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ASYMPTOTIC FORMULA IN SIMULTANEOUS APPROXIMATION FOR CERTAIN ISMAIL-MAY-BASKAKOV OPERATORS

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Abstract. The present study deals with a modification of Ismail-May operators with weights of Beta basis functions. We estimate weighted Korovkin's theorem and difference estimates between two operators and establish an asymptotic formula for the derivative of a function.

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1. THE OPERATOR

In 1978 Ismail and May [13] (see also [12]) studied certain operators of exponential type and introduced some new exponential operators. The following sequence of operators constitutes one of the families of linear positive operator proposed in [13, (3.14)]:

$$(R_n f)(x) = \sum_{\nu=0}^{\infty} r_{n,\nu}(x) F_{n,\nu}, \tag{1}$$

where $F_{n,\nu} = f\left(\frac{\nu}{n}\right)$ and

$$r_{n,\nu}(x) = e^{-(n+\nu)x/(1+x)} \frac{n(n+\nu)^{\nu-1}}{\nu!} \left(\frac{x}{1+x}\right)^{\nu}.$$

These operators are exponential type operators as

$$x(1+x)^2[(R_n f)(x)]' = \sum_{\nu=0}^{\infty} (\nu - nx) r_{n,\nu}(x) f\left(\frac{\nu}{n}\right).$$

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In recent studies, Gupta and Rassias [10] presented a collection of the moments of several linear positive operators, Agratini [2] estimated approximation properties of integral-type operators. The operators R_n are not suitable in order to approximate integral functions. Additionally, it is observed in [6] that obtaining the usual Durrmeyer variant of the operators (1) is not possible due to the non-convergence of the integral

$$\int_0^\infty r_{n,\nu}(u)du.$$

In the recent work [6], the hybrid link operators were introduced and the approximation behavior was discussed. We consider here new hybrid operators with weights $b_{n+2,\nu-1}(t)$ of slightly modified Baskakov basis functions, which for $x \in [0,\infty)$ are defined as

$$(V_n f)(x) = \sum_{\nu=0}^{\infty} r_{n,\nu}(x) G_{n,\nu}(f), \tag{2}$$

where

$$G_{n,\nu}(f) = \begin{cases} (n+1)\int_0^\infty b_{n+2,\nu-1}(u)f(u)du & : 1 \le \nu < \infty \\ f(0) & : \nu = 0 \end{cases}$$

$$b_{n,\nu}(u) = \binom{n+\nu-1}{\nu}\frac{u^\nu}{(1+u)^{n+\nu}}.$$

For simplicity, we use the notation

$$v_{n,\nu}(u) = (n+1)b_{n+2,\nu-1}(u)$$

throughout the paper.

The operators (2) reproduce linear functions. The investigation of convergence of operators in ordinary and simultaneous approximation has been an active area of research over recent years. We mention here some of the related work viz. [3], [7], [8], [9], [11]. In the present paper, we estimate direct result in weighted norm, difference of these operators with discrete operators and prove a formula of asymptotic kind for derivatives of the operators V_n.

2. SOME LEMMAS

Lemma 1. *If*

$$U_{n,q}(x) = \sum_{\nu=0}^{\infty} r_{n,\nu}(x)\left(\frac{\nu}{n} - x\right)^q, \quad q \in N \cup \{0\},$$

then the moments satisfy the relation:

$$nU_{n,q+1}(x) = x(1+x)^2\left[(U_{n,q}(x))' + qU_{n,q-1}(x)\right].$$

Also, one can observe that

$$U_{n,q}(x) = O_x(n^{-[(q+1)/2]}),$$

where $[s]$ *stands for the integral part of* s.

Furthermore, for $e_i(t)=t^i, i=0,1,2,\ldots,$ *we have*

$$n(R_ne_{q+1})(x)=x(1+x)^2[(R_ne_q)(x)]'+nx(R_ne_q)(x).$$

LEMMA 2. *If we denote* $\mu_{n,q}(x)=(V_n(e_1-xe_0)^q)(x)$ *for* $q\in N$ *and* $x\geq 0$, *then*

$$\begin{aligned}(n-q)\mu_{n,q+1}(x) &= x(1+x)^2\left[[\mu_{n,q}(x)]'+q\mu_{n,q-1}(x)\right]\\ &\quad +q\mu_{n,q}(x)+qx(1+x)\mu_{n,q-1}(x).\end{aligned}$$

In particular, we have

$$\mu_{n,0}(x)=1,\mu_{n,1}(x)=0,\mu_{n,2}(x)=\tfrac{x(1+x)(2+x)}{n-1}.$$

In general for all $x\in[0,\infty)$, *we have* $\mu_{n,q}(x)=O_x(n^{-[(q+1)/2]})$.

Proof. Using

$$x(1+x)^2r'_{n,\nu}(x)=(\nu-nx)r_{n,\nu}(x)$$

and

$$u(1+u)v'_{n,\nu}(u)=[(\nu-1)-(n+2)u]v_{n,\nu}(u),$$

we have

$$\begin{aligned}&x(1+x)^2[\mu_{n,q}(x)]'=\\ =\ &\sum_{\nu=1}^{\infty}x(1+x)^2r'_{n,\nu}(x)\int_0^\infty v_{n,\nu}(u)(e_1-xe_0)^q du\\ &+ne^{-nx/(1+x)}(-x)^{q+1}+q(-x)^q(1+x)^2e^{-nx/(1+x)}\\ &-qx(1+x)^2\sum_{\nu=1}^{\infty}r_{n,\nu}(x)\int_0^\infty v_{n,\nu}(u)(e_1-xe_0)^{q-1}du\\ =\ &\sum_{\nu=1}^{\infty}(\nu-nx)r_{n,\nu}(x)\int_0^\infty v_{n,\nu}(u)(e_1-xe_0)^q du\\ &+ne^{-nx/(1+x)}(-x)^{q+1}-qx(1+x)^2\mu_{n,q-1}(x)\\ =\ &\sum_{\nu=1}^{\infty}r_{n,\nu}(x)\int_0^\infty u(1+u)v'_{n,\nu}(u)(e_1-xe_0)^q du+(n+2)[\mu_{n,q+1}(x)\\ &-(-x)^{q+1}e^{-nx/(1+x)}]+(1+2x)[\mu_{n,q}(x)-(-x)^qe^{-nx/(1+x)}]\\ &+ne^{-nx/(1+x)}(-x)^{q+1}-qx(1+x)^2\mu_{n,q-1}(x).\end{aligned}$$

Thus, we have

$$\begin{aligned}&x(1+x)^2\left[[\mu_{n,q}(x)]'+q\mu_{n,q-1}(x)\right]=\\ =\ &\sum_{\nu=1}^{\infty}r_{n,\nu}(x)\int_0^\infty[(e_1-xe_0)^2+(1+2x)(e_1-xe_0)+x(1+x)]v'_{n,\nu}(u)(e_1-xe_0)^q du\\ &+(n+2)[\mu_{n,q+1}(x)-(-x)^{q+1}e^{-nx/(1+x)}]\\ &+(1+2x)[\mu_{n,q}(x)-(-x)^qe^{-nx/(1+x)}]+ne^{-nx/(1+x)}(-x)^{q+1}.\end{aligned}$$

Integrating by parts the last integral, we obtain that

$$\begin{aligned}&x(1+x)^2\left[[\mu_{n,q}(x)]' + q\mu_{n,q-1}(x)\right]\\ =\ & -(q+2)\mu_{n,q+1}(x) - (q+1)(1+2x)\mu_{n,q}(x) - qx(1+x)\mu_{n,q-1}(x)\\ & +(n+2)\mu_{n,q+1}(x) + (1+2x)\mu_{n,q}(x).\end{aligned}$$

This completes the proof of the recurrence relation. The other consequences follow from the recurrence relation. □

Lemma 3. *If we denote $T_{n,q}(x) = (V_n e_q)(x)$ for $q \in N$ and $x \geq 0$, then*

$$(n-q)T_{n,q+1}(x) = x(1+x)^2[T_{n,q}(x)]' + (q+nx)T_{n,q}(x).$$

In general, we have

$$T_{n,q}(x) = x^q + \tfrac{q(q-1)}{2}x^{q-1}\tfrac{(1+x)(2+x)}{n-q+1} + O_x(n^{-2}).$$

The proof of the lemma follows along the lines of Lemma 1, we thus omit the details.

Lemma 4 ([6]). *There exist the polynomials $\phi_{i,j,q}(x)$ independent of n and ν such that*

$$[x^q(1+x)^{2q}]\tfrac{\partial^q}{\partial x^q}[r_{n,\nu}(x)] = \sum_{\substack{2i+j\leq q\\ i,j\geq 0}} n^i(\nu - nx)^j\phi_{i,j,q}(x)[r_{n,\nu}(x)].$$

Lemma 5. *If $\mu_r^{J_{n,\nu}} = \sum_{i=0}^r \binom{r}{i}(-1)^i J_{n,\nu}(e_{r-i})[J_{n,\nu}(e_1)]^i$, then we have*

$$\mu_m^{F_{n,\nu}} = 0, m = 1,2,3,\ldots, \qquad \mu_2^{G_{n,\nu}} = \tfrac{\nu^2+n\nu}{n^2(n-1)}, \quad \mu_3^{G_{n,\nu}} = \tfrac{4\nu^3+6n\nu^2+2n^2\nu}{n^3(n-1)(n-2)}$$

and

$$\mu_4^{G_{n,\nu}} = \tfrac{3(6+n)\nu^4+6(6+n)n\nu^3+3(8+n)n^2\nu^2+6n^3\nu}{n^4(n-1)(n-2)(n-3)}.$$

Proof. By definition, we have

$$b^{F_{n,\nu}} = F_{n,\nu}(e_1) = \tfrac{\nu}{n}.$$

By simple computation, we get

$$\mu_m^{F_{n,\nu}} = 0, m = 1,2,3,\ldots$$

Moreover

$$\begin{aligned}G_{n,\nu}(e_q) &= (n+1)\int_0^\infty b_{n+2,\nu-1}(u)u^q du\\ &= (n+1)\tbinom{n+\nu}{\nu-1}\int_0^\infty \tfrac{u^{\nu+q-1}}{(1+u)^{n+\nu+1}}du = \tfrac{(\nu+q-1)!(n-q)!}{(\nu-1)!n!}\end{aligned}$$

Thus

$$b^{G_{n,\nu}} = G_{n,\nu}(e_1) = \tfrac{\nu}{n},$$

and

$$\mu_2^{G_{n,\nu}} = \sum_{i=0}^2 \tbinom{2}{i}(-1)^i G_{n,\nu}(e_{2-i})[G_{n,\nu}(e_1)]^i$$

$$
\begin{aligned}
&= G_{n,\nu}(e_2) - [G_{n,\nu}(e_1)]^2 \\
&= \tfrac{\nu^2+n\nu}{n^2(n-1)}.
\end{aligned}
$$

$$
\begin{aligned}
\mu_3^{G_{n,\nu}} &= \sum_{i=0}^{3} \tbinom{3}{i}(-1)^i G_{n,\nu}(e_{3-i})[G_{n,\nu}(e_1)]^i \\
&= G_{n,\nu}(e_3) - 3G_{n,\nu}(e_2)G_{n,\nu}(e_1) + 3[G_{n,\nu}(e_1)]^3 - [G_{n,\nu}(e_1)]^3 \\
&= \tfrac{1}{n^3(n-1)(n-2)}\left[n^2\nu(\nu+2)(\nu+1)\nu - 3n(n-2)(\nu+1)\nu^2 + 2(n-1)(n-2)\nu^3\right] \\
&= \tfrac{4\nu^3+6n\nu^2+2n^2\nu}{n^3(n-1)(n-2)}
\end{aligned}
$$

and

$$
\begin{aligned}
\mu_4^{G_{n,\nu}} &= \sum_{i=0}^{4} \tbinom{4}{i}(-1)^i G_{n,\nu}(e_{4-i})[G_{n,\nu}(e_1)]^i \\
&= G_{n,\nu}(e_4) - 4G_{n,\nu}(e_3)G_{n,\nu}(e_1) + 6G_{n,\nu}(e_2)[G_{n,\nu}(e_1)]^2 \\
&- 4G_{n,\nu}(e_1)[G_{n,\nu}(e_1)]^3 + [G_{n,\nu}(e_1)]^4 \\
&= \tfrac{1}{n^4(n-1)(n-2)(n-3)}\Big[n^3(\nu+3)(\nu+2)(\nu+1)\nu - 4n^2(n-3)(\nu+2)(\nu+1)\nu^2 \\
&\quad + 6n(n-2)(n-3)(\nu+1)\nu^3 - 3\nu^4(n-1)(n-2)(n-3)\Big] \\
&= \tfrac{1}{n^4(n-1)(n-2)(n-3)}\left[6n^3\nu + 3(8+n)n^2\nu^2 + 6(6+n)n\nu^3 + 3(6+n)\nu^4\right].
\end{aligned}
$$

The other estimates follow similarly, we thus omit the details. □

3. DIRECT ESTIMATES AND ASYMPTOTIC FORMULA

Let

$$B_m[0,\infty) = \{f : |f(x)| \le c_f(1+x^m), \forall x \in [0,\infty)\}, \quad m > 3$$

where c_f is an absolute constant dependent on f, but independent of x. Let

$$C_m[0,\infty) = C[0,\infty) \cap B_m[0,\infty).$$

For each $f \in C_m[0,\infty)$ we consider modulus of continuity (see [14]) as

$$\Omega(f,\delta) = \sup_{|\triangle x|<\delta, x\in R^+} \tfrac{|f(x+\triangle x)-f(x)|}{(1+h^m)(1+x^m)}.$$

Additionally, $C_m^*[0,\infty)$ denotes the subspace of continuous functions $f \in B_m[0,\infty)$, for which

$$\lim_{x\to\infty} |f(x)|(1+x^m)^{-1} < \infty.$$

We consider the norm by

$$\|f\|_m = \sup_{0\le x<\infty} \tfrac{|f(x)|}{(1+x^m)}.$$

THEOREM 6. *If* $f \in C_m^* [0, \infty)$, *then*

$$\lim_{n\to\infty} \|(V_n f) - f\|_m = 0.$$

Proof. By the weighted Korovkin theorem due to Gadjiev [4], we know that if $f \in C_m^* [0, \infty)$, satisfying

$$\lim_{n\to\infty} \|(V_n e_i) - e_i\|_m = 0, \quad i = 0, 1, 2,$$

then we have

$$\lim_{n\to\infty} \|(V_n f)(x) - f(x)\|_m = 0.$$

In order to prove the result, we use Lemma 3. Clearly, the result holds for $i = 0, 1$. Thus, in order to complete the proof we proceed for

$$\begin{aligned}\lim_{n\to\infty} \|(V_n e_2)(x) - e_2\|_2 &= \lim_{n\to\infty} \tfrac{1}{(1+x^m)} \left[x^2 + \tfrac{x(1+x)(2+x)}{n-1} - x^2\right] \\ &= \lim_{n\to\infty} \tfrac{1}{(1+x^m)} \left[\tfrac{x(1+x)(2+x)}{n-1}\right] = 0.\end{aligned}$$

This completes the proof of the theorem. ☐

THEOREM 7. *For* $f^{(s)} \in C_B[0, \infty)$, $0 \le s \le 4$, $s \in \mathbf{N}$, $x \in [0, \infty)$ *and* $n \in \mathbf{N}$, *we have*

$$\begin{aligned}&|((V_n f) - (R_n f))(x)| \le \\ \le\ & \tfrac{\|f^{iv}\|}{24(n-1)(n-2)(n-3)} \Big[6x + 3(8+n)\left(x^2 + \tfrac{x(1+x)^2}{n}\right) \\ &+6(6+n)\left(x^3 + \tfrac{3x^2(1+x)^2}{n} + \tfrac{x(1+x)^3(1+3x)}{n^2}\right) \\ &+3(6+n)\left(x^4 + \tfrac{6x^3(1+x)^2}{n} + \tfrac{x^2(1+x)^3(7+15x)}{n^2} + \tfrac{x(1+x)^4(1+10x+15x^2)}{n^3}\right)\Big] \\ &+\tfrac{\|f'''\|}{3(n-1)(n-2)} \left[2x^3 + \tfrac{6x^2(1+x)^2}{n} + \tfrac{2x(1+x)^3(1+3x)}{n^2} + 3x^2 + \tfrac{3x(1+x)^2}{n} + x\right] \\ &+\tfrac{\|f''\|}{2(n-1)} \left[x^2 + \tfrac{x(1+x)^2}{n} + x\right].\end{aligned}$$

Proof. Using Lemma 5 and Lemma 1, we have

$$\begin{aligned}\alpha(x) &= \sum_{\nu=0}^{\infty} r_{n,\nu}(x)(\mu_4^{F_{n,\nu}} + \mu_4^{G_{n,\nu}}) \\ &= \tfrac{1}{24n^4(n-1)(n-2)(n-3)} \sum_{\nu=0}^{\infty} r_{n,\nu}(x) \Big[6n^3\nu + 3(8+n)n^2\nu^2 \\ &\quad +6(6+n)n\nu^3 + 3(6+n)\nu^4\Big] \\ &= \tfrac{1}{24(n-1)(n-2)(n-3)} \Big[6(R_n e_1)(x) + 3(8+n)(R_n e_2)(x)\end{aligned}$$

$$
\begin{aligned}
&+6(6+n)(R_ne_3)(x)+3(6+n)(R_ne_4)(x)\Big]\\
=\;& \tfrac{1}{24(n-1)(n-2)(n-3)}\Big[6x+3(8+n)\left(x^2+\tfrac{x(1+x)^2}{n}\right)\\
&+6(6+n)\left(x^3+\tfrac{3x^2(1+x)^2}{n}+\tfrac{x(1+x)^3(1+3x)}{n^2}\right)\\
&+3(6+n)\left(x^4+\tfrac{6x^3(1+x)^2}{n}+\tfrac{x^2(1+x)^3(7+15x)}{n^2}+\tfrac{x(1+x)^4(1+10x+15x^2)}{n^3}\right)\Big].
\end{aligned}
$$

Additionally,

$$
\begin{aligned}
\beta(x) &= \tfrac{1}{3!}\sum_{\nu=0}^{\infty} r_{n,\nu}(x)\left|\mu_3^{F_{n,\nu}}-\mu_3^{G_{n,\nu}}\right|\\
&= \tfrac{1}{3!}\sum_{\nu=0}^{\infty} r_{n,\nu}(x)\tfrac{4\nu^3+6n\nu^2+2n^2\nu}{n^3(n-1)(n-2)}\\
&= \tfrac{1}{3(n-1)(n-2)}\left[2(R_ne_3)(x)+3(R_ne_2)(x)+(R_ne_1)(x)\right]\\
&= \tfrac{1}{3(n-1)(n-2)}\left[2x^3+\tfrac{6x^2(1+x)^2}{n}+\tfrac{2x(1+x)^3(1+3x)}{n^2}+3x^2+\tfrac{3x(1+x)^2}{n}+x\right].
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
\gamma(x) &= \tfrac{1}{2!}\sum_{\nu=0}^{\infty} r_{n,\nu}(x)\left|\mu_2^{F_{n,\nu}}-\mu_2^{G_{n,\nu}}\right|\\
&= \tfrac{1}{2!}\sum_{\nu=0}^{\infty} r_{n,\nu}(x)\tfrac{\nu^2+n\nu}{n^2(n-1)}\\
&= \tfrac{1}{2(n-1)}[(R_ne_2)(x)+(R_ne_1)(x)] = \tfrac{1}{2(n-1)}\left[x^2+\tfrac{x(1+x)^2}{n}+x\right].
\end{aligned}
$$

Finally, we get

$$
\delta^2(x) = \sum_{\nu=0}^{\infty} r_{n,\nu}(x)(b^{F_{n,\nu}}-b^{G_{n,\nu}})^2 = 0.
$$

Following [5, Theorem 3] and [1, Theorem 4], we derive that

$$
\begin{aligned}
|((V_nf)-(R_nf))(x)| \;\le\; & \|f^{iv}\|\alpha(x)+\|f'''\|\beta(x)+\|f''\|\gamma(x)\\
&+2\omega(f,\delta(x)).
\end{aligned}
$$

Substituting the values of the above estimates, we get the desired result. □

THEOREM 8. *Let $f\in C[0,\infty)$ with $|f(t)|\le C(1+t)^\gamma$ for some $\gamma>0, t\ge 0$. If $f^{(q+2)}$ exists at a point $x\in(0,\infty)$, then we have*

$$
\begin{aligned}
\lim_{n\to\infty} n(V_n^{(q)}f)(x)-f^{(q)}(x)) \;=\; & \tfrac{q(q-1)(q-2)}{2}f^{(q-1)}(x)+\tfrac{3q(q-1)(x+1)}{2}f^{(q)}(x)\\
&+\tfrac{q[(q^2+2)x^2+3(q+1)x+2]}{2}f^{(q+1)}(x)\\
&+\tfrac{x(1+x)(2+x)}{2}f^{(q+2)}(x).
\end{aligned}
$$

Proof. Applying Taylor's expansion, we have From Taylor's theorem, we have

$$\begin{aligned}
(V_n^{(q)}f)(x) &= \sum_{v=0}^{q+2} \frac{f^{(v)}(x)}{v!}\left(\frac{\partial^q}{\partial\kappa^q}(V_n(e_1-xe_0)^v)(\kappa)\right)_{\kappa=x} \\
&\quad +\left(\frac{\partial^q}{\partial\kappa^q}(V_n\psi(t,x)(e_1-xe_0)^{q+2})(\kappa)\right)_{\kappa=x} \\
&= \sum_{v=0}^{q+2} \frac{f^{(v)}(x)}{v!}\sum_{j=0}^{v}\binom{v}{j}(-x)^{v-j}\left(\frac{\partial^q}{\partial\kappa^q}T_{n,j}(\kappa)\right)_{\kappa=x} \\
&\quad +\left(\frac{\partial^q}{\partial\kappa^q}(V_n\psi(u,x)(e_1-xe_0)^{q+2})(\kappa)\right)_{\kappa=x} \\
&:= I_1+I_2
\end{aligned}$$

where the function $\psi(u,x)\to 0$ as $u\to x$. Obviously I_1 is estimated as

$$\begin{aligned}
I_1 &= \frac{f^{(q-1)}(x)}{(q-1)!}\left(\frac{\partial^q}{\partial\kappa^q}T_{n,q-1}(\kappa)\right)_{\kappa=x} \\
&\quad +\frac{f^{(q)}(x)}{q!}\left[q(-x)\left(\frac{\partial^q}{\partial\kappa^q}T_{n,q-1}(\kappa)\right)_{\kappa=x}+\left(\frac{\partial^q}{\partial\kappa^q}T_{n,q}(\kappa)\right)_{\kappa=x}\right] \\
&\quad +\frac{f^{(q+1)}(x)}{(q+1)!}\left[\frac{(q+1)q}{2}x^2\left(\frac{\partial^q}{\partial\kappa^q}T_{n,q-1}(\kappa)\right)_{\kappa=x}+(q+1)(-x)\left(\frac{\partial^q}{\partial\kappa^q}T_{n,q}(\kappa)\right)_{\kappa=x}\right. \\
&\quad \left.+\left(\frac{\partial^q}{\partial\kappa^q}T_{n,q+1}(\kappa)\right)_{\kappa=x}\right] \\
&\quad +\frac{f^{(q+2)}(x)}{(q+2)!}\left[\frac{(q+2)(q+1)q}{3!}(-x)^3\left(\frac{\partial^q}{\partial\kappa^q}T_{n,q-1}(\kappa)\right)_{\kappa=x}\right. \\
&\quad +\frac{(q+2)(q+1)}{2}x^2\left(\frac{\partial^q}{\partial\kappa^q}T_{n,q}(\kappa)\right)_{\kappa=x} \\
&\quad \left.+(q+2)(-x)\left(\frac{\partial^q}{\partial\kappa^q}T_{n,q+1}(\kappa)\right)_{\kappa=x}+\left(\frac{\partial^q}{\partial\kappa^q}T_{n,q+2}(\kappa)\right)_{\kappa=x}\right].
\end{aligned}$$

Applying Lemma 3, we have

$$\begin{aligned}
I_1 &= \frac{f^{(q-1)}(x)}{(q-1)!}\left[\frac{q!(q-1)(q-2)}{2(n-q+2)}\right]+f^{(q)}(x)\left[1+\frac{3nq(q-1)(x+1)}{2(n-q+2)(n-q+1)}+O(n^{-2})\right] \\
&\quad +f^{(q+1)}(x)\left[\frac{nq[(q^2+2)x^2+3(q+1)x+2]}{2(n-q)(n-q+2)}+O(n^{-2})\right] \\
&\quad +\frac{f^{(q+2)}(x)}{(q+2)!}\left[\frac{x^3}{2(n-q+1)}+\frac{3x^2}{2(n-q-1)}+\frac{nx}{(n-q)(n-q-1)}\right].
\end{aligned}$$

Thus

$$\begin{aligned}
\lim_{n\to\infty} n((V_n^{(q)}f)(x)-f^{(q)}(x)) &= \frac{q(q-1)(q-2)}{2}f^{(q-1)}(x)+\frac{3q(q-1)(x+1)}{2}f^{(q)}(x) \\
&\quad +\frac{q[(q^2+2)x^2+3(q+1)x+2]}{2}f^{(q+1)}(x)
\end{aligned}$$

$$+\tfrac{x(1+x)(2+x)}{2}f^{(q+2)}(x)+\lim_{n\to\infty}nI_2.$$

Next, in view of Lemma 4, we have

$$\begin{aligned}|I_2| &\le \\ &\le \left(\sum_{\nu=1}^{\infty}\sum_{\substack{2i+j\le q\\ i,j\ge 0}} n^i|\nu-n\kappa|^j\tfrac{|\phi_{i,j,q}(\kappa)|}{\kappa^q(1+\kappa)^{2q}}r_{n,\nu}(\kappa)\int_0^\infty v_{n,\nu}(u)|\psi(u,x)|.|u-x|^{q+2}du\right)_{\kappa=x}\\ &\quad+|\psi(0,x)(-x)^{q+2}|\left(\tfrac{\partial^q}{\partial\kappa^q}r_{n,0}(\kappa)\right)_{\kappa=x}\\ &:= E_1+E_2.\end{aligned}$$

Because of the fact $\psi(u,x)\to 0$ as $u\to x$, for a given $\epsilon>0$ there will exist a $\delta>0$ such that $|\psi(u,x)|<\epsilon$ whenever $|u-x|<\delta$.
Furthermore, if $m\ge\{\gamma,q+2\}$, and m a positive integer

$$|(u-x)^{q+2}\psi(u,x)|\le M|u-x|^m$$

for $|u-x|\ge\delta$. Thus

$$\begin{aligned}|E_1| &\le \sum_{\nu=1}^{\infty}\sum_{\substack{2i+j\le q\\ i,j\ge 0}} n^i|\nu-nx|^j\tfrac{|\phi_{i,j,q}(x)|}{x^q(1+x)^{2q}}r_{n,\nu}(x)\Big(\epsilon\int_{|u-x|<\delta}v_{n,\nu}(u)|u-x|^{q+2}du\\ &\quad+M\int_{|u-x|\ge\delta}v_{n,\nu}(u)|u-x|^m du\Big)\\ &:= J_1+J_2.\end{aligned}$$

Let us consider

$$K=\sup_{\substack{2i+j\le q\\ i,j\ge 0}}\tfrac{|\phi_{i,j,q}(x)|}{x^q(1+x)^{2q}}.$$

Using Schwarz's inequality, Lemma 1, Lemma 2 and Lemma 3, we deduce that

$$\begin{aligned}J_1 &= \epsilon\ K\sum_{\nu=1}^{\infty}\sum_{\substack{2i+j\le q\\ i,j\ge 0}} n^i|\nu-nx|^j r_{n,\nu}(x)\Big(\int_0^\infty v_{n,\nu}(u)du\Big)^{1/2}\\ &\quad\Big(\int_0^\infty v_{n,\nu}(u)(e_1-xe_0)^{2q+4}du\Big)^{1/2}\\ &\le \epsilon\ K\sum_{\substack{2i+j\le q\\ i,j\ge 0}} n^{i+j}\Big(\sum_{\nu=0}^{\infty}r_{n,\nu}(x)\Big(\tfrac{\nu}{n}-x\Big)^{2j}-x^{2j}r_{n,0}(x)\Big)^{1/2}\\ &\quad\Big(V_n((e_1-xe_0)^{2q+4},x)-x^{2q+4}r_{n,0}(x)\Big)^{1/2}\end{aligned}$$

$$= \epsilon \sum_{\substack{2i+j\le q\\ i,j\ge 0}} n^{i+j}\left\{O_x\left(\tfrac{1}{n^j}\right)+O_x\left(\tfrac{1}{n^s}\right)\right\}^{1/2}\left\{O_x\left(\tfrac{1}{n^{q+2}}\right)+O_x\left(\tfrac{1}{n^p}\right)\right\}^{1/2},$$

for any $s,p>0$, choosing s and p such that $s>j$ and $p>q+2$, we obtain

$$J_1 \le \epsilon \sum_{\substack{2i+j\le q\\ i,j\ge 0}} n^{i+j}O_x\left(\tfrac{1}{n^{j/2}}\right)O_x\left(\tfrac{1}{n^{q/2+1}}\right)=\epsilon.O_x(n^{-1}).$$

Since $\epsilon>0$ is arbitrary, $nJ_1\to 0$ as $n\to\infty$.

Applying again Schwarz's inequality, Lemma 1 and Lemma 3, we obtain

$$J_2 \le M_1 \sum_{\substack{2i+j\le q\\ i,j\ge 0}} n^{i+j}\left(\sum_{\nu=0}^{\infty}\left(\tfrac{\nu}{n}-x\right)^{2j}r_{n,\nu}(x)-x^{2j}r_{n,0}(x)\right)^{1/2}$$
$$\left(\sum_{\nu=1}^{\infty}r_{n,\nu}(x)\int_{|u-x|\ge\delta}v_{n,\nu}(u)(e_1-xe_0)^{2m}du\right)^{1/2}$$
$$\le M_1 \sum_{\substack{2i+j\le q\\ i,j\ge 0}} n^{i+j}\left\{O_x\left(\tfrac{1}{n^j}\right)+O_x\left(\tfrac{1}{n^s}\right)\right\}^{1/2}\left\{O_x\left(\tfrac{1}{n^m}\right)+O_x\left(\tfrac{1}{n^l}\right)\right\}^{1/2},$$

for any $s,l>0$. We now choose s,l such that $s>j, l>m$. Then, we derive that

$$J_2 \le M_1 \sum_{\substack{2i+j\le q\\ i,j\ge 0}} n^{i+j}O_x\left(\tfrac{1}{n^{j/2}}\right)O_x\left(\tfrac{1}{n^{m/2}}\right)=O_x\left(\tfrac{1}{n^{(m-q)/2}}\right),$$

which implies that $nJ_2\to 0$, as $n\to\infty$, on choosing $m>q+2$.

From the above estimates of J_1 and J_2, $nE_1\to 0$, as $n\to\infty$. Finally, we have

$$|E_2| = |\psi(0,x)(-x)^q|\left(\tfrac{\partial^q}{\partial\kappa^q}r_{n,0}(\kappa)\right)_{\kappa=x}.$$

Since $|\psi(0,x)(-x)^q|<N_1$ for some $N_1>0$, also

$$\left(\tfrac{\partial^q}{\partial\kappa^q}r_{n,0}(\kappa)\right)_{\kappa=x} = \left[\tfrac{\partial^q}{\partial\kappa^q}\left(e^{-n\kappa/(1+\kappa)}\right)\right]_{\kappa=x}$$

which yields that $nE_2\to 0$ as $n\to\infty$. By collecting the estimates of E_1 and E_2, we obtain $nI_2\to 0$ as $n\to\infty$. Therefore, combining I_1 and I_2, the required result follows. □

COROLLARY 9. *If $f\in C[0,\infty)$ and $|f(t)|\le C(1+t)^\gamma$ for some $\gamma>0, t\ge 0$ with the existence of f'' in $x\in(0,\infty)$, then we have*

$$\lim_{n\to\infty} n((V_nf)(x)-f(x)) = \tfrac{x(x^2+3x+2)}{2}f''(x).$$

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