JOURNAL OF NUMERICAL ANALYSIS AND APPROXIMATION THEORY
J. Numer. Anal. Approx. Theory, vol. 50 (2021) no. 1, pp. 27–43
ictp.acad.ro/jnaat

NONUNIFORM LOW-PASS FILTERS ON NON ARCHIMEDEAN LOCAL FIELDS*

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Abstract. In real life application all signals are not obtained from uniform shifts; so there is a natural question regarding analysis and decompositions of this types of signals by a stable mathematical tool. Gabardo and Nashed (J. Funct. Anal. 158:209-241, 1998) filled this gap by the concept of nonuniform multiresolution analysis. In this setting, the associated translation set $\Lambda = \{0, r/N\} + 2\,\mathbb{Z}$ is no longer a discrete subgroup of $\mathbb{R}$ but a spectrum associated with a certain one-dimensional spectral pair and the associated dilation is an even positive integer related to the given spectral pair. The main aim of this article is to provide the characterization of nonuniform low-pass filters on non-Archimedean local fields.

MSC 2020. 42C40; 42C15; 43A70; 11S85.

Keywords. Low pass filter, Nonuniform MRA, Fourier transform, Local field.

1. INTRODUCTION

Multiresolution analysis is an important mathematical tool since it provides a natural framework for understanding and constructing discrete wavelet systems. The concept of MRA has been extended in various ways in recent years. These concepts are generalized to $L^2(\mathbb{R}^d)$, to lattices different from $\mathbb{Z}^d$, allowing the subspaces of MRA to be generated by Riesz basis instead of orthonormal basis, admitting a finite number of scaling functions, replacing the dilation factor 2 by an integer $M \geq 2$ or by an expansive matrix $A \in GL_d(\mathbb{R})$ as long as $A \subset A\mathbb{Z}^d$. All these concepts are developed on regular lattices, that is the translation set is always a group. Recently, Gabardo and Nashed [25] considered a generalization of Mallat's [42] celebrated theory of MRA based on

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spectral pairs, in which the translation set acting on the scaling function associated with the MRA to generate the subspace V_0 is no longer a group, but is the union of $\mathbb{Z}$ and a translate of $\mathbb{Z}$. Based on one-dimensional spectral pairs, Gabardo and Yu [26] considered sets of nonuniform wavelets in $L^2(\mathbb{R})$. In real life application all signals are not obtained from uniform shifts; so there is a natural question regarding analysis and decompositions of this types of signals by a stable mathematical tool. Gabardo and Nashed [25] and Gabardo and Yu [26] filled this gap by the concept of nonuniform multiresolution analysis.

During the last two decades, there is a substantial body of work that has been concerned with the construction of wavelets on local fields. Even though the structures and metrics of local fields of zero and positive characteristics are similar, their wavelet and MRA (*multiresolution analysis*) theory are quite different. For example, R. L. Benedetto and J. J. Benedetto [18] developed a wavelet theory for local fields and related groups. They did not develop *the multiresolution analysis* (MRA) approach, their method is based on the theory of wavelet sets and only allows the construction of wavelet functions whose Fourier transforms are characteristic functions of some sets. Khrennikov, Shelkovich and Skopina [31] constructed a number of scaling functions generating an MRA of $L^2(\mathbb{Q}_p)$. But later on in [15], Albeverio, Evdokimov and Skopina proved that all these scaling functions lead to the same Haar MRA and that there exist no other orthogonal test scaling functions generating an MRA except those described in [31]. Some wavelet bases for $L^2(\mathbb{Q}_p)$ different from the Haar system were constructed in [14, 22]. These wavelet bases were obtained by relaxing the basis condition in the definition of an MRA and form Riesz bases without any dual wavelet systems. For some related works on wavelets and frames on $\mathbb{Q}_p$, we refer to [16, 30, 34, 35]. On the other hand, Lang [37, 38, 39] constructed several examples of compactly supported wavelets for the Cantor dyadic group. Farkov [23, 24] has constructed many examples of wavelets for the Vilenkin p-groups. Jiang *et al.* [29] pointed out a method for constructing orthogonal wavelets on local field $\mathbb{K}$ with a constant generating sequence and derived necessary and sufficient conditions for a solution of the refinement equation to generate a multiresolution analysis of $L^2(\mathbb{K})$. During the last two decades, p-adics has been extensively applied to a variety of problems in theoretical physics (string theory, cosmology, quantum theory, and disordered systems,) and biology (in modeling the thinking process and in genetics) [17, 33, 36, 32, 43, 45, 53, 54, 52].

Recently, Shah and Abdullah [50] have generalized the concept of multiresolution analysis on Euclidean spaces $\mathbb{R}^n$ to nonuniform multiresolution analysis on local fields of positive characteristic, in which the translation set acting on the scaling function associated with the multiresolution analysis to generate the subspace V_0 is no longer a group, but is the union of $\mathcal{Z}$ and a translate of $\mathcal{Z}$, where $\mathcal{Z} = \{u(n) : n \in \mathbb{N}_0\}$ is a complete list of (distinct) coset representation of the unit disc $\mathfrak{D}$ in the locally compact Abelian group $\mathbb{K}^+$. More precisely, this set is of the form $\Lambda = \{0, r/N\} + \mathcal{Z}$, where $N \geq 1$ is an integer and r is an odd integer such that r and N are relatively prime. They call this a *nonuniform multiresolution analysis* on local fields of positive characteristic.The notion of nonuniform wavelet frames on non-Archimedean local fields was introduced by Ahmad and Sheikh [12] and established a complete characterization of tight nonuniform wavelet frames on non-Archimedean local fields. More results in this direction can also be found in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 41, 48, 49] and the references therein.

W. Lawton [40] gave the necessary and sufficient conditions for a trigonometric polynomial to be a low-pass filter of an MRA on $L^2(\mathbb{R})$. Later, Hernandez and Weiss [28] gave a characterization of low-pass filters by using Cohen's approach. They considered certain smooth classes of low-pass filters. Then Papadakis, Sikic, and Weiss [44] gave a complete characterization by assuming only the Holder condition at the origin instead of smoothness condition. Furthermore, San Antolin [46] generalized it to a general dilation matrix. R. F. Gundy [27] gave necessary and sufficient conditions for an arbitrary periodic function to be a low-pass filter. His technique is also useful if we consider that the translates of scaling function form a Riesz basis instead of an orthonormal basis for V_0. E. Curry [20] extended this result for multivariable wavelets.

The article is organized as follows. Section 2 contains a brief introduction to local fields and Fourier analysis on such a field. In Section 3, we give some definitions and state the main theorem of this article, which gives necessary and sufficient conditions for a function to be a low-pass filter on local fields of positive characteristic. In the last section, we continue the proof of our main result via probability and martingale methods.

2. PRELIMINARIES ON NON-ARCHIMEDEAN LOCAL FIELDS

2.1. Non-Archimedean Local Fields. A non-Archimedean local field $\mathbb{K}$ is a locally compact, non-discrete and totally disconnected field. If it is of characteristic zero, then it is a field of p-adic numbers $\mathbb{Q}_p$ or its finite extension. If $\mathbb{K}$ is of positive characteristic, then $\mathbb{K}$ is a field of formal Laurent series over a finite field $GF(p^c)$. If $c = 1$, it is a p-series field, while for $c \neq 1$, it is an algebraic extension of degree c of a p-series field. Let $\mathbb{K}$ be a fixed non-Archimedean local field with the ring of integers $\mathfrak{D} = \{x \in K : |x| \leq 1\}$. Since K^+ is a locally compact Abelian group, we choose a Haar measure dx for K^+. The field K is locally compact, non-trivial, totally disconnected and complete topological field endowed with non–Archimedean norm $|\cdot| : \mathbb{K} \to \mathbb{R}^+$ satisfying

(a) $|x| = 0$ if and only if $x = 0$;
(b) $|x\,y| = |x| \cdot |y|$ for all $x, y \in \mathbb{K}$;
(c) $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in \mathbb{K}$.

Property (c) is called the ultrametric inequality. Let $\mathfrak{B} = \{x \in \mathbb{K} : |x| < 1\}$ be the prime ideal of the ring of integers $\mathfrak{D}$ in $\mathbb{K}$. Then, the residue space $\mathfrak{D}/\mathfrak{B}$ is isomorphic to a finite field $GF(q)$, where $q = p^c$ for some prime p and $c \in \mathbb{N}$.

Since K is totally disconnected and $\mathfrak{B}$ is both prime and principal ideal, so there exist a prime element $\mathfrak{p}$ of $\mathbb{K}$ such that $\mathfrak{B} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$. Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{B} = \{x \in \mathbb{K} : |x| = 1\}$. Clearly, $\mathfrak{D}^*$ is a group of units in $\mathbb{K}^*$ and if $x \neq 0$, then can write $x = \mathfrak{p}^n y, y \in \mathfrak{D}^*$. Moreover, if $\mathcal{U} = \{a_m : m = 0, 1, \ldots, q-1\}$ denotes the fixed full set of coset representatives of $\mathfrak{B}$ in $\mathfrak{D}$, then every element $x \in K$ can be expressed uniquely as $x = \sum_{\ell=k}^{\infty} c_\ell \, \mathfrak{p}^\ell$ with $c_\ell \in \mathcal{U}$. Recall that $\mathfrak{B}$ is compact and open, so each fractional ideal $\mathfrak{B}^k = \mathfrak{p}^k \mathfrak{D} = \{x \in K : |x| < q^{-k}\}$ is also compact and open and is a subgroup of K^+. We use the notation in Taibleson's book [51]. In the rest of this paper, we use the symbols $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural, non-negative integers and integers, respectively.

Let χ be a fixed character on K^+ that is trivial on $\mathfrak{D}$ but non-trivial on $\mathfrak{B}^{-1}$. Therefore, χ is constant on cosets of $\mathfrak{D}$ so if $y \in \mathfrak{B}^k$, then $\chi_y(x) = \chi(y, x), x \in K$. Suppose that χ_u is any character on K^+, then the restriction $\chi_u|\mathfrak{D}$ is a character on $\mathfrak{D}$. Moreover, as characters on $\mathfrak{D}, \chi_u = \chi_v$ if and only if $u - v \in \mathfrak{D}$. Hence, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathfrak{D}$ in K^+, then, as it was proved in [51], the set $\{\chi_{u(n)} : n \in \mathbb{N}_0\}$ of distinct characters on $\mathfrak{D}$ is a complete orthonormal system on $\mathfrak{D}$.

We now impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$. We have $\mathfrak{D}/\mathfrak{B} \cong GF(q)$ where $GF(q)$ is a c-dimensional vector space over the field $GF(p)$. We choose a set $\{1 = \zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1}\} \subset \mathfrak{D}^*$ such that $\operatorname{span}\{\zeta_j\}_{j=0}^{c-1} \cong GF(q)$. For $n \in \mathbb{N}_0$ satisfying

$$0 \leq n < q, \; n = a_0 + a_1 p + \cdots + a_{c-1} p^{c-1}, \; 0 \leq a_k < p, \text{ and } k = 0, 1, \ldots, c-1,$$

we define

$$(n) = (a_0 + a_1\zeta_1 + \cdots + a_{c-1}\zeta_{c-1})\, \mathfrak{p}^{-1}. \tag{2.1}$$

Also, for $n = b_0 + b_1 q + b_2 q^2 + \cdots + b_s q^s, \; n \in \mathbb{N}_0, \; 0 \leq b_k < q, k = 0, 1, 2, \ldots, s$, we set

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \cdots + u(b_s)\mathfrak{p}^{-s}. \tag{2.2}$$

This defines $u(n)$ for all $n \in \mathbb{N}_0$. In general, it is not true that $u(m+n) = u(m) + u(n)$. But, if $r, k \in \mathbb{N}_0$ and $0 \leq s < q^k$, then $u(rq^k + s) = u(r)\mathfrak{p}^{-k} + u(s)$. Further, it is also easy to verify that $u(n) = 0$ if and only if $n = 0$ and $\{u(\ell) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$ for a fixed $\ell \in \mathbb{N}_0$. Hereafter we use the notation $\chi_n = \chi_{u(n)}, n \geq 0$.

Let the local field $\mathbb{K}$ be of characteristic $p > 0$ and $\zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1}$ be as above. We define a character χ on K as follows:

$$\chi(\zeta_\mu \mathfrak{p}^{-j}) = \begin{cases} \exp(2\pi i/p), & \mu = 0 \text{ and } j = 1, \\ 1, & \mu = 1, \ldots, c-1 \text{ or } j \neq 1. \end{cases} \tag{2.3}$$

2.2. Fourier Transforms on Non-Archimedean Local Fields. The Fourier transform of $f \in L^1(K)$ is denoted by $\hat{f}(\xi)$ and defined by

$$\mathcal{F}\{f(x)\} = \hat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}\, dx. \tag{2.4}$$

It is noted that

$$\hat{f}(\xi) = \int_K f(x)\,\overline{\chi_\xi(x)} dx = \int_K f(x)\chi(-\xi x)\,dx.$$

The properties of Fourier transforms on non-Archimedean local field $\mathbb{K}$ are much similar to those of on the classical field $\mathbb{R}$. In fact, the Fourier transform on non-Archimedean local fields of positive characteristic have the following properties:

- The map $f \to \hat{f}$ is a bounded linear transformation of $L^1(\mathbb{K})$ into $L^\infty(\mathbb{K})$, and $\|\hat{f}\|_\infty \le \|f\|_1$.
- If $f \in L^1(\mathbb{K})$, then $\hat{f}$ is uniformly continuous.
- If $f \in L^1(\mathbb{K}) \cap L^2(\mathbb{K})$, then $\|\hat{f}\|_2 = \|f\|_2$.

The Fourier transform of a function $f \in L^2(\mathbb{K})$ is defined by

$$\hat{f}(\xi) = \lim_{k\to\infty} \hat{f}_k(\xi) = \lim_{k\to\infty} \int_{|x|\le q^k} f(x)\overline{\chi_\xi(x)}\,dx, \tag{2.5}$$

where $f_k = f\,\Phi_{-k}$ and Φ_k is the characteristic function of $\mathfrak{B}^k$. Furthermore, if $f \in L^2(\mathfrak{D})$, then we define the Fourier coefficients of f as

$$\hat{f}\big(u(n)\big) = \int_{\mathfrak{D}} f(x)\overline{\chi_{u(n)}(x)}\,dx. \tag{2.6}$$

The series $\sum_{n\in\mathbb{N}_0} \hat{f}\big(u(n)\big)\chi_{u(n)}(x)$ is called the Fourier series of f. From the standard L^2-theory for compact Abelian groups, we conclude that the Fourier series of f converges to f in $L^2(\mathfrak{D})$ and Parseval's identity holds:

$$\|f\|_2^2 = \int_{\mathfrak{D}} \left|f(x)\right|^2 dx = \sum_{n\in\mathbb{N}_0} \left|\hat{f}\big(u(n)\big)\right|^2.$$

2.3. Uniform MRA on Non-Archimedean Local Fields. In order to be able to define the concepts of uniform MRA and wavelets on non-Archimedean local fields, we need analogous notions of translation and dilation. Since $\bigcup_{j\in\mathbb{Z}} \mathfrak{p}^{-j}\mathfrak{D} = \mathbb{K}$, we can regard $\mathfrak{p}^{-1}$ as the dilation and since $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representatives of $\mathfrak{D}$ in K, the set $\mathcal{Z} = \{u(n) : n \in \mathbb{N}_0\}$ can be treated as the translation set. Note that Λ is a subgroup of $\mathbb{K}^+$ and unlike the standard wavelet theory on the real line, the translation set is not a group. Let us recall the definition of a uniform MRA on non-Archimedean local fields of positive characteristic introduced by Jiang *et al.* in [29].

Definition 2.1. *Let $\mathbb{K}$ be a non-Archimedean local field of positive characteristic $p > 0$ and $\mathfrak{p}$ be a prime element of $\mathbb{K}$. An MRA of $L^2(\mathbb{K})$ is a sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{K})$ satisfying the following properties:*

(a) $V_j \subset V_{j+1}$ *for all* $j \in \mathbb{Z}$;
(b) $\bigcup_{j\in\mathbb{Z}} V_j$ *is dense in* $L^2(\mathbb{K})$;
(c) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;

(d) $f(x) \in V_j$ *if and only if* $f(\mathfrak{p}^{-1}x) \in V_{j+1}$ *for all* $j \in \mathbb{Z}$;

(e) There exists a function $\phi \in V_0$*, such that* $\{\phi(x - u(k)) : k \in \mathbb{N}_0\}$ *forms an orthonormal basis for* V_0.

According to the standard scheme for construction of MRA-based wavelets, for each j, we define a wavelet space W_j as the orthogonal complement of V_j in V_{j+1}, *i.e.*, $V_{j+1} = V_j \oplus W_j$, $j \in \mathbb{Z}$, where $W_j \perp V_j$, $j \in \mathbb{Z}$. It is not difficult to see that

$$f(x) \in W_j \quad \text{if and only if} \quad f(\mathfrak{p}^{-1}x) \in W_{j+1}, \quad j \in \mathbb{Z}. \tag{2.7}$$

Moreover, they are mutually orthogonal, and we have the following orthogonal decompositions:

$$L^2(K) = \bigoplus_{j\in\mathbb{Z}} W_j = V_0 \oplus \left(\bigoplus_{j\geq 0} W_j \right). \tag{2.8}$$

As in the case of $\mathbb{R}^n$, we expect the existence of $q-1$ number of functions $\psi_1, \psi_2, \ldots, \psi_{q-1}$ to form a set of basic wavelets. In view of (2.7) and (2.8), it is clear that if $\{\psi_1, \psi_2, \ldots, \psi_{q-1}\}$ is a set of function such that the system $\{\psi_\ell(x - u(k)) : 1 \leq \ell \leq q-1, k \in \mathbb{N}_0\}$ forms an orthonormal basis for W_0, then $\{q^{j/2}\psi_\ell(\mathfrak{p}^{-j}x - u(k)) : 1 \leq \ell \leq q-1, j \in \mathbb{Z}, k \in \mathbb{N}_0\}$ forms an orthonormal basis for $L^2(K)$.

3. NONUNIFORM LOW-PASS FILTERS ON NON-ARCHIMEDEAN FIELDS

For an integer $N \geq 1$ and an odd integer r with $1 \leq r \leq qN-1$ such that r and N are relatively prime, we define

$$\Lambda = \left\{0, \tfrac{u(r)}{N}\right\} + \mathcal{Z}.$$

where $\mathcal{Z} = \{u(n) : n \in \mathbb{N}_0\}$. It is easy to verify that Λ is not a group on non-Archimedean local field $\mathbb{K}$, but is the union of $\mathcal{Z}$ and a translate of $\mathcal{Z}$. Following is the definition of *nonuniform multiresolution analysis* (NUMRA) on non-Archimedean local fields of positive characteristic given by Shah and Abdullah [50].

DEFINITION 3.1. *For an integer* $N \geq 1$ *and an odd integer* r *with* $1 \leq r \leq qN-1$ *such that* r *and* N *are relatively prime, an associated NUMRA on non-Archimedean local field* $\mathbb{K}$ *of positive characteristic is a sequence of closed subspaces* $\{V_j : j \in \mathbb{Z}\}$ *of* $L^2(\mathbb{K})$ *such that the following properties hold:*

(a) $V_j \subset V_{j+1}$ *for all* $j \in \mathbb{Z}$;

(b) $\bigcup_{j\in\mathbb{Z}} V_j$ *is dense in* $L^2(\mathbb{K})$;

(c) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;

(d) $f(\cdot) \in V_j$ *if and only if* $f(\mathfrak{p}^{-1}N\cdot) \in V_{j+1}$ *for all* $j \in \mathbb{Z}$;

(e) There exists a function φ *in* V_0 *such that* $\{\varphi(\cdot - \lambda) : \lambda \in \Lambda\}$*, is a complete orthonormal basis for* V_0.

It is worth noticing that, when $N = 1$, one recovers from the definition above the definition of an MRA on non-Archimedean local fields of positive characteristic $p > 0$. When, $N > 1$, the dilation is induced by $\mathfrak{p}^{-1}N$ and $|\mathfrak{p}^{-1}| = q$ ensures that $qN\Lambda \subset \mathcal{Z} \subset \Lambda$.

For every $j \in \mathbb{Z}$, define W_j to be the orthogonal complement of V_j in V_{j+1}. Then we have

$$V_{j+1} = V_j \oplus W_j \quad \text{and} \quad W_\ell \perp W_{\ell'} \quad \text{if } \ell \neq \ell'.$$

It follows that for $j > J$,

$$V_j = V_J \oplus \bigoplus_{\ell=0}^{j-J-1} W_{j-\ell}\,,$$

where all these subspaces are orthogonal. By virtue of condition (b) in the Definition 3.1, this implies

$$L^2(\mathbb{K}) = \bigoplus_{j\in\mathbb{Z}} W_j,$$

a decomposition of $L^2(\mathbb{K})$ into mutually orthogonal subspaces.

As in the standard scheme, one expects the existence of $qN - 1$ number of functions so that their translation by elements of Λ and dilations by the integral powers of $\mathfrak{p}^{-1}N$ form an orthonormal basis for $L^2(\mathbb{K})$.

Let φ be a scaling function for a NUMRA $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{K})$. For $f \in L^2(\mathbb{K})$, we define $f_{j,k}(x) = (qN)^{j/2} f\left((\mathfrak{p}^{-1}N)^j x - \lambda\right), j \in \mathbb{Z}, \lambda \in \Lambda$. Since $\varphi \in V_0 \subset V_1$, and $\{\varphi_{1,\lambda} : \lambda \in \Lambda\}$ is an orthonormal basis in V_1, we have

$$\varphi(x) = \sum_{\lambda\in\Lambda} h_\lambda (qN)^{1/2} \varphi(\mathfrak{p}^{-1}Nx - \lambda), \tag{3.1}$$

where $h_\lambda = \langle \varphi, \varphi_{1,\lambda}\rangle$ and $\{h_\lambda : \lambda \in \Lambda\} \in \ell^2(\Lambda)$. Taking Fourier transforms, we get

$$\widehat{\varphi}(\xi) = (qN)^{-1/2} \sum_{\lambda\in\Lambda} h_\lambda \overline{\chi_\lambda(\mathfrak{p}N\xi)}\widehat{\varphi}(\mathfrak{p}N\xi) = m(\mathfrak{p}N\xi)\widehat{\varphi}(\mathfrak{p}N\xi), \tag{3.2}$$

where $m(\xi) = (qN)^{-1/2} \sum_{\lambda\in\Lambda} h_\lambda \overline{\chi_\lambda(\xi)}$ is an integral-periodic function, called the nonuniform lowpass filter associated with the scaling function φ. For such a low-pass filter m we have the following relation.

$$\sum_{\ell=0}^{qN-1} |m(\xi + \mathfrak{p}Nu(\ell))|^2 = 1 \; a.e. \; \xi \in \mathbb{K}.$$

Consider two operators $\mathcal{P}$ and $\mathcal{R}$ respectively on $L^\infty(\mathfrak{D})$ and $L^1(\mathbb{K}) \cap L^\infty(K)$ defined by

$$\mathcal{P}f = \sum_{\ell=0}^{qN-1} |m(\mathfrak{p}N(\cdot + u(\ell)))|^2 f(\mathfrak{p}N(\cdot + u(\ell))),$$

$$\mathcal{R}f = |m(\mathfrak{p}N\ \cdot)|^2 f(\mathfrak{p}N\ \cdot).$$

Corresponding to the scaling function φ, the associated low-pass filter is m therefore by virtue of (3.2) $|\widehat{\varphi}(\xi)|^2$ is a fixed point of the operator $\mathcal{R}$. Define $\mathcal{J}_\varphi(\xi) = \sum_{\lambda\in\Lambda} |\widehat{\varphi}(\xi+\lambda)|^2$, therefore we have

$$\begin{aligned} \mathcal{J}_\varphi(\xi) &= \sum_{\lambda\in\Lambda} |\widehat{\varphi}(\xi+\lambda)|^2 \\ &= \sum_{\ell=0}^{qN-1} \sum_{\lambda\in\Lambda} |\widehat{\varphi}(\xi+u(\ell+qN\lambda))|^2 \\ &= \sum_{\ell=0}^{qN-1} \sum_{\lambda\in\Lambda} |\widehat{\varphi}(\xi+u(\ell)+(\mathfrak{p}^{-1}N)^{-1}\lambda)|^2 \\ &= \sum_{\ell=0}^{qN-1} \sum_{\lambda\in\Lambda} |\widehat{\varphi}(\mathfrak{p}N\xi+\mathfrak{p}Nu(l+\lambda))|^2 |m(\mathfrak{p}N\xi+\mathfrak{p}Nu(\ell)+\lambda))|^2 \\ &= \sum_{\ell=0}^{qN-1} |m(\mathfrak{p}N\xi+\mathfrak{p}Nu(\ell))|^2 \mathcal{J}_\varphi(\mathfrak{p}N(\xi+u(l))) \text{ (since m is integral-periodic)} \\ &= \mathcal{P}\mathcal{J}_\varphi(\xi) \end{aligned} \tag{3.3}$$

Therefore, $\mathcal{J}_\varphi(\xi)$ is a fixed point of the operator $\mathcal{P}$.

Definition 3.2. *Let $g \in L^1(\mathbb{K}) \cap L^\infty(\mathbb{K})$. A function f is almost everywhere g-continuous at the origin if*

$$\lim_{j\to\infty} \frac{f((\mathfrak{p}^{-1}N)^{-j}\xi)}{|g((\mathfrak{p}^{-1}N)^{-j}\xi)|^2}$$

exists and is constant almost everywhere.

Definition 3.3. *$D_\infty(\widehat{\varphi})$ is a space of function $h(\xi)$ satisfying*
(i) both $h(\xi)$ and $h^{-1}(\xi)$ belong to $L^\infty(\mathfrak{D})$.
(ii) $h(\xi)$ is almost everywhere $\widehat{\varphi}$-continuous at the origin and $\frac{h(0)}{|\widehat{\varphi}(0)|^2} = 1$.

Note that if $\varphi(x)$ is a scaling function then $\mathcal{J}_\varphi$ is almost everywhere $\widehat{\varphi}$-continuous at the origin. In fact, $\mathcal{J}_\varphi(\xi) \in D_\infty(\widehat{\varphi})$. Using this weak form of continuity, Gundy [34] has given a characterization of low-pass filter for dyadic dilations. E. Curry [24] has generalized this characterization for the multivariable case.

Definition 3.4. *We call a function φ a pre-scaling function associated with a NUMRA $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{K})$ if its translates $\{\varphi(\cdot - \lambda) : \lambda \in \Lambda\}$ form a Riesz basis for V_0.*

Let H be a closed subspace of $L^2(\mathbb{K})$. A system $\{f_k : k \in \mathbb{N}_0\}$ of functions in $L^2(\mathbb{K})$ is said to be a Riesz basis of H if for any $f \in H$, there exists a sequence $\{a_k : k \in \mathbb{N}_0\} \in \ell^2(\mathbb{N}_0)$ such that $f = \sum_{k\in\mathbb{N}_0} a_k f_k$ with convergence in $L^2(K)$ and

$$A_1 \sum_{k\in\mathbb{N}_0} |a_k|^2 \le \Big\| \sum_{k\in\mathbb{N}_0} a_k f_k \Big\|_2^2 \le A_2 \sum_{k\in\mathbb{N}_0} |a_k|^2 \tag{3.4}$$

where the constants A_1 and A_2 are independent of f .

Remark 3.5. (i) Note that if we take $A_1 = A_2 = 1$, then the Riesz basis is an orthonormal basis for H.

(ii) A function $\varphi \in L^2(\mathbb{K})$ that satisfies the refinement equation (3.1) for some scalars $\{h_k\}_{k\in\mathbb{N}_0}$ but need not satisfy the Riesz basis property (3.4) is called a *refinement function*. So, every pre-scaling function is a refinement function. □

We have the following lemma for integral-periodic unimodular functions on K. This lemma will be helpful for proving our main result.

Lemma 3.6. *Let μ be an integral-periodic unimodular function on $\mathbb{K}$. That is,*

(i) $\mu(\xi) = \mu(\xi + \lambda)$ almost everywhere for every $\lambda \in \Lambda$, and

(ii) $|\mu(\xi)| = 1$ almost everywhere on $\mathbb{K}$.

Then there is a unimodular function t on $\mathbb{K}$ such that

$$\mu(\xi) = t(\mathfrak{p}^{-1}N\xi)\overline{t(\xi)} \quad a.e.\ on\ \mathbb{K}. \tag{3.5}$$

Proof. Let $\Gamma_j = \{x \in \mathbb{K} : |x| = (qN)^j\}$. Observe that $\mathbb{K}\setminus\{0\} = \cup_{j\in\mathbb{Z}}\Gamma_j$. Let t be any measurable unimodular function defined on Γ_0. For example, we can take $t(\xi) = 1$ for all $\xi \in \Gamma_0$.

Consider $\xi \in \Gamma_1$; then $|\mathfrak{p}N\xi| = q^{-1}N|\xi| = 1$. This implies $\mathfrak{p}N\xi \in \Gamma_0$. Hence, $t(\mathfrak{p}^{-1}N\xi)$ is well defined for $\xi \in \Gamma_1$. Define

$$t(\xi) = t(\mathfrak{p}^{-1}N\xi)\mu((\mathfrak{p}^{-1}N)\xi) \tag{3.6}$$

We now proceed inductively. Suppose that t is defined for $\Gamma_1, \Gamma_2, ..., \Gamma_{n-1}$ so that equation (3.5) satisfies for $\cup_{j=0}^{n-1}\Gamma_j$. Define t by (3.6) if $\xi \in \Gamma_n$. Hence, the induction is complete.

Similarly, if $\xi \in \Gamma_{-1}$, then $\mathfrak{p}^{-1}N\xi \in \Gamma_0$. Hence, $t(\mathfrak{p}^{-1}\xi)$ is defined. Using (3.6), we define

$$t(\xi) = t(\mathfrak{p}^{-1}N\xi)\overline{\mu(\xi)} \tag{3.7}$$

Again using induction we can define t by equation (3.7) for $\Gamma_j,\ j \le -1$.

Therefore, we define $t(\xi)$ for $\xi \in \Gamma_j,\ j \ne 0$, by

$$t(\xi) = \begin{cases} t(\mathfrak{p}^{-1}N\xi)\mu(\mathfrak{p}^{-1}N\xi), & for\ \xi \in \Gamma_j,\ j \ge 1 \\ t(\mathfrak{p}^{-1}N\xi)\overline{\mu(\xi)}, & for\ \xi \in \Gamma_j,\ j \le -1, \end{cases} \tag{3.8}$$

Thus, (3.4) follows from (3.8) if we set $t(0) = 1$. □

We are now ready to present our main theorem, which gives necessary and sufficient conditions of a function to be a low-pass filter for a local field $\mathbb{K}$ of positive characteristic.

Theorem 3.7. *Let m be a low-pass filter associated with a pre-scaling function φ. Then the following hold.*

(i) m is integral-periodic, $m \in L^2(\mathfrak{D})$, and $|m(\xi)|^2$ is almost everywhere φ-continuous at the origin with

$$\lim_{j\to\infty}\left|m\left((\mathfrak{p}^{-1}N)^{-j}\xi\right)\right| = 1 \ a.e.$$

(ii) The operators $\mathcal{P}$ and $\mathcal{R}$ have nontrivial fixed points, $\mathcal{J}_\varphi(\xi) \in L^\infty(\mathfrak{D})$ and $|\widehat{\varphi}|^2 \in L^1(\mathbb{K}) \cap L^\infty(\mathbb{K})$, respectively

(iii) The fixed point $\mathcal{J}_\varphi$ of operator $\mathcal{P}$ is the unique function in the class $D_\infty(\widehat{\varphi})$.

Conversely, if a function m satisfies (i), (ii), and (iii), then m is a low-pass filter associated with a pre-scaling function φ for a NUMRA $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{K})$.

Proof. First we prove the converse part.

Suppose that the operator $\mathcal{R}$ has a fixed point $|\widehat{\varphi}(\xi)|^2$.The fixed point $\mathcal{J}_\varphi(\xi)$ of the operator $\mathcal{P}$ is the unique function in $D_\infty(\widehat{\varphi})$. Then by [4, Prop. 3.5], the ratio $|\widehat{\varphi}|/\mathcal{J}_\varphi^{1/2}$ is a scaling function for a NUMRA $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{K})$. The low-pass filter corresponding to this scaling function is

$$m_0(\xi) = |m(\xi)|\left\{\frac{S_\varphi(\xi)}{S_\varphi(\mathfrak{p}^{-1}N\xi)}\right\}^{1/2}.$$

This leads us to define

$$\widetilde{m}_0(\xi) = m(\xi)\left\{\frac{S_\varphi(\xi)}{S_\varphi(\mathfrak{p}^{-1}N\xi)}\right\}^{1/2}.$$

Note that $\widetilde{m}_0(\xi) = \operatorname{sgn} m(\xi) m_0(\xi)$

By Lemma 3.6 we can write $\operatorname{sgn} m(\xi) = t(\mathfrak{p}^{-1}N\xi)\overline{t(\xi)}$, where t is an unimodular function on K. Define

$$\begin{aligned}\widehat{\varphi}(\xi) := t(\xi)|\widehat{\varphi}(\xi)| &= t(\xi)\overline{t(\mathfrak{p}N\xi)}t(\mathfrak{p}N\xi)|m(\mathfrak{p}\xi)\widehat{\varphi}(\mathfrak{p}N\xi)| \\ &= \operatorname{sgn} m(\mathfrak{p}N\xi)|m(\mathfrak{p}N\xi)|\widehat{\varphi}(\mathfrak{p}N\xi) = m(\mathfrak{p}N\xi)\widehat{\varphi}(\mathfrak{p}N\xi).\end{aligned}$$

Since $t(\xi)$ is a unimodular function and hence, $\varphi(\xi)$ is a required pre-scaling function for NUMRA.

Now let $m(\xi)$ be a low-pass filter associated with a pre-scaling function φ for a NUMRA $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{K})$. By definition, the operator $\mathcal{R}$ has a fixed point $|\widehat{\varphi}|^2$. And also from (3.3), $\mathcal{J}_\varphi$ is a fixed point of the operator $\mathcal{P}$. Furthermore, $\mathcal{J}_\varphi^{-1} \in L^2(\mathfrak{D})$. This implies that the function $\gamma(x)$, defined by

$$|\widehat{\gamma}(\xi)|^2 = \frac{|\widehat{\varphi}(\xi)|^2}{\mathcal{J}_\varphi(\xi)}$$

is a scaling function for the same NUMRA and that

$$\sum_{\lambda\in\Lambda} |\widehat{\gamma}(\xi+\lambda)|^2 = 1.$$

By the characterization of scaling function, we have

$$1 = \lim_{j\to\infty} |\widehat{\gamma}((\mathfrak{p}^{-1}N)^{-j}\xi)|^2 = \lim_{j\to\infty} \frac{|\widehat{\varphi}((\mathfrak{p}^{-1}N)^{-j}\xi)|^2}{\mathcal{J}_\varphi((\mathfrak{p}^{-1}N)^{-j}\xi)} \text{ a.e.}$$

This shows that $\mathcal{J}_\varphi(\xi)$ is almost everywhere $\widehat{\varphi}$-continuous at zero. It only remains to prove that $\mathcal{J}_\varphi$ is the unique function in the class $D_\infty(\widehat{\varphi})$. □

4. PROOF OF THE UNIQUENESS

In this section, we want to prove that $\mathcal{J}_\varphi(\xi)$ is a unique function in $D_\infty(\widehat{\varphi})$. Suppose $h(\xi)$ is another such function. We claim that $\mathcal{J}_\varphi(\xi) = h(\xi)$ for almost every ξ. Since $\gamma(\xi)$ is a scaling function of a NUMRA, it is obvious that the Fourier transform of γ at $\xi = 0$ is 1. Also, we have $\sum_{\lambda\in\Lambda} |\widehat{\gamma}(\xi+\lambda)|^2 = 1$ for almost every $\xi \in \mathfrak{D}$ and $\lim_{j\to\infty} |\widehat{\gamma}((\mathfrak{p}^{-1}N)^{-j}\xi)|^2 = 1$ for almost every ξ on $\mathbb{K}$. Therefore, we can interpret $|\widehat{\gamma}(\xi+\lambda)|^2, \lambda \in \Lambda$, as a probability distribution on Λ for almost every $\xi \in \mathfrak{D}$.

Let μ be the low-pass filter associated with the scaling function γ. Then

$$\mu(\xi) = \frac{\widehat{\varphi}(\mathfrak{p}^{-1}N\xi)}{S_\varphi(\mathfrak{p}^{-1}N\xi)} \cdot \frac{\mathcal{J}_\varphi(\xi)}{\widehat{\varphi}(\xi)} = m(\xi)\frac{\mathcal{J}_\varphi(\xi)}{\mathcal{J}_\varphi(\mathfrak{p}^{-1}\xi)}.$$

Let $\mathcal{M}(\xi) = |\mu(\xi)|^2$. Notice that $\mathcal{M}(\xi)$ is an integral-periodic function and satisfies $\mathcal{M}(0) = 1$ and

$$\sum_{\ell=0}^{qN-1} M(\xi + \mathfrak{p}Nu(\ell)) = 1, \text{ a.e. } \xi \in \mathfrak{D}. \tag{4.1}$$

Every non-negative integer $k \in \mathbb{N}_0$ can be expressed uniquely as

$$k = \sum_{j=1}^{\infty} \omega_j(k) q^{j-1}, \ 0 \le \omega_j(k) \le qN-1$$

We identify k with the sequence $(0, \omega_1(k), \omega_2(k), ...)$ and define $\omega_0(k) = 0$. The integer zero is identified with the sequence zero. Note that each such sequence has finitely many non zero terms.

Let $D = \{1, 2, ..., qN-1\}$ and $D_0 = D \cup \{0\}$. Let $\Omega = D_0^{\mathbb{N}}$ be the set of sequences. We identify $\mathbb{N}_0$ with the subset of Ω consisting of finite sequences. Fix $\lambda \in \Lambda$. For $J \ge 1$, let $\lambda_J = \{\omega : \omega_i = \omega_i(\lambda), 0 \le i \le J)\}$, be a finite cylinder in Ω.

For each $\xi \in \mathfrak{D}$, we define probability Γ_ξ^J on the set of all such cylinders as follows.

For $0 \leq \lambda \leq (qN)^J - 1$, we set

$$\Gamma_\xi^J(\lambda) = \prod_{j=1}^{N} \mathcal{M}((\mathfrak{p}^{-1}N)^{-j}(\xi + \lambda)). \tag{4.2}$$

LEMMA 4.1. *It holds*

$$\sum_{0 \leq k \leq (qN)^J - 1} \Gamma_\xi^J(k) = 1. \tag{4.3}$$

Proof. We will prove this lemma by using induction on J. Define conditional probability by

$$\mathcal{M}((\mathfrak{p}^{-1}N)^{-j}(\xi + \lambda)) = \Gamma_\xi(\omega_j(\lambda)||\omega_{j-1}, ..., \omega_1).$$

Equation (4.3) can also be written as $\Gamma_\xi^J(\lambda_J) = 1$.
For $J = 1$,

$$\Gamma_\xi^1(\lambda) = \mathcal{M}((\mathfrak{p}^{-1}N)^{-1}(\xi + \lambda)) = \Gamma_\xi(\omega_1(\lambda))$$

Using equation (4.1), we can easily see that the result is true for $J = 1$.

$$\Gamma_\xi^1(\lambda_1) = \sum_{\omega_1 \in D_0} \Gamma_\xi(\omega_1(\lambda)) = \sum_{\lambda=0}^{qN-1} \mathcal{M}((\mathfrak{p}^{-1}N)^{-1}(\xi + \lambda)) = 1 \text{ a.e } \xi.$$

Assume that it is true for $J - 1$, *i.e.*, $\Gamma_\xi^{J-1}(\lambda_{J-1}) = 1$. Now we want to prove it is true for J. We write

$$\begin{aligned}
\Gamma_\xi^J(\lambda) =& \left(\prod_{j=1}^{J-1} \mathcal{M}((\mathfrak{p}^{-1}N)^{-j}(\xi + \lambda)) \right) \times \mathcal{M}((\mathfrak{p}^{-1}N)^{-J}(\xi + \lambda)) \\
=& \Gamma_\xi^{J-1}(\lambda) \times \Gamma_\xi\big(\omega_J(\lambda))\|\omega_{J-1}, ..., \omega_1\big), \\
\Gamma_\xi^J(\lambda_J) =& \Gamma_\xi^{J-1}(\lambda_{J-1}) \times \Gamma_\xi\big(\omega_J(\lambda_J))\|\omega_{J-1}, ..., \omega_1\big)
\end{aligned}$$

Where,

$$\begin{aligned}
&\Gamma_\xi\big(\omega_J(\lambda_J))\|\omega_{J-1}, ..., \omega_1\big) = \\
&= \sum_{\omega_J=0}^{qN-1} \mathcal{M}\big((\mathfrak{p}^{-1}N)^J(\xi + u(\omega_1) + \mathfrak{p}^{-1}Nu(\omega_2) + \cdots + (\mathfrak{p}^{-1}N)^{-J+1}u(\omega_J))\big) \\
&= \sum_{\omega_J=0}^{qN-1} \mathcal{M}((\mathfrak{p}^{-1}N)^J\xi + (\mathfrak{p}^{-1}N)^J u(\omega_1) + (\mathfrak{p}^{-1}N)^{J-1}u(\omega_2) + \cdots + \mathfrak{p}^{-1}Nu(\omega_J)).
\end{aligned}$$

Note that the summation is only on ω_J as $\omega_1, ..., \omega_{J-1}$ are given. Again using (4.1), we get

$$\Gamma_\xi\big(\omega_J(\lambda_J))\|\omega_{J-1}, ..., \omega_1\big) = 1$$

Hence, the induction is complete.

Therefore, $\Gamma_\xi^J, J \geq 1$, specifies a probability. By the basic Kolmogorov theorem, the family Γ_ξ^J extends to a probability say P_ξ on the Borel sets of Ω. If we assume that infinite product of (4.2) exists, then we have

$$\begin{aligned} 1 &= \sum_{\lambda\in\Lambda} |\widehat{\gamma}(\xi+\lambda)|^2 = \sum_{\lambda\in\Lambda} \lim_{J\to\infty} \prod_{j=1}^{J} \mathcal{M}((\mathfrak{p}^{-1}N)^{-j}(\xi+\lambda)) \\ &= \sum_{\lambda\in\Lambda} \lim_{J\to\infty} \Gamma_\xi^J(\lambda) \text{ for a.e. } \xi \end{aligned}$$

Hence, Γ_ξ^J is tight in the Prokorov sense on the set of finite sequence. Therefore, P_ξ is concentrated on finite sequences. We say $P_\xi(\Lambda) = 1$ for almost every ξ.

Consider $X_j(\omega(\lambda)) = \omega_j(\lambda)$, where $\omega_j(\lambda) \in D_0$. Define $\xi_1(\lambda) := \xi$ and $\xi_{j+1}(\lambda) := \mathfrak{p}N(\xi_j + u(\omega_j(\lambda)))$.

For $0 \leq \lambda \leq (qN)^J - 1$, we write $\lambda = \sum_{j=1}^{J} \omega_j(\lambda)(qN)^{j-1},\ 0 \leq \omega_j(\lambda) \leq qN - 1$. And

$u(\lambda) = u(\omega_1) + \mathfrak{p}Nu(\omega_2) + \cdots + (\mathfrak{p}^{-1}N)^{-J+1}u(\omega_J)$, using equation (2.2).

Also, we can write

$$\begin{aligned} &(\mathfrak{p}^{-1}N)^{-J}(\xi+\lambda) = \\ &= (\mathfrak{p}^{-1}N)^{-J}\big(\xi + u(\omega_1) + \mathfrak{p}Nu(\omega_2) + \cdots + (\mathfrak{p}^{-1}N)^{-J+1}u(\omega_J)\big) \\ &= \mathfrak{p}N\big((\mathfrak{p}^{-1}N)^{J-1}\xi + (\mathfrak{p}^{-1}N)^{J-1}u(\omega_1) \\ &\quad + (\mathfrak{p}^{-1}N)^{J-2}u(\omega_2) + \cdots + \mathfrak{p}Nu(\omega_{J-1}) + u(\omega_J)\big) \\ &= \mathfrak{p}N\big(\xi_J + u(\omega_J)\big). \end{aligned}$$

Now we can define the conditional probability of X_j given $X_{j-1}, ..., X_1$ as

$$\mathcal{M}\big((\mathfrak{p}^{-1}N)^{-1}(\xi_j + u(\omega_j(\lambda)))\big)$$

for each $j \geq 1$. Since P_ξ is concentrated on finite sequences for almost every ξ, hence, the sequence $\{X_j\}_{j\geq1}$ converges to zero relative to P_ξ.

Now

$$P_\xi(\xi_{j+1}\|\xi_j, ..., \xi_1) = \mathcal{M}\big(\mathfrak{p}N(\xi_j + u(\omega_j(\lambda)))\big).$$

By construction, $P_\xi(\xi_{j+1}\|\xi_j, ..., \xi_1) = P_\xi(\xi_{j+1}\|\xi_j)$. Thus, $\{\xi_j\}_{j\geq1}$ is a Markov process.

Since P_ξ is concentrated on a finite sequence, hence, sequence $\{\xi_j\}_{j\geq1}$ converges to zero.

Now we will come back to uniqueness question. Consider $r(\xi) = \dfrac{h(\xi)}{\mathcal{J}_\varphi(\xi)}$. We want to show that $r(\xi) = 1$ for almost every ξ. We know that $h(\xi)$ and $\mathcal{J}_\varphi(\xi)$ are fixed points of the operator $\mathcal{P}$ and $\mathcal{J}_\varphi(\xi) = 1$ almost everywhere, hence,

$r(\xi)$ satisfies

$$r(\xi) = \sum_{\ell=0}^{qN-1} |m((\mathfrak{p}^{-1}N)^{-1}(\xi + u(\ell)))|^2 r((\mathfrak{p}^{-1}N)^{-1}(\xi + u(\ell))).$$

Therefore, the composition $r(\xi_j)$ is a martingale, *i.e.*,

$$\begin{aligned} E\big(r(\xi_{j+1})\|r(\xi_j),...,r(\xi_1)\big) &= E\big(r(\mathfrak{p}N(\xi_j + u(\omega_j)))\|r(\xi_j),...,r(\xi_1)\big) \\ &= E\big(r(\mathfrak{p}N(\xi_j + u(\omega_j)))\|r(\xi_j)\big) \\ &= \sum_{\omega_j \in D_0} \mathcal{M}\big(\mathfrak{p}N(\xi_j + u(\omega_j))\big) r\big(\mathfrak{p}N(\xi_j + u(\omega_j))\big) \\ &= r(\xi_j). \end{aligned}$$

The martingale $r(\xi_j)$ is strictly positive, bounded, and converges P_ξ-almost surely to $r(0) = 1$ for almost every ξ, since $\xi_j \to 0$. By Lebesgue dominated converges theorem and for all $j \geq 1$, we get

$$r(0) = E\big(r(0)\|r(\xi_j)\big) = E\big(\lim_{n\to\infty} r(\xi_n)\|r(\xi_j)\big) = \lim_{n\to\infty} E\big(r(\xi_n)\|r(\xi_j)\big) = r(\xi_j).$$

Thus,

$$r(0) = r(\xi) = \frac{h(\xi)}{\mathcal{J}_\varphi(\xi)}$$

for almost every ξ. This gives $h(\xi) = \mathcal{J}_\varphi(\xi)$ for almost every ξ, which proves the uniqueness assertion of the theorem.

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Received by the editors: August 30, 2021; accepted: September 16, 2021; published online: November 8, 2021.