Lambda^{\beta}(e_n) \lesssim \log(n+1)\left(\sum_{k=1}^{n}\frac{A^*_{n,n-k}}{k}\right)^{\frac{\beta}{\alpha}}\left\{a^{\alpha-\beta}_{n,n}+\left(\sum_{k=1}^{n}\frac{A^*_{n,n-k}}{k^{1+\alpha}}\right)^{1-\frac{\beta}{\alpha}}\right\}.$$

The proof is completed by combining this last estimate, (9) and (28). □

4. REMARKS

The degree of approximation by $(C,1)$, Riesz and Nörlund means of trigonometric Fourier series was investigated in [18], [19], [9] and [1]. In this section we conclude from Theorems 1 and 3 results about the degree of approximation of these means of hexagonal Fourier series.

Remark 4. Let $p=(p_k)$ be a nondecreasing sequence of numbers positive real numbers. If we take

$$a_{n,k} := \begin{cases} \frac{p_k}{P_n}, & 0\leq k\leq n \\ 0, & k>n \end{cases}$$

where $P_n := \sum_{k=0}^{n} p_k$, then $A=(a_{n,k})$ satisfies (5), (6) and (7), and $T_n^{(A)}$ becomes the Riesz mean

$$R_n(p;f) = \frac{1}{P_n}\sum_{k=0}^{n} p_k S_k(f).$$

Theorem 1 gives

$$\|f-R_n(p;f)\|_{C_H(\overline{\Omega})} \lesssim \log\left(\frac{P_n}{p_0}\right)\left\{\omega_H\left(f,\frac{p_n}{P_n}\right)+\frac{1}{P_n}\sum_{k=1}^{n}\frac{Q_{n,k}\omega_H(f,1/k)}{k}\right\} \tag{29}$$

for $f\in C_H(\overline{\Omega})$, and Theorem 3 yields

$$\|f-R_n(p;f)\|_{H^{\beta}(\overline{\Omega})} \lesssim \log\left(\frac{P_n}{p_0}\right)\left(\frac{1}{P_n}\sum_{k=1}^{n}\frac{Q_{n,k}}{k}\right)^{\frac{\beta}{\alpha}}\left\{\left(\frac{p_n}{P_n}\right)^{\alpha-\beta}+\left(\frac{1}{P_n}\sum_{k=1}^{n}\frac{Q_{n,k}}{k^{1+\alpha}}\right)^{1-\frac{\beta}{\alpha}}\right\} \tag{30}$$

for $f\in H^{\alpha}(\overline{\Omega})$, $(0\leq\beta<\alpha\leq 1)$, where $Q_{n,k} := \sum_{\nu=n-k}^{n} p_\nu$. □

Remark 5. Let (p_k) be a nonincreasing sequence of positive real numbers. In this case the matrix $A=(a_{n,k})$ with entries

$$a_{n,k} = \begin{cases} \frac{p_{n-k}}{P_n}, & 0\leq k\leq n, \\ 0, & k>n, \end{cases}$$

satisfies (5), (6) and (7), and $T_n^{(A)}$ becomes the Nörlund mean

$$N_n(p;f) = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k} S_k(f).$$

By Theorem 1 we conclude

$$\|f - N_n(p;f)\|_{C_H(\overline{\Omega})} \lesssim \log\left(\frac{P_n}{p_n}\right)\left\{\omega_H\left(f, \frac{p_0}{P_n}\right) + \frac{1}{P_n}\sum_{k=1}^{n}\frac{P_k \omega_H(f,1/k)}{k}\right\} \tag{31}$$

for $f \in C_H\left(\overline{\Omega}\right)$, and by Theorem 3

$$\|f - N_n(p;f)\|_{H^\beta(\overline{\Omega})} \lesssim \log\left(\frac{P_n}{p_n}\right)\left(\frac{1}{P_n}\sum_{k=1}^{n}\frac{P_k}{k}\right)^{\frac{\beta}{\alpha}}\left\{\left(\frac{p_0}{P_n}\right)^{\alpha-\beta} + \left(\frac{1}{P_n}\sum_{k=1}^{n}\frac{P_k}{k^{1+\alpha}}\right)^{1-\frac{\beta}{\alpha}}\right\} \tag{32}$$

for $f \in H^\alpha(\overline{\Omega})$, $(0 \le \beta < \alpha \le 1)$. □

REMARK 6. f we take $p_k = 1$ $(k = 0, 1, \ldots)$, $R_n(p;f)$ and $N_n(p;f)$ become $(C,1)$ means $S_n^{(1)}(f)$, and both of (29) and (31) reduce to

$$\left\|f - S_n^{(1)}(f)\right\|_{C_H(\overline{\Omega})} \lesssim \frac{\log(n+1)}{n+1}\sum_{k=1}^{n}\omega_H\left(f, \frac{1}{k}\right)$$

for $f \in C_H(\overline{\Omega})$. Furthermore, (30) and (32) give the estimate

$$\left\|f - S_n^{(1)}(f)\right\|_{H^\beta(\overline{\Omega})} \lesssim \begin{cases} \frac{\log(n+1)}{n^{\alpha-\beta}}, & \alpha < 1 \\ \frac{(\log(n+1))^{2-\beta}}{n^{1-\beta}}, & \alpha = 1 \end{cases}$$

for $(C,1)$ means of $f \in H^\alpha(\overline{\Omega})$ $(0 \le \beta < \alpha \le 1)$. □

REFERENCES

[1] P. CHANDRA, *On the generalised Fejér means in the metric of Hölder space*, Math. Nachr., **109** (1982) no. 1, pp. 39–45.
[2] P. CHANDRA, *On the degree of approximation of a class of functions by means of Fourier series*, Acta Math. Hung., **52** (1988) nos. 3–4, pp. 199–205.
[3] B. FUGLEDE, *Commuting self-adjoint partial differential operators and a group theoretic problem*, J. Funct. Anal., **16** (1974) no. 1, pp. 101–121.
[4] A. GUVEN, *Approximation by means of hexagonal Fourier series in Hölder norms*, J. Classical. Anal., **1** (2012) no. 1, pp. 43–52.
[5] A. GUVEN, *Approximation by $(C,1)$ and Abel-Poisson means of Fourier series on hexagonal domains*, Math. Inequal. Appl., **16** (2013) no. 1, pp. 175–191.
[6] A. GUVEN, *Approximation by Riesz means of hexagonal Fourier series*, Z. Anal. Anwend., **36** (2017) no. 1, pp. 1–16.
[7] A. GUVEN, *Approximation by Nörlund means of hexagonal Fourier series*, Anal. Theory Appl., **33** (2017) no. 4, pp. 384–400.
[8] A. GUVEN, *Approximation of continuous functions on hexagonal domains*, J. Numer. Anal. Approx. Theory., **47** (2018) no. 1, pp. 42–57.

[9] A.S.B. Holland, B.N. Sahney, J. Tzimbalario, *On degree of approximation of a class of functions by means of Fourier series*, Acta Sci. Math., **38** (1976) nos. 1-2, pp. 69–72.

[10] P.D. Kathal, A.S.B. Holland, B.N. Sahney, *A class of continuous functions and their degree of approximation*, Acta Math. Acad. Sci. Hungar., **30** (1977) nos. 3–4, pp. 227–231.

[11] H. Li, J. Sun, Y. Xu, *Discrete Fourier analysis, cubature and interpolation on a hexagon and a triangle*, SIAM J. Numer. Anal., **46** (2008) no. 4, pp. 1653–1681.

[12] L. McFadden, *Absolute Nörlund summability*, Duke Math. J., **9** (1942) no. 1, pp. 168–207.

[13] R.N. Mohapatra, B.N. Sahney, *Approximation of continuous functions by their Fourier series*, Rev. Anal. Numér. Théor. Approx., **10** (1981) no. 1, pp. 81–87.

[14] R.N. Mohapatra, P. Chandra, *Degree of approximation of functions in the Hölder metric*, Acta Math. Hung., **41** (1983) nos. 1–2, pp. 67–76.

[15] F. Moricz, B.E. Rhoades, *Approximation by Nörlund means of double Fourier series for Lipschitz functions*, J. Approx. Theory, **50** (1987) no. 4, pp. 341–358.

[16] F. Moricz, B.E. Rhoades, *Approximation by Nörlund means of double Fourier series to continuous functions in two variables*, Constr. Approx., **3** (1987) no. 1, pp. 281–296.

[17] F. Moricz, X.L. Shi, *Approximation to continuous functions by Cesàro means of double Fourier series and conjugate series*, J. Approx. Theory, **49** (1987) no. 4, pp. 346–377.

[18] S. Prössdorf, *Zur konvergenz der Fourierreihen hölderstetiger funktionen*, Math. Nachr., **69** (1975) no. 1, pp. 7–14.

[19] B.N. Sahney, D.S. Goel, *On the degree of approximation of continuous functions*, Ranchii Univ. Math. J., **4** (1973), pp. 50–53.

[20] A.F. Timan, *Theory of Approximation of Functions of a Real Variable*, Pergamon Press, 1963.

[21] Y. Xu, *Fourier series and approximation on hexagonal and triangular domains*, Constr. Approx., **31** (2010) no. 1, pp. 115–138.

[22] L. Zhizhiashvili, *Trigonometric Fourier Series and Their Conjugates*, Kluwer Academic Publishers, 1996.

[23] A. Zygmund, *Trigonometric Series*, Cambridge Univ. Press, 2nd edition, 1959.

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