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SOLUTION TO UNSTEADY FRACTIONAL HEAT CONDUCTION IN THE QUARTER-PLANE *VIA* THE JOINT LAPLACE-FOURIER SINE TRANSFORMS

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Abstract. In this article, the author implemented the joint transform method, for solving the boundary value problems of time fractional heat equation. We also used methods of operational nature to solve a Fokker-Planck equation with non-constant coefficients. The results reveal that the integral transform method is reliable and efficient. Some illustrative non-trivial examples are also provided.

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1. INTRODUCTION AND PRELIMINARIES

In this study, the author provided mathematical results that are useful to the researchers in a variety of fields. This article is devoted to studying and application of the joint Laplace-Fourier sine transform for solving time fractional diffusion equation in the quarter-plane. We also consider a Fokker-Planck equation with variable coefficients. We provided methods and results for a partial fractional differential equations which arise in applications. So far, different methods of solution have been introduced to solve partial fractional differential equations, the Laplace transform method, [1] [2], [3], the Fourier transform method [10], operational method [4], [6]. We provided methods and results for a partial fractional differential equations which arise in applications. Different methods of solution have been introduced to solve partial fractional differential equations, the Laplace transform method, [1], [2], [3], the Fourier transform method [10], operational method [4], [6]. The diffusion equation describes the flow of heat, or a concentration of particles. In [11], the author considered the time fractional radial diffusion in a cylinder by using the joint Laplace-finite Hankel transforms `https://www.overleaf.com/project/6148a908e65ae2a25915fa5d` method.

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In [12], the authors considered the time-fractional diffusion-wave equation. The corresponding Green's function was obtained in closed form for arbitrary space dimension in terms of Fox H-functions.

REMARK 1.1. In [9], the authors summarize the essential definitions and notations for the Fox H-functions.They also provide for the general Green function a representation in terms of Mellin–Barnes integrals and, consequently, in terms of Fox H-functions. □

1.1. Definitions and Notations.

DEFINITION 1.2. *The left Riemann-Liouville fractional derivative of order* α $(0 < \alpha < 1)$ *of* $\phi(t)$ *is defined as follows* [10]

$$D_{a,t}^{R-L,\alpha}\phi(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t \frac{1}{(t-\xi)^\alpha}\phi(\xi)d\xi. \tag{1.1}$$

DEFINITION 1.3. *The left Caputo fractional derivative of order* α $(0 < \alpha < 1)$ *of* $\phi(t)$ *is defined as follows* [10]

$$D_{a,t}^{C,\alpha}\phi(t) = \frac{1}{\Gamma(1-\alpha)}\int_a^t \frac{1}{(t-\xi)^\alpha}\phi'(\xi)d\xi.$$

It should be pointed out that in the literature, the Riemann-Liouville and the Caputo fractional derivatives generally mean the left Riemann-Liouville and the left Caputo derivatives, respectively.

Let us recall some definitions and properties that are related to the classical continuous Fourier transform.

DEFINITION 1.4. *The Fourier transform of the function* $f(x)$*, and* $-\infty < x < +\infty$ *is defined as follows*

$$\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{i\omega x}f(x)dx := F(\omega). \tag{1.2}$$

If $\mathcal{F}\{f(x)\} = F(\omega)$*, then the inverse Fourier transform* $\mathcal{F}^{-1}\{F(\omega)\}$ *is given by*

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ix\omega}F(\omega)d\omega. \tag{1.3}$$

LEMMA 1.5. *The following identities hold true.*

(1) $\mathcal{F}^{-1}(\frac{1}{a\sqrt{2}}\exp(-\frac{\omega^2}{4a^2})) = \exp(-a^2x^2)$,
(2) $\mathcal{F}^{-1}(\sqrt{\frac{2}{\pi}}\frac{a}{a^2+\omega^2}) = \exp(-a|x|)$,
(3) $\mathcal{F}^{-1}(\sqrt{\frac{2}{\pi}}\frac{2ai\omega}{(a^2+\omega^2)^2}) = x\exp(-a|x|)$,
(4) $\mathcal{F}^{-1}(i\sqrt{\frac{2}{\pi}}\frac{\Gamma(1-\delta)}{|\omega|^{1-\delta}}\cos(\frac{\pi\delta}{2})) = |x|^{-\delta}sgn(x)$.

Proof. See [8]. □

In the sequel a new class of the inverse Fourier transforms of exponential functions involving square roots are determined. Inverse Fourier transforms involving square roots arise in many areas of applied mathematics and mathematical physics.

Lemma 1.6. *The following identity holds true*

$$\mathcal{F}^{-1}\Big[\tfrac{e^{-t\sqrt{\omega^2+2i\lambda\omega+k^2}}}{\sqrt{\omega^2+2i\lambda\omega+k^2}};\omega\to x\Big]=\sqrt{\tfrac{2}{\pi}}e^{-\lambda x}K_0\Big(\sqrt{(x^2+t^2)(k^2+\lambda^2)}\Big).$$

Proof. Let us assume that

$$F(\omega)=\tfrac{e^{-(t\sqrt{\omega^2+2i\lambda\omega+k^2})}}{\sqrt{\omega^2+2i\lambda\omega+k^2}}.$$

Let us consider the following well-known elementary integral

$$\int_0^{+\infty}e^{(-a^2x^2-\frac{b^2}{x^2})}dx=\tfrac{\sqrt{\pi}}{2a}e^{-2ab}.$$

In view of the above integral, $F(\omega)$ can be written as follows

$$F(\omega)=\tfrac{2}{\sqrt{\pi}}\Big[\tfrac{\sqrt{\pi}e^{-2(\frac{t}{2}\sqrt{\omega^2+2i\lambda\omega+k^2})}}{2\sqrt{\omega^2+2i\lambda\omega+k^2}}\Big]=\tfrac{2}{\sqrt{\pi}}\int_0^{+\infty}e^{-(\omega^2+2i\lambda\omega+k^2)\xi^2-\frac{t^2}{4\xi^2}}d\xi.$$

Upon using Fourier inversion formula, we arrive at

$$\mathcal{F}^{-1}(F(\omega))=f(x)=\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-ix\omega}\Big(\tfrac{2}{\sqrt{\pi}}\int_0^{+\infty}e^{-(\omega^2+2i\lambda\omega+k^2)\xi^2-\frac{t^2}{4\xi^2}}d\xi\Big)d\omega,$$

changing the order of integration, we get the following

$$f(x)=\tfrac{2}{\sqrt{\pi}}\int_0^{+\infty}e^{-k^2\xi^2-\frac{t^2}{4\xi^2}}\Big(\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-i(x+2\lambda\xi^2)\omega-\xi^2\omega^2}d\omega\Big)d\xi,$$

after evaluating the inner integral by using first part of the Lemma 1.5, we have

$$f(x)=\tfrac{2}{\sqrt{\pi}}\int_0^{+\infty}e^{-k^2\xi^2-\frac{t^2}{4\xi^2}}(\tfrac{1}{\xi\sqrt{2}})e^{-\frac{(x+2\lambda\xi^2)^2}{4\xi^2}}d\xi,$$

at this stage, if we make the simple change of variable $\xi^2=\theta$ in the above integral, after simplifying we obtain

$$f(x)=\tfrac{e^{-\lambda x}}{\pi}\int_0^{+\infty}e^{-(\lambda^2+k^2)\theta-\frac{0.5(t^2+x^2)}{\theta}}\tfrac{d\theta}{2\theta},$$

by means of the following integral representation for the modified Bessel's function of the second kind of order zero (Macdonald's function)

$$K_0(2\sqrt{pq})=\int_0^{+\infty}e^{-(p\xi+\frac{q}{\xi})}\tfrac{d\xi}{2\xi},$$

we have finally

$$f(x)=\sqrt{\tfrac{2}{\pi}}e^{-\lambda x}K_0\Big(\sqrt{(t^2+x^2)(\lambda^2+k^2)}\Big).\qquad\square$$

Corollary 1.7. *The following identity holds true*

$$\mathcal{F}^{-1}\left[\frac{e^{-t\sqrt{\omega^2+k^2}}}{\sqrt{\omega^2+k^2}};\omega\to x\right]=\sqrt{\tfrac{2}{\pi}}K_0\Big(k\sqrt{(x^2+t^2)}\Big).$$

Proof. In the above Lemma 1.6 let us choose $\lambda=0$ we get the desired identity. □

Corollary 1.8. *The following integral identity holds true*

$$\int_{-\infty}^{+\infty}\frac{e^{-(t\sqrt{\omega^2+2i\lambda\omega+k^2})}}{\sqrt{\omega^2+2i\lambda\omega+k^2}}d\omega=2K_0\Big(t(\sqrt{k^2+\lambda^2})\Big).$$

Proof. In the above Lemma 1.6, if we take $x=0$ we arrive at the result. □

Corollary 1.9. *The following identity holds true*

$$\mathcal{F}^{-1}\left[\frac{e^{-t\sqrt{\omega^2+2i\lambda\omega+k^2}}}{\sqrt{\omega^2+2i\lambda\omega+k^2}};\omega\to x\right]=\sqrt{\tfrac{2}{\pi}}e^{-\lambda x}\int_1^{+\infty}e^{-\eta\sqrt{(\lambda^2+k^2)(t^2+x^2)}}\frac{d\eta}{\sqrt{\eta^2-1}}.$$

Proof. Let us recall an integral representation for the modified Bessel's function of order zero

$$K_0(\xi)=\int_0^{+\infty}e^{-\xi\cosh\phi}d\phi.$$

At this point, let us choose $\xi=\sqrt{(k^2+\lambda^2)(t^2+x^2)}$ and making a change of variable $\cosh\phi=\eta$, we get

$$K_0(\xi)=K_0\Big(\sqrt{(t^2+x^2)(k^2+\lambda^2)}\Big)=\int_1^{+\infty}e^{-\eta\sqrt{(t^2+x^2)(k^2+\lambda^2)}}\frac{d\eta}{\sqrt{\eta^2-1}}.$$

By replacing the above integral on the right hand side of the Corollary 1.7 we obtain

$$\mathcal{F}^{-1}\left[\frac{e^{-t\sqrt{\omega^2+2i\lambda\omega+k^2}}}{\sqrt{\omega^2+2i\lambda\omega+k^2}};\omega\to x\right]=\sqrt{\tfrac{2}{\pi}}e^{-\lambda x}\int_1^{+\infty}e^{-\eta\sqrt{(\lambda^2+k^2)(t^2+x^2)}}\frac{d\eta}{\sqrt{\eta^2-1}}.$$ □

Remark 1.10. In the above relation, let us choose $k=0$, then we get the following relation

$$\mathcal{F}^{-1}\left[\frac{e^{-t\sqrt{\omega^2+2i\lambda\omega}}}{\sqrt{\omega^2+2i\lambda\omega}};\omega\to x\right]=\sqrt{\tfrac{2}{\pi}}e^{-\lambda x}\int_1^{+\infty}e^{-\eta\lambda\sqrt{(t^2+x^2)}}\frac{d\eta}{\sqrt{\eta^2-1}}.$$ □

Note. To the best of the author's knowledge, in the literature the same result is obtained by using complex integration around a complicated key-hole contour in the complex plane.

Definition 1.11. *The Laplace transform of the function $f(t)$, $0<t<+\infty$ is defined as follows*

$$\mathcal{L}\{f(t)\}=\int_0^{\infty}e^{-st}f(t)dt:=F(s). \tag{1.4}$$

If $\mathcal{L}\{f(t)\}=F(s)$, then $\mathcal{L}^{-1}\{F(s)\}$ is given by

$$f(t)=\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{st}F(s)ds=\sum_{k=1}^{n}\mathrm{Res}[e^{ts}F(s);s=s_k]. \tag{1.5}$$

where $F(s)$ is analytic in the region $\mathrm{Re}(s) > c$ with finite number of isolated singularities in the complex-plane and s is a complex number with positive real part. This complex integral can be evaluated by using an appropriate contour in the complex plane, with the parameter c chosen to take into account the analytic structure of the integrand, such as the presence of any poles and branch cuts. The expression in (1.5) is the inverse Laplace transform for the function $F(s)$, and is often called the Bromwich integral.

Lemma 1.12 (Gross-Levi). *Let us assume that $\mathcal{L}[f(t); t \to s] = F(s)$ and $F(s) = F(re^{i\phi})$, $|\phi| < \pi$ and $\int_0^{+\infty} |F(re^{i\phi})|^2 dr < +\infty$, then we have the following inversion formula*

$$f(t) = \frac{1}{\pi}\int_0^{+\infty} e^{-tr}\,\mathrm{Im}\Big[\lim_{\phi\to-\pi} F(re^{i\phi})\Big]dr.$$

Proof. See [5]. □

The most important use of the Caputo fractional derivative is treated in initial value problems where initial conditions are expressed in terms of integer order derivatives. In this respect, it is interesting to know the Laplace transform of this kind of derivative. In the following lemma, Laplace transform of the Caputo fractional derivatives of order non integer α is given.

Lemma 1.13. *We have the following relations*

$$\mathcal{L}\{D_{0,t}^{C,\alpha} f(t)\} = s^{\alpha}F(s) - s^{\alpha-1}f(0+), \quad 0 < \alpha < 1. \tag{1.6}$$

and generally

$$\mathcal{L}\{D_{0,t}^{C,\alpha} f(t)\} = s^{\alpha}F(s) - \sum_{k=0}^{k=m-1} s^{\alpha-1-k}f^{k}(0+), \quad m-1 < \alpha < m. \tag{1.7}$$

Proof. See [7]. □

The Laplace transform provides a useful technique for the solution of such fractional singular integro-differential equations.

Lemma 1.14. *Let $\mathcal{L}\{f(t)\} = F(s)$, then the following identities hold true.*

(1) $\mathcal{L}^{-1}[F(s^{\alpha}); s \to t)] =$
$$= \frac{1}{\pi}\int_0^{+\infty} f(u)\Big(\int_0^{+\infty} e^{-tr-ur^{\alpha}\cos\alpha\pi}\sin(ur^{\alpha}\sin\alpha\pi)dr\Big)du.$$

(2) $\mathcal{L}^{-1}(e^{-k\sqrt{s}}) = \frac{k}{(2\sqrt{\pi})}\int_0^{\infty} e^{-s\xi-\frac{k^2}{4\xi}}d\xi$

(3) $e^{-\omega s^{\alpha}} = \frac{1}{\pi}\int_0^{\infty} e^{-r^{\alpha}(\omega cos\alpha\pi)}\sin(\omega r^{\alpha}\sin\alpha\pi)\Big(\int_0^{\infty} e^{-s\tau-r\tau}d\tau\Big)dr$

Proof. See [1], [2]. □

Definition 1.15. *The Stieltjes transform (i.e. the second iterate of the Laplace transform) of a function $\psi(t) : R_+ \to C$ is defined by means of the following relation*

$$\mathcal{S}[\psi](s) = \mathcal{L}[\mathcal{L}[\psi(t); p]; s] = \int_0^{+\infty} \tfrac{\psi(t)}{t+s} dt = \Psi(s).$$

Provided that the integral exists.

We have the following inversion formula for the Stieltjes transform [5],

$$\mathcal{S}^{-1}[\Psi(s); s \to t] = \tfrac{1}{\pi} \operatorname{Im}\Big[\lim_{s \to te^{-i\pi}} \Psi(s)\Big],$$

Lemma 1.16. *The following integral identity holds true.*

$$\int_0^{+\infty} \tfrac{e^t K_\nu(t)}{(t+\xi)\sqrt{t}} dt = \tfrac{e^{-\xi}}{\sqrt{\xi}} K_\nu(\xi).$$

Proof. The left hand side of the above identity can be written in terms of the Stieltjes transform (*i.e.* the second itrate of the Laplace transform) of a function as below

$$\mathcal{L}\Big[\mathcal{L}[\tfrac{e^{-t}K_\nu(t)}{\sqrt{t}}; t \to \eta]; \eta \to \xi\Big] = \mathcal{S}\Big[\tfrac{e^{-t}K_\nu(t)}{\sqrt{t}}; t \to \xi\Big] = \tfrac{\pi e^{-\xi}}{\cos(\pi\nu)\sqrt{\xi}} K_\nu(\xi).$$

Equivalently, we need to show that

$$\tfrac{e^{-t}K_\nu(t)}{\sqrt{t}} = \mathcal{S}^{-1}\Big[\tfrac{\pi e^{-\xi}}{\cos(\pi\nu)\sqrt{\xi}} K_\nu(\xi); \xi \to t\Big].$$

At this point, let us evaluate the right hand side by means of the inversion formula for the Stieltjes transform [5] as follows

$$\mathcal{S}^{-1}\Big[\tfrac{e^{-\xi}}{\sqrt{\xi}} K_\nu(\xi); \xi \to t\Big] = \tfrac{1}{\pi} \operatorname{Im}\Big[\lim_{\xi \to te^{-i\pi}} \tfrac{\pi e^{-\xi} K_\nu(\xi)}{\cos(\pi\nu)\sqrt{\xi}}\Big],$$

after simplifying we get

$$\mathcal{S}^{-1}\Big[\tfrac{\pi e^{-\xi}}{\cos(\pi\nu)\sqrt{\xi}} K_\nu(\xi); \xi \to t\Big] = \tfrac{1}{\pi} \operatorname{Im}\Big[\tfrac{\pi e^{-t} K_\nu(-t)}{-i\cos(\pi\nu)\sqrt{t}}\Big] = \operatorname{Im}\Big[\tfrac{ie^{-t} K_\nu(-t)}{\cos(\pi\nu)\sqrt{t}}\Big].$$

Let us use the well-known identity for the Macdonal's function as below

$$K_\nu(-t) = e^{i\pi\nu} K_\nu(t) = (\cos \pi\nu + i \sin \pi\nu) K_\nu(t),$$

hence, we have

$$\mathcal{S}^{-1}\Big[\tfrac{\pi e^{-\xi}}{\cos(\pi\nu)\sqrt{\xi}} K_\nu(\xi); \xi \to t\Big] = \operatorname{Im}\Big[\tfrac{e^{-t}(i\cos \pi\nu - \sin \pi\nu) K_\nu(t)}{\cos(\pi\nu)\sqrt{t}}\Big] = \tfrac{e^{-t}K_\nu(t)}{\sqrt{t}}.$$

□

The operational methods provide a fast and universal mathematical tool for obtaining the solution of partial differential equations. The most commonly exponential operators which act on the function $\Psi(t)$ are as follows.

Lemma 1.17. *The following exponential identities hold true.*

(1) $\exp(\pm\lambda \frac{d}{dt})\Psi(t) = \Psi(t \pm \lambda)$
(2) $\exp(\pm\lambda t \frac{d}{dt})\Psi(t) = \Psi(te^{\pm\lambda})$

(3) $\exp(\lambda q(t)\frac{d}{dt})\Psi(t) = \Psi(Q(F(t)+\lambda))$

where $F(t)$ is the primitive function of $\frac{1}{q(t)}$ and $\mathcal{Q}(t)$ is the inverse function of $F(t)$.

Proof. See [6]. □

Example 1.18. Let us consider the following Fokker-Planck equation with variable coefficients as

$$\frac{\partial u}{\partial t} - \alpha t^{\alpha-1}(\frac{\partial}{\partial x}x\frac{\partial u}{\partial x}) = -\lambda u. \quad \lambda, \gamma > 0.$$

$$u(x,0) = e^{-\gamma x}.$$

Note: The operator ${}_L D_x(.) = \frac{\partial}{\partial x}x\frac{\partial}{\partial x}$ is known as Laguerre derivative [7].

Solution. The above partial differential equation can be written as below,

$$\frac{\partial u}{\partial t} = (-\lambda + \alpha t^{\alpha-1}\frac{\partial}{\partial x}x\frac{\partial}{\partial x})u,$$

in order to solve the above differential equation, we separate the variables and rewrite the above equation as follows

$$\frac{du}{u} = (-\lambda + \alpha t^{\alpha-1}\frac{\partial}{\partial x}x\frac{\partial}{\partial x})dt,$$

by integrating the above equation, we get

$$\ln u = (-\lambda t + t^{\alpha}\frac{\partial}{\partial x}x\frac{\partial}{\partial x}) + c(x),$$

or

$$u(x,t) = \exp[(-\lambda t + t^{\alpha}\frac{\partial}{\partial x}x\frac{\partial}{\partial x})]c_1(x),$$

by using the fact that $u(x,0) = e^{-\gamma x} = c_1(x) = e^{c(x)}$, we arrive at

$$u(x,t) = e^{(-\lambda t + t^{\alpha}(\frac{\partial}{\partial x}x\frac{\partial u}{\partial x}))}e^{-\gamma x}.$$

After simplifying, we obtain

$$u(x,t) = e^{-\lambda t}[e^{t^{\alpha}\frac{\partial}{\partial x}x\frac{\partial}{\partial x}}]e^{-\gamma x}.$$

Let us define the operators A and B as follows

$$A = t^{\alpha}\frac{\partial}{\partial x}, \quad B = t^{\alpha}x\frac{\partial^2}{\partial x^2}.$$

Then we have

$$t^{\alpha}\frac{\partial}{\partial x}x\frac{\partial}{\partial x} = t^{\alpha}\frac{\partial}{\partial x} + t^{\alpha}x\frac{\partial^2}{\partial x^2} = A + B,$$

with

$$[A,B] = AB - BA = t^{2\alpha}\frac{\partial^2}{\partial x^2} = A^2.$$

In this case we have the following decomposition [6]

$$e^{A+B} = (1+A)e^{B}.$$

From the above relation, we arrive at

$$u(x,t) = e^{-\lambda t}[e^{t^{\alpha}\frac{\partial}{\partial x}x\frac{\partial}{\partial x}}]e^{-\gamma x} =$$

$$= e^{-\lambda t}(1+A)[e^{B}e^{-\gamma x}] = e^{-\lambda t}(1+t^{\alpha}\frac{\partial}{\partial x})[e^{t^{\alpha}x\frac{\partial^2}{\partial x^2}}e^{-\gamma x}],$$

at this point let us recall the following integral identity

$$e^{\xi} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(s^2+2s\sqrt{\xi})}ds.$$

In the above integral let us set $\xi = t^{\alpha}x\frac{\partial^2}{\partial x^2}$ we obtain the following operational identity

$$e^{t^{\alpha}x\frac{\partial^2}{\partial x^2}} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(s^2+2s\sqrt{xt^{\alpha}}\frac{\partial}{\partial x})}ds = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-s^2}\Big[e^{-2\sqrt{t^{\alpha}}s\sqrt{x}\frac{\partial}{\partial x}}\Big]ds,$$

thus, we have

$$\begin{aligned} u(x,t) &= e^{-\lambda t}[e^{t^{\alpha}\frac{\partial}{\partial x}x\frac{\partial}{\partial x}}]e^{-\gamma x} \\ &= e^{-\lambda t}(1+t^{\alpha}\frac{\partial}{\partial x})\Big[\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-s^2}[e^{(-2s\sqrt{t^{\alpha}})\sqrt{x}\frac{\partial}{\partial x}}e^{-\gamma x}]ds\Big]. \end{aligned}$$

Then apply the identity

$$e^{\phi\sqrt{x}\frac{\partial}{\partial x}}f(x) = f\Big([\sqrt{x}+\frac{\phi}{2}]^2\Big),$$

leads to

$$u(x,t) = e^{-\lambda t}(1+t^{\alpha}\frac{\partial}{\partial x})\Big[\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-s^2}e^{-\gamma(\sqrt{x}-\sqrt{t^{\alpha}}s)^2}ds\Big].$$

After evaluation of the inner integral we have

$$u(x,t) = \frac{e^{-\lambda t}}{\sqrt{\pi}}(1+t^{\alpha}\frac{\partial}{\partial x})\Big[\frac{\sqrt{\pi}}{\sqrt{1+\gamma t^{\alpha}}}e^{-(\gamma-\frac{\gamma^2t^{\alpha}}{1+\gamma t^{\alpha}})x}\Big],$$

after simplification

$$u(x,t) = \frac{e^{-\lambda t}}{(1+\gamma t^{\alpha})^{\frac{3}{2}}}e^{-(\gamma-\frac{\gamma^2t^{\alpha}}{1+\gamma t^{\alpha}})x}.$$

Note: It is easy to verify that $u(x,0) = e^{-\gamma x}$.

In the next lemma, let us illustrate the use of Bromwich integral and residues theorem.

LEMMA 1.19. *Using Bromwich complex inversion formula to show that*

$$\mathcal{L}^{-1}\Big[\frac{K_0(a\sqrt{s})}{\sqrt{s}-b}\Big] = 2bK_0(ab)e^{b^2t} - \int_{0}^{+\infty} e^{-t\xi^2}\frac{bJ_0(a\xi)+\xi Y_0(a\xi)}{b^2+\xi^2}\xi d\xi.$$

Note. *The techniques introduce for evaluating the inverse Laplace transforms are adequate for a wide variety of routine applications involving the Laplace transform.*

Proof. Direct application of the complex inversion formula and (1.5) leads to the following

$$\mathcal{L}^{-1}\left[\frac{K_0(a\sqrt{s})}{\sqrt{s}-b}\right] = g(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\frac{K_0(a\sqrt{s})}{\sqrt{s}-b}ds =$$

$$= \sum_{k=1}^{n} \text{Res}\left[e^{ts}\frac{K_0(a\sqrt{s})}{\sqrt{s}-b}; s = s_k\right]. \tag{1.8}$$

The transform $G(s) = [\frac{K_0(a\sqrt{s})}{\sqrt{s}-b}]$ has a simple pole at $s = b^2$ and a branch point at $s = 0$. Then, the inverse Laplace transform will be obtained by the Gross-Levi method as follows [5]

$$g(t) = \lim_{s\to b^2}\left[(s-b^2)\frac{K_0(a\sqrt{s})e^{ts}}{\sqrt{s}-b}\right] + \frac{1}{\pi}\int_0^{+\infty} e^{-tr}\,\text{Im}\left[\lim_{\phi\to-\pi} G(re^{i\phi})\right]. \tag{1.9}$$

or,

$$g(t) = \lim_{s\to b^2}\left[(\sqrt{s}-b)(\sqrt{s}+b)\frac{K_0(a\sqrt{s})e^{ts}}{\sqrt{s}-b}\right] + \frac{1}{\pi}\int_0^{+\infty} e^{-tr}\,\text{Im}\left[\frac{K_0(a\sqrt{re^{-i\pi}})}{\sqrt{re^{-i\pi}}-b}\right].$$

after simplifying we get

$$g(t) = 2bK_0(ab)e^{b^2t} - \frac{1}{\pi}\int_0^{+\infty} e^{-tr}\,\text{Im}\left[\frac{K_0(-ia\sqrt{r})}{i\sqrt{r}+b}\right]dr.$$

At this stage, we use the following well-known identity for the Bessel's functions of the first and second kind [5]

$$K_0(-ia\sqrt{r}) = \frac{\pi i}{2}[J_0(a\sqrt{r}) + iY_0(a\sqrt{r})].$$

therefore,

$$g(t) = 2bK_0(ab)e^{b^2t} - \frac{1}{\pi}\int_0^{+\infty} e^{-tr}\,\text{Im}\left[\frac{\frac{i\pi}{2}(b-i\sqrt{r})(J_0(a\sqrt{r}+iY_0(a\sqrt{r})}{r+b^2}\right]dr,$$

after taking imaginary part of the fraction under integral sign, followed by making a change of variable $r = \xi^2$ and simplifying, we arrive at

$$g(t) = 2bK_0(ab)e^{b^2t} - \int_0^{+\infty} e^{-t\xi^2}\frac{bJ_0(a\xi)+\xi Y_0(a\xi)}{b^2+\xi^2}\xi d\xi.$$

Definition 1.20. *The Hankel transform of order ν of a function $f(t)$ is given by*

$$\mathcal{H}_\nu[f(t);\rho] = \int_0^{+\infty} f(t)tJ_\nu(\rho t)dt = F(\rho). \tag{1.10}$$

It is well to note that ν is not specified at this point and can be chosen to fit best the particular problem under consideration. In order for a transformation to be useful in solving boundary value problems, it must have an inverse. The inverse Hankel transform of a function $F(\rho)$ is given by [4, 5]

$$\mathcal{H}_\nu^{-1}[F(\rho);t] = \int_0^{+\infty} F(\rho)\rho J_\nu(t\rho)d\rho = f(t).$$

Lemma 1.21. *Let us define the function*

$$h_\nu(r) = \frac{1}{r^{\nu+1}}\frac{d}{dr}\Big[\frac{1}{r^\nu}y(r)\Big].$$

and apply the Hankel transform of order ν, then we obtain

1. $\mathcal{H}_\nu[h_\nu(r); r \to \rho] = -\rho^2 \mathcal{H}_\nu[y(r); r \to \rho]$.
2. $\mathcal{H}_0[h_0(r); r \to \rho] = -\rho^2 \mathcal{H}_0[y(r); r \to \rho]$.

As an example of the Hankel transform we will consider the problem of heat conduction formulated as follows.

Solution to time fractional heat equation *via* **the joint Laplace-Hankel transform.**

Problem 1.22. *Let us solve the time fractional heat conduction equation with boundary conditions as follows*

$$D_t^{C,\alpha} u = k^2 \tfrac{1}{r}\tfrac{\partial}{\partial r}(r\tfrac{\partial u}{\partial r}), \tag{1.11}$$

$$u(r,0) = f(r), \quad 0 < \alpha \le 1, \qquad 0 < r < +\infty. \tag{1.12}$$

$$\lim_{r\to+\infty} u(r,t) = 0, \tag{1.13}$$

$$\lim_{r\to 0} |u(r,t)| < +\infty. \tag{1.14}$$

Note. Notice that the fractional derivative is in the Caputo sense. The constant k^2 is the coefficient of diffusion for the substance under consideration. In the most general case the coefficient of diffusion will depend on the concentration and the coordinates of the point in question.

Solution: In order to obtain a solution for equations (1.11)–(1.14), let us define the joint Laplace-Hankel transform as follows

$$\mathcal{H}_\nu(\mathcal{L}[u(r,t); t \to s], r \to \rho]) = U(\rho, s) = \int_0^{+\infty} rJ_0(\rho r)\Big(\int_0^{+\infty} e^{-st}u(r,t)dt\Big)dr,$$

taking the joint Laplace-Hankel transform to each term of Eq. (1.11) followed by second part of the Lemma 1.19 and using boundary conditions (1.12), (1.13), (1.14) leads to

$$s^\alpha U(\rho, s) + (k^2\rho^2)U(\rho, s) = s^{\alpha-1}F(\rho), \tag{1.15}$$

from which we obtain

$$U(\rho, s) = \frac{s^{\alpha-1}F(\rho)}{s^\alpha + k^2\rho^2}.$$

At this point, taking the inverse joint Laplace-Hankel transform to obtain

$$u(r,t) = \int_0^{+\infty} \rho J_0(r\rho)F(\rho)\Big[\mathcal{L}^{-1}(\tfrac{s^{\alpha-1}}{s^\alpha+k^2\rho^2})\Big]d\rho.$$

But, using the fact that $\mathcal{L}^{-1}(\frac{s^{\alpha-1}}{s^\alpha+k^2\rho^2}) = E_{\alpha,1}(-k^2\rho^2t^\alpha)$, we get the formal solution as follows

$$u(r,t) = \int_0^{+\infty} \rho J_0(r\rho)F(\rho)E_{\alpha,1}(-k^2\rho^2t^\alpha)d\rho.$$

Note. $E_{\alpha,1}(z)$ stands for the Mittag-Leffler function in one-parameter and we have

$$E_{\alpha,1}(z) = E_\alpha(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k+1)}.$$

Let us consider the special case $\alpha = 0.5$, then we have

$$\mathcal{L}^{-1}\Big[\frac{s^{\alpha-1}}{s^\alpha+k^2\rho^2}\Big] = \mathcal{L}^{-1}\Big[\frac{1}{\sqrt{s}\sqrt{s}+k^2\rho^2}\Big] = e^{k^4\rho^4t}erfc(k^2\rho^2\sqrt{t}).$$

Therefore, our formal solution becomes

$$u(r,t) = \int_0^{+\infty} \rho J_0(r\rho)Erfc(k^2\rho^2\sqrt{t})F(\rho)d\rho.$$

Let us recall that $\mathcal{H}_0[f(r); r \to \rho] = \int_0^{+\infty} f(\xi)\xi J_0(\rho\xi)\xi = F(\rho)$ then we get finally

$$u(r,t) = \int_0^{+\infty} \xi f(\xi)\Big[\int_0^{+\infty} \rho J_0(r\rho)J_0(\xi\rho)Erfc(\sqrt{t}k^2\rho^2)d\rho\Big]d\xi.$$

The last step is to verify that $u(r,0) = f(r)$, we have

$$u(r,0) = \int_0^{+\infty} \xi f(\xi)\Big(\int_0^{+\infty} \rho J_0(\xi\rho)J_0(r\rho)d\rho\Big)d\xi.$$

But the value of the inner integral is

$$\int_0^{+\infty} \rho J_0(\xi\rho)J_0(r\rho)d\rho = \tfrac{1}{2}\delta(\tfrac{r^2-\xi^2}{4}).$$

From which we deduce that

$$u(r,0) = \int_0^{+\infty} \xi f(\xi)\Big[\tfrac{1}{2}\delta(\tfrac{r^2-\xi^2}{4})\Big]d\xi.$$

In order to evaluate the above integral, we introduce a new change of variable $\frac{r^2-\xi^2}{4} = \eta$, we get

$$u(r,0) = \int_{-\infty}^{\frac{r^2}{4}} \sqrt{r^2-4\eta}f\Big(\sqrt{r^2-4\eta}\Big)\frac{\delta(\eta)d\eta}{\sqrt{r^2-4\eta}} =$$

$$= \int_{-\infty}^{\frac{r^2}{4}} f\Big(\sqrt{r^2-4\eta}\Big)\delta(\eta)d\eta = f(r).$$

Lemma 1.23. *We have the following integral identity for the Bessel's functions*

$$J_0\Big(2|\sinh\tfrac{t}{2}|\xi\Big) = \tfrac{4}{\pi^2}\int_0^{+\infty} \sin(t\tau)\sinh(\pi\tau)K_{i\tau}(\xi)d\tau.$$

Proof. Let us start with the well-known identity for the product of the modified Bessel's functions

$$K_\nu(x)K_\nu(y) = \tfrac{\pi}{2\sin\pi\nu}\int_0^{+\infty} J_0\Big(\sqrt{2xy\cosh t - (x^2+y^2)}\Big)\sinh(\nu t)dt.$$

Let us take $x = y = \xi, \nu = i\tau$ then after simplifying we get

$$K_{i\nu}^2(\xi) = \tfrac{\pi}{2i\sinh\pi\tau}\int_0^{+\infty} J_0\Big(\sqrt{2\xi^2\cosh t - 2\xi^2}\Big) i\sin(\tau t)dt,$$

or,

$$\tfrac{2}{\pi}\sinh\pi\tau K_{i\tau}^2(\xi) = \int_0^{+\infty} J_0\Big(\xi\sqrt{2(\cosh t - 1)}\Big)\sin\tau t dt =$$

$$= \int_0^{+\infty} J_0\Big(2|\sinh(\tfrac{t}{2})|\xi\Big)\sin\tau t dt.$$

At this stage, by taking the inverse Fourier-sine transform, we have

$$\tfrac{4}{\pi^2}\int_0^{+\infty} K_{i\tau}(\xi)\sinh(\pi\tau)\sin(t\tau)d\tau = J_0\Big(2|\sinh\tfrac{t}{2}|\xi\Big).$$

□

2. SOLUTION TO UNSTEADY FRACTIONAL HEAT CONDUCTION IN THE QUARTER-PLANE VIA THE JOINT LAPLACE-FOURIER SINE TRANSFORM.

During the last three decade, many mathematical methods were widely applied in fractal analysis. Fractional calculus used to investigate fractal functions is an important tool in this fields [13]. The main physical purpose for investigating fractional diffusion equations is to describe phenomena of anomalous diffusion.

PROBLEM 2.1. *Let us solve the time fractional heat conduction equation in two dimensions with boundary conditions as follows*

$$D_t^{c,\alpha}u = \Delta u = \tfrac{\partial^2 u}{\partial x^2} + \tfrac{\partial^2 u}{\partial y^2}, \tag{2.1}$$

$$u(x,0,t) = 0, \tag{2.2}$$

$$u(x,y,0) = 0, \tag{2.3}$$

$$u(0,y,t) - u_x(0,y,t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad 0 < \alpha \le 1, 0 < x, y < +\infty. \tag{2.4}$$

$$\lim_{y\to+\infty} u(x,y,t) = 0, \tag{2.5}$$

$$\lim_{x\to+\infty} |u(x,y,t)| = 0. \tag{2.6}$$

Solution: In order to obtain a solution for equations (2.1)–(2.6), let us define the joint Laplace-Fourier sine transform as follows

$$\mathcal{F}_s(\mathcal{L}[u(x,t);t\to s],y\to\omega]) = U(x,w,s) = \sqrt{\tfrac{2}{\pi}}\int_0^{+\infty}\sin(wy)\Big(\int_0^{+\infty}e^{-st}u(x,y,t)dt\Big)dy,$$

taking the joint Laplace-Fourier sine transform of (2.1) and using boundary conditions (2.2), (2.3), (2.5) yields

$$U_{xx}(x,\omega,s) - (s^\alpha+\omega^2)U(x,\omega,s) = 0, \tag{2.7}$$

with boundary conditions as follows

$$U(0,\omega,s) - U_x(0,\omega,s) = \tfrac{1}{\omega s^\alpha}, \tag{2.8}$$

$$\lim_{x\to+\infty}|U(x,\omega,s)| = 0. \tag{2.9}$$

The solution to Eq. (2.7) that satisfies the boundary conditions (2.8), (2.7) is as follows

$$U(x,\omega,s) = \frac{e^{-x\sqrt{s^\alpha+\omega^2}}}{\omega s^\alpha(\sqrt{s^\alpha+\omega^2}+1)}.$$

Upon inverting the joint Laplace-Fourier sine transform, we obtain

$$u(x,y,t) = \sqrt{\tfrac{2}{\pi}}\int_0^{+\infty}\tfrac{\sin y\omega}{\omega}\Big(\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tfrac{e^{ts-x\sqrt{s^\alpha+\omega^2}}}{s^\alpha(\sqrt{s^\alpha+\omega^2}+1)}ds\Big)d\omega.$$

Let us evaluate first the complex inner integral, we have

$$h(x,\omega,t) = \mathcal{L}^{-1}\Big[\tfrac{e^{-x\sqrt{s^\alpha+\omega^2}}}{s^\alpha(\sqrt{s^\alpha+\omega^2}+1)}\Big] = \mathcal{L}^{-1}[G(s^\alpha)],$$

in order to evaluate $h(x,t)$, let us assume that

$$G(s) = \frac{e^{-x\sqrt{s+\omega^2}}}{s(\sqrt{s+\omega^2}+1)}.$$

We first evaluate $\mathcal{L}^{-1}[G(s);s\to t] = g(x,\omega,t)$ by means of the Gross-Levi method as follows

$$\mathcal{L}^{-1}[G(s);s\to t] = g(x,\omega,t) = \tfrac{e^{-\omega x}}{\omega+1} + \tfrac{e^{-\omega^2 t}}{\pi}\int_0^{+\infty}e^{-t\xi}\Big(\tfrac{\sqrt{\xi}\cos x\sqrt{\xi}+\sin x\sqrt{\xi}}{(1+\xi)(\xi+\omega^2)}\Big)d\xi.$$

Then we evaluate $\mathcal{L}^{-1}[G(s^\alpha);s\to t] = h(x,\omega,t)$, by means of the Lemma 1.13, we get

$$h(x,\omega,t) = \tfrac{1}{\pi}\int_0^{+\infty}g(x,\omega,\eta)\Big(\int_0^{+\infty}e^{-tr-\eta r^\alpha\cos\alpha\pi}\sin(\eta r^\alpha\sin\alpha\pi)dr\Big)d\eta.$$

Finally, we get the exact solution to diffusion equation as follows

$$u(x,y,t) = \sqrt{\tfrac{2}{\pi}}\int_0^{+\infty}\tfrac{\sin y\omega}{\omega}h(x,\omega,t)d\omega.$$

Let us consider the special case $\alpha = 0.5$, then we have

$$h(x,\omega,t) = \tfrac{1}{\pi}\int_0^{+\infty} g(x,\omega,\eta)\Big(\int_0^{+\infty} e^{-tr}\sin(\eta\sqrt{r})dr\Big)d\eta$$

$$= \tfrac{1}{2t\sqrt{\pi t}}\int_0^{+\infty} g(x,\omega,\eta)[\eta e^{-\frac{\eta^2}{4t}}]d\eta.$$

Finally, we obtain

$$u(x,y,t) = \tfrac{1}{\pi t\sqrt{2t}}\int_0^{+\infty}\tfrac{\sin(y\omega)}{\omega}\Big(\int_0^{+\infty}\eta g(x,\omega,\eta)e^{-\frac{\eta^2}{4t}}d\eta\Big)d\omega,$$

with

$$g(x,\omega,t) = \tfrac{e^{-\omega x}}{\omega+1} + \tfrac{e^{-\omega^2 t}}{\pi}\int_0^{+\infty} e^{-t\xi}\Big(\tfrac{\sqrt{\xi}\cos x\sqrt{\xi}+\sin x\sqrt{\xi}}{(1+\xi)(\xi+\omega^2)}\Big)d\xi.$$

3. CONCLUSION

The Fourier, Laplace and Hankel transforms provide a powerful method for solving certain linear differential and integral equations, and can be used for evaluating certain definite integrals. The paper is devoted to studying and application of the joint Laplace-Fourier sine transform for solving time fractional diffusion equation in the first quadrant. As a mathematical tool, the proposed method is extremely simple, attractive and concise.

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