JOURNAL OF NUMERICAL ANALYSIS AND APPROXIMATION THEORY
J. Numer. Anal. Approx. Theory, vol. 50 (2021) no. 1, pp. 52–59
ictp.acad.ro/jnaat

SOME INEQUALITIES FOR A STANCU TYPE OPERATOR *VIA* (1,1) BOX CONVEX FUNCTIONS

IOAN GAVREA* and DANIEL IANOȘI*

Abstract. In this paper we introduce a Stancu type operator and we prove inequalities of Raşa's type.

MSC 2020. 41A36, 41A17.

Keywords. Stancu operators, box convex functions.

1. INTRODUCTION

D.D. Stancu, in [12] using a probabilistic method, constructs a linear positive polynomial operators $L_{n,r}^{\alpha,\beta}$ of Bernstein type, depending on a non-negative integer parameter r and on two parameters α and β, such that $0 \leq \alpha \leq \beta$. More precisely the operators are defined by

$$\left(L_{n,r}^{\alpha,\beta} f\right)(x) = \sum_{k=0}^{n} w_{n,k,r}(x) f\left(\tfrac{k+\alpha}{n+\beta}\right) \tag{1}$$

for $f \in C[0,1]$, where

(2)

$$w_{n,k,r}(x) = \\ = \begin{cases} \binom{n-r}{k} x^k (1-x)^{n-r-k+1}, & \text{if } 0 \leq k < r, \\ \binom{n-r}{k} x^k (1-x)^{n-r-k+1} + \binom{n-r}{k-r} x^{k-r+1} (1-x)^{n-k}, & \text{if } r \leq k \leq n-r, \\ \binom{n-r}{k-r} x^{k-r+1} (1-x)^{n-k}, & \text{if } n-r < k \leq n. \end{cases}$$

Let $b_{n,k}(x)$ be the fundamental Bernstein polynomials, $k, n \in \mathbb{N}$ defined by

$$b_{n,k}(x) = \begin{cases} \binom{n}{k} x^k (1-x)^{n-k}, & \text{if } 0 \leq k \leq n \\ 0, & \text{otherwise} \end{cases}$$

*Department of Mathematics, Technical University of Cluj Napoca, Str. Memorandumului nr. 28, 400114 Cluj-Napoca, Romania, e-mail: Ioan.Gavrea@math.utcluj.ro, dan.ianosi@yahoo.com.

The operators $L_{n,r}^{\alpha,\beta}$ can be written in the following forms

$$\left(L_{n,r}^{\alpha,\beta}f\right)(x)=\sum_{k=0}^{n-r}b_{n-r,k}(x)\left[(1-x)f\left(\tfrac{k+\alpha}{n+\beta}\right)+xf\left(\tfrac{k+r+\alpha}{n+\beta}\right)\right] \tag{3}$$

$$\left(L_{n,r}^{\alpha,\beta}f\right)(x)=(1-x)\sum_{k=0}^{n-r}b_{n-r,k}(x)f\left(\tfrac{k+\alpha}{n+\beta}\right)+x\sum_{k=r}^{n}b_{n-r,k-r}(x)f\left(\tfrac{k+\alpha}{n+\beta}\right). \tag{4}$$

REMARK 1. For $r=0$, $L_{n,0}^{\alpha,\beta}=L_n^{\alpha,\beta}$ the operator was introduced by D.D. Stancu in [11]. For $r=0,\alpha=\beta=0$, $L_n^{0,0}$ coincides with the Bernstein operator B_n. □

Inspired by (4) we construct the following operator.

Let φ be an increasing differential function, $\varphi:[0,1]\to[0,1], r,n\in\mathbb{N}$ such that $n>2r, 0\le\alpha\le\beta\le 1$.

Then for $f\in C[0,1]$ we defined $L_{n,r,\varphi}^{\alpha,\beta}$ by:

(5)

$$\left(L_{n,r,\varphi}^{\alpha,\beta}f\right)(x)=(1-\varphi(x))\sum_{k=0}^{n-r}b_{n-r,k}(x)f\left(\tfrac{k+\alpha}{n+\beta}\right)+\varphi(x)\sum_{k=r}^{n}b_{n-r,k-r}(x)f\left(\tfrac{k+\alpha}{n+\beta}\right),$$

$x\in[0,1]$.

The operator defined by (5) can be also written as:

$$\left(L_{n,r,\varphi}^{\alpha,\beta}f\right)(x)=\sum_{k=0}^{n}w_{n,r,k,\varphi}(x)f\left(\tfrac{k+\alpha}{n+\beta}\right), \tag{6}$$

where

(7)

$$w_{n,r,k,\varphi}(x)=\begin{cases}(1-\varphi(x))\,b_{n-r,k}(x), & \text{if } 0\le k<r\\ (1-\varphi(x))\,b_{n-r,k}(x)+\varphi(x)b_{n-r,k-r}(x), & \text{if } r\le k\le n-r\\ \varphi(x)b_{n-r,k-r}(x), & \text{if } n-r<k\le n.\end{cases}$$

The sequence $(L_{n,r,\varphi}^{\alpha,\beta}f)_{n\in\mathbb{N}}$ is a linear positive sequence and for $\varphi=x$ it becomes Stancu's sequence $(L_n^{\alpha,\beta}f)_{n\in\mathbb{N}}$.

In [10], Ioan Raşa recalled his 25-year-old problem relative to preservation of convexity by the Bernstein-Schnabl operators.

Prove or disprove that

$$\sum_{i,j=0}^{n}\left(b_{n,i}(x)b_{n,j}(x)+b_{n,i}(y)b_{n,j}(y)-2b_{n,i}(x)b_{n,j}(y)\right)f\left(\tfrac{i+j}{2n}\right)\ge 0 \tag{8}$$

for each convex function $f\in C[0,1]$ and for all $x,y\in[0,1]$.

In [8], J. Mrowiec, T. Rajba and S. Wąsowicz prove that (8) holds. In the proof of (8) they use the probability theory. As a tool they applied stochastic convex ordering.

In [1], U. Abel gave an elementary proof of (8). An extension of (8) was considered in [5], where $b_{n,k}(x)$ were replaced by more general functions and the functional evaluations were replaced by $A_{\frac{i+j}{2n}}(f)$, where $\{A_t\}_{t\geq 0}$ is a set of linear positive functionals defined on a linear space of functions satisfying certain assumptions (see [5] for more details).

In [6], Raşa's conjecture (8) was studied for the case of Baskakov-Mastroianni operators.

Given $f \in C[0,1]$, denote

$$\Delta_h^1 f(x) := \Delta_h f(x) := \begin{cases} f(x+h) - f(x), & x, x+h \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

and for $q \geq 1$

$$\Delta_h^{q+1} f(x) := \Delta_h^q \left(\Delta_h f(x)\right).$$

A function f defined on $[0,1]$ is called q-monotone if $\Delta_h^q f(x) \geq 0$, for all $h \geq 0$ (see [2]).

Abel and Leviatan in [2] proved the following result.

THEOREM A. *Let $q, n \in \mathbb{N}$. If $f \in C[0,1]$ is a q-monotone function, then for all $x, y \in [0,1]$*

$$\operatorname{sgn}(x-y)^q \times$$
$$\sum_{\partial_1,\dots,\partial_q=0}^{n} \sum_{j=0}^{q} (-1)^{q-j} \tbinom{q}{j} \Big(\prod_{i=1}^{j} b_{n,\partial_i}(x)\Big)\Big(\prod_{i=j+1}^{q} b_{n,\partial_i}(y)\Big) \int_0^1 f\left(\tfrac{\partial_1+\ldots+\partial_q+\alpha t}{qn+\alpha}\right) dt \geq 0.$$

In [7], it was proved that Theorem A follows from the fact that the tensorial product of Bernstein polynomials, *i.e.*

$$\left(B_{n,m} f\right)(x,y) = \sum_{i=0}^{n} \sum_{j=0}^{m} b_{n,i}(x) b_{m,j}(y) f\left(\tfrac{i}{n}, \tfrac{j}{m}\right)$$

preserves (q,s)-convexity (see [4]).

The aim of this article is to prove an inequality of type (8) for operators $L_{n,r,\varphi}^{\alpha,\beta}$.

2. MAIN RESULTS

If $f(x,y)$ is a function defined on the rectangle $I \times J, I = [a,b], J = [c,d], x_1, \dots, x_m$ are distinct points from I and $y_1, \dots, y_n$ are distinct points from J, then the double divided difference is defined by

$$\begin{bmatrix} x_1, \dots, x_m \\ y_1, \dots, y_n \end{bmatrix}; f \Bigg] = \left[x_1, \dots, x_m; [y_1, \dots, y_n; f(x,\cdot)]\right]$$
$$= \left[y_1, \dots, y_n; [x_1, \dots, x_m; f(\cdot, y)]\right].$$

Here, for the distinct points z_k, $k = \overline{1,n}$ and a function g defined on a set that contains these points, $[z_1, \dots, z_n; g]$ denote the classical divided difference given by

$$[z_1; g] = g(z_1)$$

$$[z_1, z_2; g] = \frac{g(z_1) - g(z_2)}{z_1 - z_2}$$
$$[z_1, z_2, \ldots, z_{n-1}, z_n; g] = \frac{[z_1, \ldots, z_{n-1}; g] - [z_2, \ldots, z_n; g]}{z_1 - z_n}.$$

T. Popoviciu introduced the notion of (m, n) convexity in [9, p. 78]. In [4], S. Gal and P. Niculescu used box convexity of order (m, n) for (m, n) convexity. We will use the terminology introduced in [4] and say that the function $f : I \times J \to \mathbb{R}$ is box convex of order (m, n) if all divided differences

$$\begin{bmatrix} x_1, & \ldots, & x_{m+1} \\ y_1, & \ldots, & y_{n+1} \end{bmatrix}; f \Bigg]$$

are non-negative for any choice of distinct points $x_1, \ldots, x_{m+1}$ and $y_1, \ldots, y_{n+1}$.

For the remaining of this paper, we will consider $I = J = [0, 1]$.

In particular, let us observe that when $m = n = 1$ for $f : I \times I \to \mathbb{R}$, box convexity of order $(1, 1)$ is equivalent to

$$\left[\begin{matrix} x_1, & x_2 \\ y_1, & y_2 \end{matrix}; f\right] = \tfrac{1}{(x_1-x_2)(y_1-y_2)} \Big[f(x_1, y_1) - f(x_1, y_2) - f(x_2, y_1) + f(x_2, y_2)\Big] \geq 0.$$

Lemma 2 ([9]). *If $f \in C^{q+s}(I \times I)$ and $\frac{\partial^{q+s} f}{\partial x^q \partial y^s} \geq 0$ on $I \times I$, then f is (q, s) box convex.*

In this paper we will use Lemma 2 for the case $q = s = 1$.

Definition 3. *Let U, V be two operators $U, V : C[0, 1] \to C[0, 1]$ defined by*

$$(Uf)(x) = \sum_{i=0}^{n} a_i(x) f(x_i)$$
$$(Vf)(y) = \sum_{j=0}^{m} b_j(y) f(y_j).$$

for all $x, y \in I, x_i, y_j \in I, i = 0, 1, \ldots, n, j = 0, 1, \ldots, m$.

The operator denoted by $UV, UV : C(I \times I) \to C(I \times I)$ defined by

$$UVf(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{m} f(x_i, y_j) a_i(x) b_j(y) \tag{9}$$

is called the tensor product of the operators U and V.

The following results will be proved in this paper.

Theorem 4. *Let $\varphi, \psi : [0, 1] \to [0, 1]$ be two increasing differentiable functions and let $L = L^{\alpha,\beta}_{n,r_1,\varphi} L^{\gamma,\delta}_{m,r_2,\psi}$ the tensor product of operators $L^{\alpha,\beta}_{n,r_1,\varphi}, L^{\gamma,\delta}_{m,r_2,\psi}$ defined as in (6) i.e.*

$$(Lf)(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{m} w_{n,r_1,i,\varphi}(x) w_{m,r_2,j,\psi}(y) f\left(\tfrac{i+\alpha}{n+\beta}, \tfrac{j+\gamma}{m+\delta}\right). \tag{10}$$

If f is $(1,1)$ box convex function, then Lf is $(1,1)$ box convex.

THEOREM 5. *Let f be a $(1,1)$ box convex function on $I\times I$ and $x, x_1, y, y_1 \in [0,1]$. The following inequality holds:*

(11)
$$\operatorname{sgn}\left[(x-x_1)(y-y_1)\right]\cdot \cdot\sum_{i=0}^{n}\sum_{j=0}^{m}\left(w_{n,r_1,\varphi}(x)-w_{n,r_1,\varphi}(x_1)\right)\left(w_{m,r_2,\psi}(y)-w_{m,r_2,\psi}(y_1)\right)f\left(\tfrac{i+\alpha}{n+\beta},\tfrac{j+\gamma}{m+\delta}\right)\geq 0.$$

COROLLARY 6. *If f is a $(1,1)$ box convex function then*

(12)
$$\sum_{i=0}^{n}\sum_{j=0}^{m}\Big[w_{n,r_1,\varphi}(x)w_{m,r_2,\psi}(x)+w_{n,r_1,\varphi}(y)w_{m,r_2,\psi}(y) - w_{n,r_1,\varphi}(x)w_{m,r_2,\psi}(y)-w_{n,r_1,\varphi}(y)w_{m,r_2,\psi}(x)\Big]f\left(\tfrac{i+\alpha}{n+\beta},\tfrac{j+\gamma}{m+\delta}\right)\geq 0.$$

COROLLARY 7. *If $f:[0,1]\to\mathbb{R}$ is a convex function, then for all $x, x_1, y, y_1 \in [0,1]$ the following inequality holds:*

(13)
$$\operatorname{sgn}\left[(x-x_1)(y-y_1)\right]\cdot \cdot\sum_{i=0}^{n}\sum_{j=0}^{m}\left[w_{n,r_1,\varphi}(x)-w_{n,r_1,\varphi}(x_1)\right]\left[w_{m,r_2,\psi}(y)-w_{m,r_2,\psi}(y_1)\right]f\left(\tfrac{i+\alpha}{2(n+\beta)}+\tfrac{j+\gamma}{2(m+\delta)}\right)\geq 0.$$

3. PROOFS

Proof of Theorem 4. From (5) we have

(14)
$$\begin{aligned}
&L'_{n,r_1,\varphi}(f)(x)=\\
&=\varphi'(x)\sum_{i=0}^{n-r_1}b_{n-r_1,i}(x)\left(f\left(\tfrac{i+\alpha+r}{n+\beta}\right)-f\left(\tfrac{i+\alpha}{n+\beta}\right)\right)\\
&\quad+(1-\varphi(x))(n-r_1)\sum_{i=0}^{n-r_1-1}b_{n-r_1-1,i}(x)\left(f\left(\tfrac{i+\alpha+1}{n+\beta}\right)-f\left(\tfrac{i+\alpha}{n+\beta}\right)\right)\\
&\quad+\varphi(x)(n-r_1)\sum_{i=0}^{n-r_1-1}b_{n-r_1-1,i}(x)\left(f\left(\tfrac{i+r_1+\alpha+1}{n+\beta}\right)-f\left(\tfrac{i+r_1+\alpha}{n+\beta}\right)\right)\\
&=\varphi'(x)\tfrac{r_1}{n+\beta}\sum_{i=0}^{n-r_1}b_{n-r_1,i}(x)\left[\tfrac{i+\alpha}{n+\beta},\tfrac{i+\alpha+r_1}{n+\beta};f\right]\\
&\quad+(1-\varphi(x))\tfrac{n-r_1}{n+\beta}\sum_{i=0}^{n-r_1-1}b_{n-r_1-1,i}(x)\left[\tfrac{i+\alpha}{n+\beta},\tfrac{i+\alpha+1}{n+\beta};f\right]
\end{aligned}$$

$$+\varphi(x)\tfrac{n-r_1}{n+\beta}\sum_{i=0}^{n-r_1-1} b_{n-r_1-1,i}(x)\left[\tfrac{i+r_1+\alpha}{n+\beta},\tfrac{i+r_1+\alpha+1}{n+\beta};f\right]$$

and

$$\begin{aligned}(15)\quad L'_{m,r_2,\psi}(f)(y) =&\psi'(y)\tfrac{r_2}{m+\delta}\sum_{j=0}^{m-r_2} b_{m-r_2,j}(y)\left[\tfrac{j+\gamma}{m+\delta},\tfrac{j+\gamma+r_2}{m+\delta};f\right]\\ &+(1-\psi(y))\tfrac{m-r_2}{m+\delta}\sum_{j=0}^{m-r_2-1} b_{m-r_2-1,j}(y)\left[\tfrac{j+\gamma}{m+\delta},\tfrac{j+\gamma+1}{m+\delta};f\right]\\ &+\psi(y)\tfrac{m-r_2}{m+\delta}\sum_{j=0}^{m-r_2-1} b_{m-r_2-1,j}(y)\left[\tfrac{j+r_2+\gamma}{m+\delta},\tfrac{j+r_2+\gamma+1}{m+\delta};f\right]\end{aligned}$$

From (14) and (15) we get

$$\begin{aligned}(16)\quad &\frac{\partial^2 L}{\partial x\partial y}(x,y)=\\ &=\varphi'(x)\psi'(y)\tfrac{r_1}{n+\beta}\tfrac{r_2}{m+\delta}\sum_{i=0}^{n-r_1}\sum_{j=0}^{m-r_2} b_{n-r_1,i}(x)b_{m-r_2,j}(y)\begin{bmatrix}\frac{i+\alpha}{n+\beta}, & \frac{i+\alpha+r_1}{n+\beta} \\ \frac{j+\gamma}{m+\delta}, & \frac{j+\gamma+r_2}{m+\delta}\end{bmatrix};f\\ &+(1-\varphi(x))(1-\psi(y))\tfrac{(n-r_1)(m-r_2)}{(n+\beta)(m+\delta)}\\ &\times\sum_{i=0}^{n-r_1-1}\sum_{j=0}^{m-r_2-1} b_{n-r_1-1,i}(x)b_{m-r_2-1,j}(y)\begin{bmatrix}\frac{i+\alpha}{n+\beta}, & \frac{i+\alpha+1}{n+\beta} \\ \frac{j+\gamma}{m+\delta}, & \frac{j+\gamma+1}{m+\delta}\end{bmatrix};f\\ &+\varphi(x)\psi(y)\tfrac{(n-r_1)(m-r_2)}{(n+\beta)(m+\delta)}\\ &\times\sum_{i=0}^{n-r_1-1}\sum_{j=0}^{m-r_2-1} b_{n-r_1-1,i}(x)b_{m-r_2-1,j}(y)\begin{bmatrix}\frac{i+r_1+\alpha}{n+\beta}, & \frac{i+r_1+\alpha+1}{n+\beta} \\ \frac{j+r_2+\gamma}{m+\delta}, & \frac{j+r_2+\gamma+1}{m+\delta}\end{bmatrix};f,\end{aligned}$$

f being a $(1,1)$ box convex function, from (16) we obtain

$$\frac{\partial^2 L}{\partial x\partial y}(x,y)\geq 0.$$

From the last inequality and Lemma 2 the theorem is proved. □

Proof of Theorem 5. If $x=x_1$ or $y=y_1$, then the relation (11) is trivial.

Let us suppose that $x>x_1$ and $y=y_1$. Then we have

$$\begin{aligned}&\sum_{i=0}^{n}\sum_{j=0}^{m}[w_{n,r_1,\varphi}(x)-w_{n,r_1,\varphi}(x_1)][w_{m,r_2,\psi}(y)-w_{m,r_2,\psi}(y_1)]f\left(\tfrac{i+\alpha}{n+\beta},\tfrac{j+\gamma}{m+\delta}\right)=\\ &=\int_{x_1}^{x}\int_{y_1}^{y}\left(\sum_{i=0}^{n}\sum_{j=0}^{m}w'_{n,r_1,\varphi}(u)w'_{m,r_2,\psi}(v)f\left(\tfrac{i+\alpha}{n+\beta},\tfrac{j+\gamma}{m+\delta}\right)\right)dudv\\ &=\int_{x_1}^{x}\int_{y_1}^{y}\tfrac{\partial^2 L}{\partial x\partial y}(u,v)dudv.\end{aligned}$$

From Theorem 4 we have $\frac{\partial^2 L}{\partial u \partial v}(u,v) \geq 0$, for all $(u,v) \in I \times I$. □

Proof of Corollary 6. Inequality (12) follows from the inequality

$$\sum_{i=0}^{n}\sum_{j=0}^{m}\left(w_{n,r_1,\varphi}(x)-w_{n,r_1,\varphi}(y)\right)\left(w_{m,r_2,\psi}(x)-w_{m,r_2,\psi}(y)\right) f\left(\tfrac{i+\alpha}{n+\beta},\tfrac{j+\gamma}{m+\delta}\right) \geq 0,$$

which follows from (11). □

Proof of Corollary 7. If the function f is a convex function then the function $g : I \times I \to \mathbb{R}, g(x,y) = f\left(\frac{x+y}{2}\right)$ is a $(1,1)$ box convex function and so inequality (13) is inequality (11) for the function g. □

REMARK 8. If f is a $(1,1)$ box convex function, in (11) the evaluation functional $f\left(\frac{i+\alpha}{n+\beta},\frac{j+\gamma}{m+\delta}\right)$ can be replaced by

$$\int_0^1\int_0^1 f\left(\tfrac{i+\alpha u}{n+\beta},\tfrac{j+\gamma v}{m+\delta}\right) du dv, \qquad i=0,1,\ldots,n, j=0,1,\ldots,m. \quad \square$$

REMARK 9. For $m=n, r=0, \varphi(x)=\psi(x)=x$ the inequality (13) becomes (8). □

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Received by the editors: September 20, 2021; accepted: October 16, 2021; published online: November 8, 2021.