JOURNAL OF NUMERICAL ANALYSIS AND APPROXIMATION THEORY
J. Numer. Anal. Approx. Theory, vol. 50 (2021) no. 2, pp. 180–193
ictp.acad.ro/jnaat

ON THE DEVELOPMENT AND EXTENSIONS OF SOME CLASSES OF OPTIMAL THREE–POINT ITERATIONS FOR SOLVING NONLINEAR EQUATIONS

T. ZHANLAV[†] and KH. OTGONDORJ[†,*]

Abstract. We develop new families of optimal eight–order methods for solving nonlinear equations. We also extend some classes of optimal methods for any suitable choice of iteration parameter. Such development and extension was made using sufficient convergence conditions given in [20]. Numerical examples are considered to check the convergence order of new families and extensions of some well-known methods.

MSC 2010. 65H05.
Keywords. Multipoint methods; order of convergence; nonlinear equations.

1. INTRODUCTION

Finding solution of nonlinear equations $f(x) = 0$ is an important problem in science and engineering. In last years, many optimal eight–order iterative methods were developed, see [1–6, 8, 10–14, 17, 19–23] and references therein. But many of them work only for special choices of iteration parameter and absolutely not clear how changed the structure of iterations for another choice of parameter. Therefore, it is very desirable to construct the optimal iterations that work well for any suitable choice of parameter. Our aim is to develop and to extend some classes of optimal three–point iterations using sufficient convergence conditions given in [20].

We consider the following standard three–point iterative methods:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \bar{\tau}_n \frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \alpha_n \frac{f(z_n)}{f'(x_n)}, \ \ n = 0, 1 \dots \end{aligned} \tag{1}$$

[†]Institute of Mathematics and Digital Technology, Mongolian Academy of Sciences, Mongolia e-mail: tzhanlav@yahoo.com.
[*]School of Applied Sciences, Mongolian University of Science and Technology, Mongolia e-mail: otgondorj@gmail.com.

In [20] was proven that the order of convergence iterations (1) is eight if and only if the parameters $\bar{\tau}_n$ and α_n satisfy the conditions

$$\bar{\tau}_n = 1 + 2\theta_n + \widetilde{\beta}\theta_n^2 + \widetilde{\gamma}\theta_n^3 + \dots, \ \theta_n = \frac{f(y_n)}{f(x_n)} \tag{2}$$

and

$$\begin{aligned} \alpha_n = & 1 + 2\theta_n + (\widetilde{\beta}+1)\theta_n^2 + (2\widetilde{\beta}+\widetilde{\gamma}-4)\theta_n^3 \\ & + (1+4\theta_n)\upsilon_n + O(f(x_n)^4), \end{aligned} \tag{3}$$

where $\upsilon_n = \frac{f(z_n)}{f(y_n)}$. The optimal methods (1) distinguish each other only by choices of parameters $\bar{\tau}_n$ and α_n. It should be pointed out that to establish the convergence order of iterative methods often used either the error equation, see for example [1–5, 8–15, 17–19], or the nonlinear residuals [20–23]. An more detailed explanation of various aspects of convergence order based on error analysis, corrections and nonlinear residuals was given in the excellent surveys [6, 7]. In this paper, we propose a new family of optimal three–point methods and extensions of some classes of optimal methods. The rest of this paper is organized as follows.

In Section 2, we propose new families of optimal three-point methods. In Section 3, we suggested extension of classes of optimal eighth-order methods. The numerical experiments and dynamic behavior of methods are discussed in Section 4. Finally, short conclusions are included in Section 5.

2. DEVELOPMENT OF THE NEW FAMILIES OF OPTIMAL THREE–POINT METHODS

First, we consider iterations (1) with parameter α_n given by

$$\alpha_n = \frac{f'(x_n)}{f[y_n, z_n] + 2(f[x_n, z_n] - f[x_n, y_n]) + (y_n - z_n)f[y_n, x_n, x_n]}, \tag{4}$$

where

$$f[y_n, x_n, x_n] = \frac{f[y_n, x_n] - f'(x_n)}{y_n - x_n}. \tag{5}$$

To show the convergence analysis of methods (1), (4), the following results is proven.

Theorem 1. *Let the function $f(x)$ be sufficiently smooth and have a simple root x^* on the open interval $I \subset R$. Furthermore, let the initial approximation x_0 be sufficiently close to x^* and the parameter $\bar{\tau}_n$ in (1) satisfies the condition (2). Then the order of convergence of the methods (1), (4) is eight.*

Proof. Using the relations

$$\begin{aligned} f[x_n, y_n] &= f'(x_n)(1-\theta_n), \\ f[y_n, z_n] &= f'(x_n)\frac{1-\upsilon_n}{\bar{\tau}_n}, \end{aligned} \tag{6}$$

Table 1. The choices of parameter $\bar{\tau}_n$

Cases	Methods	Special case of (10)	$\bar{\tau}_n$	$\widetilde{\beta}$	$\widetilde{\gamma}$
i	Potra-Ptack's	$c=\omega=1,\ d=b=0$	$1+2\theta_n+\theta_n^2$	1	0
ii	Maheshwari's	$c=1,\ b=0,\ d=\omega=-1$	$\frac{1+\theta_n-\theta_n^2}{1-\theta_n}$	1	1
iii	Kung-Traub's	$c=b=1,\ d=-2,\ \omega=0$	$\frac{1}{(1-\theta_n)^2}$	3	4
iv	King's type	$c=1,\ \omega=b=0,\ d=\beta-2$	$\frac{1+\beta\theta_n}{1+(\beta-2)\theta_n}$	$2(2-\beta)$	$2(2-\beta)^2$

$$f[x_n, z_n] = f'(x_n)\frac{1-\theta_n\upsilon_n}{1+\bar{\tau}_n\theta_n},$$

in (4), we obtain

$$\alpha_n = \frac{\bar{\tau}_n}{1-\upsilon_n+2\frac{\bar{\tau}_n\theta_n}{1+\bar{\tau}_n\theta_n}(1-\upsilon_n-\bar{\tau}_n+\bar{\tau}_n\theta_n)+\bar{\tau}_n^2\theta_n^2}. \tag{7}$$

Using (2) and well-known expansion

$$\frac{1}{1-x} = 1+x+x^2+x^3+\ldots,\ |x|<1, \tag{8}$$

in (7) we obtain

$$\alpha_n = \bar{\tau}_n(1+\theta_n^2-(6-2\widetilde{\beta})\theta_n^3+(1+2\theta_n)\upsilon_n)+\mathcal{O}(f(x_n)^4). \tag{9}$$

From (2) and (9) it follows that α_n defined by (9) satisfies the condition (3) that completes the proof of theorem. □

Of course, there are many possibility for choice $\bar{\tau}_n$ in (1) satisfying the condition (2). In particular, we give $\bar{\tau}_n$ as

$$\bar{\tau}_n = \frac{c+(2c+d)\theta_n+\omega\theta_n^2}{c+d\theta_n+b\theta_n^2},\ c+d+b\neq 0, \tag{10}$$

that includes four free parameters. In Table 1, we list some well-known choices.

Note that similar theorem for iteration (1), (4) for Kung-Traub's type iteration were proved by Petković *et al.* [11] and by Zhanlav *et al.* [22] for Kings type iteration and by Wang *et al.* [19] for Ostrowski's type method. Thus, theorem 1 extend essentially the class of families of optimal eight-order iterations (1), (4). Now we consider the iterations (1) with α_n given by

$$\alpha_n = \frac{f'(x_n)(1+A\theta_n+B\theta_n^2+C\theta_n^3+(\delta+\Delta\theta_n)\upsilon_n)}{\omega_1 f[x_n,z_n]+\omega_2 f[z_n,y_n]+\omega_3 f[x_n,y_n]}, \tag{11}$$

where $\omega_1+\omega_2+\omega_3=1$ and $A, B, C, \delta, \Delta, \omega_1, \omega_2$ and ω_3 are free parameters to be determined such that the iterations (1) with α_n given by (11) has optimal eight-order of convergence. Namely we can prove

THEOREM 2. *Let all assumptions of Theorem 1 be fulfilled. Then the order of convergence of the iterations* (1), (11) *is eight when*

$$\begin{aligned} &A=\delta=1-\omega_2,\ B=(\widetilde{\beta}-2)(1-\omega_2)+1-\omega_1,\ \Delta=3-\omega_1-\omega_2,\\ &C=\widetilde{\gamma}(1-\omega_2)+\widetilde{\beta}(1+\omega_2-\omega_1)+\omega_1-\omega_2-5. \end{aligned} \tag{12}$$

Proof. The proof is the same as that of Theorem 1. For convenience, here we only give the main step of proof. As before, using (6) and (8) after some manipulations we obtain

$$\begin{aligned} \alpha_n =& \frac{f'(x_n)}{\omega_1 f[x_n,z_n]+\omega_2 f[z_n,y_n]+\omega_3 f[x_n,y_n]}\\ =&1+(1+\omega_2)\theta_n+(\omega_1+\widetilde{\beta}\omega_2+(1-\omega_2)^2)\theta_n^2\\ &+(\widetilde{\gamma}\omega_2+\widetilde{\beta}(2(1-\omega_2)^2-\omega_1-2\omega_3)+\omega_1\\ &+2(1-\omega_2)(2-\omega_1)-2(1-\omega_2)^2-(1-\omega_2)^3)\theta_n^3\\ &+(\omega_2+(\omega_1+2\omega_2^2)\theta_n)\upsilon_n+O(f(x_n)^4). \end{aligned} \tag{13}$$

Substituting (13) into (11) and comparing (11) with (3) we arrive at (12). □

The expression in the numerator of (11) can be expressed through first order divided differences $f[x_n,y_n]$, $f[x_n,z_n]$ and $f[z_n,y_n]$ within accuracy $\mathcal{O}(f(x_n)^4)$. Indeed using the iterations

$$f[x_n,y_n]-f[x_n,z_n]=f'(x_n)(\theta_n^2+(\widetilde{\beta}-3)\theta_n^3+\theta_n\upsilon_n), \tag{14}$$

and

$$\begin{aligned} f[z_n,x_n]-f[y_n,z_n]=&f'(x_n)(\theta_n+(\widetilde{\beta}-5)\theta_n^2\\ &+(\widetilde{\gamma}-5\widetilde{\beta}+11)\theta_n^3+(1-3\theta_n)\upsilon_n), \end{aligned} \tag{15}$$

in (11) we obtain

$$\alpha_n=\frac{(3\omega_2+\omega_1-5)(f[x_n,z_n]-f[x_n,y_n])+F_n+Q_n}{\omega_1 f[x_n,z_n]+\omega_2 f[z_n,y_n]+\omega_3 f[x_n,y_n]}, \tag{16}$$

where

$$Q_n=f'(x_n)(1+(\omega_2-2)\theta_n^2+2(1-\omega_1-\omega_2)\theta_n^3)$$

and

$$F_n=(1-\omega_2)(f[x_n,y_n]-f[y_n,z_n]).$$

From (11), (12) we see that α_n includes two free parameters ω_1 and ω_2. Thus, we develop the class of optimal eight-order iterations (1), (11), (12). We consider some choices of parameters ω_1 and ω_2.

(1) Let $\omega_1=\omega_3=0$, $\omega_2=1$. Then (16) converted to

$$\alpha_n=\frac{(3+\theta_n)f[x_n,y_n]-2f[z_n,x_n]}{f[y_n,z_n]}. \tag{17}$$

(2) Let $\omega_1 = 1$, $\omega_2 = \omega_3 = 0$. Then (16) converted to

$$\alpha_n = \frac{5f[x_n, y_n] - 4f[z_n, x_n] - f[y_n, z_n] + f'(x_n)(1 - 2\theta_n^2)}{f[x_n, z_n]}, \tag{18}$$

(3) Let $\omega_1 = \omega_2 = 0$, $\omega_3 = 1$. Then (16) converted to

$$\alpha_n = \frac{6f[x_n, y_n] - 5f[z_n, x_n] - f[y_n, z_n] + f'(x_n)(1 - 2\theta_n^2 + 2\theta_n^3)}{f[y_n, x_n]}, \tag{19}$$

(4) Let $\omega_1 = -1$, $\omega_2 = 2$, $\omega_3 = 0$. Then (16) converted to

$$\alpha_n = \frac{f[z_n, y_n] - f[x_n, y_n] + f'(x_n)}{2f[z_n, y_n] - f[z_n, x_n]}. \tag{20}$$

The iteration (1), (20) can be considered as another variant of iterations given by Sharma *et al.* in [12–14] and given by Zhanlav *et al.* in [23].

(5) Let $\omega_1 = \omega_2 = 1$, $\omega_3 = -1$. Then (16) converted to

$$\alpha_n = \frac{f[y_n, x_n] - f[z_n, x_n] + f'(x_n)(1 - \theta_n^2 - 2\theta_n^3)}{f[z_n, x_n] + f[z_n, y_n] - f[y_n, x_n]}. \tag{21}$$

It can be rewritten as:

$$\alpha_n \approx \frac{1}{\left(1 - \frac{f(z_n)}{f(x_n)}\right)\left(1 + (5 - \widetilde{\beta})\left(\frac{f(y_n)}{f(x_n)}\right)^3\right)} \frac{f'(x_n)}{f[z_n, x_n] + f[z_n, y_n] - f[y_n, x_n]}.$$

It is worth to note that similar results for derivative-free case and for some choices of $\bar{\tau}_n$ were obtained by Thukral in [18] and by Khattri *et al.* in [9]. We also note that the iteration (1), (21) for $\widetilde{\beta} = 4$ was considered by Sharma *et al.* in [15].

(6) Let $\omega_1 = -1$, $\omega_2 = \omega_3 = 1$. Then (16) converted to

$$\alpha_n = \frac{3f[x_n, y_n] - 3f[z_n, x_n] + f'(x_n)(1 - \theta_n^2 + 2\theta_n^3)}{f[y_n, x_n] + f[z_n, y_n] - f[z_n, x_n]}. \tag{22}$$

In each iteration step the methods (1), (4) and (1), (11) require three function evaluations and one evaluation of first derivative. Based on the conjecture of Kung and Traub, the methods reached the optimality with higher efficiency index $E = 8^{1/4} = 1.68179$. One of main advantageous of the proposed iterative methods (1), (4) and (1), (11) is that they work well for any choice of parameter $\bar{\tau}_n$ satisfying the condition (2).

3. EXTENSIONS OF SOME CLASSES OF OPTIMAL EIGHT-ORDER METHODS

Now we consider the iterations (1) with parameter α_n given by

$$\alpha_n = (p(t_n) + \hat{\gamma}\theta_n^3) \frac{f'(x_n)}{f[z_n, y_n] + (z_n - y_n) f[z_n, x_n, x_n]}, \tag{23}$$

where $t_n = \frac{f(z_n)}{f(x_n)} = \theta_n \upsilon_n$ and $\hat{\gamma}$ constant and $p(t_n)$ some function of t.

Namely, we have

THEOREM 3. *Let all assumptions of Theorem 1 be fulfilled. Then the order of convergence of the iterations (1) and (23) is eight when*

$$p(0)=1,\ p'(0)=2,\ \hat{\gamma}=2(\widetilde{\beta}-5). \tag{24}$$

Proof. We denote the second factor in (23) by $\hat{\alpha}_n$. That is

$$\hat{\alpha}_n=\frac{f'(x_n)}{f[z_n,y_n]+(z_n-y_n)f[z_n,x_n,x_n]}. \tag{25}$$

Then using the relations (6) we obtain

$$\hat{\alpha}_n=\frac{\bar{\tau}_n}{1-\upsilon_n-\bar{\tau}_n\left(\frac{\bar{\tau}_n\theta_n}{1+\bar{\tau}_n\theta_n}\right)^2}. \tag{26}$$

Using the expansion (8) in (26) and taking into account $\upsilon_n=\mathcal{O}(f(x_n)^2)$, $\theta_n=\mathcal{O}(f(x_n))$, we obtain

$$\hat{\alpha}_n=\bar{\tau}_n\left(1+\upsilon_n+\bar{\tau}_n\left(\frac{\bar{\tau}_n\theta_n}{1+\bar{\tau}_n\theta_n}\right)^2\right)+\mathcal{O}(f(x_n)^4). \tag{27}$$

By using (2) and (8) it is easy to show that

$$\left(\frac{\bar{\tau}_n\theta_n}{1+\bar{\tau}_n\theta_n}\right)^2=\theta_n^2+2\theta_n^3+\mathcal{O}(f(x_n)^4). \tag{28}$$

Substituting (2) and (28) into (27) we get

$$\hat{\alpha}_n=1+2\theta_n+(\widetilde{\beta}+1)\theta_n^2+(\widetilde{\gamma}+6)\theta_n^3+(1+2\theta_n)\upsilon_n+\mathcal{O}(f(x_n)^4). \tag{29}$$

Then (23) is written as

$$\begin{aligned}\alpha_n=&(p(t_n)+\hat{\gamma}\theta_n^3)(1+2\theta_n+(\widetilde{\beta}+1)\theta_n^2\\&+(\widetilde{\gamma}+6)\theta_n^3+(1+2\theta_n)\upsilon_n)+\mathcal{O}(f(x_n)^4),\end{aligned} \tag{30}$$

which satisfies the condition (3) provided that (24). □

Thus, we develop the family of optimal three-point iterative methods (1) with α_n given by

$$\alpha_n=(p(t_n)+2(\widetilde{\beta}-5)\theta_n^3)\frac{f'(x_n)}{f[z_n,y_n]+(z_n-y_n)f[z_n,x_n,x_n]}. \tag{31}$$

Similar results were obtained in [2, 3] for the iterations (1) with

$$\bar{\tau}_n=\frac{1-\theta_n/2}{1-5\theta_n/2}, \tag{32}$$

and

$$\bar{\tau}_n=\frac{1}{1-2\theta_n-\theta_n^2-\theta_n^3/2}, \tag{33}$$

respectively. The parameters $\bar{\tau}_n$ given by (32) and (33) satisfy the condition (2) with $\widetilde{\beta} = 5$. In this case $\hat{\gamma} = 0$ by (24) and the α_n given by (23) leads to

$$\alpha_n = p(t_n)\frac{f'(x_n)}{f[z_n, y_n] + (z_n - y_n)f[z_n, x_n, x_n]}. \tag{34}$$

This means that our iterations (1) and (31) include the iterations proposed by Bi *et al.* [2] and by Cordero *et al.* [3] as particular cases.

Now we consider the expression

$$\tilde{\alpha}_n = \frac{f'(x_n)}{f[z_n, y_n] + (z_n - y_n)f[y_n, x_n, x_n]}. \tag{35}$$

As before, using the relations (6) in (35) we obtain

$$\tilde{\alpha}_n = \frac{\bar{\tau}_n}{1 - \upsilon_n - \bar{\tau}_n^2\theta_n^2} = \bar{\tau}_n(1 + \upsilon_n + \bar{\tau}_n^2\theta_n^2) + \mathcal{O}(f(x_n)^4). \tag{36}$$

By (2) one can easily to check that

$$\tilde{\alpha}_n = \hat{\alpha}_n + \mathcal{O}(f(x_n)^4). \tag{37}$$

It means that instead of (23) one can also use

$$\alpha_n = (p(t_n) + \hat{\gamma}\theta_n^3)\frac{f'(x_n)}{f[z_n, y_n] + (z_n - y_n)f[y_n, x_n, x_n]}, \tag{38}$$

Therefore, Theorem 3 holds true for iterations (1), (38). Similar extension can be done for all optimal eight-order iterations. As examples, we present in Table 2 some of methods and their extension $\tilde{\alpha}_n = \xi_n \cdot \alpha_n$ with extension factor ξ_n.

Table 2. The extraneous fixed points

N	Methods	$\bar{\tau}_n$	$\widetilde{\beta}$	$\widetilde{\gamma}$	α_n	Extension ξ_n
1	Sharma *et al.* [12]	$\frac{1}{1-2\theta_n}$	4	8	$\frac{W(t_n)f[x_n,y_n]f'(x_n)}{f[x_n,z_n]f[y_n,z_n]}$ $W(0)=1, W'(0)=1$	$1+(\widetilde{\beta}-4)\theta_n^3$
2	DP8 [11]	$\frac{1}{1-2\theta_n}$	4	8	$\frac{1}{(1-2\theta_n-\theta_n^2)(1-\upsilon_n)(1-2\theta_n\upsilon_n)}$	$1+(\widetilde{\beta}-4)\theta_n^2$ $+(\widetilde{\gamma}-8)\theta_n^3$
3	GK8 [8]	$\frac{1+\beta\theta_n+\lambda\theta_n^2}{1+(\beta-2)\theta_n+\mu\theta_n^2}$ $\mu=-\frac{3\beta}{2}, \lambda=-1+\frac{\beta}{2}$	3	6	$\frac{1}{1-2\theta_n-\upsilon_n}$	$1+(\widetilde{\beta}-3)\theta_n^2$ $+(\widetilde{\gamma}-6)\theta_n^3$
4	Chun *et al.* [4]	$\frac{1}{(1-\theta_n)^2}$	3	4	$\frac{1}{(1-H(\theta_n)-J(t_n)-P(\upsilon_n))^2}$	$1+(\widetilde{\beta}-3)\theta_n^2$ $+(\widetilde{\gamma}-4)\theta_n^3$
5	Thukral-Petković [17]	$\frac{1+\beta\theta_n}{1+(\beta-2)\theta_n}$	$2(2-\beta)$	$2(2-\beta)^2$	$\phi(\theta_n)+\frac{t_n}{\theta_n-at_n}+4t_n$	$1+(\widetilde{\beta}-2(2-\beta))\theta_n^2$ $+(\widetilde{\gamma}-2(2-\beta)^2)\theta_n^3$
6	Lotfi *et al.* [10]	$\frac{1}{1-2\theta_n}$	4	8	$\frac{H(\theta_n)+K(t_n)}{G(\upsilon_n)}$	$1+(\widetilde{\beta}-4)\theta_n^2$ $+(\widetilde{\gamma}-8)\theta_n^3$

Thus, we obtain extensions of some well-known optimal methods that work well for any suitable parameter $\bar{\tau}_n$, satisfying the condition (2). This allows us to expand the applicability of the original methods.

4. NUMERICAL EXPERIMENTS

In order to show the convergence behavior and to check the validity of theoretical results of the presented family (1) with parameters $\bar{\tau}_n$ and α_n, we make some numerical experiments. We also compare our methods with existing methods of same order in [13], [14] and [23] that denoted by (SAWN8) and (ZO8). Here all the computations are performed using the programming package MATHEMATICA with multiple-precision arithmetic and 1000 significant digits. As a test, we consider the following sample functions.

$$f_1(x) = e^{x^3-3x}\sin x + \log(x^2+1), \quad x^* = 0,$$
$$f_2(x) = x^2 - \exp(x) - 3x + 2, \quad x^* \approx 0.25.$$

In Tables 3–5, we present the necessary iterations (n), absolute error $|x_n - x^*|$ and computational order of convergence, which is calculated by the following formula [11, 16]:

$$\rho \approx \frac{\ln(|x_{n-1} - x_n|/|x_n - x_{n-1}|)}{\ln(|x_n - x_{n-1}|/|x_{n-2} - x_{n-1}|)},$$

where x_n, x_{n-1}, x_{n-2} are three consecutive approximations of iterations. The convergence orders and their computational variants have been thoroughly treated in [6, 7]. Outcomes of the numerical experiments are calculated so as to satisfy the criterion $|x_n - x^*| < 10^{-30}$. For $\bar{\tau}_n$ parameter, we choose the cases i–iv listed in Table 1. Table 3 gives some numerical results in order to show convergence behaviour of method (1) with α_n parameter given by (4), (17)–(22). We observe from Table 3 that the methods (1) with parameters $\bar{\tau}_n$ given by case iv and α_n given by (4), (21) produce approximations of higher accuracy compared to the eight-order methods SAWN8, ZO8.

The results corresponding to the same kind of experiments for the extension of methods can be found in Table 4-5. In Table 4, we present the numerical results of iteration (1) with parameter α_n given (23) and (38), in which we used function $p(t) = \frac{1}{(1-t)^2}$. In Table 5, we present the numerical results of the extension of some methods that work well any parameters $\bar{\tau}_n$ satisfies condition (2).

From the results displayed in Table 3–5, we see that the calculated values of the computational order of convergence are in complete agreement with the theoretical orders proved in Section 2, 3.

Additionally, we analyze the basin of attraction of our methods to find out what is the best choice for the parameters. To generate basin attraction for complex polynomials using the methods, we take a grid of 400×400 points z_0 in the square $[-3,3] \times [-3,3] \subset C$. We have used the method (1) for cubic polynomial $p(z) = z^3 - 1$ having three simple zeros.

Table 3. The numerical result for $f_i(x)$ by the methods (1) with $\bar{\tau}_n$ and α_n

α_n	$\bar{\tau}_n$	n	$\|x^* - x_n\|$	ρ	n	$\|x^* - x_n\|$	ρ
			$f_1(x)$, $x_0 = 0.5$			$f_2(x)$, $x_0 = 2$	
(4)	case i	3	0.5407e-222	8.00	3	0.1220e-231	8.00
	case ii	3	0.4264e-222	8.00	3	0.4412e-189	7.99
	case iii	3	0.2180e-233	8.00	3	0.2054e-245	8.00
	case iv, $\beta = 0$	3	0.2111e-226	8.00	3	0.4836e-229	7.99
(17)	case i	3	0.5182e-200	8.00	3	0.3202e-231	7.99
	case ii	3	0.7800e-203	8.00	2	0.6607e-31	8.00
	case iii	3	0.4084e-132	8.00	3	0.2183e-205	7.99
	case iv, $\beta = 0$	3	0.5805e-127	8.00	3	0.3391e-235	8.00
(18)	case i	3	0.3401e-122	8.00	3	0.3173e-229	7.99
	case ii	3	0.5182e-200	8.00	3	0.3976e-247	7.99
	case iii	3	0.4084e-132	7.99	3	0.9235e-207	7.99
	case iv, $\beta = 0$	3	0.5805e-127	8.00	3	0.1166e-232	7.99
(19)	case i	3	0.8671e-118	8.00	3	0.2475e-221	7.99
	case ii	3	0.1363e-116	8.00	3	0.3905e-241	7.99
	case iii	3	0.1213e-121	8.00	3	0.2152e-202	7.99
	case iv, $\beta = 0$	3	0.3621e-119	8.00	3	0.1296e-226	7.99
(20)	case i	3	0.1024e-139	7.99	3	0.1307e-234	7.99
	case ii	3	0.4756e-142	7.99	2	0.7314e-33	7.93
	case iii	3	0.1560e-146	7.99	3	0.3104e-204	8.00
	case iv, $\beta = 0$	3	0.5207e-153	8.00	3	0.1715e-172	8.00
(21)	case i	3	0.2195e-222	8.00	3	0.4879e-248	7.99
	case ii	3	0.4844e-243	8.00	2	0.9301e-33	7.97
	case iii	3	0.2720e-219	8.00	3	0.8760e-212	7.99
	case iv, $\beta = 0$	3	0.8966e-220	8.00	3	0.4439e-247	8.00
(22)	case i	3	0.1236e-187	8.00	3	0.1047e-221	7.99
	case ii	3	0.9877e-182	8.00	3	0.1326e-245	7.99
	case iii	3	0.1077e-184	8.00	3	0.9069e-201	8.00
	case iv, $\beta = 0$	3	0.1073e-184	8.00	3	0.1044e-227	8.00
SAWN8 [14]		3	0.5207e-153	8.00	3	0.1715e-172	8.00
ZO8 [23]		3	0.1036e-137	7.99	3	0.3317e-178	8.00

In Figure 4.1–4.3, the yellow, red and blue colors are assigned for the attraction basin of the three zeros and the roots of function are marked with white points. Black color is shown lack of convergence to any of the roots. In this cases, the stopping criterion $\varepsilon = 10^{-3}$ and maximum of 25 iterations are used.

Based on Figure 4.1–4.3 for $p(z)$, we can see that the method (1) with $\bar{\tau}_n$ given by case iii and α_n given by (4) is the best one and have fewer diverging points that other cases of parameters.

Table 4. The numerical result for $f_i(x)$ by the methods (1) with $\bar{\tau}_n$ and α_n

α_n	$\bar{\tau}_n$	n	$\lvert x^* - x_n\rvert$	ρ	n	$\lvert x^* - x_n\rvert$	ρ
			$f_1(x),\quad x_0 = 0.5$			$f_2(x),\quad x_0 = 2$	
(23)	case i	3	0.6992e-190	8.00	3	0.2459e-223	7.99
	case ii	3	0.1915e-190	8.00	3	0.3996e-244	7.99
	case iii	3	0.2945e-186	8.00	3	0.1124e-181	7.99
	case iv, $\beta = 0$	3	0.1334e-189	8.00	3	0.8535e-221	8.00
(38)	case i	3	0.3186e-194	8.00	3	0.1093e-200	7.99
	case ii	3	0.1273e-194	8.00	3	0.1051e-217	7.99
	case iii	3	0.2945e-186	8.00	3	0.3833e-158	7.99
	case iv, $\beta = 0$	3	0.6747e-194	8.00	3	0.5685e-198	8.00

Table 5. The numerical results of extension of methods for $f_2(x)$

Methods	Extension factor ξ_n	$\bar{\tau}_n$	n	$\lvert x^* - x_n\rvert$	ρ
Sharma [12]	$1 + (\widetilde{\beta} - 4)\theta_n^3$	Case i	3	0.3159e-214	8.00
		Case ii	3	0.3453e-212	8.00
		Case iv, $\beta = 0$	3	0.8671e-210	7.99
DP8 [11]	$1 + (\widetilde{\beta} - 4)\theta_n^2 + (\widetilde{\gamma} - 8)\theta_n^3$	Case i	3	0.2781e-229	7.99
		Case ii	3	0.1848e-238	7.99
		Case iv, $\beta = 0$	3	0.1196e-162	7.99
GK8 [8]	$1 + (\widetilde{\beta} - 3)\theta_n^2 + (\widetilde{\gamma} - 6)\theta_n^3$	Case i	3	0.2433e-214	7.99
		Case ii	3	0.5841e-236	7.99
		Case iv, $\beta = 1$	3	0.6505e-172	7.99
Chun [4]	$1 + (\widetilde{\beta} - 3)\theta_n^2 + (\widetilde{\gamma} - 4)\theta_n^3$	Case i	3	0.2100e-114	8.00
		Case ii	3	0.2547e-114	8.00
		Case iii	3	0.7827e-120	8.00
Thukral [17]	$1 + (\widetilde{\beta} - 2(2 - \beta))\theta_n^2$ $+(\widetilde{\gamma} - 2(2 - \beta)^2)\theta_n^3$	Case i	3	0.5103e-156	8.00
		Case ii	3	0.1603e-155	8.00
		Case iv, $\beta = 0$	3	0.5103e-156	8.00
Lotfi [10]	$1 + (\widetilde{\beta} - 4)\theta_n^2 + (\widetilde{\gamma} - 8)\theta_n^3$	Case i	3	0.1915e-234	8.00
		Case ii	3	0.1797e-233	8.00
		Case iv, $\beta = 0$	3	0.2567e-157	8.00

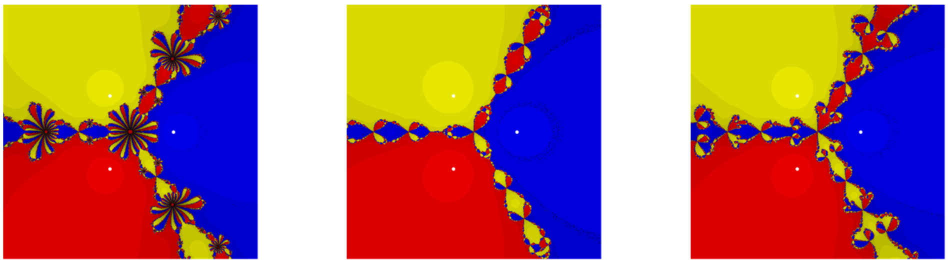
5. CONCLUSION

The main contributions of this work are:

The development of wide class of optimal eight-order iterative methods and extensions of some optimal methods that work well for any suitable choice of parameter $\bar{\tau}_n$ satisfying the condition (2).

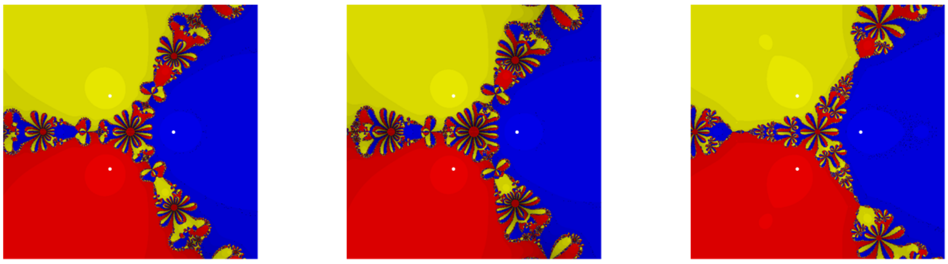
The proposed iterative methods can be regarded as an advancement in the topic and can compete with other well-known methods.

ACKNOWLEDGEMENTS. The authors wish to thank the anonymous referees for their valuable suggestions and comments, which improved paper.



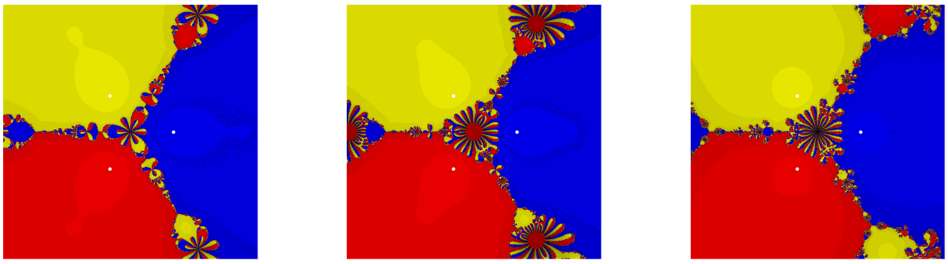
(a) (1)-ii-(4) (b) (1)-iii-(4) (c) (1)-iv-(4)

Fig. 4.1. Basins of attraction of methods for $z^3 - 1$.



(a) (1)-iv-(17) (b) (1)-iv-(18) (c) (1)-iii-(20)

Fig. 4.2. Basins of attraction of methods for $z^3 - 1$.



(a) (1)-iv-(20) (b) (1)-iv-(21) (c) (1)-iv-(31)

Fig. 4.3. Basins of attraction of methods for $z^3 - 1$.

REFERENCES

[1] W. Bi, H. Ren, Q. Wu, *Three-step iterative methods with eight-order convergence for solving nonlinear equations*, J. Comput. Appl. Math., **225** (2009), 105–112, `https://doi.org/10.1016/j.cam.2008.07.004`.

[2] W. Bi, Q. Wu, H. Ren, *A new family of eighth–order iterative methods for solving nonlinear equations*, Appl. Math. Comput., **214** (2009), 236–245, `https://doi.org/10.1016/j.amc.2009.03.077`.

[3] A. Cordero, M. Fardi, M. Ghasemi, J.R. Torregrosa, *Accelerated iterative methods for finding solutions of nonlinear equations and their dynamical behavior*, Calcolo, **51** (2014), 17–30, `http://dx.doi.org/10.1007/s10092-012-0073-1`.

[4] C. Chun, M.Y. Lee, *A new optimal eighth-order family of iterative methods for the solution of nonlinear equations*, Appl. Math. Comput., **223** (2013), 506–519, `https://doi.org/10.1016/j.amc.2013.08.033`.

[5] C. Chun, B. Neta, *Comparative study of eighth–Order methods for finding simple roots of nonlinear equations*, Numer. Algor., **74** (2017), 1169–1201, `https://link.springer.com/article/10.1007/s11075-016-0191-y`.

[6] E. Cătinaş, *A survey on the high convergence orders and computational convergence orders of sequences*, Appl. Math. Comput., **343** (2019), pp. 1-20, `https://doi.org/10.1016/j.amc.2018.08.006`.

[7] E. Cătinaş, *How many steps still left to x^*?*, SIAM Review, **63**(3) (2021), pp. 585-624, `https://doi.org/10.1137/19M1244858`.

[8] Y.H. Geum, Y.I. Kim, *A uniparamtric family of three-step eighth-order multipoint iterative methods for simple roots.* Appl. Math. Lett., **24** (2011), 929–935, `https://doi.org/10.1016/j.aml.2011.01.002`.

[9] S.K. Khattri, R.P. Agarwal, *Derivative-free optimal iterative methods*, Comput. Methods. Appl. Math., **10** (2010), 368–375, `https://doi.org/10.2478/cmam-2010-0022`.

[10] T. Lotfi, S. Sharifi, M. Salimi, S. Siegmund, *new class of three-point methods with optimal convergence order eight and its dynamics*, Numer. Algor., **68** (2015), 261–288, `https://doi.org/10.1007/s11075-014-9843-y`.

[11] M.S. Petković, B. Neta, L.D. Petković, J. Džunić, *Multipoint methods for solving nonlinear equations.* Elsevier, (2013).

[12] J.R. Sharma, R. Sharma, *A new family of modified Ostrowski's method with accelerated eighth–order convergence*, Numer. Algorithms, **54** (2010), 445–458, `https://doi.org/10.1007/s11075-009-9345-5`.

[13] J. R. Sharma, H. Arora, *A new family of optimal eighth order methods with dynamics for nonlinear equations.* Appl. Math. Comput., **273** (2016), 924–933, `http://dx.doi.org/10.5923/j.ajcam.20180801.02`.

[14] J. R. Sharma, H. Arora, *An efficient family of weighted-Newton methods with optimal eighth order convergence*, Appl. Math. Lett., **29** (2014), 1–6, `https://www.academia.edu/22799830/`.

[15] R. Sharma, A. Bahl, *Optimal eighth order convergent iteration scheme based on Lagrange interpolation*, Acta Mathematicae Applicatae Sinica, English Series **33** (2017), 1093–1102, `https://doi.org/10.1007/s10255-017-0722-x`.

[16] J. W. Schmidt, *On the R-Order of Coupled Sequences*, Computing, **26** (1981), 333—342.

[17] R. Thukral, M.S. Petković, *Family of three-point methods of optimal order for solving nonlinear equations*, J. Comput. Appl. Math., **233** (2010), 2278–2284, `https://doi.org/10.1016/j.cam.2009.10.012`.

[18] R. Thukral, Eighth–order iterative Methods without derivatives for solving nonlinear equations, ISRN. Appl. Math., Vol (2011), Article ID 693787, 12 pages. `https://doi.org/10.5402/2011/693787.`.

[19] X. Wang, L. Liu, *Modified Ostrowski's method with eighth-order convergence and high efficiency index*, Appl. Math. Lett,. **23** (2010), 549–554, `https://doi.org/10.1016/j.aml.2010.01.009`.

[20] T. Zhanlav, V. Ulziibayar, O. Chuluunbaatar, *The necessary and sufficient conditions for two and three-point iterative method of Newton's type*, Comput. Math. Math. Phys., **57**: 1093–1102, `http://dx.doi.org/10.1134/S0965542517070120`.

[21] T. Zhanlav, O. Chuluunbaatar, V. Ulziibayar, *Generating function method for constructing new iterations*, Appl. Math. Comput., **315** (2017), 414–423, https://doi.org/10.1016/j.amc.2017.07.078.

[22] T. Zhanlav, V. Ulziibayar, *Modified King's methods with optimal eighth-order of convergence and high efficiency index*, Amer. J. Comput. Appl. Math., **6** (2016), 177–181 http://dx.doi.org/10.5923/j.ajcam.20160605.01.

[23] T. Zhanlav, Kh. Otgondorj, *A new family of optimal eighth-order methods for solving nonlinear equations*, Amer. J. Comput. Appl. Math., **8** (2018), 15–19, http://dx.doi.org/10.5923/j.ajcam.20180801.02.

Received by the editors: August 20, 2021; accepted: January 4, 2021; published online: February 17, 2022.