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A TWO-POINT EIGHTH-ORDER METHOD BASED ON THE WEIGHT FUNCTION FOR SOLVING NONLINEAR EQUATIONS

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Abstract. In this work, we have designed a family of with-memory methods with eighth-order convergence. We have used the weight function technique. The proposed methods have three parameters. Three self-accelerating parameters are calculated in each iterative step employing only information from the current and all previous iteration. Numerical experiments are carried out to demonstrate the convergence and the efficiency of our iterative method.

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1. INTRODUCTION

Nonlinearity is of interest to physicists and mathematicians, since most physical systems are inherently nonlinear in nature. One of the most important problem in computational mathematics is solving nonlinear equations. For example, nonlinear optimization aims to find a minimum or maximum of a given nonlinear function. Nonlinear equations are difficult to solve in general. The best way to solve these equations is using iterative methods. One of the classical method to solve nonlinear equation is Newton's method which has convergence order equal to 2. It can be said that the Secant method is the oldest with memory methods that have been studied so far. The Secant method obtain by approximating the derivative in Newton's method *via* a finite divided difference $f'(x_k) = \frac{f(x_{k-1})-f(x_k)}{x_{k-1}-x_k}$.

The method is given as

$$x_{k+1} = x_k - \frac{x_{k-1} - x_k}{f(x_{k-1}) - f(x_k)} f(x_k). \tag{1}$$

In the continuation of this work, we will first define the efficiency index (EI) of an iterative method by Ostrowski [22]:

$$EI = r^{\frac{1}{\theta_f}} \tag{2}$$

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The Q-order of convergence r and the number of function evaluations θ_f per iteration. The efficiency index of the Secant method is 1.6803. Traub in [31] proposed the following with-memory method (TM)

$$(3)\qquad \begin{cases} \gamma_k = -\frac{1}{f[x_k, x_{k-1}]}, & k = 1, 2, 3, \ldots, \\ w_k = x_k + \gamma f(x_k), \quad x_{k+1} = x_k - \frac{f(x_k)}{f[x_k, w_k]}, & k = 0, 1, 2, \ldots, \end{cases}$$

with the order of convergence 2.41421. In the following, Neta proposed (NM) three step with memory method which has the order of convergence 10.81525 [20]:

$$(4)\qquad \begin{cases} w_k = x_k + \frac{f(x_k)}{f'(x_k)}(f(w_{k-1})\phi_z - f(z_{k-1})\phi_w)(\frac{f(x_k)^2}{f(w_{k-1} - f(z_{k-1})}), \quad k = 1, 2, \ldots, \\ z_k = x_k + \frac{f(x_k)}{f'(x_k)}(f(w_k)\phi_z - f(z_{k-1})\psi_w)(\frac{f(x_k)^2}{f(w_k - f(z_{k-1})}), \quad k = 1, 2, 3, \ldots, \\ x_{k+1} = x_k + \frac{f(x_k)}{f'(x_k)}(f(w_k)\psi_z - f(z_k)\psi_w)(\frac{f(x_k)^2}{f(w_k - f(z_k)}), \quad k = 0, 1, 2, \ldots \end{cases}$$

He used inverse interpolation. Neta increased the convergence order from 8 to 10.81. Therefore, he has improved the degree of convergence by 35%. Also, Traub improved by 20.71% by increasing the degree of convergence from 2 to 2.41. Bassiri *et al.* [2] also increased the degree of convergence of a two-step method from 4 to 7.22. Therefore, the convergence order improvement of their proposed method is 80%.

The remaining materials of this paper are uncovered as follows. Section 2 is devoted to modifications of the two-steps method proposed by Bassiri *et al.* [2]. Further accelerations of convergence speed are attained in Section 3. This self-accelerating parameter is calculated by the Newton interpolating polynomial. The corresponding Q-order of convergence [8] is increased from 4 to $7.53113, 7.94449, 7.99315$ and $7.99915 \approx 8$.

Numerical examples are given in Section 4 to confirm theoretical results.

Finally, Section 5 is devoted to the main conclusions of this work.

2. WITHOUT MEMORY METHODS

Bassiri *et al.* proposed the following optimal iterative without memory method [2]:

$$(5)\qquad \begin{cases} w_k = x_k + \gamma f(x_k), \quad y_k = x_k - \frac{f(x_k)}{f[x_k, w_k] + \beta f(w_k)}, \quad k = 0, 1, \ldots, \\ s_k = \frac{f(y_k)}{f(x_k)}, \quad x_{k+1} = y_k - H(s_k)\frac{f(y_k)}{f[w_k, y_k] + \beta f(w_k) + \lambda(y_k - x_k)(y_k - w_k)}, \ k = 0, 1, \ldots \end{cases}$$

This method achieves order convergence 4 when the weight functions satisfy the conditions

$$(6)\qquad H(0) = H'(0) = H''(0) = 1.$$

And its error expression is

$$e_{k+1}=((1+\gamma f'(\alpha))^2(\beta+c_2)(2\lambda+f'(\alpha)\beta^2(1+\gamma f'(\alpha))+f'(\alpha)c_2(2\beta(3+\gamma f'(\alpha)) \\ + (5+\gamma f'(\alpha))c_2) - 2f'(\alpha)c_3))(-2f'(\alpha))^{-1}e_k^4 + \mathcal{O}(e_k^5). \tag{7}$$

where $c_k = \frac{f^{(k)}(\alpha)}{k!f'(\alpha)}$ for $k = 2, 3, \ldots$ If the weight function is not used, the order convergence of method (5) will be as follows:

$$e_{k+1} = ((1+\gamma f'(\alpha))^2(\beta+c_2)^2 e_k^3 + \mathcal{O}(e_k^4). \tag{8}$$

In this case, the optimality of method (5) disappears. For maintaining optimality, it must be four until it reaches an optimal without-memory method by according to Kung-Traub's conjecture [12]. One way to increase the degree of convergence is by using the weight function. Refer to [3, 5, 6, 10, 13, 14, 15, 25, 26] for further study. Some concrete weight functions that satisfy the conditions are

$$H_1(s) = 1 + s + \tfrac{s^2}{2}, \quad H_2(s) = e^s, \quad H_3(s) = \tfrac{2+s}{2-s}, \quad H_4(s) = \tfrac{1}{1-s-\frac{s^2}{2}}. \tag{9}$$

Here it should be noted that under the functions of weight $H_1(s), H_2(s)$ and $H_3(s)$ the error equation is (7). But the error equation of this method for the weight function $H_4(s)$ is as follows:

$$e_{k+1} = ((1+\gamma f'(\alpha))^2(\beta+c_2)(f'(\alpha)\beta^2(1+\gamma f'(\alpha)) - 2\lambda + f'(\alpha)c_2(2\beta(-1+ \\ \gamma f'(\alpha)) + (-3+\gamma f'(\alpha))c_2) + 2f'(\alpha)c_3))(-2f'(\alpha))^{-1}e_k^4 + \mathcal{O}(e_k^5). \tag{10}$$

It is also necessary to note that any weight function that applies only in conditions $H(0) = 1$ and $H'(0) = 1$ will converge to convergence order 4. In addition, weight function $H_4(s)$ does not apply in terms of (6), and $H_4''(0) = 3$.

Error relations (10) plays the key role in our study of the convergence acceleration. For method (5), we have the following result.

THEOREM 1. *For a sufficiently good initial approximation x_0 of a simple zero α of the function f, the family of two-point methods (5) obtains the order at least four if the weight function H satisfies conditions (6) Then the error relation for the family (5) is given by (10).*

Proof. By using Taylor's expansion of $f(x)$ about α and taking into account that $f(\alpha) = 0$, we obtain

$$f(x_k) = f'(\alpha)(e_k + c_2e_k^2 + c_3e_k^3 + c_4e_k^4 + \mathcal{O}(e_k^5)). \tag{11}$$

Then, computing $e_{k,w} = w_k - \alpha$, we attain $w_k = x_k + \gamma f(x_k)$

$$e_{k,w} = f'(\alpha)(1+\gamma f'(\alpha))e_k + \gamma f'(\alpha)c_2e_k^2 + \gamma f'(\alpha)c_3e_k^3 + \gamma f'(\alpha)c_4e_k^4 + \mathcal{O}(e_k^5), \tag{12}$$

and

$$f(w_k) = f'(\alpha)(1+\gamma f'(\alpha))e_k + f'(\alpha)(1+\gamma f'(\alpha)(3+\lambda f'(\alpha)))c_2e_k^2 \\ +f'(\alpha)(2\gamma f'(\alpha)(1+\gamma f'(\alpha))c_2^2 + \gamma f'(\alpha)c_3 + (1+\gamma f'(\alpha))^3$$

$$c_3)e_k^3 + f'(\alpha)(c_4 + \gamma f'(\alpha)(\gamma f'(\alpha)c_2^3 + (1+\gamma f'(\alpha))(5+3\gamma f'(\alpha))$$
$$c_2c_3 + (5+\gamma f'(\alpha)(6+\gamma f'(\alpha)(4+\gamma f'(\alpha))))c_4))e_k^4 + \mathcal{O}(e_k^5). \tag{13}$$

Now by the Eqs. (11) and (13), we get that

$$f[x_k, w_k] = \tag{14}$$
$$= f'(\alpha) + f'(\alpha)(2+\gamma f'(\alpha))c_2e_k + f'(\alpha)(\gamma f'(\alpha)c_2^2 + (3+\gamma f'(\alpha)$$
$$(3+\gamma f'(\alpha)))c_3)e_k^2 + f'(\alpha)(2+\gamma f'(\alpha)(2+\gamma f'(\alpha))c_2c_3 + (2+\gamma f'(\alpha)(2$$
$$+\gamma f'(\alpha)))c_4)e_k^3 + f'(\alpha)(5c_5 + \gamma f'(\alpha)(\lambda f'(\alpha)c_2^2c_3 + (3+2\gamma f'(\alpha))$$
$$c_3^2 + (7+\gamma f'(\alpha)(8+3\gamma f'(\alpha)))c_2c_4 + (10+\gamma f'(\alpha)(10+\gamma f'(\alpha)$$
$$(5+\gamma f'(\alpha))))c_5))e_k^4 + \mathcal{O}(e_k^5).$$

Furthermore, we have

$$\frac{f(x_k)}{f[x_k,w_k]+\beta f(w_k)} =$$
$$= e_k - (1+\gamma f'(\alpha))(\beta + c_2)e_k^2 + ((\beta+\beta\gamma f'(\alpha))^2 + (2+\gamma f'(\alpha)$$
$$(2+\gamma f'(\alpha)))c_2(\beta+c_2) - 2c_3 - \gamma f'(\alpha)(3+\gamma f'(\alpha))c_3)e_k^3 +$$
$$+ (-(\beta+\beta\gamma f'(\alpha))^3 - \beta(5+\gamma f'(\alpha)(7+\gamma f'(\alpha)(4+\gamma f'(\alpha))))c_2^2$$
$$-(4+\gamma f'(\alpha)(5+\gamma f'(\alpha)(3+\gamma f'(\alpha))))c_2^3 + \beta(4+\gamma f'(\alpha)(7+$$
$$+\gamma f'(\alpha)(5+\gamma f'(\alpha))))c_3 + c_2(-\beta^2(1+\gamma f'(\alpha))(3+\gamma f'(\alpha)(2$$
$$+\gamma f'(\alpha))) + (7+\gamma f'(\alpha)(10+\gamma f'(\alpha)(7+2\gamma f'(\alpha))))c^3$$
$$-(1+\gamma f'(\alpha))(3+\gamma f'(\alpha)(3+\gamma f'(\alpha)))c_4)e_k^4 + \mathcal{O}(e_k^5). \tag{15}$$

Using the second step of (5) and $e_{k,y} = y_k - \alpha$, we get:

$$y_k = \tag{16}$$
$$= \alpha + (1+\beta f'(\alpha))(\beta+c_2)e_k^2 + (-(\beta+\beta\gamma f'(\alpha))^2 - (2+\lambda f'(\alpha)(2+\gamma f'(\alpha)))$$
$$c_2(\beta+c_2) + 2c_3 + \gamma f'(\alpha)(3+\gamma f'(\alpha))c_3e_k^3 + ((\beta+\beta\gamma f'(\alpha))^3 + \beta(5+\gamma$$
$$f'(\alpha)(7+\lambda f'(\alpha)(4+\lambda f'(\alpha))))c_2^2 + (4+\gamma f'(\alpha)(5+\gamma f'(\alpha)(3+\gamma f'(\alpha))))$$
$$c_2^3 - \beta(4+\gamma f'(\alpha)(7+\gamma f'(\alpha)(5+\gamma f'(\alpha))))c_3 + c_2(\beta^2(1+\gamma f'(\alpha))(3+\gamma$$
$$f'(\alpha))(2+\gamma f'(\alpha))) - (7+\gamma f'(\alpha)(10+\gamma f'(\alpha)(7+2\gamma f'(\alpha))))c_3)+$$
$$+ (1+\gamma f'(\alpha))(3+\gamma f'(\alpha)(3+\gamma f'(\alpha)))c_4)e_k^4 + \mathcal{O}(e_k^5).$$

For $f(y_k)$, we also have

$$f(y_k) \ = \ f'(\alpha)(1+\gamma f'(\alpha))(\beta+c_2)e_k^2 - f'(\alpha)((\beta+\beta\gamma f'(\alpha))^2(2+\gamma f'(\alpha)$$

$$(2+\gamma f'(\alpha)))c_2(\beta+c_2)-2c_3+\gamma f'(\alpha)(3+\gamma f'(\alpha))c_3)e_k^3+f'(\alpha)$$
$$((\beta+\beta\gamma f'(\alpha))^3+\beta(7+\gamma f'(\alpha)(11+\gamma f'(\alpha)(6+\gamma f'(\alpha))))c_2^2$$
$$+(5+\gamma f'(\alpha)(7+\gamma f'(\alpha)(4+\gamma f'(\alpha))))c_2^3-\beta(4+\gamma f'(\alpha)(7+$$
$$\gamma f'(\alpha)(5+\gamma f'(\alpha))))c_3+c_2(\beta^2(1+\gamma f'(\alpha))(4+\gamma f'(\alpha)(3+\gamma f'(\alpha)))$$
$$-(7+\gamma f'(\alpha)(10+\gamma f'(\alpha)(7+2\gamma f'(\alpha))))c_3)(1+\gamma f'(\alpha))(3+\gamma f'(\alpha)$$
$$(3+\gamma f'(\alpha)))c_4)e_k^4+\mathcal{O}(e_k^5). \tag{17}$$

Additionally, by using relations (12), (13), (16) and (17), we gain

$$\frac{f(y_k)}{f[y_k,w_k]+\beta f(w_k)+\lambda(y_k-w_k)(y_k-x_k)}$$
$$=(1+\gamma f'(\alpha))(\beta+c_2)e_k^2+(-2(\beta+\beta\gamma f'(\alpha))^2-\beta(4+3\gamma f'(\alpha)(2+$$
$$\gamma f'(\alpha)))c_2-(3+2\gamma f'(\alpha)(2+\gamma f'(\alpha)))c_2^2+(1+\gamma f'(\alpha))(2+\gamma f'(\alpha))$$
$$c_3)e_k^3+(\tfrac{1}{f'(\alpha)})(\beta f'(\alpha)(11+\gamma f'(\alpha)(19+\lambda f'(\alpha)(14+5\gamma f'(\alpha))))c_2^2$$
$$+f'(\alpha)(7+\gamma f'(\alpha)(11+\gamma f'(\alpha)(8+3\gamma f'(\alpha))))c_2^3-\beta f'(\alpha)(7+3\gamma f'(\alpha)$$
$$(5+\gamma f'(\alpha)(4+\gamma f'(\alpha))))c_3+c_2((1+\gamma f'(\alpha))(\beta^2 f'(\alpha)(8+\gamma f'(\alpha)(9+$$
$$5\gamma f'(\alpha)))-(1+\gamma f'(\alpha))\lambda)-2f'(\alpha)(5+\gamma f'(\alpha)(9+\gamma f'(\alpha)(7+2\gamma f'(\alpha))))$$
$$c_3)+(1+\gamma f'(\alpha))(\beta(1+\gamma f'(\alpha))(3\beta^2 f'(\alpha)(1+\gamma f'(\alpha))-\lambda)$$
$$+f'(\alpha)(3+\gamma f'(\alpha)(3+\gamma f'(\alpha)))c_4))e_k^4+\mathcal{O}(e_k^5). \tag{18}$$

Dividing these two relations (17) and (11) on each other gives us

$$s_k=\frac{f(y_k)}{f(x_k)}= \tag{19}$$
$$=(1+\gamma f'(\alpha))(\beta+c_2)e_k+(-(\beta+\beta\gamma f'(\alpha))^2-(3+\gamma f'(\alpha)$$
$$(3+\gamma f'(\alpha)))c_2(\beta+c_2)+2c_3+\gamma f'(\alpha)(3+\gamma f'(\alpha))c_3)e_k^2+$$
$$((\beta+\gamma f'(\alpha))^3+\beta(10+\gamma f'(\alpha)(14+\gamma f'(\alpha)(7+\gamma f'(\alpha))))$$
$$c_2^2+(2+\gamma f'(\alpha))(4+\gamma f'(\alpha)(3+\gamma f'(\alpha)))c_2^3-\beta(5+\gamma f'(\alpha)$$
$$(8+\gamma f'(\alpha)(5+\gamma f'(\alpha))))c_3+c_2(\beta^2(1+\gamma f'(\alpha))(5+\gamma f'(\alpha)$$
$$(4+\gamma f'(\alpha)-2(5+\gamma f'(\alpha)(7+\gamma f'(\alpha)(4+\gamma f'(\alpha))))c_3))\gamma f'(\alpha))^4$$
$$(1+\gamma f'(\alpha))(3+\gamma f'(\alpha)(3+\gamma f'(\alpha)))c_4)e_k^3(-(\beta+\beta$$
$$-\beta(30+\gamma f'(\alpha)(50+\gamma f'(\alpha)(34\gamma f'(\alpha)(11+\gamma f'(\alpha)))))$$
$$c_2^3-(2+\gamma f'(\alpha)(10+\gamma f'(\alpha)(10+\gamma f'(\alpha)(5+\gamma f'(\alpha)))))c_2^4+\beta^2$$

$$(1+\gamma f'(\alpha))(7+\gamma f'(\alpha)(10+\gamma f'(\alpha)(6+\gamma f'(\alpha))))c_3-(8+\gamma f'(\alpha)$$
$$(15+\gamma f'(\alpha)(13+\gamma f'(\alpha)(6+\gamma f'(\alpha)))))c_3^2+c_2^2(-\beta^2(20+\gamma f'(\alpha)$$
$$(41+\gamma f'(\alpha)(32+\gamma f'(\alpha)(11+\gamma f'(\alpha)))))+(37+\gamma f'(\alpha)(3+$$
$$\gamma f'(\alpha))(20+\gamma f'(\alpha)(8+3\gamma f'(\alpha))))c_3)-\beta(7+\gamma f'(\alpha)(3+\gamma$$
$$f'(\alpha))(5+\gamma f'(\alpha)(3+\gamma f'(\alpha))))c_4+c_2(-\beta^3(1+\gamma f'(\alpha))^2(7+\gamma f'(\alpha)$$
$$(5+\gamma f'(\alpha)))+\beta(2+\gamma f'(\alpha))(3+\gamma f'(\alpha))(5+\gamma f'(\alpha)(5+2\gamma$$
$$f'(\alpha)))c_3-(14+\gamma f'(\alpha)(5+2\gamma f'(\alpha))(5+\gamma f'(\alpha)(2+\gamma f'(\alpha)))c_3)$$
$$-(14+\gamma f'(\alpha)(5+2\gamma f'(\alpha))(5+\gamma f'(\alpha)(2+\gamma f'(\alpha))))c_4)+$$
$$(1+\gamma f'(\alpha))(2+\gamma f'(\alpha))(2+\gamma f'(\alpha)(2+\gamma f'(\alpha)))c_5)e_k^4+\mathcal{O}(e_k^5).$$

Expanding H at 0 yields

$$H(s_k)=H(0)+H'(0))s_k+H''(0)\tfrac{s_k^2}{2}; \tag{20}$$

and

$$H(s_k)=H(0)+H'(0)(1+\gamma f'(\alpha))(\beta+c_2)e_k+(\tfrac{1}{2}(H''(0)(1+\gamma f'(\alpha))^2$$
$$(\beta+c_2)^2-H'(0)((\beta+\beta\gamma f'(\alpha))^2+(3+\gamma f'(\alpha)(3+\gamma f'(\alpha)))$$
$$c_2(\beta)+c_2)-2c_3-\gamma f'(\alpha)(3+\gamma f'(\alpha))c_3))e_k^2+(-H''(0)(1+\gamma$$
$$f'(\alpha))(\beta+c_2)((\beta+\beta\gamma f'(\alpha))^2+(3+\gamma f'(\alpha)(3+\gamma f'(\alpha)))c_2$$
$$(\beta+c_2)-2c_3-\gamma f'(\alpha)(3+\gamma f'(\alpha))c_3)+H'(0)((\beta+\beta\lambda f'(\alpha))^3$$
$$+\beta(10+\gamma f'(\alpha)(14+\gamma f'(\alpha)(7+\gamma f'(\alpha))))c_2^2+(2+\gamma f'(\alpha))(4+$$
$$\gamma f'(\alpha)(3+\lambda f'(\alpha)))c_2^3-\beta(5+\gamma f'(\alpha)(8+\gamma f'(\alpha)(5+\gamma f'(\alpha))))c_3$$
$$+c_2(\beta^2(1+\gamma f'(\alpha))(5+\gamma f'(\alpha)(4+\gamma f'(\alpha)))-2(5+\gamma f'(\alpha)(7+$$
$$\gamma f'(\alpha)(4+\gamma f'(\alpha))))c_3)(1+\gamma f'(\alpha))(3+\gamma f'(\alpha)(3+\gamma f'(\alpha)))c_4))$$
$$e_k^3+\tfrac{1}{2}(-(2H'(0)-3H''(0))(\beta+\beta\gamma f'(\alpha))^4+2\beta(H''(0)(3+\gamma$$
$$f'(\alpha)(3+\gamma f'(\alpha)))(9+\gamma f'(\alpha)(11+3\gamma f'(\alpha)))-H'(0)(30+\gamma f'(\alpha)$$
$$(50+\gamma f'(\alpha)(34+\gamma f'(\alpha)(11+\gamma f'(\alpha))))))c_2^3+(H''(0)(25+3\gamma f'(\alpha)$$
$$(3+\gamma f'(\alpha))(6+\gamma f'(\alpha)(3+\gamma f'(\alpha))))-2H'(0)(2+\gamma f'(\alpha))(10+$$
$$\gamma f'(\alpha)(10+\gamma f'(\alpha)(5+\gamma f'(\alpha)))))c_2^4+2\beta^2(1+\gamma f'(\alpha))(H'(0)(7+$$
$$\gamma f'(\alpha)(10+\gamma f'(\alpha)(6+\gamma f'(\alpha))))-H''(0)(7+\gamma f'(\alpha)(13+\gamma f'(\alpha)$$
$$(9+2\gamma f'(\alpha)))))c_3+(H''(0)(1+\gamma f'(\alpha))^2(2+\gamma f'(\alpha))^2-2H'(0)(8$$
$$+\gamma f'(\alpha)(15+\gamma f'(\alpha)(13+\gamma f'(\alpha)(6+\gamma f'(\alpha))))))c_3^2+c_2^2(\beta^2$$
$$(-2H'(0)(20+\gamma f'(\alpha)(41+\gamma f'(\alpha)(32+\gamma f'(\alpha)(11+\gamma f'(\alpha)))))$$

$$
\begin{aligned}
&+H''(0)(45+\gamma f'(\alpha)(112+\gamma f'(\alpha)(105\gamma f'(\alpha)(44+7\gamma f'(\alpha))))))\\
&+2(H'(0)(37+\gamma f'(\alpha)(3+\gamma f'(\alpha))(20+\gamma f'(\alpha)(8+3\gamma f'(\alpha))))-\\
&H''(0)(1+\gamma f'(\alpha))(16+\gamma f'(\alpha)(23+\gamma f'(\alpha)(13+3\gamma f'(\alpha)f'(\alpha)))))c_3)\\
&+2\beta(H''(0)(1+\gamma f'(\alpha))^2(3+\gamma f'(\alpha)(3+\gamma f'(\alpha)))-H'(0)(7+\gamma f'(\alpha)\\
&(3+\gamma f'(\alpha))(5+\gamma f'(\alpha)(3+\gamma f'(\alpha))))c_4-2c_2(\beta^3(1+\gamma f'(\alpha))^2\\
&(-3H''(0)(3+\gamma f'(\alpha)(3+\gamma f'(\alpha)))+H'(0)(7+\gamma f'(\alpha)(5+\gamma f'(\alpha))))+\\
&\beta(H'(0)(2+\gamma f'(\alpha))(3+\gamma f'(\alpha))(5+\gamma f'(\alpha)(5+2\gamma f'(\alpha)))-H''(0)\\
&(1+\gamma f'(\alpha))(21+\gamma f'(\alpha)(31+2\gamma f'(\alpha)(9+2\gamma f'(\alpha)))))c_3|1+H''(0)(1+\gamma\\
&f'(\alpha))^2(3+\gamma f'(\alpha)(3+\gamma f'(\alpha)))-H'(0)(14+\gamma f'(\alpha)(5+2\gamma f'(\alpha))\\
&H'(0)(5+\gamma f'(\alpha))(2+\gamma f'(\alpha)))))c_4)+2H'(0)(1+\gamma f'(\alpha))\\
&(2+\gamma f'(\alpha))(2+\gamma f'(\alpha)(2+\gamma f'(\alpha)))c_5)e_k^4+\mathcal{O}(e_k^5).
\end{aligned}
$$

Finally, by placing $H(0) = H'(0) = H''(0) = 1$ as well as using equations (16), (18) and (2), the error equation of the memoryless method (5) will be as follows:

$$
\begin{aligned}
e_{k+1} \;=\;& (2f'(\alpha))^{-1}((1+\gamma f'(\alpha))^2(\beta+c_2)(f'(\alpha)\beta^2(1+\gamma f'(\alpha))+2\lambda+f'(\alpha)c_2\\
& (2\beta(3+\gamma f'(\alpha))+(5+\gamma f'(\alpha))+c_2-2f'(\alpha)c_3))e_k^4+\mathcal{O}(e_k^5)
\end{aligned}
\tag{21}
$$

which finishes the proof of the theorem. □

3. ACCELERATION OF THE TWO-POINT METHOD

It is easy to recognize from (21) that the order of convergence of (5) is four when $\gamma \neq \frac{-1}{f'(\alpha)}, \beta \neq -\frac{f''(\alpha)}{2f'(\alpha)}$ and $\lambda \neq \frac{f'''(\alpha)}{6}$. By taking the value of $\gamma_k = \frac{-1}{f'(\alpha)}, \beta_k = -\frac{f''(\alpha)}{2f'(\alpha)}$ and $\lambda_k = \frac{f'''(\alpha)}{6}$, it can be established that the order of the method (5) would be 6, 7, 7.22 and 7.53. For this type of acceleration of convergence and in actual fact the exact values of $f'(\alpha), f''(\alpha)$ and $f'''(\alpha)$ are not obtainable. We could replace the parameters γ, β and λ by γ_k, β_k and λ_k. In the remainder of this chapter, we consider the following two-parametric methods:

(a) If we only interpolate parameter γ_k using the Newton method, a procedure by six order with memory is obtained.

$$
\begin{cases}
\gamma_k = -\frac{1}{N_3'(x_k)}, k = 1, 2, 3, \dots,\\
H(0) = 1, H'(0) = 1, |H''(0)| < \infty, s_k = \frac{f(y_k)}{f(x_k)}, k = 0, 1, 2, \dots,\\
y_k = x_k - \frac{f(x_k)}{f[x_k, w_k] + \beta f(w_k)}, w_k = x_k + \gamma_k f(x_k),\\
x_{k+1} = y_k - H(s_k)\frac{f(y_k)}{f[y_k, w_k] + \beta f(w_k) + \lambda(y_k - x_k)(y_k - w_k)}.
\end{cases}
\tag{22}
$$

(b) We attempt to prove that the method with memory (5) has convergence order seven provided that we use accelerators γ_k, λ_k.

$$
\begin{cases}
\gamma_k = -\frac{1}{N_3'(x_k)}, \beta_k = -\frac{N_4'(w_k)}{2N_4''(w_k)}, k = 1, 2, 3, \ldots, \\
H(0) = 1, H'(0) = 1, |H''(0)| < \infty, s_k = \frac{f(y_k)}{f(x_k)}, k = 0, 1, 2, \ldots, \\
y_k = x_k - \frac{f(x_k)}{f[x_k, w_k] + \beta_k f(w_k)}, w_k = x_k + \gamma_k f(x_k), \\
x_{k+1} = y_k - H(s_k) \frac{f(y_k)}{f[y_k, w_k] + \beta_k f(w_k) + \lambda(y_k - x_k)(y_k - w_k)}.
\end{cases} \tag{23}
$$

(c) Bassiri *et al.* approximated self-accelerator parameters as

$$
\begin{cases}
\gamma_k = -\frac{1}{N_3'(x_k)} \simeq \frac{-1}{f'(\alpha)}, \\
\beta_k = -\frac{N_4''(w_k)}{2N_4'(w_k)} \simeq -\frac{f''(\alpha)}{2f'(\alpha)}, \\
\lambda_k = \frac{N_5'''(w_k)}{6} \simeq f'(\alpha) c_3 = \frac{f'''(\alpha)}{6},
\end{cases} \tag{24}
$$

and thus three parameters family with memory is given by (BBAM)

$$
\begin{cases}
\gamma_k = -\frac{1}{N_3'(x_k)}, \beta_k = -\frac{N_4'(w_k)}{2N_4''(w_k)}, \lambda_k = \frac{N_5'''(w_k)}{6}, k = 1, 2, 3, \ldots, \\
H(0) = 1, H'(0) = 1, |H''(0)| < \infty, s_k = \frac{f(y_k)}{f(x_k)}, k = 0, 1, 2, \ldots, \\
y_k = x_k - \frac{f(x_k)}{f[x_k, w_k] + \beta_k f(w_k)}, w_k = x_k + \gamma_k f(x_k), \\
x_{k+1} = y_k - H(s_k) \frac{f(y_k)}{f[y_k, w_k] + \beta_k f(w_k) + \lambda_k(y_k - x_k)(y_k - w_k)}.
\end{cases} \tag{25}
$$

(d) The self-accelerating parameters γ_k, β_k and λ_k are calculated by using of the formula:

$$
\begin{cases}
\gamma_k = -\frac{1}{N_3'(x_k)} \simeq \frac{-1}{f'(\alpha)}, \\
\beta_k = -\frac{N_4''(w_k)}{2N_4'(w_k)} \simeq -\frac{f''(\alpha)}{2f'(\alpha)}, \\
\lambda_k = \frac{N_5'''(y_k)}{6} \simeq f'(\alpha) c_3 = \frac{f'''(\alpha)}{6},
\end{cases} \tag{26}
$$

where $N_3(x_k); N_4(w_k)$ and $N_5(y_k)$ defined by:

$$
\begin{cases}
N_3(x_k) = N_3(t; x_k, x_{k-1}, w_{k-1}, y_{k-1}), \\
N_4(w_k) = N_4(t; w_k, x_k, x_{k-1}, w_{k-1}, y_{k-1}), \\
N_5(y_k) = N_5(t; y_k, w_k, x_k, x_{k-1}, w_{k-1}, y_{k-1}).
\end{cases} \tag{27}
$$

Now, we obtain the following three-parameter iterative with memory(TM1)method :

$$
\begin{cases}
\gamma_k = -\frac{1}{N_3'(x_k)}, \beta_k = -\frac{N_4'(w_k)}{2N_4''(w_k)}, \lambda_k = \frac{N_5'''(y_k)}{6}, k = 1, 2, 3, \ldots, \\
H(0) = 1, H'(0) = 1, |H''(0)| < \infty, s_k = \frac{f(y_k)}{f(x_k)}, k = 0, 1, 2, \ldots, \\
y_k = x_k - \frac{f(x_k)}{f[x_k, w_k] + \beta_k f(w_k)}, w_k = x_k + \gamma_k f(x_k), \\
x_{k+1} = y_k - H(s_k) \frac{f(y_k)}{f[y_k, w_k] + \beta_k f(w_k) + \lambda_k(y_k - x_k)(y_k - w_k)}.
\end{cases} \tag{28}
$$

It should be noted that the convergence order varies as the iteration go ahead. First, we need the following lemma:

Lemma 2. *If* $\gamma_k = -\frac{1}{N_3'(x_k)}, \beta_k = -\frac{N_4''(w_k)}{2N_4'(w_k)}$, *and* $\lambda_k = \frac{N_5'''(y_k)}{6}$, *then*

$$(1 + \gamma_k f'(\alpha)) \sim e_{k-1}e_{k-1,w}e_{k-1,y}, \tag{29}$$

$$(c_2 + \beta_k) \sim e_{k-1}e_{k-1,w}e_{k-1,y}, \tag{30}$$

$$\begin{aligned}(f'(\alpha)\beta^2(1+\gamma f'(\alpha)) + 2\lambda + f'(\alpha)c_2(2\beta(3+\gamma f'(\alpha))+ \\ +(5+\gamma f'(\alpha)) + c_2 - 2f'(\alpha)c_3)) \sim e_{k-1}e_{k-1,w}e_{k-1,y}\end{aligned} \tag{31}$$

where $e_k = x_k - \alpha, e_{k,w} = w_k - \alpha, e_{k,y} = y_k - \alpha$.

Proof. The proof is similar to Lemma 3.1 in [30]. □

Now we state the following convergence theorem:

Theorem 3. *If an initial approximation* x_0 *is sufficiently close to the zero* α *of* $f(x)$ *and the parameters* γ_k, β_k *and* λ_k *in the iterative schemes* (22), (23), (25) *and* (28) *are recursively calculated by the forms given in* (24) *and* (26). *Then, the R-order of convergence of the three-point methods* (22), (23), (25) *and* (28) *with the corresponding expressions* γ_k, β_k *and* λ_k *are at least* 6, 7, 7.22 *and* 7.53.

Proof. Here, we obtain the convergence order of 6 and 7.5 for the methods (22) and (28). Bassiri and his colleagues [2] have achieved the degree of convergence of the method mentioned in Equation (25). Proof of convergence of method (23) is similar to these three cases.

First we assume that the C-order of convergence of sequence x_k, w_k, y_k is at least r, p and q, respectively. Hence:

$$e_{k+1} \sim e_k^r \sim e_{k-1}^{r^2}, \tag{32}$$

$$e_{k,w} \sim e_k^p \sim e_{k-1}^{rp}, \tag{33}$$

and

$$e_{k,y} \sim e_k^q \sim e_{k-1}^{rq}. \tag{34}$$

By (32), (33), (34), and lemma(2), we obtain

$$1 + \gamma_k f'(\alpha) \sim e_{k-1}^{p+q+1}. \tag{35}$$

On the other hand, we get

$$e_{k,w} \sim (1 + \gamma_k f'(\alpha))e_k, \tag{36}$$

$$e_{k,y} \sim (1 + \gamma_k f'(\alpha))e_k^2, \tag{37}$$

$$e_{k+1} \sim (1 + \gamma_k f'(\alpha))^2 e_k^4. \tag{38}$$

Combining (32)–(38), (33)–(36), and (34)–(37), we conclude

$$e_{k,w} \sim e_{k-1}^{(1+p+q)+r}, \tag{39}$$

$$e_{k,y} \sim e_{k-1}^{(1+p+q)+2r}, \tag{40}$$

and

$$e_{k+1} \sim e_{k-1}^{2(1+p+q)+4r}. \tag{41}$$

Equating the powers of error exponents of e_{k-1} in pair relations (32)–(41), (33)–(39), and (34)–(40), we have

$$\begin{cases} rp - r - (p+q+1) = 0, \\ rq - 2r - (p+q+1) = 0, \\ r^2 - 4r - 2(p+q+1) = 0. \end{cases} \tag{42}$$

This system has the solution $p = 2, q = 4$ and $r = 6$ which specifies the C-order of convergence of the derivative-free scheme with memory (22). Varying parameters γ_k, β_k and λ_k in (28) using (26), we obtain the family of two-point with-memory methods of order 7.53, which is the improvement of the convergence rate of 88.25%. Similar to the first part of the Theorem 3:

$$e_{k+1} \sim e_k^r \sim e_{k-1}^{r^2}, \tag{43}$$

$$e_{k,w} \sim e_k^p \sim e_{k-1}^{rp}, \tag{44}$$

and

$$e_{k,y} \sim e_k^q \sim e_{k-1}^{rq}. \tag{45}$$

By (43), (44), (45), and Lemma 2, we obtain

$$1 + \gamma_k f'(\alpha) \sim e_{k-1}^{p+q+1}. \tag{46}$$

On the other hand, we get

$$e_{k,w} \sim (1 + \gamma_k f'(\alpha))e_k, \tag{47}$$

$$e_{k,y} \sim (1 + \gamma_k f'(\alpha))(\beta_k + c_2)e_k^2, \tag{48}$$

$$\begin{aligned} e_{k+1} \sim\ & ((1+\gamma f'(\alpha))^2(\beta + c_2)(f'(\alpha)\beta^2(1+\gamma f'(\alpha)) + 2\lambda + f'(\alpha)c_2 \\ & (2\beta(3+\gamma f'(\alpha)) + (5+\gamma f'(\alpha)) + c_2 - 2f'(\alpha)c_3))e_k^4 \end{aligned} \tag{49}$$

Combining (46)–(47), (46)–(48), and (46)–(49), we conclude

$$e_{k,w} \sim e_{k-1}^{(1+p+q)+r}, \tag{50}$$

$$e_{k,y} \sim e_{k-1}^{2(1+p+q)+2r}, \tag{51}$$

and

$$e_{k+1} \sim e_{k-1}^{4(1+p+q)+4r}. \tag{52}$$

Considering the error equations of e_{k-1}, e_w, e_y in pair relations of (44), (50), (45), (51), and (43), (52), we have

$$(53)\qquad \begin{cases} rp - r - (p+q+1) = 0, \\ rq - 2r - 2(p+q+1) = 0, \\ r^2 - 4r - 4(p+q+1) = 0. \end{cases}$$

Th positive solution for the system is $p = \frac{1}{8}(7+\sqrt{65}) \approx 1.88278, q = \frac{1}{4}(7+\sqrt{65}) \approx 3.76556$ and $r = \frac{1}{2}(7+\sqrt{65}) \approx 7.53113$. It specifies that the C-order for convergence of the derivative-free scheme with memory (28) is 7.53113. □

4. MAXIMUM IMPROVEMENT OF CONVERGENCE ORDER

Now, we can propose our iteration schemes (TM2) with memory,
(54)

$$\begin{cases} \gamma_k = -\frac{1}{N_6'(x_k)}, \beta_k = -\frac{N_7''(w_k)}{2N_7'(w_k)}, \lambda_k = \frac{N_8'''(y_k)}{6}, k = 2,3,4,\ldots, \\ w_k = x_k + \gamma_k f(x_k), y_k = x_k - \frac{f(x_k)}{f[x_k,w_k]+\beta_k f(w_k)}, s_k = \frac{f(y_k)}{f(x_k)}, k = 0,1,2,\ldots, \\ x_{k+1} = y_k - H(s_k)\frac{f(y_k)}{f[w_k,y_k]+\beta_k f(w_k)+\lambda_k(y_k-x_k)(y_k-w_k)}, k = 0,1,2,\ldots, \end{cases}$$

and, similarly, the following ones with better interpolation degrees (TM3):
(55)

$$\begin{cases} \gamma_k = -\frac{1}{N_9'(x_k)}, \beta_k = -\frac{N_{10}''(w_k)}{2N_{10}'(w_k)}, \lambda_k = \frac{N_{11}'''(y_k)}{6}, k = 3,4,5,\ldots, \\ w_k = x_k + \gamma_k f(x_k), y_k = x_k - \frac{f(x_k)}{f[x_k,w_k]+\beta_k f(w_k)}, s_k = \frac{f(y_k)}{f(x_k)}, k = 0,1,2,\ldots, \\ x_{k+1} = y_k - H(s_k)\frac{f(y_k)}{f[w_k,y_k]+\beta_k f(w_k)+\lambda_k(y_k-x_k)(y_k-w_k)}, k = 0,1,2,\ldots, \end{cases}$$

Also, we get our proposed two-step iterative method given for $k = 4; 5; 6; \ldots$, by(denoted (TM4))

$$(56)\qquad \begin{cases} \gamma_k = -\frac{1}{N_{12}'(x_k)}, \beta_k = -\frac{N_{13}''(w_k)}{2N_{13}'(w_k)}, \lambda_k = \frac{N_{14}'''(y_k)}{6}, k = 4,5,6,\ldots, \\ y_k = x_k - \frac{f(x_k)}{f[x_k,w_k]+\beta_k f(w_k)}, s_k = \frac{f(y_k)}{f(x_k)}, k = 0,1,2,\ldots, \\ x_{k+1} = y_k - H(s_k)\frac{f(y_k)}{f[w_k,y_k]+\beta_k f(w_k)+\lambda_k(y_k-x_k)(y_k-w_k)}, k = 0,1,2,\ldots. \end{cases}$$

As an illustration, here we can also define:

- $N_6(t) = N_6(t; x_k, x_{k-1}, w_{k-1}, y_{k-1}, x_{k-2}, w_{k-2}, y_{k-2})$, as an interpolation polynomial of sixth degree, passing through the best seven saved points
  $x_k, x_{k-1}, w_{k-1}, y_{k-1}, x_{k-2}, w_{k-2}, y_{k-2}$, for any $k \in \{2,3,4,\ldots\}$.
- $N_7(t) = N_7(t; w_k, x_k, x_{k-1}, w_{k-1}, y_{k-1}, x_{k-2}, w_{k-2}, y_{k-2})$, as an interpolation polynomial of seventh degree, passing through the best eight saved points $w_k, x_k, x_{k-1}, w_{k-1}, y_{k-1}, x_{k-2}, w_{k-2}, y_{k-2}$, for any $k \in \{2,3,4,\ldots\}$.
- $N_8(t) = N_8(t; y_k, w_k, x_k, x_{k-1}, w_{k-1}, y_{k-1}, x_{k-2}, w_{k-2}, y_{k-2})$, as an interpolation polynomial of eighth degree, passing through the best nine

saved points $y_k, w_k, x_k, x_{k-1}, w_{k-1}, y_{k-1}, x_{k-2}, w_{k-2}, y_{k-2}$, for any $k \in \{2, 3, 4, ...\}$.

The convergence analysis of (30), (31) and (32) can now be established as follows.

THEOREM 4. *Consider the same assumptions as in Theorem 3. Then, the C-order of the improved Steffensen's methods with memory (54), (55) and (56) are* 7.94449, 7.99315 *and* 7.99915, *respectively.*

Proof. The convergence of each of the methods mentioned in equations (54), (55) and (56) is given below separately.

Method TM2:

Let $\{x_k\}, \{w_k\}$, and $\{y_k\}$, be convergent with orders r, p, and q, respectively. Then:

$$e_{k+1} \sim e_k^r \sim e_{k-1}^{r^2} \sim e_{k-2}^{r^3} \sim e_{k-3}^{r^4} \sim e_{k-4}^{r^5}, \tag{57}$$

$$e_{k,w} \sim e_k^p \sim e_{k-1}^{pr} \sim e_{k-2}^{r^2 p} \sim e_{k-3}^{r^3 p} \sim e_{k-4}^{r^4 p}, \tag{58}$$

$$e_{k,y} \sim e_k^q \sim e_{k-1}^{qr} \sim e_{k-2}^{r^2 q} \sim e_{k-3}^{r^3 q} \sim e_{k-4}^{r^4 q}. \tag{59}$$

Using Theorem (1) and error equations (12), (16) and (21) we obtain:

$$e_{k,w} \sim (1 + \gamma_k f'(\alpha))e_k, \tag{60}$$

$$e_{k,y} \sim (1 + \gamma_k f'(\alpha))(\beta_k + c_2)e_k^2, \tag{61}$$

$$\begin{aligned} e_{k+1} \sim\ & ((1 + \gamma f'(\alpha))^2(\beta + c_2)(f'(\alpha)\beta^2(1 + \gamma f'(\alpha)) + 2\lambda + f'(\alpha)c_2 \\ & (2\beta(3 + \gamma f'(\alpha)) + (5 + \gamma f'(\alpha)) + c_2 - 2f'(\alpha)c_3))e_k^4. \end{aligned} \tag{62}$$

And

$$(1 + \gamma_k f'(\alpha)) \sim e_{k-2}e_{k-1}e_{k-2,w}e_{k-1,w}e_{k-2,y}e_{k-1,y}, \tag{63}$$

$$(\beta_k + c_2) \sim e_{k-2}e_{k-1}e_{k-2,w}e_{k-1,w}e_{k-2,y}e_{k-1,y}, \tag{64}$$

$$\begin{aligned} (f'(\alpha)\beta^2(1 + \gamma f'(\alpha)) + 2\lambda + f'(\alpha)c_2(2\beta(3 + \gamma f'(\alpha)) + \\ + (5 + \gamma f'(\alpha)) + c_2 - 2f'(\alpha)c_3)) \\ \sim e_{k-2}e_{k-1}e_{k-2,w}e_{k-1,w}e_{k-2,y}e_{k-1,y}. \end{aligned} \tag{65}$$

Combining (58), (60), (63) and (59), (61), (63), (64) also (57), (62), (63), (64), (65) we get

$$e_{k,w} \sim e_{k-2}^{1+r+p+rp+q+qr}, \tag{66}$$

and

$$e_{k,y} \sim e_{k-2}^{2(1+r+p+rp+q+qr)}, \tag{67}$$

also

$$e_{k+1} \sim e_{k-2}^{4(1+r+p+rp+q+qr)}, \tag{68}$$

Comparing the right and left side of error equations (58), (66) and (59), (67), and (60), (68), we have:

$$\begin{cases} r^2p = (1+r+r^2+p+pr+q+qr), \\ r^2q = 2(1+r+r^2+p+pr+q+qr), \\ r^3 = 4(1+r+r^2+p+pr+q+qr). \end{cases} \tag{69}$$

The positive real solution of this system is $p_1 \simeq 1.98612, p_2 \simeq 3.97225$ and $r \simeq 7.94449$. We conclude that the C-order of the methods with memory (54) is at least 7.94449.

Method TM3:

Similar to the previous method:

$$(1+\gamma_k f'(\alpha)) \sim e_{k-3}e_{k-2}e_{k-1}e_{k-3,w}e_{k-2,w}e_{k-1,w}e_{k-3,y}e_{k-2,y}e_{k-1,y}, \tag{70}$$

$$(\beta_k + c_2) \sim e_{k-3}e_{k-2}e_{k-1}e_{k-3,w}e_{k-2,w}e_{k-1,w}e_{k-3,y}e_{k-2,y}e_{k-1,y}, \tag{71}$$

$$\begin{aligned}(f'(\alpha)\beta^2(1+\gamma f'(\alpha))+ \\ &+ 2\lambda + f'(\alpha)c_2(2\beta(3+\gamma f'(\alpha))+ \\ &+ (5+\gamma f'(\alpha)) + c_2 - 2f'(\alpha)c_3)) \\ &\sim e_{k-3}e_{k-2}e_{k-1}e_{k-3,w}e_{k-2,w}e_{k-1,w}e_{k-3,y}e_{k-2,y}e_{k-1,y}.\end{aligned} \tag{72}$$

Combining (58), (60), (70) and (59), (61), (70), (71) also (57), (62), (63), (64), (65), (70), (71), (72) we get

$$e_{k,w} \sim e_{k-3}^{1+r+r^2+p+rp+r^2p+q+qr+r^2q}, \tag{73}$$

and

$$e_{k,y} \sim e_{k-3}^{1+r+r^2+p+rp+r^2p+q+qr+r^2q} \tag{74}$$

also

$$e_{k+1} \sim e_{k-3}^{1+r+r^2+p+rp+r^2p+q+qr+r^2q} \tag{75}$$

Comparing the right and left side of error equations (60), (73) and (61), (74), also (62), (75), we obtained the following system of equations:

$$\begin{cases} r^3p = (1+r+r^2+p+pr+r^2p+q+qr+r^2q), \\ r^3q = 2(1+r+r^2+p+pr+r^2p+q+qr+r^2q), \\ r^4 = 4(1+r+r^2+p+pr+r^2p+q+qr+r^2q). \end{cases} \tag{76}$$

Positive solution of this system is $p \simeq 1.99829, q \simeq 3.99657$ and $r \simeq 7.99315$. Therefore, the C-order of the methods with memory (54) is at least 7.99315.

Method TM4

Using the result of the two methods TM2, TM3 and lemma (2) we have

$$
\begin{aligned}
&(1+\gamma_k f'(\alpha)) \sim \prod_{s=0}^{k-4} e_s e_{s,w} e_{s,y},\\
&(\beta_k + c_2) \sim \prod_{s=0}^{k-4} e_s e_{s,w} e_{s,y},\\
&(f'(\alpha)\beta^2(1+\gamma f'(\alpha)) + 2\lambda + f'(\alpha)c_2(2\beta(3+\gamma f'(\alpha))+\\
&(5+\gamma f'(\alpha)) + c_2 - 2f'(\alpha)c_3))\\
&\sim \prod_{s=0}^{k-4} e_s e_{s,w} e_{s,y}. \qquad (77)
\end{aligned}
$$

Compare the right and left side of error equations (60), (77) and (61), (77), and (62), (77), we have:

$$e_{k,w} \sim e_{k-4}^{1+r+r^2+r^3+p+pr+r^2p+r^3p+q+qr+r^2q+r^3q}, \qquad (78)$$

and

$$e_{k,y} \sim e_{k-4}^{1+r+r^2+r^3+p+pr+r^2p+r^3p+q+qr+r^2q+r^3q} \qquad (79)$$

also

$$e_{k+1} \sim e_{k-4}^{1+r+r^2+r^3+p+pr+r^2p+r^3p+q+qr+r^2q+r^3q} \qquad (80)$$

Comparing the right and left side of error equations (60), (78) and (61), (79), also (62), (80), we have. In the similar fashion we find the final system equation which is given by

(81)

$$
\begin{cases}
r^4 p = (1+r+r^2+r^3+p+pr+r^2p+r^3p+q+qr+r^2q+r^3q),\\
r^4 q = 2(1+r+r^2+r^3+p+pr+r^2p+r^3p+q+qr+r^2q+r^3q),\\
r^5 = 4(1+r+r^2+r^3+p+pr+r^2p+r^3p+q+qr+r^2q+r^3q).
\end{cases}
$$

Since positive solution of this system is $p = 1.99979, q = 3.99957$ and, $r = 7.99915 \approx 8$, and therefore, we conclude that the C-order of the methods with memory (56) is at least $7.99915 \approx 8$.

Therefore, the proof of the Theorem is completed. □

5. NUMERICAL RESULTS AND COMPARISONS

Let $e_k = x_k - \alpha$ be the error in the k^{th} iteration, we call the relation

$$e_{k+1} = Ce_k^p + \mathcal{O}(e_k^{p+1}) \qquad (82)$$

as the error equation. If we can obtain error equation for any iterative method, then the value of p is the order of convergence.

Suppose that x_k, x_{k-1} and x_{k+1} are three successive iterations closer to the root α. Then, the computational order of convergence ρ (see [8]) is approximated by using (8) as

$$\rho = \frac{\log \frac{|x_{k+1}-\alpha|}{|x_k-\alpha|}}{\log \frac{|x_k-\alpha|}{|x_{k-1}-\alpha|}}. \tag{83}$$

We now present some examples to comparison the performance of present methods $TM1, TM2, TM3, TM4, TM, NM, BBAM$. And two-step with memory methods of Soleymani *et al.* (SLTKM) with convergence order of 7.22 [27] are defined as:

$$\begin{cases} \gamma_k = -\frac{1}{N_3'(x_k)}, \beta_k = -\frac{N_4''(w_k)}{2N_4'(w_k)}, \lambda_k = \frac{N_5'''(w_k)}{6}, k = 1, 2, 3, \ldots, \\ y_k = x_k - \frac{f(x_k)}{f[x_k, w_k] + \beta_k f(w_k)}, k = 0, 1, 2, \ldots, \\ x_{k+1} = y_k - \frac{f(y_k)}{f[w_k, y_k] + \beta_k f(w_k) + \lambda_k (y_k - x_k)(y_k - w_k)} (1 + \frac{f(y_k)}{f(x_k)}). \end{cases} \tag{84}$$

Computations are performed using MATHEMATICA software. A comparison between without memory and with memory methods in terms of the maximum convergence order alongside the number of steps per cycle are given in Fig. 5.1

Table 1. Test functions

Nonlinear function	Zero	Initial guess
$f_1(x) = t\log(1 + x\sin(x)) + e^{-1+x^2+x\cos(x)}\sin(\pi x)$	$\alpha = 0$	$x_0 = 0.6$
$f_2(x) = 1 + \frac{1}{x^4} - \frac{1}{x} - x^2$	$\alpha = 1$	$x_0 = 1.4$
$f_3(x) = e^{x^3-x} - \cos(x^2-1) + x^3 + 1$	$\alpha = -1$	$x_0 = -1.5$
$f_4(x) = \frac{-5x^2}{2} + x^4 + x^5 + \frac{1}{1+x^2}$	$\alpha = 1$	$x_0 = 1.4$
$f_5(x) = \log(1 + x^2) + e^{-3x+x^2}\sin(x)$	$\alpha = 0$	$x_0 = 0.4$
$f_6(x) = \frac{8}{17} - \sqrt{6} + \frac{x^3}{1+x^4} + \sqrt{8+x^4}\sin(\frac{\pi}{2+x^2})$	$\alpha = \sqrt{\pi}$	$x_0 = 1.7$

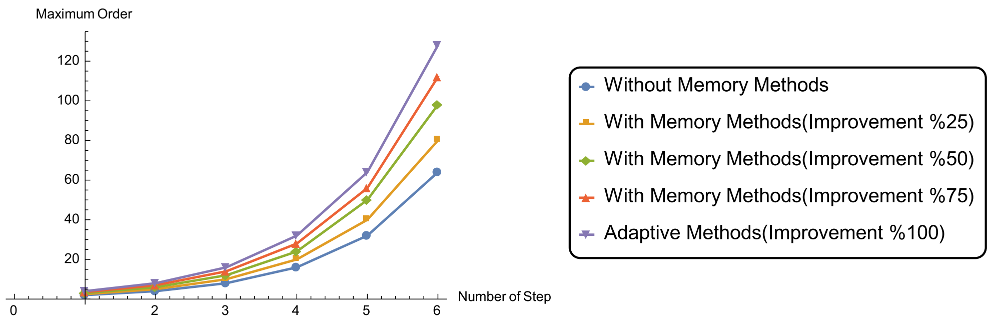

Fig. 5.1. Comparison of methods without memory, with memory and adaptive (%25, %50, %75, and %100 of improvements) in terms of the highest possible convergence order.

Table 2. Numerical results for $f_1(x), f_2(x)$

Methods	$\lvert x_1-\alpha\rvert$	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$	ρ
$f_1(x), \beta_0=\gamma_0=\lambda_0=0.1$				
BBAM (25), H_1	0.21313(0)	0.39355(−5)	0.10883(−39)	7.2456
BBAM (25), H_2	0.21414(0)	0.38636(−5)	0.93320(−40)	7.2457
BBAM (25), H_3	0.21455(0)	0.38340(−5)	0.87536(−40)	7.2457
BBAM (25), H_4	0.20150(0)	0.45682(−5)	0.37936(−39)	7.2453
TM1 (28), H_1	0.21313(0)	0.31807(−5)	0.12473(−43)	7.5517
TM4 (32), H_1	0.21313(0)	0.42385(−5)	0.16123(−42)	8.0000
TM1 (28), H_2	0.21414(0)	0.31172(−5)	0.10374(−43)	7.5518
TM4 (32), H_2	0.21414(0)	0.41544(−5)	0.13741(−42)	8.0000
TM1 (28), H_3	0.21455(0)	0.30912(−5)	0.96118(−44)	7.3753
TM4 (32), H_3	0.21455(0)	0.41199(−5)	0.12855(−42)	8.0000
TM1 (28), H_4	0.20150(0)	0.37754(−5)	0.54962(−43)	7.5514
TM4 (32), H_4	0.20150(0)	0.50018(−5)	0.60479(−42)	8.0000
$f_2(x), \beta_0=\gamma_0=\lambda_0=0.1$				
BBAM (25), H_1	0.46835(−2)	0.60268(−15)	0.27605(−108)	7.2388
BBAM (25), H_2	0.46282(−2)	0.58207(−15)	0.21389(−108)	7.2389
BBAM (25), H_3	0.46040(−2)	0.57310(−15)	0.19088(−108)	7.2389
BBAM (25), H_4	0.60717(−2)	0.11532(−14)	0.31089(−106)	7.2389
TM1 (28), H_1	0.46835(−2)	0.82793(−16)	0.13403(−119)	7.5317
TM4 (32), H_1	0.46835(−2)	0.60206(−15)	0.93782(−120)	8.0000
TM1 (28), H_2	0.46282(−2)	0.79699(−16)	0.10053(−119)	7.5317
TM4 (32), H_2	0.46282(−2)	0.58147(−15)	0.70999(−120)	8.0000
TM1 (28), H_3	0.46040(−2)	0.78359(−16)	0.88431(−120)	7.5220
TM4 (32), H_3	0.46040(−2)	0.57252(−15)	0.62711(−120)	8.0000
TM1 (28), H_4	0.60711(−2)	0.17133(−15)	0.29944(−117)	7.5318
TM4 (32), H_4	0.60711(−2)	0.11520(−14)	0.16842(−117)	8.0000

Table 3. Numerical results for $f_3(x), f_4(x)$

Methods	$\|x_1-\alpha\|$	$\|x_2-\alpha\|$	$\|x_3-\alpha\|$	COC
$f_3(x), \beta_0=\gamma_0=\lambda_0=0.1$				
BBAM (25), H_1	0.56205(−4)	0.57879(−24)	0.14139(−179)	7.2417
BBAM (25), H_2	0.56128(−4)	0.57562(−24)	0.13570(−179)	7.2417
BBAM (25), H_3	0.56090(−4)	0.57408(−24)	0.13299(−179)	7.2417
BBAM (25), H_4	0.70544(−4)	0.14312(−23)	0.12585(−175)	7.2416
TM1 (28), H_1	0.56205(−4)	0.12031(−25)	0.46980(−200)	7.5335
TM4 (32), H_1	0.56205(−4)	0.57879(−24)	0.11038(−194)	8.0000
TM1 (28), H_2	0.56128(−4)	0.11966(−25)	0.44966(−200)	7.5335
TM4 (32), H_2	0.56128(−4)	0.57562(−24)	0.10564(−194)	8.0000
TM1 (28), H_3	0.56090(−4)	0.11934(−25)	0.44013(−200)	7.4971
TM4 (32), H_3	0.56090(−4)	0.57408(−24)	0.10340(−194)	8.0000
TM1 (28), H_4	0.70544(−4)	0.29684(−25)	0.65051(−197)	7.5334
TM4 (32), H_4	0.70544(−4)	0.14312(−23)	0.15426(−191)	8.0000
$f_4(x), \beta_0=\gamma_0=\lambda_0=0.1$				
BBAM (25), H_1	0.19289(0)	0.37539(−3)	0.46525(−24)	7.2489
BBAM (25), H_2	0.19024(0)	0.34800(−3)	0.24511(−24)	7.2498
BBAM (25), H_3	0.18803(0)	0.32640(−3)	0.14220(−24)	7.2506
BBAM (25), H_4	0.18331(−1)	0.68016(−10)	0.48337(−73)	7.2407
TM1 (28), H_1	0.19289(0)	0.35581(−3)	0.57300(−24)	7.5060
TM4 (32), H_1	0.19289(0)	0.32894(−3)	0.25124(−24)	8.0000
TM1 (28), H_2	0.19024(0)	0.33030(−3)	0.31618(−24)	7.5065
TM4 (32), H_2	0.19024(0)	0.30520(−3)	0.13800(−24)	8.0000
TM1 (28), H_3	0.18803(0)	0.31015(−3)	0.19123(−24)	7.7585
TM4 (32), H_3	0.18803(0)	0.28646(−3)	0.83136(−25)	8.0000
TM1 (28), H_4	0.18331(−1)	0.12512(−9)	0.28649(−75)	7.5310
TM4 (32), H_4	0.18331(−1)	0.58632(−10)	0.25626(−78)	8.0000

6. CONCLUSION

We have proposed the new two-step tri-parametric family of iterative methods by using four weight functions. In all these techniques, the proper initial guess is necessary for the convergent [29]. These techniques have been used in Tables 2-4. There are two main advantages of the adaptive-type methods. Firstly, these new methods have the best efficiency index (equal 2), and second, the convergence order of eight for the proposed methods have been formulated with only three function evaluations compared to other with and without memory methods. From Tables 2,3,4, and 5, it can obtain that the adaptive-methods perform better than the considered methods in this study, while the degree of convergence improved up to 100% [1, 2, 4, 7, 9, 11, 13, 14, 15, 16, 17, 18, 19, 24, 27, 30, 31, 32, 33, 34].

Table 4. Numerical results for $f_5(x), f_6(x)$

Methods	$\lvert x_1-\alpha\rvert$	$\lvert x_2-\alpha\rvert$	$\lvert x_3-\alpha\rvert$	COC
$f_5(x), \beta_0=\gamma_0=\lambda_0=0.1$				
BBAM (25), H_1	$0.72896(-2)$	$0.23509(-15)$	$0.55235(-112)$	7.2400
BBAM (25), H_2	$0.72908(-2)$	$0.23480(-15)$	$0.54729(-112)$	7.2400
BBAM (25), H_3	$0.72915(-2)$	$0.23465(-15)$	$0.54465(-112)$	7.2400
BBAM (25), H_4	$0.74155(-2)$	$0.20479(-15)$	$0.19725(-112)$	7.2173
TM1 (28), H_1	$0.72896(-2)$	$0.49295(-16)$	$0.12794(-122)$	7.5330
TM4 (32), H_1	$0.72896(-2)$	$0.23502(-15)$	$0.27176(-122)$	8.0000
TM1 (28), H_2	$0.72908(-2)$	$0.49241(-16)$	$0.12677(-122)$	7.5330
TM4 (32), H_2	$0.72908(-2)$	$0.23473(-15)$	$0.26909(-122)$	8.0000
TM1 (28), H_3	$0.72915(-2)$	$0.49213(-16)$	$0.12616(-122)$	7.5056
TM4 (32), H_3	$0.72915(-2)$	$0.23458(-15)$	$0.26870(-122)$	8.0000
TM1 (28), H_4	$0.74155(-2)$	$0.43552(16)$	$0.45392(-123)$	7.5050
TM4 (32), H_4	$0.74155(-2)$	$0.20474(-15)$	$0.90133(-123)$	8.0000
$f_6(x), \beta_0=\gamma_0=\lambda_0=0.1$				
BBAM (25), H_1	$0.51844(-4)$	$0.24676(-30)$	$0.12709(-229)$	7.2389
BBAM (25), H_2	$0.51846(-4)$	$0.24680(-30)$	$0.12724(-223)$	7.2389
BBAM (25), H_3	$0.51847(-4)$	$0.24682(-30)$	$0.12731(-223)$	7.2389
BBAM (25), H_4	$0.50848(-4)$	$0.23052(-30)$	$0.75805(-224)$	7.2389
TM1 (28), H_1	$0.51844(-4)$	$0.43704(-33)$	$0.29624(-251)$	7.5307
TM4 (32), H_1	$0.51844(-4)$	$0.24676(-30)$	$0.10414(-245)$	8.0000
TM1 (28), H_2	$0.51846(-4)$	$0.43707(-33)$	$0.29651(-251)$	7.5307
TM4 (32), H_2	$0.51846(-4)$	$0.24680(-530)$	$0.10427(-245)$	8.0000
TM1 (28), H_3	$0.51847(-4)$	$0.43709(-33)$	$0.29664(-251)$	7.5370
TM4 (32), H_3	$0.51847(-4)$	$0.24682(-30)$	$0.10433(-245)$	8.0000
TM1 (28), H_4	$0.50848(-4)$	$0.42256(-33)$	$0.19722(-251)$	7.5365
TM4 (32), H_4	$0.50848(-4)$	$0.23052(-30)$	$0.60406(-246)$	8.0000

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Table 5. Comparison improvement of convergence order the proposed method with other schemes

with memory methods	optimal order	p	percentage increase
BBAM[2]	4.000	7.240	%81
CPJM[4]	4.000	4.560	%14
CPJM[4]	4.000	4.790	%19.75
CPJM[4]	4.000	5.000	%20
CLTAMM[7]	4.000	7.000	%75
EM[9]	8.000	12.000	%50
JM[11]	4.000	7.000	%75
JM[11]	8.000	14.000	%75
LMBSM[13]	4.000	6.000	%50
LMBSM[13]	4.000	5.200	%30
LAM[15]	4.000	5.240	%31
LAM[15]	4.000	6.000	%50
LSGAM[16]	4.000	7.770	%94.25
LSGAM[16]	8.000	12.000	%50
LSNKKM[17]	4.000	6.000	%50
LSNKKM[17]	8.000	12.000	%50
MLAM[19]	4.000	5.700	%42.5
MLAM[19]	4.000	5.950	%48.75
NM[20]	8.000	10.815	%35.19
PIDM[24]	4.000	4.449	%11.23
SLTKM[27]	8.000	12.000	%50
SLTKM[27]	4.000	7.240	%81
SM[28]	8.000	10.000	%81
TKM[30]	4.000	7.000	%75
TKM[30]	8.000	14.000	%75
TKM[30]	16.000	28.000	%75
TM[31]	2.000	2.410	%20.5
WM[32]	4.000	4.449	%11.23
WM[33]	4.000	5.702	%42.55
TM1(28)	4.000	7.531	%83.78
TM2(30)	4.000	7.945	%98.63
TM3(31)	4.000	7.993	%99.83
TM4(32)	4.000	8.000	%100

REFERENCES

[1] I.K. Argyros, *Approximating solutions of equations using Newton's method with a modified Newton's method iterate as a starting point*, J. Numer. Anal. Approx. Theory, **36** (2007), pp. 123–138, `https://ictp.acad.ro/jnaat/journal/article/view/2007-vol36-no2-art2`.

[2] P. Bassiri, P. Bakhtiari, S. Abbasbandy, *A new iterative with memory class for solving nonlinear equations*, Int. J. Ind. Math., **8** (2016), pp. 225–229.

[3] R. Behl, S.S. Motsa, *Higher order two-point efficient family of Halley type methods for simple roots*, Intl. J. Comput. Meth., **13** (2016), pp. 1–16, `https://doi.org/10.1142/s0219876216410164`.

[4] N. Choubey, B. Panday, J.P. Jaiswal, *Several two-point with memory iterative methods for solving nonlinear equations*, Afr. Mat., (2018), pp. 1–15, https://doi.org/10.1007/s13370-018-0552-x.

[5] C. Chun, *Some fourth-order iterative methods for solving nonlinear equationss*, Appl. Math. Comput., **195** (2008), pp. 454–459, https://doi.org/10.1016/j.amc.2007.04.105.

[6] A. Cordero, T. Lotfi, K. Mahdiani, J.R. Torregrosa, *Two optimal general classes of iterative methods with eighth-order*, Acta Appl. Math., **134** (2014), pp. 61–74, https://doi.org/10.1007/s10440-014-9869-0.

[7] A. Cordero, T. Lotfi, J.R. Torregrosa, P. Assari, K. Mahdiani, *Some new bi-accelarator two-point methods for solving nonlinear equations*, Comput. Appl. Math., **35** (2016), pp. 251–267, https://doi.org/10.1007/s40314-014-0192-1.

[8] E. Catinas, *A survey on the convergence orders and the computational convergence orders of sequences*, Appl. Math. Comput., **343** (2019), pp. 1–20, https://doi.org/10.1016/j.amc.2018.08.006.

[9] T. Eftekhari, *An efficient class of multipoint root-solvers with and without memory for nonlinear equations*, Acta. Math. Viet., **2015** (2015) pp. 1–13, https://doi.org/10.1007/s40306-015-0132-1.

[10] Y.H. Geum, Y.I. Kim, *A biparametric family optimally convergent sixteenth-order multipoint methods with their fourth-step weighting function as a sum of a rational and a generic two-variable function*, J. Comput. Appl. Math., **235** (2011), pp. 3178–3188, https://doi.org/10.1016/j.cam.2011.01.003.

[11] J.P. Jaiswal, *Two efficient bi-parametric derivative free with memory methods for finding simple roots of nonlinear equations*, J. Adv. Appl. Math., **1** (2016), pp. 203–210, https://doi.org/10.22606/jaam.2016.14001.

[12] H.T. Kung, J.F. Traub, *Optimal order of one-point and multipoint iteration*, J. Assoc. Comput. Mach., **21** (1974), pp. 643–651, https://doi.org/10.1145/321850.321860.

[13] T. Lotfi, K. Mahdiani, P. Bakhtiari, F. Soleymani, *Constructing two-step iterative methods with and without memory*, Comput. Math. Math. Phys., **55** (2015), pp. 183–193, https://doi.org/10.1134/s0965542515020189.

[14] T. Lotfi, F. Soleymani, S. Sharifi, S. Shateyi, F.K. Haghani, *Multipoint iterative methods for finding all the simple zeros in an interval*, J. Appl. Math., (2014), pp. 1–14, https://doi.org/10.1155/2014/601205.

[15] T. Lotfi, P. Assari, *A new two-step class of methods with memory for solving nonlinear equations with high efficiency index*, Int. J. Math. Model. Comput., **4** (2014), pp. 277–288.

[16] T. Lotfi, F. Soleymani, M. Ghorbanzadeh, P. Assari, *On the cunstruction of some tri-parametric iterative methods with memory*, Numer. Algor., **70** (2015), pp. 835–845, https://doi.org/10.1007/s11075-015-9976-7.

[17] T. Lotfi, F. Soleymani, Z. Noori, A. Kilicman, F. Khaksar Haghani, *Efficient iterative methods with and without memory possessing high efficiency indices*, Disc. Dyna. Natu. Soci., **2014** (2014), pp. 1–9, https://doi.org/10.1155/2014/912796.

[18] A.K. Maheshwari, *A fourth order iterative method for solving nonlinear equations*, Appl. Math. Comput., **211** (2009), pp. 383–391, https://doi.org/10.1016/j.amc.2009.01.047.

[19] M. Mohamadi Zadeh, T. Lotfi, M. Amirfakhrian, *Developing two efficient adaptive Newton-type methods with memory*, Math. Meth. Appl. Sci., **2018** (2018), pp. 1–9.

[20] B. Neta, *A new family of higher order methods for solving equations*, Int. J. Comput. Math., **14** (1983), pp. 191–195, https://doi.org/10.1080/00207168308803384.

[21] J.M. Ortega, W.C. Rheinboldt, *Iterative Solutions of Nonlinear Equations in Several Variables*, Academic Press, New York, 1970.

[22] A.M. Ostrowski, *Solution of Equations and Systems of Equations*, Academic Press, New York, 1960.

[23] M.S. Petkovic, B. Neta, L.D. Petkovic, J. Dzunic, *Multipoint Methods for Solving Nonlinear Equations*, Elsevier, Amsterdam, 2013.

[24] M.S. Petkovic, S. Ilic, J. Dzunic, *Derivative free two-point methods with and without memory for solving nonlinear equations*, Appl. Math. Comput., **217** (2010), pp. 1887–1895, `https://doi.org/10.1016/j.amc.2010.06.043`.

[25] S. Sharifi, M. Salimi, S. Siegmund, T. Lotfi, *A new class of optimal four-point methods with convergence order 16 for solving nonlinear equations*, Math. Comput. Simul., **2015** (2015), pp. 1–26.

[26] F. Soleymani, *On a bi-parametric classes of optimal eight-order derivative-free methods*, Int. J. Pure Appl. Math., **72** (2011), pp. 27–37.

[27] F. Soleymani, T. Lotfi, E. Tavakoli, F. Khaksar Haghani, *Several iterative methods with memory using self-accelerators*, Appl. Math. Comput., **254** (2015), pp. 452–458, `https://doi.org/10.1016/j.amc.2015.01.045`.

[28] F. Soleymani, *Some optimal iterative methods and their with memory variants*, J. Egypt. Math. Soc., **2013** (2013), pp. 1–9.

[29] V. Torkashvand, T. Lotfi, M.A. Fariborzi Araghi, *A new family of adaptive methods with memory for solving nonlinear equations*, Math. Sci., **13** (2019), pp. 1–20.

[30] V. Torkashvand, M. Kazemi, *On an efficient family with memory with high order of convergence for solving nonlinear equations*, Int. J. Ind. Math., **12** (2020), pp. 209–224.

[31] J.F. Traub, *Iterative Methods for the Solution of Equations*, Prentice Hall, New York, USA, 1964.

[32] X. Wang, *A new accelerating technique applied to a variant of Cordero-Torregrosa method*, J. Comput. Appl. Math., **330** (2018), pp. 695–709, `https://doi.org/10.1016/j.cam.2017.07.003`.

[33] X. Wang, T. Zhang, *High-order Newton-type iterative methods with memory for solving nonlinear equations*, Math. Commun., **19** (2014), pp. 91–109 `https://hrcak.srce.hr/121828`.

[34] T. Zhanlav, O. Chuluunbaatar, V. Ulziibayar, *Accelerating the convergence of Newton-type iterations*, J. Numer. Anal. Approx. Theory, **46** (2017), pp. 162–180, `https://ictp.acad.ro/jnaat/journal/article/view/1105`.

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