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APPROXIMATION BY MATRIX TRANSFORM IN GENERALIZED GRAND LEBESGUE SPACES WITH VARIABLE EXPONENT

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Abstract. In this work, the Lipschitz subclass of the generalized grand Lebesgue space with variable exponent is defined and the error of approximation by matrix transforms in this subclass is estimated.

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1. INTRODUCTION

Let $\mathbb{T} := [0, 2\pi]$ and let $p(\cdot) : \mathbb{T} \to [0, \infty)$ be a Lebesgue measurable 2π periodic function satisfying the conditions

$$1 \leq p_- := \operatorname{ess\,inf}_{x\in\mathbb{T}}\ p(x) \leq \operatorname{ess\,sup}_{x\in\mathbb{T}}\ p(x) := p^+ < \infty,$$

$$|p(x) - p(y)| \ln\left(\tfrac{2\pi}{|x-y|}\right) \leq c, \qquad x, y \in [0, 2\pi] \text{ and } |x-y| \leq 1/2, x \neq y$$

with some positive constant $c = c(p)$. From now on, the class of such functions $p(\cdot)$ we denote by $\mathcal{P}(\mathbb{T})$. We also denote $\mathcal{P}_0(\mathbb{T}) := \{p(\cdot) \in \mathcal{P}(\mathbb{T}) : p_- > 1\}$. The Lebesgue space $L^{p(\cdot)}(\mathbb{T})$ with variable exponent is defined as the set of all Lebesgue measurable 2π periodic functions f such that

$$\rho_{p(\cdot)}(f) := \int_0^{2\pi} |f(x)|^{p(x)}\, \mathrm{d}x < \infty.$$

If $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, then equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \leq 1\right\}$$

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$L^{p(\cdot)}(\mathbb{T})$ becomes a Banach space. Let $\theta \geqslant 0$ and $p \in \mathcal{P}_0(\mathbb{T})$. The generalized grand Lebesgue space with variable exponent $L^{p(\cdot),\theta}(\mathbb{T})$ is the set of all measurable 2π periodic functions $f : \mathbb{T} \to \mathbb{R}$ such that

$$\|f\|_{p(\cdot),\theta} = \sup_{0<\varepsilon<p_--1} \varepsilon^{\frac{\theta}{p_--\varepsilon}} \|f\|_{p(\cdot)-\varepsilon} < \infty.$$

It is easily seen that

$$L^{p(\cdot)}(\mathbb{T}) \subset L^{p(\cdot),\theta}(\mathbb{T}) \subset L^{p(\cdot)-\varepsilon}(\mathbb{T}), \qquad 0 < \varepsilon < p_- - 1.$$

When $\theta = 0$ and $p(\cdot) = p =$*const*, the spaces $L^{p,0}(\mathbb{T})$ reduce to classical Lebesgue spaces $L^p(\mathbb{T})$ and when $\theta = 0$ and $p(\cdot) \neq$ *const*, the spaces $L^{p(\cdot),0}(\mathbb{T})$ reduce to the variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{T})$, investigated in detail in the monograph [8, 4]. When $p =$ *const* and $\theta > 0$ it was introduced by Iwaniec and Sbordone in [19] (for $\theta = 1$) and by Greco, Iwaniec and Sbordone in [9] (for $\theta > 1$).

The grand and generalized grand Lebesgue spaces have been applied in the various fields; in particular in theory PDE [20, 32, 33]. The fundamental problems of the spaces $L^{p(\cdot),\theta}(\mathbb{T})$ in view of potential theory, maximal and singular operator theory were investigated in the monograph [26].

Regarding to grand spaces, it will be observed that these spaces in general are not separable; in particular, Lebesgue space is not dense in grand Lebesgue space. Similar situations are also valid in the case of variable exponents. The closure of $L^{p(\cdot)}(\mathbb{T})$ in $L^{p(\cdot),\theta}(\mathbb{T})$ doesn't coincide with $L^{p(\cdot),\theta}(\mathbb{T})$ [27]. Henceforth, the closure of $L^{p(\cdot)}(\mathbb{T})$ in $L^{p(\cdot),\theta}(\mathbb{T})$ we denote by $L_0^{p(\cdot),\theta}(\mathbb{T})$. Then $L_0^{p(\cdot),\theta}(\mathbb{T})$ comprises the set of functions f such that

$$\lim_{\varepsilon\to 0} \varepsilon^{\frac{\theta}{p_--\varepsilon}} \|f\|_{p(\cdot)-\varepsilon} = 0.$$

Note that in the generalizations of classical Lebesgue spaces different problems of approximation theory are also investigated. In particular, in [14, 34, 35, 12, 13, 15, 23] the approximation properties of different summation methods in the variable exponent Lebesgue spaces were studied. Similar investigations have been done also in the grand and generalized grand spaces, with constant p and variable exponent $p(\cdot)$ in [5, 6, 16, 17, 22].

Let $f \in L^{p(\cdot),\theta}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ and $\theta > 0$. For $\alpha \in (0, 1]$ we define the Lipschitz class

$$\operatorname{Lip}(\alpha, p(\cdot), \theta) := \left\{ f \in L_0^{p(\cdot),\theta}(\mathbb{T}) : \Omega(f, \delta)_{p(\cdot),\theta} = \mathcal{O}(\delta^\alpha), \quad \delta > 0 \right\},$$

where $\Omega(f, \delta)_{p(\cdot),\theta}$ is the modulus of smoothness for $f \in L_0^{p(\cdot),\theta}(\mathbb{T})$, defined as

$$\Omega(f, \delta)_{p(\cdot),\theta} := \sup_{|h|\leq\delta} \left\| \frac{1}{h} \int_0^h |f(x+t) - f(x)| \,\mathrm{d}t \right\|_{p(\cdot),\theta}, \quad \delta > 0.$$

Let $f \in L^1(\mathbb{T})$ and

$$f(x) \sim \tfrac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

be its Fourier series representation with the Fourier coefficients

$$a_k := \tfrac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \cos(kt)\,\mathrm{d}t \text{ and } b_k := \tfrac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \sin(kt)\,\mathrm{d}t$$

and let

$$S_n(f)(x) = \sum_{k=0}^{n} u_k(f)(x), \quad n = 1, 2, \ldots,$$

be the nth partial sums of the Fourier series of f, where

$$u_k(f)(x) := a_0/2 \text{ and } u_k(f)(x) := (a_k \cos kx + b_k \sin kx), \ k = 1, 2, \ldots$$

Let $A = (a_{n,k})$ be a lower triangular infinite matrix of real elements $a_{n,k}$ such that

$$a_{n,k} \geq 0 \quad \text{for } k \leq n, \text{ and } a_{n,k} = 0 \text{ for } k > n,$$

where $k = 0, 1, 2, \ldots$, and let

$$s_n^{(A)} := \sum_{k=0}^{n} a_{n,k} = 1, \quad n = 0, 1, 2, \ldots$$

Unless otherwise indicate, we assume that $A = (a_{n,k})$ is a matrix that the summation of row elements equal to one throughout this work. For a given matrix $A = (a_{n,k})$, the matrix transform of Fourier series of f is defined as

$$T_n^{(A)}(f)(x) := \sum_{k=0}^{n} a_{n,n-k} S_k(f)(x), \quad n = 0, 1, 2, \ldots$$

Let $P_n = \sum_{k=0}^{n} p_k$ and (p_n) be a sequence of positive real numbers. If $a_{n,k} = p_k/P_n$, then $T_n^{(A)}(f)$ coincides with the Nörlund means

$$N_n(f)(x) = \tfrac{1}{P_n} \sum_{k=0}^{n} p_{n-k} S_k(f)(x), \quad n = 0, 1, 2, \ldots,$$

which in the case of $p_n = 1$, for all $n = 0, 1, 2, \ldots$, reduce to the Cesàro means

$$\sigma_n(f)(x) = \tfrac{1}{n+1} \sum_{k=0}^{n} S_k(f)(x).$$

We consider the matrix transforms $T_n^{(A)}(f)$ as approximating trigonometric polynomials to f and study approximation properties of $T_n^{(A)}(f)$ in the generalized grand Lebesgue space with variable exponent, exactly in the Lipschitz classes defined above. The required conditions on the sequences $(a_{n,k})$ are crucial points to arrive better approach when matrix transform $T_n^{(A)}(f)$

is constructed with respect to a given matrix $A = (a_{n,k})$. We mention the notations and definitions of some classes of sequences consisting of nonnegative numbers to explain the amongst important relations.

A nonnegative sequence $(a_{n,k})$ is called *almost monotone increasing (decreasing) sequence* if there exists a constant $K_1 := K_1(a_{n,k})$ $(K_2 := K_2(a_{n,k}))$ depending only the sequence $(a_{n,k})$ such that

$$a_{n,k} \le K_1 a_{n,m} \quad (a_{n,m} \le K_2 a_{n,k})$$

for all $0 \le k \le m \le n$. If $(a_{n,k})$ is *almost monotone decreasing sequence,* then we write $(a_{n,k}) \in AMDS$ and if $(a_{n,k})$ is *almost monotone increasing sequence* then we write $(a_{n,k}) \in AMIS$.

Let

$$A_{n,k} = \tfrac{1}{k+1} \sum_{j=n-k}^{n} a_{n,j}.$$

If $(A_{n,k}) \in AMDS$, then $(A_{n,k})$ is called *almost monotone decreasing upper mean sequence* and we write $(A_{n,k}) \in AMDUMS$. If $(A_{n,k}) \in AMIS$, then $(A_{n,k})$ is called *almost monotone increasing upper mean sequence* and then we write $(A_{n,k}) \in AMIUMS$.

There exist following embedding relations between these sequence classes

$$NIS \subset AMDS \subset AMIUMS$$

and

$$NDS \subset AMIS \subset AMDUMS,$$

where NIS is the class of nonnegative and nondecreasing sequences, NDS is the class of nonnegative and nonincreasing sequences [36].

We use also the notations

$$\Delta a_k = a_k - a_{k+1}, \quad \Delta_k a_{n,k} = a_{n,k} - a_{n,k+1}.$$

We write $m = \mathcal{O}(n)$ if there exists a positive constant K_3 such that $m \le K_3 n$.

Let now $L^p(\mathbb{T})$ be the classical Lebesgue space and let

$$\omega(f,\delta) := \sup_{0<h\le\delta} \|f(\cdot + h) - f\|_p, \quad \delta > 0$$

is the modulus of continuity for $f \in L^p(\mathbb{T})$, where $\|\cdot\|_p := \rho_p^{1/p}(\cdot)$ for $p(\cdot) := p$ is *const.*

In this case Lipschitz classes can be defined as

$$\mathrm{Lip}(\alpha,p) := \{f \in L^p(\mathbb{T}) : \omega(f,\delta) = \mathcal{O}(\delta^\alpha),\ \delta > 0\},$$

where $\alpha \in (0,1]$.

The rate of trigonometric approximation in the Lipschitz classes $\mathrm{Lip}(\alpha,p)$, $\alpha \in (0,1]$ and $1 < p < \infty$, were investigated by a great number of authors. Initially, degree of approximation by $\sigma_n(f)$, when $f \in \mathrm{Lip}(\alpha,p)$ was studied by Quade. He proved [31] that if $f \in \mathrm{Lip}(\alpha,p)$ for $\alpha \in (0,1]$ and $1 < p < \infty$, then $\|f - \sigma_n(f)\|_p = \mathcal{O}(n^{-\alpha})$. There have been various generalizations of this result [29, 2, 3, 36, 10, 11, 12, 38, 25, 21, 24, 7]. Especially, Chandra [3] generalizes the result of Quade and prove that if $f \in \operatorname{Lip}(\alpha, p)$ for $\alpha \in (0,1]$ and $1 < p < \infty$, then $\|f - N_n(f)\|_p = \mathcal{O}(n^{-\alpha})$, when (p_n) is a monotonic sequence of positive real numbers such that $(n+1)p_n = \mathcal{O}(P_n)$. Later, Leindler [28] generalizes the result of Chandra, imposing weaker assumptions on the sequence (p_n). Let's give some results that are as close as possible to our work in the case of $1 < p < \infty$.

Theorem 1. [28] *Let $f \in \operatorname{Lip}(\alpha, p)$ for $\alpha \in (0,1]$, $1 < p < \infty$ and let (p_n) be a sequence of positive real numbers. If one of the conditions*

i) $0 < \alpha < 1$ *and* $(p_n) \in AMDS$,

ii) $0 < \alpha < 1$, $(p_n) \in AMIS$ *and* $(n+1)p_n = \mathcal{O}(P_n)$,

iii) $\alpha = 1$ *and* $\sum\limits_{k=1}^{n-1} k|\Delta p_k| = \mathcal{O}(P_n)$,

iv) $\alpha = 1$, $\sum\limits_{k=0}^{n-1} |\Delta p_k| = \mathcal{O}(P_n)$ *and* $(n+1)p_n = \mathcal{O}(P_n)$,

holds, then

$$\|f - N_n(f)\|_p = \mathcal{O}(n^{-\alpha}).$$

Mittal and his collaborators [30] extend the results of Leindler by using matrix transforms of functions in $\operatorname{Lip}(\alpha, p)$, when the matrix $A = (a_{n,k})$ such that $(a_{n,k}) \in NIS$ or $(a_{n,k}) \in NDS$. In the weighted case Guven [11] proves this result using more general matrix transforms, namely for the matrices $A = (a_{n,k})$ such that $(a_{n,k}) \in AMIS$ or $(a_{n,k}) \in AMDS$. As a result we can deduce the following theorem from *Theorems 1 and 2* proved in [11] for the matrix transforms defined as:

$$T_n(f, A) := \sum_{k=0}^{n} a_{n,k} S_k(f)(x), \quad n = 0, 1, 2, \ldots$$

Theorem 2 ([11]). *Let $f \in \operatorname{Lip}(\alpha, p)$ for $\alpha \in (0,1]$, $1 < p < \infty$ and $A = (a_{n,k})$ be a lower triangular matrix with non-negative entries such that $\left|s_n^{(A)} - 1\right| = \mathcal{O}(n^{-\alpha})$. If one of the conditions*

i) $0 < \alpha < 1$, $(a_{n,k}) \in AMDS$ *and* $(n+1)a_{n,0} = \mathcal{O}(1)$,

ii) $0 < \alpha < 1$, $(a_{n,k}) \in AMIS$ *and* $(n+1)a_{n,r} = \mathcal{O}(1)$ *where r is integer part of $n/2$,*

iii) $\alpha = 1$ *and* $\sum\limits_{k=1}^{n-1} |\Delta a_{n,k-1}| = \mathcal{O}(n^{-1})$,

holds, then

$$\|f - T_n(f, A)\|_p = \mathcal{O}(n^{-\alpha}).$$

This theorem was generalized to weighted and nonweighted Lebesgue space with variable exponent in [18, 37, 13], respectively. At the same time, it can be observed [36] that theorem similar to Theorem 2 is also true under the assumption $s_n^{(A)} = 1$.

THEOREM 3 ([36]). *Let* $f \in \mathrm{Lip}\,(\alpha, p)$ *for* $\alpha \in (0,1]$, $1 < p < \infty$ *and* $A = (a_{n,k})$ *be a lower triangular matrix with non-negative entries and* $s_n^{(A)} = 1$. *If one of the conditions*

i) $0 < \alpha < 1$, $(a_{n,k}) \in AMDUMS$,

ii) $0 < \alpha < 1$, $(a_{n,k}) \in AMIUMS$ *and* $(n+1)\, a_{n,n} = \mathcal{O}(1)$,

iii) $\alpha = 1$ *and* $\sum\limits_{k=0}^{n-2} |\Delta_k A_{n,k}| = \mathcal{O}(n^{-1})$,

holds, then

$$\left\| f - T_n^{(A)}(f) \right\|_p = \mathcal{O}(n^{-\alpha}).$$

2. MAIN RESULTS

In this work, we estimate the error of trigonometric approximation by matrix transforms $T_n^{(A)}(f)$ in $f \in \mathrm{Lip}\,(\alpha, p(\cdot), \theta)$, when $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, $\alpha \in (0,1]$ and $\theta > 0$. We obtain the generalization of the above mentioned results. Theorem proved by us is stronger than the previous ones, because we prove it by imposing weaker assumptions. Main results are following.

THEOREM 4. *Let* $f \in \mathrm{Lip}\,(\alpha, p(\cdot), \theta)$ *for* $\alpha \in (0,1]$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T}), \theta > 0$ *and* $A = (a_{n,k})$ *be a lower triangular matrix with non-negative entries and* $s_n^{(A)} = 1$. *If one of the conditions*

i) $0 < \alpha < 1$, $(a_{n,k}) \in AMDUMS$,

ii) $0 < \alpha < 1$, $(a_{n,k}) \in AMIUMS$ *and* $(n+1)\, a_{n,n} = \mathcal{O}(1)$,

iii) $0 < \alpha < 1$ *and* $\sum\limits_{k=-1}^{n-1} |\Delta a_{n,k}| = \mathcal{O}(n^{-1})$ *where* $a_{n,-1} = 0$,

iv) $\alpha = 1$ *and* $\sum\limits_{k=0}^{n-2} |\Delta_k A_{n,k}| = \mathcal{O}(n^{-1})$,

holds, then

$$\left\| f - T_n^{(A)}(f) \right\|_{p(\cdot),\theta} = \mathcal{O}(n^{-\alpha}).$$

Theorem 4 is more general than the corresponding theorems given in [37]. In the case of $p(\cdot) = const$ and $\theta = 0$ it was proved in [36].

Let $P_n = \sum_{k=0}^{n} p_k$, for a sequence (p_n) of positive real numbers and let

$$P_{n,k} = \tfrac{1}{k+1} \sum_{j=n-k}^{n} p_j.$$

If $a_{n,k} := p_k / P_n$, then Theorem 4 implies

COROLLARY 5. *Let* $f \in \mathrm{Lip}\,(\alpha, p(\cdot), \theta)$ *for* $\alpha \in (0,1]$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T}), \theta > 0$ *and* (p_n) *be a sequence of positive real numbers. If one of the conditions*

i) $0 < \alpha < 1$, $(p_k) \in AMDUMS$,

ii) $0 < \alpha < 1$, $(p_k) \in AMIUMS$ *and* $(n+1)\, p_n = \mathcal{O}(P_n)$,

iii) $0 < \alpha < 1$ *and* $\sum\limits_{k=-1}^{n-1} |\Delta p_k| = \mathcal{O}(P_n/n)$ *where* $p_{-1} = 0$,

iv) $\alpha = 1$ *and* $\sum_{k=0}^{n-2} |\Delta_k P_{n,k}| = \mathcal{O}(P_n/n)$,

holds, then

$$\|f - N_n(f)\|_{p(\cdot),\theta} = \mathcal{O}(n^{-\alpha}).$$

If $p_n := A_n^{\beta-1}$ for some $\beta > 0$, where

$$A_0^\beta = 1, \quad A_k^\beta = \tfrac{(\beta+1)(\beta+2)\ldots(\beta+k)}{k!}, \quad k \geq 1$$

then $N_n(f)$ coincides with the generalized Cesàro means

$$\sigma_n^\beta(f)(x) := \tfrac{1}{A_n^\beta} \sum_{k=0}^{n} A_{n-k}^{\beta-1} S_k(f)(x) \ , \quad n = 0, 1, 2, \ldots$$

Hence, Corollary 5 implies

Corollary 6. *Let* $f \in \mathrm{Lip}(\alpha, p(\cdot), \theta)$ *for* $\alpha \in (0,1]$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T}), \theta > 0$ *and* $\beta > 0$. *Then*

$$\left\| f - \sigma_n^\beta(f) \right\|_{p(\cdot),\theta} = \mathcal{O}(n^{-\alpha}).$$

Corollary 6 in the case of $p(\cdot) =$*const* and $\theta = 0$ was proved in [3] (non-weighted case) and in [10] (weighted case).

3. AUXILIARY RESULTS

Let

$$E_n(f)_{p(\cdot),\theta} := \inf \left\{ \|f - T_n\|_{p(\cdot),\theta} : T_n \in \Pi_n \right\}$$

be the best approximation number of $f \in L_0^{p(\cdot),\theta}(\mathbb{T})$ in the class Π_n of trigonometric polynomials of degree not exceeding n.

Lemma 7 ([37]). *If* $f \in L_0^{p(\cdot),\theta}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ *and* $\theta > 0$, *then the inequality*

$$E_n(f)_{p(\cdot),\theta} = \mathcal{O}\left(\Omega(f, 1/n)_{p(\cdot),\theta}\right), \quad n = 1, 2, \ldots,$$

holds.

Lemma 8 ([37]). *Let* $p(\cdot) \in \mathcal{P}_0(\mathbb{T}), \theta > 0$, $\alpha \in (0,1]$. *If* $f \in \mathrm{Lip}(\alpha, p(\cdot), \theta)$, *then the inequality*

$$\|f - S_n(f)\|_{p(\cdot),\theta} = \mathcal{O}(n^{-\alpha}), \quad n = 1, 2, 3 \ldots,$$

holds.

Lemma 9 ([37]). *Let* $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ *and* $\theta > 0$. *If* $f \in \mathrm{Lip}(1, p(\cdot), \theta)$, *then the inequality*

$$\|S_n(f) - \sigma_n(f)\|_{p(\cdot),\theta} = \mathcal{O}\left(n^{-1}\right), \quad n = 1, 2, 3 \ldots,$$

holds.

Lemma 10. *Let $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ and $\theta > 0$. If $f \in \mathrm{Lip}(\alpha, p(\cdot), \theta)$, $\alpha \in (0,1)$, then the inequality*

$$\|f - \sigma_n(f)\|_{p(\cdot),\theta} = \mathcal{O}(n^{-\alpha}), \quad n = 1, 2, 3 \ldots,$$

holds.

Proof. Let $f \in \mathrm{Lip}(\alpha, p(\cdot), \theta)$, $\alpha \in (0,1)$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ and $\theta > 0$. The conjugate function of $f \in L^1(\mathbb{T})$ is defined as

$$\widetilde{f}(x) := \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(t)}{2 \tan \frac{t-x}{2}} \mathrm{d}t.$$

Since $f \in L_0^{p(\cdot),\theta}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ and $\theta > 0$, *Theorem 2.1* given in [5] provides that there exists a constant $c_1(p,\theta) > 0$, depending on $p(\cdot)$ and θ such that

$$\left\|\widetilde{f}\right\|_{p(\cdot),\theta} \le c_1(p,\theta) \|f\|_{p(\cdot),\theta}.$$

Using this inequality and the standard technics developed in [39] we obtain the inequality

$$\|\sigma_n(f)\|_{p(\cdot),\theta} \le c_2(p,\theta) \|f\|_{p(\cdot),\theta} \tag{1}$$

for some constant $c_2(p,\theta) > 0$.

Let $T_n^0(f)$ be the *best approximation trigonometric polynomial* to f in $L_0^{p(\cdot),\theta}(\mathbb{T})$, that is $\|f - T_n^0(f)\|_{p(\cdot),\theta} = E_n(f)_{p(\cdot),\theta}$ for $n = 0, 1, 2 \ldots$. Applying (1), the Minkowski inequality and Lemma 7 we have

$$\begin{aligned}
\|f - \sigma_n(f)\|_{p(\cdot),\theta} &\le \left\|f - T_n^0(f)\right\|_{p(\cdot),\theta} + \left\|T_n^0(f) - \sigma_n(f)\right\|_{p(\cdot),\theta} \\
&= E_n(f)_{p(\cdot),\theta} + \left\|\sigma_n\left(T_n^0(f) - f\right)\right\|_{p(\cdot),\theta} \\
&= E_n(f)_{p(\cdot),\theta} + \mathcal{O}(\left\|T_n^0(f) - f\right\|_{p(\cdot),\theta}) \\
&= \mathcal{O}(E_n(f)_{p(\cdot),\theta}) = \mathcal{O}\left(\Omega(f, 1/n)_{p(\cdot),\theta}\right) \\
&= \mathcal{O}(n^{-\alpha}).
\end{aligned}$$

Thus, lemma is proved. □

Lemma 11 ([36]). *Let $0 < \alpha < 1$ and $A = (a_{n,k})$ be infinite lower triangular matrix with non-negative entries and $s_n^{(A)} = 1$. If one of the conditions*

i) $(a_{n,k}) \in AMDUMS$,

ii) $(a_{n,k}) \in AMIUMS$ *and* $(n+1)a_{n,n} = \mathcal{O}(1)$,

holds then

$$\sum_{k=0}^{n} (k+1)^{-\alpha} a_{n,n-k} = \mathcal{O}\left((n+1)^{-\alpha}\right).$$

4. PROOF OF MAIN RESULTS

Proof of Theorem 4. Let $f \in \operatorname{Lip}(\alpha, p(\cdot), \theta)$, $\alpha \in (0,1]$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T}), \theta > 0$ and $A = (a_{n,k})$ be a lower triangular matrix with non-negative entries such that $s_n^{(A)} = 1$. We suppose that the conditions either i) or ii) holds. Since $0 < \alpha < 1$ and

$$f(x) - T_n^{(A)}(f)(x) = \sum_{k=0}^{n} a_{n,n-k}(f(x) - S_k(f)(x)),$$

applying the Minkowski inequality, Lemma 8 and Lemma 11, respectively we have that

$$\begin{aligned}\left\| f - T_n^{(A)}(f) \right\|_{p(\cdot),\theta} &\leq \sum_{k=0}^{n} a_{n,n-k} \| f - S_k(f) \|_{p(\cdot),\theta} \\ &\leq c_3 \sum_{k=0}^{n} a_{n,n-k}(k+1)^{-\alpha} = \mathcal{O}(n^{-\alpha}).\end{aligned}$$

Therefore, we proved Theorem 4 in the case of i) and ii).

Let $0 < \alpha < 1$ and $\sum\limits_{k=-1}^{n-1} |\Delta a_{n,k}| = \mathcal{O}(n^{-1})$, where $a_{n,-1} = 0$. By Abel transformation ([1, p. 1]) and definition of $\sigma_n(f)$ we have

$$\begin{aligned}f(x) - T_n^{(A)}(f)(x) &= \sum_{k=0}^{n} a_{n,n-k}(f(x) - S_k(f)(x)) \\ &= \sum_{k=0}^{n-1} (a_{n,n-k} - a_{n,n-k-1}) \sum_{j=0}^{k} (f(x) - S_j(f)(x)) \\ &\quad + a_{n,0} \sum_{k=0}^{n} (f(x) - S_k(f)(x)) \\ &= \sum_{k=0}^{n-1} (a_{n,n-k} - a_{n,n-k-1})(k+1)(f(x) - \sigma_k(f)(x)) \\ &\quad + a_{n,0}(n+1)(f(x) - \sigma_n(f)(x)).\end{aligned}$$

Using Minkowski's inequality, Lemma 10 we obtain that

$$\begin{aligned}\left\| f - T_n^{(A)}(f) \right\|_{p(\cdot),\theta} &\leq \sum_{k=0}^{n-1} |a_{n,n-k} - a_{n,n-k-1}|(k+1) \| f - \sigma_k(f) \|_{p(\cdot),\theta} \\ &\quad + a_{n,0}(n+1) \| f - \sigma_n(f) \|_{p(\cdot),\theta} \\ &\leq c_4 \left\{ \sum_{k=0}^{n-1} |a_{n,n-k} - a_{n,n-k-1}|(k+1)^{1-\alpha} + a_{n,0}(n+1)^{1-\alpha} \right\}\end{aligned}$$

$$\leq c_5 (n+1)^{1-\alpha} \left\{ \sum_{k=0}^{n-1} |a_{n,n-k} - a_{n,n-k-1}| + a_{n,0} \right\}$$
$$= c_6 (n+1)^{1-\alpha} \sum_{k=-1}^{n-1} |\Delta a_{n,k}| = \mathcal{O}(n^{-\alpha}).$$

Hence, the iii) part of Theorem 4 is also proved. Finally, we prove the last part of theorem. Let $\alpha = 1$ and $\sum_{k=0}^{n-2} |\Delta_k A_{n,k}| = \mathcal{O}(n^{-1})$. Using twice Abel's transformation we have

$$f(x) - T_n^{(A)}(f)(x) = \sum_{k=0}^{n} a_{n,n-k} (f(x) - S_k(f)(x))$$

$$= \sum_{k=0}^{n-1} (S_{k+1}(f)(x) - S_k(f)(x)) \sum_{j=0}^{k} a_{n,n-j} + (f(x) - S_n(f)(x)) \sum_{k=0}^{n} a_{n,n-k}$$
$$= \sum_{k=0}^{n-1} (S_{k+1}(f)(x) - S_k(f)(x)) \sum_{j=n-k}^{n} a_{n,j} + (f(x) - S_n(f)(x)) s_n^{(A)}$$
$$= f(x) - S_n(f)(x) + \sum_{k=0}^{n-1} u_{k+1}(f)(x)(k+1) A_{n,k}$$
$$= f(x) - S_n(f)(x) + \sum_{k=0}^{n-2} (A_{n,k} - A_{n,k+1}) \sum_{j=0}^{k} (j+1) u_{j+1}(f)(x)$$
$$+ A_{n,n-1} \sum_{k=0}^{n-1} (k+1) u_{k+1}(f)(x)$$
$$= f(x) - S_n(f)(x) + \sum_{k=0}^{n-2} (A_{n,k} - A_{n,k+1}) \sum_{j=0}^{k} (j+1) u_{j+1}(f)(x)$$
$$+ \frac{1}{n} \sum_{j=1}^{n} a_{n,j} \sum_{k=0}^{n-1} (k+1) u_{k+1}(f)(x).$$

Hence, applying the Minkowski inequality we get

$$\left\| f - T_n^{(A)}(f) \right\|_{p(\cdot),\theta} \leq \| f - S_n(f) \|_{p(\cdot),\theta} + \sum_{k=0}^{n-2} |A_{n,k} - A_{n,k+1}| \left\| \sum_{j=1}^{k+1} j u_j(f) \right\|_{p(\cdot),\theta}$$
$$+ \frac{1}{n} s_n^{(A)} \left\| \sum_{k=1}^{n} k u_k(f) \right\|_{p(\cdot),\theta}$$
$$= \| f - S_n(f) \|_{p(\cdot),\theta} + \sum_{k=0}^{n-2} |\Delta_k A_{n,k}| \left\| \sum_{j=1}^{k+1} j u_j(f) \right\|_{p(\cdot),\theta}$$

$$+\frac{1}{n}\left\|\sum_{k=1}^{n}ku_k\left(f\right)\right\|_{p(\cdot),\theta}. \tag{2}$$

Since

$$\begin{aligned}S_n\left(f\right)\left(x\right)-\sigma_n\left(f\right)\left(x\right) &= \sum_{k=0}^{n}\left[u_k\left(f\right)\left(x\right)-\tfrac{1}{n+1}\sum_{\nu=0}^{k}u_\nu\left(f\right)\left(x\right)\right] \\ &= \sum_{k=0}^{n}\left(1-\tfrac{n+1-k}{n+1}\right)u_k\left(f\right)\left(x\right) \\ &= \tfrac{1}{n+1}\sum_{k=0}^{n}ku_k\left(f\right)\left(x\right),\end{aligned}$$

using Lemma 9 we have

$$\left\|\sum_{k=1}^{n}ku_k\left(f\right)\right\|_{p(\cdot),\theta}=\left(n+1\right)\left\|S_n\left(f\right)-\sigma_n\left(f\right)\right\|_{p(\cdot),\theta}=\mathcal{O}\left(1\right). \tag{3}$$

Finally, combining (2), (3) and Lemma 8 for $\alpha=1$ we obtain that

$$\left\|f-T_n^{(A)}\left(f\right)\right\|_{p(\cdot),\theta}\leq\mathcal{O}(n^{-1})+\sum_{k=0}^{n-2}\left|\Delta_kA_{n,k}\right|\mathcal{O}\left(1\right)+\tfrac{1}{n}\mathcal{O}\left(1\right)=\mathcal{O}(n^{-1}).$$

Thereby, all parts of Theorem 4 are proved. □

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