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ON THE RATE OF CONVERGENCE OF MODIFIED α-BERNSTEIN OPERATORS BASED ON q-INTEGERS

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Abstract. In the present paper we define a q-analogue of the modified α-Bernstein operators introduced by Kajla and Acar (*Ann. Funct. Anal.*, 10 (2019), 570–582). We study uniform convergence theorem and the Voronovskaja type asymptotic theorem. We determine the estimate of error in the approximation by these operators by virtue of second order modulus of continuity via the approach of Steklov means and the technique of Peetre's K-functional. Next, we investigate the Grüss-Voronovskaya type theorem. Further, we define a bivariate tensor product of these operatos and derive the convergence estimates by utilizing the partial and total moduli of continuity. The approximation degree by means of Peetre's K-functional, the Voronovskaja and Grüss-Voronovskaja type theorems are also investigated. Finally, we construct the associated *GBS* (Generalized Boolean Sum) operator and examine its convergence behavior by virtue of the mixed modulus of smoothness.

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1. INTRODUCTION

In 1912, Bernstein [9] gave a simple and elegant proof of the Weierstrass approximation theorem by defining a sequence of positive linear operators as follows.

For $\zeta \in \mathcal{C}(\mathcal{I}) = \{f : \mathcal{I} \to \mathbb{R}|\ f \textit{ is continuous}\}$, $\mathcal{I} = [0,1]$, the n-th Bernstein polynomial $\mathcal{B}_n : \mathcal{C}(\mathcal{I}) \to \mathcal{C}(\mathcal{I})$ is defined by

$$\mathcal{B}_n(\zeta; x) = \sum_{k=0}^{n} \tbinom{n}{k} x^k (1-x)^{n-k} \zeta\big(\tfrac{k}{n}\big), \qquad \forall x \in \mathcal{I} \textit{ and } n \in \mathbb{N}. \tag{1.1}$$

Later several authors introduced various generalizations of these polynomials and studied their approximation properties (see [22]).

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In the past two decades, the development of q-calculus has been an active area of research. In 1987, Lupas [29] was the first person who introduced a q-analogue of the Bernstein polynomials and established some approximation results. A decade later, Phillips [35] gave another q-analogue, $\mathcal{L}_{n,q}:\mathcal{C}(\mathcal{I})\to\mathcal{C}(\mathcal{I})$ of the Bernstein polynomials which became very popular. He defined it as

$$\mathcal{L}_{n,q}(\zeta;x)=\sum_{k=0}^{n}\zeta\left(\frac{[k]_q}{[n]_q}\right)\mathfrak{p}_{n,k}(q;x), \tag{1.2}$$

where $0<q<1$ and $\mathfrak{p}_{n,k}(q;x)=\binom{n}{k}_q x^k(1-x)_q^{n-k},\ x\in\mathcal{I}$. Several generalizations have been studied for the definition of q-Bernstein polynomials given by (1.2), for instance, Muraru [31] proposed a q-analogue of the Bernstein-Schurer operators and examined convergence behaviour by virtue of the modulus of continuity. Agrawal *et al.* [2] defined Bernstein-Schurer-Stancu operators based on q-integers and discussed the local and global results. Dalmanoğlu [33] gave a q-analogue of the Bernstein-Kantorovich polynomials. Gupta [22] introduced a sequence of Bernstein-Durrmeyer operators based on q-integers and established some approximation theorems.

Gupta and Radu [24] discussed statistical approximation properties of another kind of q-Baskakov-Kantorovich operators. Mahmudov [30] introduced an alternate form of the q-analogue of Bernstein-Kantorovich operators considered in [33]. Aliaga and Baxhaku [4] defined a bivariate extension of the q-Bernstein type operators involving parameter λ and examined their degree of approximation.

Recently, Mursaleen *et al.* [32] proposed a sequence of generalized Bernstein operators based on q-integers. Cai [12] introduced another generalization of Bernstein operators based on q-integers and derived some convergence theorems and shape preserving properties. Based on the Phillips q-Bernstein polynomials [35], generalized Bézier curves and surfaces were introduced in ([15], [16], [34]). For a detailed account of the work in this direction, we refer the reader to the book [5]. Before proceeding further, let us mention some important basic definitions and notations of q-calculus. For $q>0$, and each nonnegative integer l, the q-integer $[l]_q$ and the q-factorial $[l]_q!$ are, respectively, given by

$$[l]_q=\begin{cases}\frac{(1-q^l)}{(1-q)}, & q\neq 1,\\ l, & q=1.\end{cases}$$

and

$$[l]_q!=\begin{cases}[l]_q[l-1]_q\ldots[1]_q, & l\geq 1,\\ 1, & l=0.\end{cases}$$

For the non-negative integers n,l satisfying $l\leq n$, the q-binomial coefficients are defined by

$$\binom{n}{l}_q:=\frac{[n]_q!}{[l]_q![n-l]_q!}.$$

In 2017, Chen *et al.* [13] introduced generalized Bernstein operators (1.1) involving a real parameter 'α' satisfying $0 \leq \alpha \leq 1$, as

$$\mathcal{L}_{n,\alpha}(\zeta;x) = \sum_{k=0}^{n} P_{n,k,\alpha}(x)\zeta(\tfrac{k}{n}),\ x \in \mathcal{I}, \tag{1.3}$$

where the α-Bernstein basis function $P_{n,k,\alpha}(x)$, for $n \geq 2$, is given by

$$\begin{aligned} P_{n,k,\alpha}(x) = &\tbinom{n-2}{k}(1-\alpha)x^k(1-x)^{n-k-1} \\ &+ \tbinom{n-2}{k-2}(1-\alpha)x^{k-1}(1-x)^{n-k} \\ &+ \tbinom{n}{k}\alpha x^k(1-x)^{n-k} \end{aligned} \tag{1.4}$$

with $\binom{n-2}{-2} = \binom{n-2}{-1} = 0$. Clearly $P_{n,k,\alpha}(x)$, verifies the following recurrence relation

$$P_{n,k,\alpha}(x) = (1-x)P_{n-1,k,\alpha}(x) + xP_{n-1,k-1,\alpha}(x),\ \forall\ \ 0 < k < n,\ \textit{and}\ \ n \geq 3.$$

The authors [13] established the uniform convergence theorem and the Voronovskaja-type asymptotic formula etc. For the special case $\alpha = 1$, the operators (1.3) reduce to (1.1).

For any $\zeta \in \mathcal{C}(\mathcal{I})$, Khosravian-Arab *et al.* [27] presented a new family of Bernstein operators as follows:

$$B_n^{M,1}(\zeta;x) \ = \ \sum_{m=0}^{n} P_{n,m}^{M,1}(x)\zeta(\tfrac{m}{n}),\ \ x \in \mathcal{I}, \tag{1.5}$$

where

$$\begin{aligned} &P_{n,m}^{M,1}(x) = a(x,n)P_{n-1,m}^{M,1}(x) + a(1-x,n)P_{n-1,m-1}^{M,1}(x),\ \ 1 \leq m \leq n-1, \\ &P_{n,0}^{M,1}(x) = a(x,n)(1-x)^{n-1}, \\ &P_{n,n}^{M,1}(x) = a(1-x,n)x^{n-1}, \end{aligned}$$

and

$$a(x,n) = a_0(n) + x\ a_1(n),\ \ n = 0,1,2,3\dots \tag{1.6}$$

$a_0(n)$ and $a_1(n)$ being two unknown sequences, which may be defined in an appropriate manner. If $a_0(n) = 1$, and $a_1(n) = -1$, then (1.5) reduces to (1.1).

Recently, Kajla and Acar [26] for any $\zeta \in \mathcal{C}(\mathcal{I})$, constructed a new family of α-Bernstein operators defined by

$$\mathcal{G}_{n,\alpha}^{M,1}(\zeta;x) = \sum_{k=0}^{n} P_{n,k,\alpha}^{M,1}(x)\zeta\left(\frac{k}{n}\right), \tag{1.7}$$

where $P_{n,k,\alpha}^{M,1}(x) = a(x,n)P_{n-1,k,\alpha}(x) + a(1-x,n)P_{n-1,k-1,\alpha}(x)$ and $a(x,n) = a_1(n)x + a_0(n)$, and investigated some approximation properties such as Korovkin type theorem and a Voronovskaja type asymptotic formula. Clearly, for $a_0(n) = 1$, and $a_1(n) = -1$, (1.7) includes (1.3).

In 2019, Gupta *et al.* [23] presented a Kantorovich variant of the operators (1.5) and determined various approximation results.

Motivated by the above research, for $\zeta \in \mathcal{C}(\mathcal{I})$ endowed with the uniform norm $\|.\|$, we define the q-analogue of the operators (1.7) as follows:

$$\mathcal{G}_{n,\alpha}^{M,1}(\zeta,q;x)=\sum_{k=0}^{n}P_{n,k,\alpha}^{M,1}(q;x)\zeta\left(\tfrac{[k]_q}{[n]_q}\right), \tag{1.8}$$

where $P_{n,k,\alpha}^{M,1}(q;x)=a(x,n,q)P_{n-1,k,\alpha}(q;x)+a(1-x,n,q)P_{n-1,k-1,\alpha}(q,x)$ with $P_{n,k,\alpha}(q;x)=\binom{n-2}{k}_q(1-\alpha)(qx)^k(1-x)_q^{n-k-1}+\binom{n-2}{k-2}_q(1-\alpha)q^{k-2}x^{k-1}(1-qx)_q^{n-k}+\binom{n}{k}_q\alpha(qx)^k(1-qx)_q^{n-k}$ and $a(x,n,q)=a_1(n,q)x+a_0(n,q)$. For $a_1(n,q)=-1$ and $a_0(n,q)=1$ and $\alpha=1$, we get q-Bernstein polynomials given by (1.2).

The purpose of the given article is to examine the convergence behavior of the operators (1.8) by virtue of the Lipschitz class and the K-functional. Next, we study a bivariate extension of these operators and determine the convergence estimates by virtue of the moduli of continuity and the K-functional. Further, we study the Voronovskaja and Grüss Voronovskaja type theorems and establish a quantitative result for functions in the Lipschitz class. Lastly, we extend our study to the corresponding GBS operators.

2. PRELIMINARIES

Throughout this paper, we assume that $2a_0(n,q)+a_1(n,q)=1$.

LEMMA 2.1. *For $e_j(\hbar)=\hbar^j, j=0,1,2,$ the moments of the operators (1.8), are given by*

i) $\mathcal{G}_{n,\alpha}^{M,1}(e_0;q;x)=1,$

ii) $\mathcal{G}_{n,\alpha}^{M,1}(e_1;q;x)=\frac{1}{[n]_q}\Big[a(x,n,q)\{(1-\alpha)[n-3]_q qx(1-x)+(1-\alpha)x([2]_q+q^3[n-3]_qx)+\alpha[n-1]_qqx\}+a(1-x,n,q)\{(1-\alpha)(1-x)(1+q^2[n-3]_qx)+(1-\alpha)x([3]_q+q^4[n-3]_qx)+\alpha(1+q^2[n-1]_qx)\}\Big],$

iii) $\mathcal{G}_{n,\alpha}^{M,1}(e_2,q;x)=\frac{a(x,n,q)}{[n]_q^2}\Big[(1-\alpha)[n-3]_qqx(1-x)+(1-\alpha)[n-3]_q[n-4]_qq^3x^2(1-x)+(1-\alpha)x[2]_q^2+2[2]_q(1-\alpha)q^3x^2[n-3]_q+(1-\alpha)q^5x^2[n-3]_q+(1-\alpha)q^7x^3[n-3]_q[n-4]_q+\alpha[n-1]_qqx+\alpha[n-1]_q[n-2]_qq^3x\}+\frac{a(1-x,n,q)}{[n]_q^2}\Big[(1-\alpha)(1-x)+2(1-\alpha)q^2x(1-x)[n-3]_q+(1-\alpha)q^3x(1-x)[n-3]_q+(1-\alpha)q^5x^2(1-x)[n-3]_q[n-4]_q+(1-\alpha)x[3]_q^2+2(1-\alpha)[3]_qq^4x^2[n-3]_q+q^7(1-\alpha)x^2[n-3]_q+q^9(1-\alpha)x^3[n-3]_q[n-4]_q+\alpha+2q^2\alpha x[n-1]_q+\alpha q^3x[n-1]_q+\alpha q^5x^2[n-1]_q[n-2]_q\Big].$

From the above lemma, one can obtain:

LEMMA 2.2. *The central moments of the operators* (1.8), *for* $\phi_x^j(\hbar) = (\hbar - x)^j$, *where j=1,2 are given by*

i) $\mathcal{G}_{n,\alpha}^{M,1}(\phi_x^1; q; x) = \frac{1}{[n]_q}\Big[a(x,n,q)\{(1-\alpha)[n-3]_q qx(1-x) + (1-\alpha)x([2]_q + q^3[n-3]_q x) + \alpha[n-1]_q qx\} + a(1-x,n,q)\{(1-\alpha)(1-x)(1+q^2[n-3]_q x) + (1-\alpha)x([3]_q + q^4[n-3]_q x) + \alpha(1+q^2[n-1]_q x)\} - [n]_q x\Big]$,

ii) $\mathcal{G}_{n,\alpha}^{M,1}(\phi_x^2; q; x) = \frac{a(x,n,q)}{[n]_q^2}\Big[(1-\alpha)[n-3]_q 2x(1-x) + (1-\alpha)[n-3]_q[n-4]_q q^3x^2(1-x) + (1-\alpha)x[2]_q^2 + 2[2]_q(1-\alpha)q^3x^2[n-3]_q + (1-\alpha)q^5x^2[n-3]_q + (1-\alpha)q^7x^3[n-3]_q[n-4]_q + \alpha[n-1]_q qx + \alpha[n-1]_q[n-2]_q q^3x^2 - 2x^2(1-x)q(1-\alpha)[n-3]_q[n]_q - 2(1-\alpha)x^2[n]_q([2]_q + q^3[n-3]_q x) - 2\alpha[n]_q[n-1]_q qx^2 + [n]_q^2x^2\Big] + \frac{a(1-x,n,q)}{[n]_q^2}\Big[(1-\alpha)(1-x) + 2(1-\alpha)q^2x(1-x)[n-3]_q + (1-\alpha)q^3x(1-x)[n-3]_q + (1-\alpha)q^5x^2(1-x)[n-3]_q[n-4]_q + (1-\alpha)x[3]_q^2 + 2(1-\alpha)[3]_q q^4x^2[n-3]_q + q^7(1-\alpha)x^2[n-3]_q + q^9(1-\alpha)x^3[n-3]_q[n-4]_q + \alpha + 2q^2\alpha x[n-1]_q + \alpha q^3x[n-1]_q + \alpha q^5x^2[n-1]_q[n-2]_q - 2x[n]_q(1-\alpha)(1-x)(1+q^2[n-3]_q x) - 2x^2(1-\alpha)([3]_q + q^4[n-3]_q x)[n]_q - 2x\alpha(1+q^2[n-1]_q x)[n]_q + [n]_q^2x^2\Big]$.

In what follows, let (q_n) be a sequence in $(0,1)$ such that

$$\lim_{n\to\infty} q_n = 1, \quad \lim_{n\to\infty} q_n^n = c, \quad 0 \le c < 1,$$
$$\lim_{n\to\infty} a_1(n,q_n) = p_1 \textit{ and } \lim_{n\to\infty} a_0(n,q_n) = p_0.$$

REMARK 2.3. From Lemma 2.2, after simple calculations, one has

(i) $\lim_{n\to\infty}[n]_{q_n}\mathcal{G}_{n,\alpha}^{M,1}(\phi_x^1; q_n; x) =$

$$= (1-c)\Big[x^2(p_1(2\alpha-1) - 4p_0(1-\alpha)) + x((1-2\alpha) + p_0(1+\alpha))\Big] + (p_0+p_1)(1-2x),$$

(ii) $\lim_{n\to\infty}[n]_{q_n}\mathcal{G}_{n,\alpha}^{M,1}(\phi_x^2; q_n; x) =$

$$= p_1\Big\{4(1-c)x^3(1-\alpha)(1-x) + 6x^4 - 7x^3 + 6\alpha x^3(1-x) + x^2 + x(1-x)^2\Big\} + p_0\Big\{6\alpha x^2(1-x) + 6x^3 - 8x^2 + 2x + 4(1-c)x^2(1-\alpha)(1-x)\Big\}.$$

Similarly, by carrying out further calculations it can be seen that

$$\mathcal{G}_{n,\alpha}^{M,1}(\phi_x^4; q_n; x) = \mathcal{O}\left([n]_{q_n}^{-2}\right), \ \textit{as } n \to \infty. \qquad \square$$

LEMMA 2.4. *For* $\zeta \in \mathcal{C}(\mathcal{I})$ *and* $x \in \mathcal{I}$, *we have* $|\mathcal{G}_{n,\alpha}^{M,1}(\zeta, q_n; x)| \le \|\zeta\|$.

Proof. Using Lemma 2.1, for every $x \in \mathcal{I}$ we have,

$$|\mathcal{G}_{n,\alpha}^{M,1}(\zeta, q_n; x)| = \sum_{k=0}^{n} p_{n,k,\alpha}^{M,1}(q; x)\Big|\zeta\left(\frac{[k]_{q_n}}{[n]_{q_n}}\right)\Big| \leq \|\zeta\|\mathcal{G}_{n,\alpha}^{M,1}(1, q_n; x) = \|\zeta\|.$$

□

3. RATE OF CONVERGENCE OF THE OPERATORS $\mathcal{G}_{n,\alpha}^{M,1}(., q_n; x)$

First, we prove the uniform convergence theorem for the operators (1.8).

THEOREM 3.1. *Let $\zeta \in \mathcal{C}(\mathcal{I})$. Then $\lim_{n\to\infty} \mathcal{G}_{n,\alpha}^{M,1}(\zeta, q_n; x) = \zeta(x)$, uniformly in $x \in \mathcal{I}$.*

Proof. From Lemma 2.1, it follows that $\mathcal{G}_{n,\alpha}^{M,1}(e_i, q; x) \to e_i(x)$, as n $\to \infty$, uniformly in $\mathcal{I}$, for $i = 0, 1, 2$. Hence applying Bohman-Korovkin Theorem [21], we obtain the required result. □

Next, we establish a Voronovskaja type asymptotic result for the operators (1.8).

THEOREM 3.2. *If $\zeta \in \mathcal{C}^2(\mathcal{I})$, then*

$$\lim_{n\to\infty} [n]_{q_n}[\mathcal{G}_{n,\alpha}^{M,1}(\zeta, q_n; x) - \zeta(x)] =$$

$$= \Bigg\{(1-c)\Bigg(x^2(p_1(2\alpha - 1) - 4p_0(1-\alpha)) + x((1-2\alpha) + p_0(1+\alpha))\Bigg)$$

$$+ (p_0 + p_1)(1-2x)\Bigg\}\zeta'(x) + \Bigg[p_1\Bigg\{4(1-c)x^3(1-\alpha)(1-x) + 6x^4 - 7x^3$$

$$+6\alpha x^3(1-x)+x^2+x(1-x)^2\Bigg\} + p_0\Bigg\{4(1-c)x^2(1-\alpha)(1-x) + 6x^3 - 8x^2$$

$$+ 6\alpha x^2(1-x) + 2x\Bigg\}\Bigg]\frac{\zeta''(x)}{2},$$

uniformly in $x \in \mathcal{I}$.

Proof. By the Taylor's expansion of ζ about the point $\hbar = x$, we have

$$\zeta(\hbar) = \zeta(x) + \zeta'(x)(\hbar - x) + \tfrac{1}{2}\zeta''(x)(\hbar - x)^2 + \tfrac{1}{2}(\hbar - x)^2\{\zeta''(\theta) - \zeta''(x)\},$$

where θ lies between $\hbar$ and x. Operating by $\mathcal{G}_{n,\alpha}^{M,1}(., q_n; x)$ in the above equation, we obtain

$$\mathcal{G}_{n,\alpha}^{M,1}(\zeta, q_n; x) - \zeta(x) = \mathcal{G}_{n,\alpha}^{M,1}((\hbar - x); q_n; x)\zeta'(x) + \mathcal{G}_{n,\alpha}^{M,1}\left((\hbar - x)^2; q_n; x\right)\tfrac{1}{2}\zeta''(x)$$

$$+ \tfrac{1}{2}\mathcal{G}_{n,\alpha}^{M,1}\left((\hbar - x)^2\{\zeta''(\theta) - \zeta''(x)\}; q_n; x\right). \tag{3.1}$$

Using the well known properties of modulus of continuity, for any $\rho > 0$, we have

$$|\zeta''(\theta) - \zeta''(x)| \leq \omega(\zeta''; |\theta - x|) \leq \left(1 + \frac{|\hbar - x|}{\rho}\right)\omega(\zeta''; \rho).$$

Therefore,

$$\begin{aligned}&\left|\mathcal{G}_{n,\alpha}^{M,1}\left((\hbar-x)^2\{\zeta''(\theta)-\zeta''(x)\};q_n;x\right)\right|\leq\\&\leq\mathcal{G}_{n,\alpha}^{M,1}\left((\hbar-x)^2|\zeta''(\theta)-\zeta''(x)|;q_n;x\right)\\&\leq\omega(\zeta'';\rho)\mathcal{G}_{n,\alpha}^{M,1}\left(((\hbar-x)^2+\tfrac{1}{\rho}|\hbar-x|^3);q_n;x\right),\rho>0.\end{aligned}$$

Using Cauchy-Schwarz inequality, Remark 2.3 and choosing $\rho=[n]_{q_n}^{-\frac{1}{2}}$, we get

$$\begin{aligned}&\left|\mathcal{G}_{n,\alpha}^{M,1}\left((\hbar-x)^2\{\zeta''(\theta)-\zeta''(x)\};q_n;x\right)\right|\leq\\&\leq\omega(\zeta'';\rho)\Big[\mathcal{G}_{n,\alpha}^{M,1}\left((\hbar-x)^2;q_n;x\right)\\&\quad+\tfrac{1}{\rho}\sqrt{\mathcal{G}_{n,\alpha}^{M,1}((\hbar-x)^2;q_n;x)}\sqrt{\mathcal{G}_{n,\alpha}^{M,1}((\hbar-x)^4;q_n;x)}\Big]\\&=\omega(\zeta'';\rho)\left[\mathcal{O}\left(\tfrac{1}{[n]_{q_n}}\right)+\tfrac{1}{\rho}\mathcal{O}\left(\frac{1}{[n]_{q_n}^{\frac{1}{2}}}\right)\mathcal{O}\left(\tfrac{1}{[n]_{q_n}}\right)\right]\\&=\omega(\zeta'';[n]_{q_n}^{-\frac{1}{2}})\mathcal{O}\left(\tfrac{1}{[n]_{q_n}}\right),\end{aligned}$$

as $n\to\infty$, uniformly in $x\in\mathcal{I}$.

Hence,

$$[n]_{q_n}\left|\mathcal{G}_{n,\alpha}^{M,1}\left((\hbar-x)^2\{\zeta''(\theta)-\zeta''(x)\};q_n;x\right)\right|=\omega\left(\zeta'';[n]_{q_n}^{-\frac{1}{2}}\right)\mathcal{O}(1),$$

as $n\to\infty$, uniformly in $x\in\mathcal{I}$.

Consequently,

$$\lim_{n\to\infty}[n]_{q_n}\mathcal{G}_{n,\alpha}^{M,1}\left((\hbar-x)^2\{\zeta''(\theta)-\zeta''(x)\};q_n;x\right)=0,$$

uniformly in $x\in\mathcal{I}$. Thus, from equation (3.1) and Remark 2.3, we get

$$\begin{aligned}&\lim_{n\to\infty}[n]_{q_n}\left[\mathcal{G}_{n,\alpha}^{M,1}(\zeta,q_n;x)-\zeta(x)\right]=\\&=\lim_{n\to\infty}[n]_{q_n}\Big\{\mathcal{G}_{n,\alpha}^{M,1}((\hbar-x),q_n;x)\zeta'(x)+\mathcal{G}_{n,\alpha}^{M,1}\left((\hbar-x)^2,q_n;x\right)\tfrac{1}{2}\zeta''(x)\\&\quad+\tfrac{1}{2}\mathcal{G}_{n,\alpha}^{M,1}\left((\hbar-x)^2\{\zeta''(\theta)-\zeta''(x)\};q_n;x\right)\Big\}\\&=\Big\{(1-c)\Big(x^2(p_1(2\alpha-1)-4p_0(1-\alpha))+x((1-2\alpha)+p_0(1+\alpha))\Big)\\&\quad+(p_0+p_1)(1-2x)\Big\}\zeta'(x)+\Big[p_1\Big\{4(1-c)x^3(1-\alpha)(1-x)+6x^4-7x^3\\&\quad+6\alpha x^3(1-x)+x^2+x(1-x)^2\Big\}+p_0\Big\{4(1-c)x^2(1-\alpha)(1-x)+6x^3-8x^2\end{aligned}$$

$$+ 6\alpha x^2(1-x) + 2x\Big\}\Big]\frac{\zeta''(x)}{2},$$

uniformly in $x \in \mathcal{I}$, which completes the proof. □

Now, we obtain an approximation theorem for the operators (1.8) by virtue of the second order modulus of continuity by using a smoothing process, *e.g.* Steklov mean. For $\zeta \in \mathcal{C}(\mathcal{I})$, the Steklov mean is defined as

$$\zeta_h(x) = \frac{4}{h^2}\int_0^{\frac{h}{2}}\int_0^{\frac{h}{2}}[2\zeta(x+u+v) - \zeta(x+2(u+v))]dudv. \tag{3.2}$$

It is known that for the function $\zeta_h(x)$, there hold the following properties:

i) $\|\zeta_h - \zeta\| \leq \omega_2(\zeta, h)$

ii) $\zeta_h', \zeta_h'' \in \mathcal{C}(\mathcal{I})$ *and* $\|\zeta_h'\| \leq \frac{5}{h}\omega(\zeta, h)$, $\|\zeta_h''\| \leq \frac{9}{h^2}\omega_2(\zeta, h)$,

where $\omega(\zeta; h)$ and $\omega_2(\zeta; h)$ denote the first and second order modulus of continuity. In what follows, for all $x \in \mathcal{I}$, let $\mu_{n,q_n}^{\alpha,m}(x) := \mathcal{G}_{n,\alpha}^{M,1}\left(\phi_x^m(\hbar); q_n; x\right), m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\nu_{n,q_n}^{\alpha,m} := \sup_{x\in\mathcal{I}}\left|\mu_{n,q_n}^{\alpha,m}(x)\right|$, $m \in \mathbb{N}$.

Theorem 3.3. *Let $\zeta \in \mathcal{C}(\mathcal{I})$. Then following inequality holds:*

$$\|\mathcal{G}_{n,\alpha}^{M,1}(\zeta; q_n.) - \zeta\| \leq 5\omega\left(\zeta; \sqrt{\nu_{n,q_n}^{\alpha,2}}\right) + \frac{13}{2}\omega_2\left(\zeta; \sqrt{\nu_{n,q_n}^{\alpha,2}}\right).$$

Proof. Using the Steklov mean $\zeta_h(x)$ given by (3.2), we may write

$$|\mathcal{G}_{n,\alpha}^{M,1}(\zeta; q_n; x) - \zeta(x)| \leq \mathcal{G}_{n,\alpha}^{M,1}(|\zeta - \zeta_h|; q_n; x+ \\ + |\mathcal{G}_{n,\alpha}^{M,1}(\zeta_h; q_n; x) - \zeta_h(x)| + |\zeta_h(x) - \zeta(x)|. \tag{3.3}$$

Hence using Lemma 2.4 and property (a) of Steklov mean, we have

$$\mathcal{G}_{n,\alpha}^{M,1}(|\zeta - \zeta_h|; q_n; x) \leq \|\zeta - \zeta_h\| \leq \omega_2(\zeta, h). \tag{3.4}$$

Since $\zeta_h'' \in \mathcal{C}(\mathcal{I})$, by Taylor's expansion we have

$$\zeta_h(\hbar) = \zeta_h(x) + (\hbar - x)\zeta_h'(x) + \frac{(\hbar-x)^2}{2!}\zeta''(\theta),$$

where θ lies between $\hbar$ and x. Then, applying Cauchy-Schwarz inequality

$$|\mathcal{G}_{n,\alpha}^{M,1}(\zeta_h(\hbar) - \zeta_h(x); q_n; x)| \leq \|\zeta_h'\|\sqrt{\nu_{n,q_n}^{\alpha,2}} + \frac{1}{2}\|\zeta_h''\|\nu_{n,q_n}^{\alpha,2}.$$

Now, applying property (b) of Steklov mean, we obtain

$$|\mathcal{G}_{n,\alpha}^{M,1}(\zeta_h(\hbar) - \zeta_h(x); q_n; x)| \leq \frac{5}{h}\omega(\zeta, h)\sqrt{\nu_{n,q_n}^{\alpha,2}} + \frac{9}{2h^2}\omega_2(\zeta, h)\nu_{n,q_n}^{\alpha,2}. \tag{3.5}$$

Choosing $h = \sqrt{\nu_{n,q_n}^{\alpha,2}}$ and combining the equations (3.3)–(3.5), we get the desired result. □

Theorem 3.4. *For any $\zeta' \in \mathcal{C}(\mathcal{I})$, we have*

$$\|\mathcal{G}_{n,\alpha}^{M,1}(\zeta) - \zeta\| \leq |\nu_{n,q_n}^{\alpha,1}|\|\zeta'\| + 2\sqrt{\nu_{n,q_n}^{\alpha,2}}\,\omega\left(\zeta', \frac{1}{2}\sqrt{\nu_{n,q_n}^{\alpha,2}}\right).$$

Proof. Since $\zeta^{'} \in \mathcal{C}(\mathcal{I})$, for any $\hbar, x \in \mathcal{I}$, we can write

$$\zeta(\hbar) - \zeta(x) = \zeta'(x)(\hbar - x) + \int_x^{\hbar} (\zeta'(u) - \zeta'(x))du.$$

Applying the operator $\mathcal{G}_{n,\alpha}^{M,1}(.;q_n;x)$ on both sides of the above relation, we get

$$\mathcal{G}_{n,\alpha}^{M,1}(\zeta(\hbar) - \zeta(x); q_n; x) = \zeta'(x)\mathcal{G}_{n,\alpha}^{M,1}((\hbar - x); q_n; x) + \mathcal{G}_{n,\alpha}^{M,1}\left(\int_x^{\hbar}(\zeta'(u) - \zeta'(x))du; q_n; x\right).$$

Using the well known property of modulus of continuity

$$|\zeta'(u) - \zeta'(x)| \leq \omega(\zeta', \rho)\left(\tfrac{|u-x|}{\rho} + 1\right), \rho > 0,$$

we obtain

$$\left|\int_x^{\hbar}(\zeta'(u) - \zeta'(x))du\right| \leq \left|\int_x^{\hbar} |\zeta'(u) - \zeta'(x)|du\right| \leq \left|\int_x^{\hbar}\left(1 + \tfrac{|u-x|}{\rho}\right)\omega(\zeta', \rho)du\right| = \omega(\zeta'; \rho)\left(|\hbar - x| + \tfrac{(\hbar-x)^2}{2\rho}\right).$$

Therefore,

$$|\mathcal{G}_{n,\alpha}^{M,1}(\zeta; q_n; x) - \zeta(x)| \leq \leq |\zeta'(x)||\mathcal{G}_{n,\alpha}^{M,1}((\hbar - x), q_n; x)| + \omega(\zeta', \rho)\left\{\tfrac{1}{2\rho}\mathcal{G}_{n,\alpha}^{M,1}((\hbar - x)^2, q_n; x) + \mathcal{G}_{n,\alpha}^{M,1}(|\hbar - x|; q_n; x)\right\}.$$

Using Cauchy-Schwarz inequality, we have

$$|\mathcal{G}_{n,\alpha}^{M,1}(\zeta; q_n; x) - \zeta(x)| \leq \leq |\zeta'(x)||\mathcal{G}_{n,\alpha}^{M,1}((\hbar - x); q_n; x)| + \omega(\zeta'; \rho)\left\{\tfrac{1}{2\rho}\sqrt{\mathcal{G}_{n,\alpha}^{M,1}((\hbar - x)^2, q_n; x)} + 1\right\}\sqrt{\mathcal{G}_{n,\alpha}^{M,1}((\hbar - x)^2, q_n; x)}.$$

Choosing $\rho = \frac{1}{2}\sqrt{\nu_{n,q_n}^{\alpha,2}}$, the required assertion is obtained. □

The following theorem is concerned with an estimate of error in the approximation by the operators (1.8) by means of the Peetre's K-functional.

THEOREM 3.5. *For all $\zeta \in \mathcal{C}(\mathcal{I})$ and $n \in \mathbb{N}$, there exists a constant $C > 0$ such that*

$$|\mathcal{G}_{n,\alpha}^{M,1}(\zeta; q_n; x) - \zeta(x)| \leq C\omega_2\left(\zeta; \frac{\sqrt{\phi_{n,q_n}^{\alpha}(x)}}{2}\right) + \omega\left(\zeta; \left|\mu_{n,q_n}^{\alpha,1}(x)\right|\right),$$

where $\phi^{\alpha}_{n,q_n}(x) = \frac{1}{2}\left\{\mu^{\alpha,2}_{n,q_n}(x) + (\mu^{\alpha,1}_{n,q_n}(x))^2\right\}$.

Proof. For $x \in \mathcal{I}$, consider the following auxiliary operator given by

$$\mathcal{G}^{M,1*}_{n,\alpha}(\zeta; q_n; x) \quad = \mathcal{G}^{M,1}_{n,\alpha}(\zeta; q_n; x) + \zeta(x) - \zeta\left(\mathcal{G}^{M,1}_{n,\alpha}(\hbar; q_n; x)\right). \tag{3.6}$$

It is clear that the operator $\mathcal{G}^{M,1*}_{n,\alpha}(\zeta; q_n; x)$ is linear, $\mathcal{G}^{M,1*}_{n,\alpha}(1; q_n; x) = 1$ and

$$\mathcal{G}^{M,1*}_{n,\alpha}(\hbar - x; q_n; x) = \mathcal{G}^{M,1}_{n,\alpha}(\hbar - x; q_n; x) - \left(\mathcal{G}^{M,1}_{n,\alpha}(\hbar; q_n; x) - x\right) = 0.$$

For every $\varphi \in \mathcal{C}^2(\mathcal{I})$ and $\hbar, x \in \mathcal{I}$, from the Taylor's theorem, one can write

$$\varphi(\hbar) = \varphi(x) + (\hbar - x)\varphi'(x) + \int_x^{\hbar} (\hbar - u)\varphi''(u)du.$$

Applying $\mathcal{G}^{M,1*}_{n,\alpha}(.; q; x)$ to the above equation and using (3.6), we obtain

$$\begin{aligned}\mathcal{G}^{M,1*}_{n,\alpha}(\varphi; q_n; x) =& \varphi(x) + \mathcal{G}^{M,1}_{n,\alpha}\left(\int_x^{\hbar}(\hbar - u)\varphi''(u)du; q_n; x\right)\\ &- \int_x^{\mathcal{G}^{M,1}_{n,\alpha}(\hbar;q_n;x)}\left(\mathcal{G}^{M,1}_{n,\alpha}(\hbar; q_n; x) - u\right)\varphi''(u)du.\end{aligned}$$

Thus, for all $x \in \mathcal{I}$, we have

$$\begin{aligned}&|\mathcal{G}^{M,1*}_{n,\alpha}(\varphi; q; x) - \varphi(x)| \le\\ &\le \left|\mathcal{G}^{M,1}_{n,\alpha}\left(\int_x^{\hbar}(\hbar - u)\varphi''(u)du; q_n; x\right)\right|\\ &\quad + \left|\int_x^{\mathcal{G}^{M,1}_{n,\alpha}(\hbar;q_n;x)}\left(\mathcal{G}^{M,1}_{n,\alpha}(\hbar; q_n; x) - u\right)\varphi''(u)du\right|\\ &\le \mathcal{G}^{M,1}_{n,\alpha}\left(\left|\int_x^{\hbar}|\hbar - u||\varphi''(u)|du\right|; q_n; x\right)\\ &\quad + \left|\int_x^{\mathcal{G}^{M,1}_{n,\alpha}(\hbar;q_n;x)}\left|\mathcal{G}^{M,1}_{n,\alpha}(\hbar; q_n; x) - u\right| \cdot \left|\varphi''(u)\right|du\right|\end{aligned}$$

$$\le \frac{\|\varphi''\|}{2}\left[\mathcal{G}^{M,1}_{n,\alpha}((\hbar - x)^2; q; x) + \left(\mathcal{G}^{M,1}_{n,\alpha}(\hbar; q_n; x) - x\right)^2\right] = \|\varphi''\|\phi^{\alpha}_{n,q_n}(x). \tag{3.7}$$

Further, for all $x \in \mathcal{I}$, in view of Lemma 2.4, we get

(3.8)

$$|\mathcal{G}^{M,1*}_{n,\alpha}(\zeta; q_n; x)| \le |\mathcal{G}^{M,1}_{n,\alpha}(\zeta; q_n; x)| + |\zeta(x)| + \left|\zeta\left(\mathcal{G}^{M,1}_{n,\alpha}(\hbar; q_n; x)\right)\right| \le 3\|\zeta\|.$$

Now, for $\zeta \in \mathcal{C}(\mathcal{I})$ and any $\varphi \in \mathcal{C}^2(\mathcal{I})$, using (3.6)–(3.8), we obtain

$|\mathcal{G}^{M,1}_{n,\alpha}(\zeta; q_n; x) - \zeta(x)| \le |\mathcal{G}^{M,1*}_{n,\alpha}(\zeta - \varphi; q_n; x) - (\zeta - \varphi)(x)|$

$$+\left|\zeta\left(\mathcal{G}_{n,\alpha}^{M,1}(\hbar;q_n;x)\right)-\zeta(x)\right|+|\mathcal{G}_{n,\alpha}^{M,1*}(\varphi;q_n;x)-\varphi(x)|$$
$$\leq 4\left(\|\zeta-\varphi\|+\tfrac{1}{4}\phi_{n,q_n}^{\alpha}(x)\|\varphi''\|\right)+\omega\left(\zeta;|\mu_{n,q_n}^{\alpha,1}(x)|\right).$$

Now, taking the infimum on the right hand side over all $\varphi\in\mathcal{C}^2(\mathcal{I})$,

$$|\mathcal{G}_{n,\alpha}^{M,1}(\zeta;q_n;x)-\zeta(x)|\leq 4K_2\left(\zeta;\frac{\phi_{n,q_n}^{\alpha}(x)}{4}\right)+\omega\left(\zeta;|\mu_{n,q_n}^{\alpha,1}(x)|\right),$$

where $K_2(\zeta,\rho)=\inf\left\{\|\zeta-\varphi\|+\rho\,\|\varphi''\|:\varphi\in\mathcal{C}^2(\mathcal{I})\right\}.$
Finally, using the relation between K-functional and second order modulus of continuity [14] given by

$$K_2(\zeta,\rho)\leq C'\omega_2(\zeta;\sqrt{\rho}),$$

C' being some constant, we get

$$|\mathcal{G}_{n,\alpha}^{M,1}(\zeta;q_n;x)-\zeta(x)|\leq C\omega_2\left(\zeta;\frac{\sqrt{\phi_{n,q_n}^{\alpha}(x)}}{2}\right)+\omega\left(\zeta;|\mu_{n,q_n}^{\alpha,1}(x)|\right),$$

This completes the proof. □

In our next result, we discuss a convergence estimate by the operators (1.8) for the continuous functions on $\mathcal{I}$ belonging to the Lipschitz class. Let

$$\mathrm{Lip}_\theta\, M=\left\{\zeta\in\mathcal{C}(\mathcal{I}):|\zeta(\hbar)-\zeta(x)|\leq M|\hbar-x|^{\theta},\forall\hbar,x\in\mathcal{I},0<\theta\leq 1,M>0\right\}$$

be the Lipschitz class of continuous functions.

THEOREM 3.6. *Let $\zeta\in\mathrm{Lip}_\theta\, M$. Then for all $x\in\mathcal{I}$,*

$$\left|\mathcal{G}_{n,\alpha}^{M,1}(\zeta;q_n;x)-\zeta(x)\right|\leq M\left(\mu_{n,q_n}^{\alpha,2}(x)\right)^{\frac{\theta}{2}}.$$

Proof. By the definition of Lipschitz class, we have

$$\begin{aligned}|\mathcal{G}_{n,\alpha}^{M,1}(\zeta,q_n;x)-\zeta(x)|&\leq\mathcal{G}_{n,\alpha}^{M,1}(|\zeta(\hbar)-\zeta(x)|;q_n;x)\\&\leq\sum_{k=0}^{n}P_{n,k,\alpha}^{M,1}(q_n;x)\left|\zeta\left(\tfrac{[k]_{q_n}}{[n]_{q_n}}\right)-\zeta(x)\right|\\&\leq M\sum_{k=0}^{n}P_{n,k,\alpha}^{M,1}(q_n;x)\left|\tfrac{[k]_{q_n}}{[n]_{q_n}}-x\right|^{\theta}.\end{aligned}$$

Applying Hölder's inequality, we get

$$\begin{aligned}|\mathcal{G}_{n,\alpha}^{M,1}(\zeta;q_n;x)-\zeta(x)|&\leq M\left(\sum_{k=0}^{n}P_{n,k,\alpha}^{M,1}(q_n;x)\left|\tfrac{[k]_{q_n}}{[n]_{q_n}}-x\right|^{2}\right)^{\frac{\theta}{2}}\\&=M\{\mathcal{G}_{n,\alpha}^{M,1}((\hbar-x)^2;q_n;x)\}^{\frac{\theta}{2}}=M\left(\mu_{n,q_n}^{\alpha,2}(x)\right)^{\frac{\theta}{2}},\end{aligned}$$

which proves the required result. □

Finally, we investigate Grüss-Voronovskaya type theorem for the operators (1.8).

THEOREM 3.7. *For $\zeta'', \nu'' \in \mathcal{C}(\mathcal{I})$, there holds the following equality:*

$$\lim_{n\to\infty}[n]_{q_n}\{\mathcal{G}_{n,\alpha}^{M,1}(\zeta\nu, q_n; x) - \mathcal{G}_{n,\alpha}^{M,1}(\zeta, q_n; x)\mathcal{G}_{n,\alpha}^{M,1}(\nu, q_n; x)\} =$$
$$= \Big(p_1\{4(1-c)x^3(1-\alpha)(1-x)+6x^4 - 7x^3 + 6\alpha x^3(1-x) + x^2 + x(1-x)^2\}$$
$$+ p_0\{4(1-c)x^2(1-\alpha)(1-x)+6x^3 - 8x^2 + 6\alpha x^2(1-x)+2x\}\Big)\zeta'(x)\nu'(x),$$

uniformly in $x \in \mathcal{I}$.

Proof. For the operators $\mathcal{G}_{n,\alpha}^{M,1}(., q_n; x)$, by our hypothesis we may write

$$[n]_{q_n}\Big\{\mathcal{G}_{n,\alpha}^{M,1}(\zeta\nu, q_n; x) - \mathcal{G}_{n,\alpha}^{M,1}(\zeta, q_n; x)\mathcal{G}_{n,\alpha}^{M,1}(\nu, q_n; x)\Big\} =$$
$$= [n]_{q_n}\Big\{\mathcal{G}_{n,\alpha}^{M,1}(\zeta\nu, q_n; x)$$
$$- \mathcal{G}_{n,\alpha}^{M,1}((\hbar - x), q_n; x)(\zeta\nu)'(x) - \zeta(x)\nu(x) - \frac{\mathcal{G}_{n,\alpha}^{M,1}((\hbar-x)^2, q_n; x)}{2!}(\zeta\nu)''(x)$$
$$-\nu(x)\left[\mathcal{G}_{n,\alpha}^{M,1}(\zeta, q_n; x) - \zeta(x) - \mathcal{G}_{n,\alpha}^{M,1}((\hbar - x), q_n; x)\zeta'(x) - \frac{\mathcal{G}_{n,\alpha}^{M,1}((\hbar-x)^2, q_n; x)}{2!}\zeta''(x)\right]$$
$$-\mathcal{G}_{n,\alpha}^{M,1}(\zeta, q_n; x)\left[\mathcal{G}_{n,\alpha}^{M,1}(\nu, q_n; x) - \nu(x) - \mathcal{G}_{n,\alpha}^{M,1}((\hbar-x), q_n; x)\nu'(x) - \frac{\mathcal{G}_{n,\alpha}^{M,1}((\hbar-x)^2, q_n; x)}{2!}\nu''(x)\right]$$
$$+ 2\frac{\mathcal{G}_{n,\alpha}^{M,1}((\hbar-x)^2, q_n; x)}{2!}\zeta'(x)\nu'(x) + \nu''(x)\frac{\mathcal{G}_{n,\alpha}^{M,1}((\hbar-x)^2, q_n; x)}{2!}\Big(\zeta(x) - \mathcal{G}_{n,\alpha}^{M,1}(\zeta, q_n; x)\Big)$$
$$+ \nu'(x)\mathcal{G}_{n,\alpha}^{M,1}((\hbar - x), q_n; x)\Big(\zeta(x) - \mathcal{G}_{n,\alpha}^{M,1}(\zeta, q_n; x)\Big)\Big\}.$$

Now, in view of Theorem 3.1, for any $\zeta \in \mathcal{C}(\mathcal{I})$, it follows that $\mathcal{G}_{n,\alpha}^{M,1}(\zeta, q_n; x) \to \zeta(x)$, as $n \to \infty$, uniformly in $x \in \mathcal{I}$. Further, following the proof of Theorem 3.2, for any $\zeta \in \mathcal{C}^2(\mathcal{I})$ we get
$[n]_{q_n}\Big\{\mathcal{G}_{n,\alpha}^{M,1}(\zeta, q_n; x) - \zeta(x) - \mathcal{G}_{n,\alpha}^{M,1}((\hbar-x), q_n; x)\zeta'(x) - \frac{\mathcal{G}_{n,\alpha}^{M,1}((\hbar-x)^2, q_n; x)}{2!}\zeta''(x)\Big\} \to 0$, *as* $n \to \infty$, uniformly in $x \in \mathcal{I}$.

Hence, using Remark 2.3, we get

$$\lim_{n\to\infty}[n]_{q_n}\{\mathcal{G}_{n,\alpha}^{M,1}(\zeta\nu, q_n; x) - \mathcal{G}_{n,\alpha}^{M,1}(\zeta, q_n; x)\mathcal{G}_{n,\alpha}^{M,1}(\nu, q_n; x)\} =$$
$$= \Big(p_1\{4(1-c)x^3(1-\alpha)(1-x)+6x^4-7x^3 + 6\alpha x^3(1-x) + x^2 + x(1-x)^2\}$$
$$+ p_0\{4(1-c)x^2(1-\alpha)(1-x)+6x^3 - 8x^2 + 6\alpha x^2(1-x)+2x\}\Big)\zeta'(x)\nu'(x),$$

which completes the proof. □

4. BIVARIATE GENERALIZATION OF THE OPERATORS $\mathcal{G}_{n,\alpha}^{M,1}(.,q_n;x)$

For $\mathcal{I}^2=\mathcal{I}\times\mathcal{I}$, let $\mathcal{C}(\mathcal{I}^2)$ be the space of all continuous functions on $\mathcal{I}^2$, equipped with the norm given by $\|\zeta\|_{\mathcal{C}(\mathcal{I}^2)}=\sup_{(x_1,x_2)\in\mathcal{I}^2}|\zeta(x_1,x_2)|$.

For $\zeta\in\mathcal{C}(\mathcal{I}^2)$ and $0\le\alpha_1,\alpha_2\le 1$, the bivariate generalization of the operator (1.8) is defined by

(4.1)
$$\mathcal{G}_{n_1,n_2,\alpha_1,\alpha_2}^{q_{n_1},q_{n_2}}(\zeta;x_1,x_2)=\sum_{k=0}^{n_1}\sum_{j=0}^{n_2}\mathfrak{P}_{n_1,n_2,k,j}^{\alpha_1,\alpha_2}(q_{n_1},q_{n_2};x_1,x_2)\zeta\left(\frac{[k]_{q_{n_1}}}{[n_1]_{q_{n_1}}},\frac{[j]_{q_{n_2}}}{[n_2]_{q_{n_2}}}\right).$$

where $\{q_{n_i}\}_{n_i\in\mathbb{N}}$ is a sequence in $(0,1)$ satisfying $\lim\limits_{n_i\to\infty}q_{n_i}=1$, $\lim\limits_{n_i\to\infty}q_{n_i}^{n_i}=c_i$, $\lim_{n_i\to\infty}a_1(n_i,q_{n_i})=\beta_i$, *and* $\lim_{n_i\to\infty}a_0(n_i,q_{n_i})=\gamma_i$, $\forall\ i=1,2$.

Further, $\mathfrak{P}_{n_1,n_2,k,j}^{\alpha_1,\alpha_2}(q_{n_1},q_{n_2};x_1,x_2)=P_{n_1,k,\alpha_1}^{M,1}(q_{n_1};x_1)P_{n_2,j,\alpha_2}^{M,1}(q_{n_2};x_2)$, where $P_{n_1,k,\alpha_1}^{M,1}(q_{n_1};x_1)$ and $P_{n_2,k,\alpha_2}^{M,1}(q_{n_2};x_2)$ are defined similarly as $P_{n,k,\alpha}^{M,1}(q;x)$ in (1.8).

Clearly, $\mathcal{G}_{n_1,n_2,\alpha_1,\alpha_2}^{q_{n_1},q_{n_2}}(\zeta;;x_1,x_2)$ is a linear positive operator. Further, we note that

$$\begin{aligned}&\mathcal{G}_{n_1,n_2,\alpha_1,\alpha_2}^{q_{n_1},q_{n_2}}\left((\hbar_1-x_1)^u(\hbar_2-x_2)^v;x_1,x_2\right)=\\&=\mathcal{G}_{n_1,\alpha_1}^{q_{n_1}}\left((\hbar_1-x_1)^u,q_{n_1};x_1\right)\mathcal{G}_{n_2,\alpha_2}^{q_{n_2}}\left((\hbar_2-x_2)^v,q_{n_2};x_2\right)\\&=\mu_{n_1,q_{n_1}}^{\alpha_1,u}(x_1)\mu_{n_2,q_{n_2}}^{\alpha_2,v}(x_2),\forall\ u,v\in\mathbb{N}_0.\end{aligned}$$

Let $e_{pq}(x_1,x_2)=x_1^px_2^q$, $(p,q)\in\mathbb{N}_0\times\mathbb{N}_0$, with $p+q\le 2$. In order to establish the results of this section, we require the following Lemmas:

LEMMA 4.1. *For the operators* $\mathcal{G}_{n_1,n_2,\alpha_1,\alpha_2}^{q_{n_1},q_{n_2}}$, *we have*

i) $\mathcal{G}_{n_1,n_2,\alpha_1,\alpha_2}^{q_{n_1},q_{n_2}}(1;x_1,x_2)=1$;

ii) $\mathcal{G}_{n_1,n_2,\alpha_1,\alpha_2}^{q_{n_1},q_{n_2}}(e_{10};x_1,x_2)=\frac{1}{[n_1]_{q_{n_1}}}\Big[a(x_1,n_1,q_{n_1})\{(1-\alpha_1)[n_1-3]_{q_{n_1}}q_{n_1}x_1\cdot(1-x_1)+(1-\alpha_1)x_1([2]_{q_{n_1}}+q_{n_1}^3[n-3]_{q_{n_1}}x_1)+\alpha_1[n-1]_{q_{n_1}}q_{n_1}x_1\}+a(1-x_1,n_1,q_{n_1})\{(1-\alpha_1)(1-x_1)(1+q_{n_1}^2[n-3]_{q_{n_1}}x_1)+(1-\alpha_1)x_1([3]_{q_{n_1}}+q_{n_1}^4[n-3]_{q_{n_1}}x_1)+\alpha_1(1+q_{n_1}^2[n-1]_{q_{n_1}}x_1)\}\Big]$;

iii) $\mathcal{G}_{n_1,n_2,\alpha_1,\alpha_2}^{q_{n_1},q_{n_2}}(e_{20};x_1,x_2)=\frac{a(x_1,n_1,q_{n_1})}{[n]_{q_{n_1}}^2}\Big[(1-\alpha_1)[n-3]_{q_{n_1}}q_{n_1}x_1(1-x_1)+(1-\alpha_1)[n-3]_{q_{n_1}}[n-4]_{q_{n_1}}q_{n_1}^3x_1^2(1-x_1)+(1-\alpha_1)x_1[2]_{q_{n_1}}^2+2[2]_{q_{n_1}}(1-\alpha_1)q^3x_1^2[n-3]_{q_{n_1}}+(1-\alpha_1)q^5x_1^2[n-3]_{q_{n_1}}+(1-\alpha_1)q^7x_1^3[n-3]_{q_{n_1}}[n-4]_{q_{n_1}}+\alpha_1[n-1]_{q_{n_1}}q_{n_1}x_1+\alpha_1[n_1-1]_{q_{n_1}}[n_1-2]_{q_{n_1}}q_{n_1}^3x_1\Big]+\frac{a(1-x_1,n_1,q_{n_1})}{[n_1]_{q_{n_1}}^2}\cdot\Big[(1-\alpha_1)(1-x_1)+2(1-\alpha_1)q_{n_1}^2x_1(1-x_1)[n_1-3]_{q_{n_1}}+(1-\alpha_1)q_{n_1}^3x_1(1-x_1)[n_1-3]_{q_{n_1}}+(1-\alpha_1)q_{n_1}^5x_1^2(1-x_1)[n_1-3]_{q_{n_1}}[n_1-4]_{q_{n_1}}+(1-$

$\alpha_1)x_1[3]^2_{q_{n_1}} + 2(1-\alpha_1)[3]_{q_{n_1}} q^4_{n_1} x_1^2[n_1-3]_{q_{n_1}} + q^7_{n_1}(1-\alpha_1)x_1^2[n_1-3]_{q_{n_1}} + q^9_{n_1}(1-\alpha_1)x_1^3[n_1-3]_{q_{n_1}}[n_1-4]_{q_{n_1}} + \alpha_1 + 2q^2_{n_1}\alpha_1 x_1[n_1-1]_{q_{n_1}} + \alpha_1 q^3_{n_1} x_1 \cdot [n_1-1]_{q_{n_1}} + \alpha_1 q^5_{n_1} x_1^2[n_1-1]_{q_{n_1}}[n_1-2]_{q_{n_1}}\Big]$;

iv) $\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(e_{01};x_1,x_2) = \frac{1}{[n_2]_{q_{n_2}}}\Big[a(x_2,n_2,q_{n_2})\{(1-\alpha_2)[n_2-3]_{q_{n_2}} q_{n_2} x_2 \cdot(1-x_2) + (1-\alpha_2)x_2([2]_{q_{n_2}} + q^3_{n_2}[n-3]_{q_{n_2}} x_2) + \alpha_2[n_2-1]_{q_{n_2}} q_{n_2} x_2\} + a(1-x_2,n_2,q_{n_2})\{(1-\alpha_2)(1-x_2)(1+q^2_{n_2}[n_2-3]_{q_{n_2}} x_2) + (1-\alpha_2)x_2([3]_{q_{n_2}} + q^4_{n_2}[n-3]_{q_{n_2}} x_2) + \alpha_2(1+q^2_{n_2}[n-1]_{q_{n_2}} x_2)\}\Big]$;

v) $\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(e_{02};x_1,x_2) = \frac{a(x_2,n_2,q_{n_2})}{[n_2]^2_{q_{n_2}}}\Big[(1-\alpha_2)[n_2-3]_{q_{n_2}} q_{n_2} x_2(1-x_2) + (1-\alpha_2)[n_2-3]_{q_{n_2}}[n_2-4]_{q_{n_2}} q^3_{n_2} x_2^2(1-x_2) + (1-\alpha_2)x_2[2]^2_{q_{n_2}} + 2[2]_{q_{n_2}}(1-\alpha_2)q^3_{n_2} x_2^2[n_2-3]_{q_{n_2}} + (1-\alpha_2)q^5_{n_2} x_2^2[n_2-3]_{q_{n_2}} + (1-\alpha_2)q^7_{n_2} x_2^3[n_2-3]_{q_{n_2}}[n_2-4]_{q_{n_2}} + \alpha_2[n_2-1]_{q_{n_2}} q_{n_2} x_2 + \alpha_2[n_2-1]_{q_{n_2}}[n_2-2]_{q_{n_2}} q^3_{n_2} x_2\Big] + \frac{a(1-x_2,n_2,q_{n_2})}{[n_2]^2_{q_{n_2}}}\Big[(1-\alpha_2)(1-x_2) + 2(1-\alpha_2)q^2_{n_2} x_2(1-x_2)[n_2-3]_{q_{n_2}} + (1-\alpha_2)q^3_{n_2} x_2(1-x_2)[n_2-3]_{q_{n_2}} + (1-\alpha_2)q^5_{n_2} x_2^2(1-x_2)[n_2-3]_{q_{n_2}}[n_2-4]_{q_{n_2}} + (1-\alpha_2)x_2[3]^2_{q_{n_2}} + 2(1-\alpha_2)[3]_{q_{n_2}} q^4_{n_2} x_2^2[n_2-3]_{q_{n_2}} + q^7_{n_2}(1-\alpha_2)x_2^2[n_2-3]q_{n_2} + q^9_{n_2}(1-\alpha_2)x_2^3[n_2-3]_{q_{n_2}}[n_2-4]_{q_{n_2}} + \alpha_2 + 2q^2_{n_2}\alpha_2 x_2[n_2-1]_{q_{n_2}} + \alpha_2 q^3_{n_2} x_2[n_2-1]_{q_{n_2}} + \alpha_2 q^5_{n_2} x_2^2[n_2-1]_{q_{n_2}}[n_2-2]_{q_{n_2}}\Big]$.

Lemma 4.2. *From Lemma 4.1, by a simple calculation we have*

i) $\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}((\hbar_1-x_1);x_1,x_2) = \frac{1}{[n_1]_{q_{n_1}}}\Big[a(x_1,n_1,q_{n_1})\{(1-\alpha_1)[n_1-3]_{q_{n_1}} \cdot q_{n_1} x_1(1-x_1) + (1-\alpha_1)x_1([2]_{q_{n_1}} + q^3_{n_1}[n_1-3]_{q_{n_1}} x_1) + \alpha_1[n_1-1]_{q_{n_1}} q_{n_1} x_1\} + a(1-x_1,n_1,q_{n_1})\{(1-\alpha_1)(1-x_1)(1+q^2_{n_1}[n_1-3]_{q_{n_1}} x_1) + (1-\alpha_1)x_1([3]_{q_{n_1}} + q^4_{n_1}[n_1-3]_{q_{n_1}} x_1) + \alpha_1(1+q^2_{n_1}[n_1-1]_{q_{n_1}} x_1)\} - [n_1]_{q_{n_1}} x_1\Big]$;

ii) $\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}((\hbar_1-x_1)^2;x_1;x_2) = \frac{a(x_1,n_1,q_{n_1})}{[n_1]^2_{q_{n_1}}}\Big[(1-\alpha_1)[n_1-3]_{q_{n_1}} 2x_1(1-x_1) + (1-\alpha_1)[n_1-3]_{q_{n_1}}[n_1-4]_{q_{n_1}} q^3_{n_1} x_1^2(1-x_1) + (1-\alpha_1)x_1[2]^2_{q_{n_1}} + 2[2]_{q_{n_1}}(1-\alpha_1)q^3_{n_1} x_1^2[n_1-3]_{q_{n_1}} + (1-\alpha_1)q^5_{n_1} x_1^2[n_1-3]_{q_{n_1}} + (1-\alpha_1)q^7_{n_1} x_1^3 \cdot[n_1-3]_{q_{n_1}}[n_1-4]_{q_{n_1}} + \alpha_1[n_1-1]_{q_{n_1}} q_{n_1} x_1 + \alpha_1[n_1-1]_{q_{n_1}} q_{n_1} x_1 + \alpha_1[n_1-1]_{q_{n_1}}[n_1-2]_{q_{n_1}} q^3_{n_1} x_1^2 - 2x_1^2(1-x_1)q_{n_1}(1-\alpha_1)[n_1-3]_{q_{n_1}}[n_1]_{q_{n_1}} - 2(1-\alpha_1)x_1^2[n_1]_{q_{n_1}}([2]_{q_{n_1}} + q^3_{n_1}[n_1-3]_{q_{n_1}} x_1) - 2\alpha_1[n_1]_{q_{n_1}}[n_1-1]_{q_{n_1}} q_{n_1} x_1^2 + [n_1]^2_{q_{n_1}} x_1^2\Big] + \frac{a(1-x_1,n_1,q_{n_1})}{[n_1]^2}\Big[(1-\alpha_1)(1-x_1) + 2(1-\alpha_1)q^2_{n_1} x_1(1-x_1)[n_1-3]_{q_{n_1}} + (1-\alpha_1)q^3_{n_1} x_1(1-x_1)[n_1-3]_{q_{n_1}} + (1-\alpha_1)q^5_{n_1} x_1^2(1-x_1)[n_1-$

$3]_{q_{n_1}}[n_1-4]_{q_{n_1}}+(1-\alpha_1)x_1[3]_{q_{n_1}}^2+2(1-\alpha_1)[3]_{q_{n_1}}q_{n_1}^4x_1{q_{n_1}}^2[n_1-3]_{q_{n_1}}+q_{n_1}^7(1-\alpha_1)x_1^2[n_1-3]_{q_{n_1}}+q_{n_1}^9(1-\alpha_1)x_1^3[n_1-3]_{q_{n_1}}[n_1-4]_{q_{n_1}}+\alpha_1+2q_{n_1}^2\alpha_1x_1[n_1-1]_{q_{n_1}}+\alpha_1q_{n_1}^3x_1[n_1-1]_{q_{n_1}}+\alpha_1q_{n_1}^5x_1^2[n_1-1]_{q_{n_1}}[n_1-2]_{q_{n_1}}-2x_1[n_1]_{q_{n_1}}(1-\alpha_1)(1-x_1)(1+q_{n_1}^2[n_1-3]_{q_{n_1}}x_1)-2x_1^2(1-\alpha_1)([3]_{q_{n_1}}+q_{n_1}^4[n_1-3]_{q_{n_1}}x_1)[n_1]q_{n_1}-2x_1\alpha_1(1+q_{n_1}^2[n_1-1]_{q_{n_1}}x_1)[n_1]_{q_{n_1}}+[n_1]_{q_{n_1}}^2x_1^2\Big]$;

iii) $\mathcal{G}_{n_1,n_2,\alpha_1,\alpha_2}^{q_{n_1},q_{n_2}}((\hbar_2-x_2);x_1,x_2)=\frac{1}{[n_2]_{q_{n_2}}}\Big[a(x_2,n_2,q_{n_2})\{(1-\alpha_2)[n_2-3]_{q_{n_2}}\cdot q_{n_2}x_2(1-x_2)+(1-\alpha_2)x_2([2]_{q_{n_2}}+q_{n_2}^3[n_2-3]_{q_{n_2}}x_2)+\alpha_2[n_2-1]_{q_{n_2}}q_{n_2}x_2\}+a(1-x_2,n_2,q_{n_2})\{(1-\alpha_2)(1-x_2)(1+q_{n_2}^2[n_2-3]_{q_{n_2}}x_2)+(1-\alpha_2)x_2([3]_{q_{n_2}}+q_{n_2}^4[n_2-3]_{q_{n_2}}x_2)+\alpha_2(1+q_{n_2}^2[n_2-1]_{q_{n_2}}x_2)\}-[n_2]_{q_{n_2}}x_2\Big]$;

iv) $\mathcal{G}_{n_1,n_2,\alpha_1,\alpha_2}^{q_{n_1},q_{n_2}}((\hbar_2-x_2)^2;x_1;x_2)=\frac{a(x_2,n_2,q_{n_2})}{[n_2]_{q_{n_2}}^2}\Big[(1-\alpha_2)[n_2-3]_{q_{n_2}}2x_2(1-x_2)+(1-\alpha_2)[n_2-3]_{q_{n_2}}[n_2-4]_{q_{n_2}}q_{n_2}^3x_2^2(1-x_2)+(1-\alpha_2)x_2[2]_{q_{n_2}}^2+2[2]_{q_{n_2}}(1-\alpha_2)q_{n_2}^3x_2^2[n_2-3]_{q_{n_2}}+(1-\alpha_2)q_{n_2}^5x_2^2[n_2-3]_{q_{n_2}}+(1-\alpha_2)q_{n_2}^7\cdot x_2^3[n_2-3]_{q_{n_2}}[n_2-4]_{q_{n_2}}+\alpha_2[n_2-1]_{q_{n_2}}q_{n_2}x_2+\alpha_2[n_2-1]_{q_{n_2}}q_{n_2}x_2+\alpha_2[n_2-1]_{q_{n_2}}[n_2-2]_{q_{n_2}}q_{n_2}^3x_2^2-2x_2^2(1-x_2)q_{n_2}(1-\alpha_2)[n_2-3]_{q_{n_2}}[n_2]_{q_{n_2}}-2(1-\alpha_2)x_2^2[n_2]_{q_{n_2}}([2]_{q_{n_2}}+q_{n_2}^3[n_2-3]_{q_{n_2}}x_2)-2\alpha_2[n_2]_{q_{n_2}}[n_2-1]_{q_{n_2}}q_{n_2}x_2^2+[n_2]_{q_{n_2}}^2x_2^2\Big]+\frac{a(1-x_2,n_2,q_{n_2})}{[n_2]_{q_{n_2}}^2}\Big[(1-\alpha_2)(1-x_2)+2(1-\alpha_2)q_{n_2}^2x_2(1-x_2)[n_2-3]_{q_{n_2}}+(1-\alpha_2)q_{n_2}^3x_2(1-x_2)[n_2-3]_{q_{n_2}}+(1-\alpha_2)q_{n_2}^5x_2^2(1-x_2)[n_2-3]_{q_{n_2}}[n_2-4]_{q_{n_2}}+(1-\alpha_2)x_2[3]_{q_{n_2}}^2+2(1-\alpha_2)[3]_{q_{n_2}}q_{n_2}^4x_2{q_{n_2}}^2[n_2-3]_{q_{n_2}}+q_{n_2}^7(1-\alpha_2)x_2^2[n_2-3]_{q_{n_2}}+q_{n_2}^9(1-\alpha_2)x_2^3[n_2-3]_{q_{n_2}}[n_2-4]_{q_{n_2}}+\alpha_2+2q_{n_2}^2\alpha_2x_2[n_2-1]_{q_{n_2}}+\alpha_2q_{n_2}^3x_2[n_2-1]_{q_{n_2}}+\alpha_2q_{n_2}^5x_2^2[n_2-1]_{q_{n_2}}[n_2-2]_{q_{n_2}}-2x_2[n_2]_{q_{n_2}}(1-\alpha_2)(1-x_2)(1+q_{n_2}^2[n_2-3]_{q_{n_2}}x_2)-2x_2^2(1-\alpha_2)([3]_{q_{n_2}}+q_{n_2}^4[n_2-3]_{q_{n_2}}x_2)[n_2]_{q_{n_2}}-2x_2\alpha_2(1+q_{n_2}^2[n_2-1]_{q_{n_2}}x_2)[n_2]_{q_{n_2}}+[n_2]_{q_{n_2}}^2x_2^2\Big]$.

Lemma 4.3. *From Lemma 4.2, one has*

(i) $\lim_{n_1\to\infty}[n_1]_{q_{n_1}}\mathcal{G}_{n_1,n_2,\alpha_1,\alpha_2}^{q_{n_1},q_{n_2}}((\hbar_1-x_1);x_1,x_2)=$

$$=(1-c_1)\Big[x_1^2(\beta_1(2\alpha_1-1)-4\gamma_1(1-\alpha_1))+x_1((1-2\alpha_1)+\gamma_1(1+\alpha_1))\Big]+(\beta_1+\gamma_1)(1-2x_1);$$

(ii) $\lim_{n_2\to\infty}[n_2]_{q_{n_2}}\mathcal{G}_{n_1,n_2,\alpha_1,\alpha_2}^{q_{n_1},q_{n_2}}((\hbar_2-x_2);x_1,x_2)=$

$$=(1-c_2)\Big[x_2^2(\beta_2(2\alpha_2-1)-4\gamma_2(1-\alpha_2))+x_2((1-2\alpha_2)+\gamma_2(1+\alpha_2))\Big]+(\beta_2+\gamma_2)(1-2x_2);$$

(iii) $\lim_{n_1\to\infty}[n_1]_{q_{n_1}}\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}((\hbar_1-x_1)^2;x_1,x_2)=$

$$=\beta_1\{4(1-c_1)x_1^3(1-\alpha_1)(1-x_1)+6x_1^4-7x_1^3+6\alpha_1x_1^3(1-x_1)+x_1^2 \\ +x_1(1-x_1)^2\}+\gamma_1\{4(1-c_1)x_1^2(1-\alpha_1)(1-x_1) \\ +6x_1^3-8x_1^2+6\alpha_1x_1^2(1-x_1)+2x_1\};$$

(iv) $\lim_{n_2\to\infty}[n_2]_{q_{n_2}}\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}((\hbar_2-x_2)^2;x_1,x_2)=$

$$=\beta_2\{4(1-c_2)x_2^3(1-\alpha_2)(1-x_2)+6x_2^4-7x_2^3+6\alpha_2x_2^3(1-x_2)+x_2^2 \\ +x_2(1-x_2)^2\}+\gamma_2\{4(1-c_2)x_2^2(1-\alpha_2)(1-x_2)+6x_2^3-8x_2^2 \\ +6\alpha_2x_2^2(1-x_2)+2x_2\}.$$

5. CONVERGENCE ESTIMATES FOR THE BIVARIATE OPERATORS

In the following result we show the uniform convergence of $\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\zeta)$ to ζ, if $\zeta\in\mathcal{C}(\mathcal{I}^2)$.

THEOREM 5.1. *If* $\zeta\in\mathcal{C}(\mathcal{I}^2)$, *then*

$$\lim_{n_1,n_2\to\infty}\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\zeta;x_1,x_2)=\zeta(x_1,x_2),$$

uniformly in $(x_1,x_2)\in\mathcal{I}^2$.

Proof. From Lemma 4.1, obviously

$$\lim_{n_1,n_2\to\infty}\|\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(e_{ij})-e_{ij}\|_{\mathcal{C}(\mathcal{I}^2)}=0, for\ (i,j)\in\{(0,0),(1,0),(0,1)\},$$

$$\lim_{n_1,n_2\to\infty}\|\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(e_{20}+e_{02})-(e_{20}+e_{02})\|_{\mathcal{C}(\mathcal{I}^2)}=0,$$

hence applying the well known theorem given by Volkov [37], the required result follows. □

For $\zeta\in\mathcal{C}(\mathcal{I}^2)$ and $\rho>0$, the total modulus of continuity is given by:

$$\widetilde{\omega}(\zeta;\rho)=\sup\{|\zeta(\hbar_1,\hbar_2)-\zeta(x_1,x_2)|:\sqrt{(\hbar_1-x_1)^2+(\hbar_2-x_2)^2}\leq\rho\}.$$

Hence,

$$|\zeta(\hbar_1,\hbar_2)-\zeta(x_1,x_2)|\leq\widetilde{\omega}(\zeta;\sqrt{(\hbar_1-x_1)^2+(\hbar_2-x_2)^2})\leq\widetilde{\omega}(\zeta;\rho),$$

whenever $\sqrt{(\hbar_1-x_1)^2+(\hbar_2-x_2)^2}\leq\rho,\rho>0$, and for any $\lambda>0$,

$$\widetilde{\omega}(\zeta;\lambda\rho)\leq(1+\lambda)\widetilde{\omega}(\zeta;\rho).$$

The partial moduli of continuity with respect to x_1 and x_2 are defined as

(5.1)

$$\widetilde{\omega_1}(\zeta;\rho)=\sup\left\{|\zeta(x_{11},x_2)-\zeta(x_{12},x_2)|:x_2\in J\ and\ |x_{11}-x_{12}|\leq\rho,\ \rho>0\right\},$$

$$\widetilde{\omega_2}(\zeta;\rho)=\sup\left\{|\zeta(x_1,x_{21})-\zeta(x_1,x_{22})|:x_1\in J\ and\ |x_{21}-x_{22}|\leq\rho,\ \rho>0\right\}.$$

For more details on the moduli of continuity for functions of two variables, we refer to [24].

Let $\mathcal{C}^2(\mathcal{I}^2)=\{\zeta\in\mathcal{C}(\mathcal{I}^2): \frac{\partial^2\zeta}{\partial x_1^i\partial x_2^j}\in\mathcal{C}(\mathcal{I}^2),\ 0\leq i+j\leq 2\ i,j\in\mathbb{N}_0\}$ equipped with the norm

$$||\zeta||_{C^2(\mathcal{I}^2)}=||\zeta||_{\mathcal{C}(\mathcal{I}^2)}+\sum_{i=1}^{2}\left(\left\|\frac{\partial^i\zeta}{\partial x_1^i}\right\|_{\mathcal{C}(\mathcal{I}^2)}+\left\|\frac{\partial^i\zeta}{\partial x_2^i}\right\|_{\mathcal{C}(\mathcal{I}^2)}\right)+\left\|\frac{\partial^2\zeta}{\partial x_1\partial x_2}\right\|_{\mathcal{C}(\mathcal{I}^2)}.$$

The appropriate Peetre's K-functional for the function $\zeta\in\mathcal{C}(\mathcal{I}^2)$ is given by

$$\mathcal{K}(\zeta;\rho)=\inf_{\tau\in\mathcal{C}^2(\mathcal{I}^2)}\{||\zeta-\tau||_{\mathcal{C}(\mathcal{I}^2)}+\rho||\tau||_{\mathcal{C}^2(\mathcal{I}^2)}\},\ \rho>0.$$

From [11, p. 192], we have

$$\mathcal{K}(\zeta;\rho)\leq M\{\overline{\omega_2}(\zeta;\sqrt{\rho})+\min(1,\rho)||\zeta||_{\mathcal{C}(\mathcal{I}^2)}\},\rho>0, \tag{5.2}$$

where the constant $M>0$, is independent of ζ and ρ and $\overline{\omega_2}(\zeta;\sqrt{\rho})$ is the second order modulus of smoothness.

In our further consideration, let

$$\sup_{(x_1,x_2)\in\mathcal{I}^2}\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}((\hbar_1-x_1)^2;x_1,x_2)=\theta^{q_{n_1}}_{n_1,\alpha_1},$$
$$\sup_{(x_1,x_2)\in\mathcal{I}^2}|\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}((\hbar_1-x_1);x_1,x_2)|=\beta^{q_{n_1}}_{n_1,\alpha_1},$$
$$\sup_{(x_1,x_2)\in\mathcal{I}^2}\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}((\hbar_2-x_2)^2;x_1,x_2)=\gamma^{q_{n_2}}_{n_2,\alpha_2},$$
$$\sup_{(x_1,x_2)\in\mathcal{I}^2}|\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}((\hbar_2-x_2);x_1,x_2)|=\rho^{q_{n_2}}_{n_2,\alpha_2}.$$

THEOREM 5.2. *Let $\zeta\in\mathcal{C}(\mathcal{I}^2)$, then we have*

$$\|\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\zeta)-\zeta\|_{\mathcal{C}(\mathcal{I}^2)}\leq 2\widetilde{\omega}(\zeta;\rho).$$

where $\rho=(\theta^{q_{n_1}}_{n_1,\alpha_1}+\gamma^{q_{n_2}}_{n_2,\alpha_2})^{\frac{1}{2}}$.

Proof. Taking into account the Cauchy-Schwarz inequality, we may write

$$|\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\zeta;x_1,x_2)-\zeta(x_1,x_2)|\leq$$
$$\leq\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(|\zeta(\hbar_1,\hbar_2)-\zeta(x_1,x_2)|;x_1,x_2)$$
$$\leq\widetilde{\omega}(\zeta,\rho)\,\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}\left((1+\frac{\sqrt{(\hbar_1-x_1)^2+(\hbar_2-x_2)^2}}{\rho};x_1,x_2\right)$$
$$\leq\widetilde{\omega}(\zeta,\rho)\left(1+\frac{1}{\rho}\sqrt{\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}((\hbar_1-x_1)^2;x_1,x_2)+\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}((\hbar_2-x_2)^2;x_1,x_2)}\right).$$

Now, choosing $\rho=(\theta^{q_{n_1}}_{n_1,\alpha_1}+\gamma^{q_{n_2}}_{n_2,\alpha_2})^{\frac{1}{2}}$, the required result is proved. □

THEOREM 5.3. *For $\zeta\in\mathcal{C}(\mathcal{I}^2)$, the operator $\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}$ verifies the following inequality:*

$$\|\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\zeta)-\zeta\|_{\mathcal{C}(\mathcal{I}^2)}\leq 2(\widetilde{\omega_1}(\zeta;\sqrt{\theta^{q_{n_1}}_{n_1,,\alpha_1}})+\widetilde{\omega_2}(\zeta;\sqrt{\gamma^{q_{n_2}}_{n_2,\alpha_2}}).$$

Proof. The proof of the theorem is a direct consequence of the definitions of the partial moduli of continuity, Cauchy-Schwarz inequality and Lemma 4.1. □

Next, we determine the rate of convergence for the operators $\mathcal{G}_{n_1,n_2,\alpha_1,\alpha_2}^{q_{n_1},q_{n_2}}(\zeta)$ in terms of Peetre's K-functional for any $\zeta \in \mathcal{C}(\mathcal{I}^2)$.

THEOREM 5.4. *For $\zeta \in \mathcal{C}(\mathcal{I}^2)$, there holds the following inequality:*

$$\begin{aligned}&\|\mathcal{G}_{n_1,n_2,\alpha_1,\alpha_2}^{q_{n_1},q_{n_2}}(\zeta)-\zeta\|_{\mathcal{C}(\mathcal{I}^2)} \leq \\ &\leq M\left\{\overline{\omega_2}\left(\zeta;\tfrac{1}{2}\sqrt{C_{n_1,n_2,\alpha_1,\alpha_2}^{q_{n_1},q_{n_2}}}\right)+\min\left\{1,\frac{C_{n_1,n_2,\alpha_1,\alpha_2}^{q_{n_1},q_{n_2}}}{4}\right\}\|\zeta\|_{\mathcal{C}(\mathcal{I}^2)}\right\}\\ &\quad+\widetilde{\omega}\left(\zeta;\sqrt{\left(\beta_{n_1,\alpha_1}^{q_{n_1}}\right)^2+\left(\rho_{n_2,\alpha_2}^{q_{n_2}}\right)^2}\right),\end{aligned}$$

where $C_{n_1,n_2,\alpha_1,\alpha_2}^{q_{n_1},q_{n_2}}=\frac{1}{2}\left\{(\sqrt{\theta_{n_1,\alpha_1}^{q_{n_1}}}+\sqrt{\gamma_{n_2,\alpha_2}^{q_{n_2}}})^2+(\beta_{n_1,,\alpha_1}^{q_{n_1}}+\rho_{n_2,\alpha_2}^{q_{n_2}})^2\right\}$.

Proof. Consider the following auxiliary operator defined as:

$$\begin{aligned}\overline{\mathcal{G}_{n_1,n_2,\alpha_1,\alpha_2}^{q_{n_1},q_{n_2}}}(\zeta;x_1,x_2)=&\mathcal{G}_{n_1,n_2,\alpha_1,\alpha_2}^{q_{n_1},q_{n_2}}(\zeta;x_1,x_2)+\zeta(x_1,x_2)\\ &-\zeta\left(\mathcal{G}_{n_1,n_2,\alpha_1,\alpha_2}^{q_{n_1},q_{n_2}}(e_{10};x_1,x_2),\mathcal{G}_{n_1,n_2,\alpha_1,\alpha_2}^{q_{n_1},q_{n_2}}(e_{01};x_1,x_2)\right).\end{aligned}\tag{5.3}$$

Then, in view of Lemma 4.1, we have $\overline{\mathcal{G}_{n_1,n_2,\alpha_1,\alpha_2}^{q_{n_1},q_{n_2}}}(1;x_1,x_2)=1$, $\overline{\mathcal{G}_{n_1,n_2,\alpha_1,\alpha_2}^{q_{n_1},q_{n_2}}}((\hbar_1-x_1);x_1,x_2)=0$ *and* $\overline{\mathcal{G}_{n_1,n_2,\alpha_1,\alpha_2}^{q_{n_1},q_{n_2}}}((\hbar_2-x_2);x_1,x_2)=0$.
Let $\tau \in \mathcal{C}^2(\mathcal{I}^2)$ and $(x_1,x_2)\in\mathcal{I}^2$ be arbitrary. By Taylor's expansion, we can write

$$\begin{aligned}\tau(\hbar_1,\hbar_2)-\tau(x_1,x_2)=&\int_{x_1}^{\hbar_1}(\hbar_1-\phi)\frac{\partial^2\tau(\phi,x_2)}{\partial\phi^2}d\phi+\frac{\partial\tau(x_1,x_2)}{\partial x_2}(\hbar_2-x_2)\\ +\frac{\partial\tau(x_1,x_2)}{\partial x_1}(\hbar_1-x_1)&+\int_{x_2}^{\hbar_2}(\hbar_2-\psi)\frac{\partial^2\tau(x_1,\psi)d\psi}{\partial\psi^2}+\int_{x_1}^{\hbar_1}\int_{x_2}^{\hbar_2}\frac{\partial^2\tau d\phi\, d\psi}{\partial\phi\partial\psi}.\end{aligned}\tag{5.4}$$

Applying $\overline{\mathcal{G}_{n_1,n_2,\alpha_1,\alpha_2}^{q_{n_1},q_{n_2}}}(.;x_1,x_2)$ on both sides of the equation (5.4) and using (5.3), we obtain

$$\begin{aligned}&\overline{\mathcal{G}_{n_1,n_2,\alpha_1,\alpha_2}^{q_{n_1},q_{n_2}}}(\tau;x_1,x_2)-\tau(x_1,x_2)=\\ &=\overline{\mathcal{G}_{n_1,n_2,\alpha_1,\alpha_2}^{q_{n_1},q_{n_2}}}\left(\int_{x_1}^{\hbar_1}(\hbar_1-\phi)\frac{\partial^2\tau(\phi,x_2)}{\partial\phi^2}d\phi;x_1,x_2\right)\\ &\quad+\overline{\mathcal{G}_{n_1,n_2,\alpha_1,\alpha_2}^{q_{n_1},q_{n_2}}}\left(\int_{x_2}^{\hbar_2}(\hbar_2-\psi)\frac{\partial^2\tau(x_1,\psi)}{\partial\psi^2}d\psi;x_1,x_2\right)\\ &\quad+\overline{\mathcal{G}_{n_1,n_2,\alpha_1,\alpha_2}^{q_{n_1},q_{n_2}}}\left(\int_{x_1}^{\hbar_1}\int_{x_2}^{\hbar_2}\frac{\partial^2\tau}{\partial\phi\partial\psi}d\phi\, d\psi;x_1,x_2\right)\\ &=\mathcal{G}_{n_1,n_2,\alpha_1,\alpha_2}^{q_{n_1},q_{n_2}}\left(\int_{x_1}^{\hbar_1}(\hbar_1-\phi)\frac{\partial^2\tau(\phi,x_2)}{\partial\phi^2}d\phi;x_1,x_2\right)\end{aligned}$$

$$
\begin{aligned}
&-\int_{x_1}^{\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(e_{10};x_1,x_2)}\left(\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(e_{10};x_1,x_2)-\phi\right)\frac{\partial^2\tau(\phi,x_2)}{\partial\phi^2}d\phi\\
&+\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}\left(\int_{x_2}^{\hbar_2}(\hbar_2-\psi)\frac{\partial^2\tau(x_1,\psi)}{\partial\psi^2}d\psi;x_1,x_2\right)\\
&-\int_{x_2}^{\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(e_{01};x_1,x_2)}\left(\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(e_{01};x_1,x_2)-\psi\right)\frac{\partial^2\tau(x_1,\psi)}{\partial\psi^2}d\psi\\
&+\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}\left(\int_{x_1}^{\hbar_1}\int_{x_2}^{\hbar_2}\frac{\partial^2\tau}{\partial\phi\partial\psi}d\phi\,d\psi;x_1,x_2\right)\\
&+\int_{x_1}^{\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(e_{10};x_1,x_2)}\int_{x_2}^{\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(e_{01};x_1,x_2)}\frac{\partial^2\tau}{\partial\phi\partial\psi}d\phi\,d\psi.
\end{aligned}
$$

Hence, applying Cauchy-Schwarz inequality

(5.5)

$$
\begin{aligned}
&\left|\overline{\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}}(\tau;x_1,x_2)-\tau(x_1,x_2)\right|\leq\\
&\leq\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}\left(\left|\int_{x_1}^{\hbar_1}|\hbar_1-\phi|\left|\frac{\partial^2\tau(\phi,x_2)}{\partial\phi^2}\right|d\phi\right|;x_1,x_2\right)\\
&\quad+\left|\int_{x_1}^{\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(e_{10};x_1,x_2)}\left|\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(e_{10};x_1,x_2)-\phi\right|\left|\frac{\partial^2\tau(\phi,x_2)}{\partial\phi^2}\right|d\phi\right|\\
&\quad+\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}\left(\left|\int_{x_2}^{\hbar_2}|\hbar_2-\psi|\left|\frac{\partial^2\tau(x_1,\psi)}{\partial\psi^2}\right|d\psi\right|;x_1,x_2\right)\\
&\quad+\left|\int_{x_2}^{\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(e_{10};x_1,x_2)}\left|\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(e_{10};x_1,x_2)-\psi\right|\left|\frac{\partial^2\tau(x_1,\psi)}{\partial\psi^2}\right|d\psi\right|\\
&\quad+\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}\left(\left|\int_{x_1}^{\hbar_1}\int_{x_2}^{\hbar_2}\left|\frac{\partial^2\tau}{\partial\phi\partial\psi}\right|d\phi\,d\psi\right|;x_1,x_2\right)\\
&\quad+\left|\int_{x_1}^{\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(e_{10};x_1,x_2)}\int_{x_2}^{\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(e_{01};x_1,x_2)}\frac{\partial^2\tau}{\partial\phi\partial\psi}d\phi\,d\psi\right|\\
&\leq\frac{1}{2}\left\{\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}((\hbar_1-x_1)^2;x_1,x_2)+\left(\mu^{\alpha_1,1}_{n_1,q_{n_1}}(x_1)\right)^2\right.\\
&\quad\left.+\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}((\hbar_2-x_2)^2;x_1,x_2)+\left(\mu^{\alpha_2,1}_{n_2,q_{n_2}}(x_2)\right)^2\right\}||\tau||_{\mathcal{C}^2(\mathcal{I}^2)}\\
&\quad+\left\{\sqrt{\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}((\hbar_1-x_1)^2;x_1,x_2)}\sqrt{\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}((\hbar_2-x_2)^2;x_1,x_2)}\right.\\
&\quad\left.+\left|\left(\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(e_{10};x_1,x_2)-x_1\right)\right|\left|\left(\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(e_{01};x_1,x_2)-x_2\right)\right|\right\}||\tau||_{\mathcal{C}^2(\mathcal{I}^2)}
\end{aligned}
$$

$$\leq \frac{1}{2}\left\{(\sqrt{\theta^{q_{n_1}}_{n_1,\alpha_1}}+\sqrt{\gamma^{q_{n_2}}_{n_2,\alpha_2}})^2+(\beta^{q_{n_1}}_{n_1,,\alpha_1}+\rho^{q_{n_2}}_{n_2,\alpha_2})^2\right\}||\tau||_{\mathcal{C}^2(\mathcal{I}^2)}=C^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}||\tau||_{\mathcal{C}^2(\mathcal{I}^2)}.$$

Also, from (5.3), we have

$$\begin{aligned}
|\overline{\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}}(\zeta;x_1,x_2)| &\leq |\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\zeta;x_1,x_2)| + |\zeta(x_1,x_2)|+\\
&+\left|\zeta\left(\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(e_{10};x_1,x_2),\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(e_{01};x_1,x_2)\right)\right|\\
&\leq 3\|\zeta\|_{\mathcal{C}(\mathcal{I}^2)}. \qquad (5.6)
\end{aligned}$$

Now, for $\zeta \in \mathcal{C}(\mathcal{I}^2)$ and any $\tau \in \mathcal{C}^2(\mathcal{I}^2)$, using (5.5) and (5.6), we may write

$$\begin{aligned}
&\left|\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\zeta;x_1,x_2)-\zeta(x_1,x_2)\right| \leq\\
&\leq |\overline{\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}}(\zeta-\tau;x_1,x_2)|\\
&\quad + |\overline{\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}}(\tau;x_1,x_2)-\tau(x_1,x_2)| + |\tau(x_1,x_2)-\zeta(x_1,x_2)|\\
&\quad + \left|\zeta\left(\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(e_{10};x_1,x_2),\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(e_{01};x_1,x_2)\right)-\zeta(x_1,x_2)\right|\\
&< 4\|\zeta-\tau\|_{\mathcal{C}(\mathcal{I}^2)} + C^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}||\tau||_{\mathcal{C}^2(\mathcal{I}^2)}\\
&\quad + \left|\zeta\left(\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(e_{10};x_1,x_2),\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(e_{01};x_1,x_2)\right)-\zeta(x_1,x_2)\right|\\
&\leq 4\left(\|\zeta-\tau\|_{\mathcal{C}(\mathcal{I}^2)} + \frac{C^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}}{4}\|\tau\|_{\mathcal{C}^2(\mathcal{I}^2)}\right) + \widetilde{\omega}\left(\zeta;\sqrt{\left(\beta^{q_{n_1}}_{n_1,,\alpha_1}\right)^2+\left(\rho^{q_{n_2}}_{n_2,\alpha_2}\right)^2}\right).
\end{aligned}$$

Taking the infimum over all $\tau \in \mathcal{C}^2(\mathcal{I}^2)$ on the right hand side of the above equation and using (5.2), we get

$$\begin{aligned}
&|\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\zeta;x_1,x_2)-\zeta(x_1,x_2)| \leq\\
&\leq 4\mathcal{K}\left(\zeta;\frac{C^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}}{4}\right) + \widetilde{\omega}\left(\zeta;\sqrt{\beta^{2q_{n_1}}_{n_1,,\alpha_1}+\rho^{2q_{n_2}}_{n_2,\alpha_2}}\right)\\
&\leq M\left\{\overline{\omega_2}\left(\zeta;\frac{1}{2}\sqrt{C^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}}\right)+\min\{1,\frac{C^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}}{4}\}\|\zeta\|_{\mathcal{C}(\mathcal{I}^2)}\right\}\\
&\quad + \widetilde{\omega}\left(\zeta;\sqrt{\left(\beta^{q_{n_1}}_{n_1,,\alpha_1}\right)^2+\left(\rho^{q_{n_2}}_{n_2,\alpha_2}\right)^2}\right),\ \forall\,(x_1,x_2)\in\mathcal{I}^2,
\end{aligned}$$

which leads us to the desired assertion. □

The next result provides a convergence estimate for functions in $\mathcal{C}^1(\mathcal{I}^2) = \{\zeta \in \mathcal{C}(\mathcal{I}^2) : \frac{\partial\zeta}{\partial x_1}, \frac{\partial\zeta}{\partial x_2} \in \mathcal{C}(\mathcal{I}^2)\}$ by the operators (4.1).

THEOREM 5.5. *If $\zeta \in \mathcal{C}^1(\mathcal{I}^2)$, then there holds*

$$\|\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\zeta)-\zeta\|_{\mathcal{C}(\mathcal{I}^2)} \leq \|\ \zeta'_{x_1}\ \|_{\mathcal{C}(\mathcal{I}^2)}\sqrt{\theta^{q_{n_1}}_{n_1,\alpha_1}} + \|\ \zeta'_{x_2}\ \|_{\mathcal{C}(\mathcal{I}^2)}\sqrt{\gamma^{q_{n_2}}_{n_2,\alpha_2}}.$$

Proof. Let $(x_1,x_2)\in\mathcal{I}^2$, be an arbitrary but fixed point. Then, we may write

$$\zeta(\hbar_1,\hbar_2)-\zeta(x_1,x_2)=\int_{x_1}^{\hbar_1}\zeta'_{\theta_1}(\theta_1,x_2)d\theta_1+\int_{x_2}^{\hbar_2}\zeta'_{\theta_2}(x_1,\theta_2)d\theta_2.$$

Now, applying $\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(.;x_1,x_2)$ on both sides of the above equation, we get

$$\begin{aligned}&|\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\zeta(\hbar_1,\hbar_2);x_1,x_2)-\zeta(x_1,x_2)|\leq\\&\leq\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}\left(\left|\int_{x_1}^{\hbar_1}\zeta'_{\theta_1}(\theta_1,x_2)d\theta_1\right|;x_1,x_2\right)\\&\quad+\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}\left(\left|\int_{x_2}^{\hbar_2}\zeta'_{\theta_2}(x_1,\theta_2)d\theta_2\right|;x_1,x_2\right).\end{aligned}$$

By applying the inequalities

$$\left|\int_{x_1}^{\hbar_1}\zeta'_{\theta_1}(\theta_1,x_2)d\theta_1\right|\leq\|\zeta'_{x_1}\|_{\mathcal{C}(\mathcal{I}^2)}|\hbar_1-x_1|,$$

and

$$\left|\int_{x_2}^{\hbar_2}\zeta'_{\theta_2}(x_1,\theta_2)d\theta_2\right|\leq\|\zeta'_{x_2}\|_{\mathcal{C}(\mathcal{I}^2)}|\hbar_2-x_2|,$$

we get

$$\begin{aligned}&|\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\zeta(\hbar_1,\hbar_2);x_1,x_2)-\zeta(x_1,x_2)|\leq\\&\leq\|\zeta'_{x_1}\|_{\mathcal{C}(\mathcal{I}^2)}\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(|\hbar_1-x_1|;x_1,x_2)\\&\quad+\|\zeta'_{x_2}\|_{\mathcal{C}(\mathcal{I}^2)}\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(|\hbar_2-x_2|;x_1,x_2).\end{aligned}$$

Applying Cauchy-Schwarz inequality and Lemma 4.2, we obtain

$$\begin{aligned}\|\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\zeta)-\zeta\|_{\mathcal{C}(\mathcal{I}^2)}&\leq\|\zeta'_{x_1}\|_{\mathcal{C}(\mathcal{I}^2)}\sqrt{\|\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}((\hbar_1-.)^2)\|_{\mathcal{C}(\mathcal{I}^2)}}\\&\quad+\|\zeta'_{x_2}\|_{\mathcal{C}(\mathcal{I}^2)}\sqrt{\|\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}((\hbar_2-.)^2)\|_{\mathcal{C}(\mathcal{I}^2)}}\\&=\|\zeta'_{x_1}\|_{\mathcal{C}(\mathcal{I}^2)}\sqrt{\theta^{q_{n_1}}_{n_1,\alpha_1}}+\|\zeta'_{x_2}\|_{\mathcal{C}(\mathcal{I}^2)}\sqrt{\gamma^{q_{n_2}}_{n_2,\alpha_2}}.\end{aligned}$$

This completes the proof. □

In the next result, we discuss the convergence behaviour of $\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\zeta)$ to ζ by virtue of the partial moduli of continuity of the partial derivatives of ζ.

THEOREM 5.6. *Let $\zeta\in\mathcal{C}^1(\mathcal{I}^2)$, then for sufficiently large n_1 and n_2, there holds the following inequality:*

$$\|\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\zeta)-\zeta\|_{\mathcal{C}(\mathcal{I}^2)}\leq C\sum_{i=1}^{2}[n_i]_{q_{n_i}}^{-\frac{1}{2}}\left(1+2\widetilde{\omega_i}\left(\zeta'_{x_i};[n_i]_{q_{n_i}}^{-\frac{1}{2}}\right)\right),$$

where $\widetilde{\omega_i}\left(\zeta'_{x_i};.\right)$ *are the partial moduli of continuity of* ζ'_{x_i} *for* $i=1,2$ *and* C *is some positive constant.*

Proof. Using the mean value theorem in the following form, we obtain

$$\begin{aligned}\zeta(\hbar_1,\hbar_2)-\zeta(x_1,x_2) &= (\hbar_1-x_1)\zeta'_{x_1}(u,x_2)+(\hbar_2-x_2)\zeta'_{x_2}(x_1,v)\\ &= (\hbar_1-x_1)\zeta'_{x_1}(x_1,x_2)+(\hbar_1-x_1)\left(\zeta'_{x_1}(u,x_2)-\zeta'_{x_1}(x_1,x_2)\right)\\ &\quad+(\hbar_2-x_2)\zeta'_{x_2}(x_1,x_2)+(\hbar_2-x_2)\left(\zeta'_{x_2}(x_1,v)-\zeta'_{x_2}(x_1,x_2)\right)\end{aligned}$$

where u and v lie between $\hbar_1,x_1$ and $\hbar_2,x_2$ respectively. Applying $\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(.;x_1,x_2)$ to the above equation, we obtain

$$\begin{aligned}&\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\zeta;x_1,x_2)-\zeta(x_1,x_2)=\\ &=\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}\left((\hbar_1-x_1);x_1,x_2\right)\zeta'_{x_1}(x_1,x_2)\\ &\quad+\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}\left((\hbar_1-x_1)\left(\zeta'_{x_1}(u,x_2)-\zeta'_{x_1}(x_1,x_2)\right);x_1,x_2\right)\\ &\quad+\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}\left((\hbar_2-x_2);x_1,x_2\right)\zeta'_{x_2}(x_1,x_2)\\ &\quad+\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}\left((\hbar_2-x_2)\left(\zeta'_{x_2}(x_1,v)-\zeta'_{x_2}(x_1,x_2)\right);x_1,x_2\right).\end{aligned}$$

Since ζ'_{x_1} and ζ'_{x_1} are continuous on $\mathcal{I}^2$, they are bounded therein, therefore there exist positive constants C_1 and C_2 such that $|\zeta'_{x_1}|\le C_1$ and $|\zeta'_{x_2}|\le C_2$, for all $(x_1,x_2)\in\mathcal{I}^2$. Hence applying Cauchy-Schwarz inequality, for any $\rho_1,\rho_2>0$, we obtain

$$\begin{aligned}&|\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\zeta;x_1,x_2)-\zeta(x_1,x_2)|\le\sum_{i=1}^{2}|\zeta'_{x_i}|\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}\left(|\hbar_i-x_i|;x_1,x_2\right)\\ &\quad+\sum_{i=1}^{2}\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}\left(|\hbar_i-x_i|\left(1+\tfrac{|\hbar_i-x_i|}{\rho_i}\right);x_1,x_2\right)\widetilde{\omega_i}(\zeta'_{x_i};\rho_i)\\ &\le\sum_{i=1}^{2}\Big(C_i\{\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}\left((\hbar_i-x_i)^2;x_1,x_2\right)\}^{\frac12}+\widetilde{\omega_i}(\zeta'_{x_i};\rho_i)\\ &\quad\times\left[\left\{\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}((\hbar_i-x_i)^2;x_1,x_2)\right\}^{\frac12}+\tfrac{1}{\rho_i}\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}\left((\hbar_i-x_i)^2;x_1,x_2\right)\right]\Big).\end{aligned}$$

Choosing $\rho_i=([n_i]_{q_{n_i}})^{-\frac12}, i=1,2$ and applying Lemma 4.3, the required result is proved. □

Now, we establish a convergence estimate for the operators $\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}$ with the aid of the Lipschitz class functions.

For $\zeta:\mathcal{I}^2\to\mathbb{R}$ and $0<\theta\le1$, the function ζ is said to be in Lipschitz class $\mathrm{Lip}_{\mathcal{M}}(\theta)$, if $\exists$ a positive constant $\mathcal{M}$ such that

$$\mathrm{Lip}_{\mathcal{M}}(\theta)=\{\zeta:|\zeta(\hbar_1,\hbar_2)-\zeta(x_1,x_2)|\le\mathcal{M}\|r-x\|^{\theta}\},$$

$\forall r=(\hbar_1,\hbar_2), x=(x_1,x_2)\in\mathcal{I}^2$, where $\|r-x\|=\{(\hbar_1-x_1)^2+(\hbar_2-x_2)^2\}^{\frac12}$ is the Euclidean norm.

Theorem 5.7. *Let $\zeta \in \mathrm{Lip}_{\mathcal{M}}(\theta)$, $0<\theta\le 1$. Then for sufficiently large n_1 and n_2, the operators (4.1) verify the following relation:*

$$\|\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\zeta)-\zeta\|_{\mathcal{C}(\mathcal{I}^2)} \le \mathcal{K}\{[n_1]^{-1}_{q_{n_1}}+[n_2]^{-1}_{q_{n_2}}\}^{\frac{\theta}{2}},$$

where $\mathcal{K}$ is some positive constant.

Proof. From hypothesis, we have

$$\begin{aligned}|\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\zeta;x_1,x_2)-\zeta(x_1,x_2)| &\le \mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(|\zeta(\hbar_1,\hbar_2)-\zeta(x_1,x_2)|\,;x_1,x_2)\\ &\le \mathcal{M}\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\|r-x\|^{\theta};x_1,x_2),\end{aligned}$$

where $r=(\hbar_1,\hbar_2)$, $x=(x_1,x_2)\in\mathcal{I}^2$. Applying Hölder's inequality , we obtain

$$\begin{aligned}|\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\zeta;x_1,x_2)-(\zeta;x_1,x_2)| &\le \mathcal{M}\{\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\|r-x\|^2;x_1,x_2)\}^{\frac{\theta}{2}}\\ &\le \mathcal{K}\{\theta^{q_{n_1}}_{n_1,\alpha_1}+\gamma^{q_{n_2}}_{n_2,\alpha_2}\}^{\frac{\theta}{2}},\ \forall\, x_1,x_2\in\mathcal{I}^2,\end{aligned}$$

hence using Lemma 4.2, the required assertion is proved. □

Next, we discuss a Voronovskaja type asymptotic theorem.

Theorem 5.8. *Let $\zeta\in\mathcal{C}^2(\mathcal{I}^2)$. Then*

$$\begin{aligned}&\lim_{n\to\infty}[n]_{q_n}\left(\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}(\zeta;x_1,x_2)-\zeta(x_1,x_2)\right)=\\ &=\Big[(1-c)\{x_1^2(p_1(2\alpha_1-1)-4p_0(1-\alpha_1))+x_1((1-2\alpha_1)+p_0(1+\alpha_1))\}\\ &\quad+(p_0+p_1)(1-2x_1)\Big]\zeta'_{x_1}(x_1,x_2)+\Big[(1-c)\{x_2^2(p_1(2\alpha_2-1)-4p_0(1-\alpha_2))\\ &\quad+x_2((1-2\alpha_2)+p_0(1+\alpha_2))\}+(p_0+p_1)(1-2x_2)\Big]\zeta'_{x_2}(x_1,x_2)\\ &\quad+\tfrac{1}{2}\Big[p_1\{4(1-c)x_1^3(1-\alpha_1)(1-x_1)+6x_1^4-7x_1^3+6\alpha_1x_1^3(1-x_1)+x_1^2\\ &\quad+x_1(1-x_1)^2\}+p_0\{4(1-c)x_1^2(1-\alpha_1)(1-x_1)\\ &\quad+6x_1^3-8x_1^2+6\alpha_1x_1^2(1-x_1)+2x_1\}\Big]\zeta''_{x_1x_1}(x_1,x_2)\\ &\quad+\tfrac{1}{2}\Big[p_1\{4(1-c)x_2^3(1-\alpha_2)(1-x_2)+6x_2^4-7x_2^3+6\alpha_2x_2^3(1-x_2)+x_2^2\\ &\quad+x_2(1-x_2)^2\}+p_0\{4(1-c)x_2^2(1-\alpha_2)(1-x_2)+6x_2^3-8x_2^2\\ &\quad+6\alpha_2x_2^2(1-x_2)+2x_2\}\Big]\zeta''_{x_2x_2}(x_1,x_2),\end{aligned}$$

uniformly in $(x_1,x_2)\in\mathcal{I}^2$.

Proof. Let $(x_1, x_2) \in \mathcal{I}^2$ be an arbitrary but fixed point. Using Taylor's theorem, we have

$$\begin{aligned}\zeta(\hbar_1, \hbar_2) &= \zeta(x_1, x_2) + \zeta'_{x_1}(x_1, x_2)(\hbar_1 - x_1) + \tfrac{1}{2}\{\zeta''_{x_1x_1}(x_1, x_2)(\hbar_1 - x_1)^2 + \\ &\quad + 2\zeta''_{x_1x_2}(x_1, x_2)(\hbar_1 - x_1)(\hbar_2 - x_2) + \zeta''_{x_2x_2}(x_1, x_2)(\hbar_2 - x_2)^2\} \\ &\quad + \varpi(\hbar_1, \hbar_2; x_1, x_2)\sqrt{((\hbar_1 - x_1)^4 + (\hbar_2 - x_2)^4)} + \zeta'_{x_2}(x_1, x_2)(\hbar_2 - x_2),\end{aligned} \tag{5.7}$$

where $\varpi(\hbar_1, \hbar_2; x_1, x_2) \in \mathcal{C}(\mathcal{I}^2)$ and $\varpi(\hbar_1, \hbar_2; x_1, x_2) \to 0$, as $(\hbar_1, \hbar_2) \to (x_1, x_2)$.
Operating $\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}(.; x_1, x_2)$ on both sides of (5.7), we have

$$\begin{aligned}\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}(\zeta; x_1, x_2) &= \zeta'_{x_1}(x_1, x_2)\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_1 - x_1); x_1, x_2) + \\ &+ \zeta'_{x_2}(x_1, x_2)\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_2 - x_2); x_1, x_2) \\ &+ \tfrac{1}{2}\{\zeta''_{x_1x_1}(x_1, x_2)\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_1 - x_1)^2; x_1, x_2) \\ &+ 2\zeta''_{x_1x_2}(x_1, x_2)\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_1 - x_1)(\hbar_2 - x_2); x_1, x_2) \\ &+ \zeta''_{x_2x_2}(x_1, x_2)\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_2 - x_2)^2; x_1, x_2)\} + \zeta(x_1, x_2) \\ &+ \mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}(\varpi(\hbar_1, \hbar_2; x_1, x_2)\sqrt{((\hbar_1 - x_1)^4 + (\hbar_2 - x_2)^4)}; x_1, x_2).\end{aligned} \tag{5.8}$$

Applying Cauchy-Schwarz inequality to the last term of (5.8), we obtain

$$\begin{aligned}&\left|\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}\left(\varpi(\hbar_1, \hbar_2; x_1, x_2)\sqrt{((\hbar_1 - x_1)^4 + (\hbar_2 - x_2)^4)}; x_1, x_2\right)\right| \leq \\ &\leq \{\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}(\varpi^2(\hbar_1, \hbar_2; x_1, x_2); x_1, x_2)\}^{1/2} \\ &\quad \times \left\{\sqrt{\{\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_1 - x_1)^4; x_1, x_2) + \mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_2 - x_2)^4; x_1, x_2)\}}\right\}.\end{aligned}$$

Since $\varpi(., .; x_1, x_2) \in \mathcal{C}(\mathcal{I}^2)$ and $\varpi(\hbar_1, \hbar_2; x_1, x_2) \to 0$, as $(\hbar_1, \hbar_2) \to (x_1, x_2)$, applying Theorem 5.1, we obtain

$$\lim_{n\to\infty}[n]_{q_n}\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}(\varpi(\hbar_1, \hbar_2; x_1, x_2)\sqrt{((\hbar_1 - x_1)^4 + (\hbar_2 - x_2)^4)}; x_1, x_2) = 0,$$

uniformly in $(x_1, x_2) \in \mathcal{I}^2$.

Now, using Lemma 4.3 and by the above equation, from (5.8) we reach to the required result. □

Grüss [20] determined the difference between the integral of a product of two functions and the product of integrals of the two functions. Later, Gal and Gonska [18], studied the Grüss Voronovskaya type theorem for Bernstein and Paltanea operators with the aid of Grüss inequality which deals with the non-multiplicavity of the operators. For more details in this direction, one can see ([1], [28]) and the references therein. In the following theorem, we examine the non-multiplicativity of the operators $\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}$.

THEOREM 5.9 (Grüss Voronovskaja type theorem). *For $\zeta, \tau \in \mathcal{C}^2(\mathcal{I}^2)$, there holds*

$$\lim_{n\to\infty}[n]_{q_n}\{\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}(\zeta\tau;x_1,x_2)-\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}(\zeta;x_1,x_2)\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}(\tau;x_1,x_2)\}=$$
$$=\Big[p_1\{4(1-c)x_1^3(1-\alpha_1)(1-x_1)+6x_1^4-7x_1^3+6\alpha_1x_1^3(1-x_1)+x_1^2$$
$$+x_1(1-x_1)^2\}+p_0\{4(1-c)x_1^2(1-\alpha_1)(1-x_1)$$
$$+6x_1^3-8x_1^2+6\alpha_1x_1^2(1-x_1)+2x_1\}\Big]\zeta'_{x_1}(x_1,x_2)\tau'_{x_1}(x_1,x_2)$$
$$+\Big[p_1\{4(1-c)x_2^3(1-\alpha_2)(1-x_2)+6x_2^4-7x_2^3+6\alpha_2x_2^3(1-x_2)+x_2^2$$
$$+x_2(1-x_2)^2\}+p_0\{4(1-c)x_2^2(1-\alpha_2)(1-x_2)+6x_2^3-8x_2^2$$
$$+6\alpha_2x_2^2(1-x_2)+2x_2\}\Big]\zeta'_{x_2}(x_1,x_2)\tau'_{x_2}(x_1,x_2),$$

uniformly in $(x_1,x_2)\in\mathcal{I}^2$.

Proof. By our hypothesis, we obtain

$$[n]_{q_n}\{\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}(\zeta\tau;x_1,x_2)-\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}(\zeta;x_1,x_2)\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}(\tau;x_1,x_2)\}=$$
$$=[n]_{q_n}\Big(\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}(\zeta\tau;x_1,x_2)-\zeta(x_1,x_2)\tau(x_1,x_2)-(\zeta(x_1,x_2)\tau'_{x_1}(x_1,x_2)$$
$$+\tau(x_1,x_2)\zeta'_{x_1}(x_1,x_2))\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_1-x_1);x_1,x_2)-(\zeta(x_1,x_2)\tau_{x_2}(x_1,x_2)$$
$$+\tau(x_1,x_2)\zeta'_{x_2}(x_1,x_2))\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_2-x_2);x_1,x_2)-\tfrac{1}{2}(\zeta(x_1,x_2)\tau''_{x_1x_1}(x_1,x_2)$$
$$+2\zeta'_{x_1}(x_1,x_2)\tau'_{x_1}(x_1,x_2)+\tau(x_1,x_2)\zeta''_{x_1x_1}(x_1,x_2))\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_1-x_1)^2;x_1,x_2)$$
$$-(\zeta(x_1,x_2)\tau''_{x_1x_2}(x_1,x_2)+\zeta'_{x_1}(x_1,x_2)\tau'_{x_2}(x_1,x_2)+\zeta'_{x_2}(x_1,x_2)\tau'_{x_1}(x_1,x_2)$$
$$+\tau(x_1,x_2)\zeta''_{x_1x_2}(x_1,x_2))\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_1-x_1)(\hbar_2-x_2);x_1,x_2)$$
$$-\tfrac{1}{2}(\zeta(x_1,x_2)\tau''_{x_2x_2}(x_1,x_2)+2\zeta'_{x_2}(x_1,x_2)\tau'_{x_2}(x_1,x_2)$$
$$+\tau(x_1,x_2)\zeta''_{x_2x_2}(x_1,x_2))\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_2-x_2)^2;x_1,x_2)-\tau(x_1,x_2)$$
$$\times\Big(\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}(\zeta;x_1,x_2)-\zeta(x_1,x_2)-\zeta'_{x_1}(x_1,x_2)\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_1-x_1);x_1,x_2)$$
$$-\zeta'_{x_2}(x_1,x_2)\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_2-x_2);x_1,x_2)$$
$$-\tfrac{1}{2}\zeta''_{x_1x_1}(x_1,x_2)\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_1-x_1)^2;x_1,x_2)$$
$$-\zeta''_{x_1x_2}(x_1,x_2)\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_1-x_1)(\hbar_2-x_2);x_1,x_2)$$
$$-\tfrac{1}{2}\zeta''_{x_2x_2}(x_1,x_2)\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_2-x_2)^2;x_1,x_2)\Big)-\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}(\zeta;x_1,x_2)$$
$$\times\Big(\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}(\tau;x_1,x_2)-\tau(x_1,x_2)-\tau'_{x_1}(x_1,x_2)\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_1-x_1);x_1,x_2)$$

$$
\begin{aligned}
&- \tau'_{x_2}(x_1,x_2)\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_2-x_2);x_1,x_2)\\
&- \tfrac{1}{2}\tau''_{x_1x_1}(x_1,x_2)\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_1-x_1)^2;x_1,x_2)\\
&- \tau''_{x_1x_2}(x_1,x_2)\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_1-x_1)(\hbar_2-x_2);x_1,x_2)\\
&- \tfrac{1}{2}\tau''_{x_2x_2}(x_1,x_2)\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_2-x_2)^2;x_1,x_2)\Big)\\
&+ \tau'_{x_1}(x_1,x_2)\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_1-x_1);x_1,x_2)(\zeta(x_1,x_2)-\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}(\zeta;x_1,x_2))\\
&+ \tau'_{x_2}(x_1,x_2)\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_2-x_2);x_1,x_2)(\zeta(x_1,x_2))-\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}(\zeta;x_1,x_2))\\
&+ \tau''_{x_1x_1}(x_1,x_2)\tfrac{\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_1-x_1)^2;x_1,x_2)}{2}(\zeta(x_1,x_2)-\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}(\zeta;x_1,x_2))\\
&+ \tau''_{x_2x_2}(x_1,x_2)\tfrac{\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_2-x_2)^2;x_1,x_2)}{2}(\zeta(x_1,x_2)-\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}(\zeta;x_1,x_2))\\
&+ \tau''_{x_1x_2}(x_1,x_2)\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_1-x_1)(\hbar_2-x_2);x_1,x_2)\\
\times &(\zeta(x_1,x_2)- \quad -\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}(\zeta;x_1,x_2))\\
&+ \zeta'_{x_1}(x_1,x_2)\tau'_{x_1}(x_1,x_2)\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_1-x_1)^2;x_1,x_2)\\
&+ \zeta'_{x_1}(x_1,x_2)\tau'_{x_2}(x_1,x_2)\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_1-x_1)(\hbar_2-x_2);x_1,x_2)\\
&+ \zeta'_{x_2}(x_1,x_2)\tau'_{x_1}(x_1,x_2)\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_1-x_1)(\hbar_2-x_2);x_1,x_2)\\
&+ \zeta'_{x_2}(x_1,x_2)\tau'_{x_2}(x_1,x_2)\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_2-x_2)^2;x_1,x_2)\Big).
\end{aligned}
$$

From Theorem 5.1, for all $\zeta \in \mathcal{C}(\mathcal{I}^2)$, it follows that $\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}(\zeta;x_1,x_2) \to \zeta(x_1,x_2)$, as $n\to\infty$, uniformly in $(x_1,x_2)\in\mathcal{I}^2$, and by Theorem 5.8, for every $\zeta\in\mathcal{C}^2(\mathcal{I}^2)$, we have

$$
\begin{aligned}
&\lim_{n\to\infty}[n]_{q_n}\Big[\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}(\zeta;x_1,x_2)-\zeta(x_1,x_2)-\\
&-\zeta'_{x_1}(x_1,x_2)\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_1-x_1);x_1,x_2)\\
&-\zeta'_{x_2}(x_1,x_2)\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_2-x_2);x_1,x_2)\\
&-\tfrac{1}{2}\{\zeta''_{x_1x_1}(x_1,x_2)\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_1-x_1)^2;x_1,x_2)\\
&\quad+2\zeta''_{x_1x_2}\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_1-x_1)(\hbar_2-x_2);x_1,x_2)\\
&\quad+\zeta''_{x_2x_2}\mathcal{G}^{q_n,q_n}_{n,n,\alpha_1,\alpha_2}((\hbar_2-x_2)^2;x_1,x_2)\}\Big]=0,
\end{aligned}
$$

uniformly in $(x_1,x_2)\in\mathcal{I}^2$.

Hence, in view of the fact that $\zeta,\tau\in\mathcal{C}^2(\mathcal{I}^2)$, using Lemma 4.3, we reach the desired assertion. □

6. CONSTRUCTION OF GBS OPERATORS FOR THE BIVARIATE OPERATORS $\mathcal{G}_{n_1,n_2,\alpha_1,\alpha_2}^{q_{n_1},q_{n_2}}(\,\cdot\,;x_1,x_2)$

In the past decade, the study of *GBS* (Generalized Boolean Sum) operators associated with the positive linear operators has been an active area of research in the field of approximation theory. The concepts of Bögel continuous and Bögel differentiable functions were first given by Bögel [10]. Dobrescu and Matei [17] showed that any Bögel continuous function on a bounded interval can be uniformly approximated by the boolean sum of bivariate Bernstein polynomials. Badea *et al.* [6] obtained a Korovkin type theorem for the Bögel continuous functions. Agrawal *et al.* [3] studied convergence estimates for the GBS case of the bivariate Lupas-Durrmeyer type operators based on Polya distribution. Barbosu *et al.* [8] proposed GBS operators of Durrmeyer-Stancu type based on q-integers and examined the approximation degree by using Lipschitz class and the mixed modulus of smoothness. Kajla and Miclaus [25] determined the convergence behaviour of GBS operators of Bernstein-Durrmeyer type for Bögel continuous and Bögel differentiable functions. Agrawal and Chauhan [36] introduced the sequence of GBS operators of Bernstein-Durrmeyer type on a triangle and investigated the rate of convergence by virtue of the mixed modulus of smoothness for Bögel continuous and Bögel differentiable functions. For a detailed account of the research in this direction, one can see [19] and the references therein.

A function $\zeta:\mathcal{I}^2\to\mathbb{R}$, is called B-continuous (Bögel continuous) at a point $(x_1,x_2)\in\mathcal{I}^2$ if

$$\lim_{(\hbar_1,\hbar_2)\to(x_1,x_2)}\Delta_{(x_1,x_2)}\zeta[(\hbar_1,\hbar_2);(x_1,x_2)]=0,$$

where $\Delta_{(x_1,x_2)}\zeta[(\hbar_1,\hbar_2);(x_1,x_2)]=\zeta(\hbar_1,\hbar_2)-\zeta(\hbar_1,x_2)-\zeta(x_1,\hbar_2)+\zeta(x_1,x_2)$. Further, a function $\zeta:\mathcal{I}^2\to\mathbb{R}$, is said to be B- continuous on $\mathcal{I}^2$, if is B-continuous $\forall(x_1,x_2)\in\mathcal{I}^2$.

A function $\zeta:\mathcal{I}^2\to\mathbb{R}$, is called B- differentiable (Bögel differentiable) on $\mathcal{I}^2$, if for every $(x_1,x_2)\in\mathcal{I}^2$,

$$\lim_{(\hbar_1,\hbar_2)\to(x_1,x_2)}\frac{\Delta_{(x_1,x_2)}\zeta[(\hbar_1,\hbar_2);(x_1,x_2)]}{(\hbar_1-x_1)(\hbar_2-x_2)}=D_B\zeta(x_1,x_2)<\infty.$$

The function $\zeta:\mathcal{I}^2\to\mathbb{R}$ is said to be B-bounded on $\mathcal{I}^2$ if $\exists$ some $K>0$, such that $|\Delta_{(x_1,x_2)}\zeta[(\hbar_1,\hbar_2);(x_1,x_2)]|\leq K$, for every $(\hbar_1,\hbar_2)$, $(x_1,x_2)\in\mathcal{I}^2$. The space of B-bounded functions is denoted by $B_b(\mathcal{I}^2)$, the space of B-continuous functions is denoted by $C_b(\mathcal{I}^2)$ and the space of all B-differentiable functions is denoted by $D_b(\mathcal{I}^2)$. Further, let $B(\mathcal{I}^2)$ be the space of bounded functions on $\mathcal{I}^2$ endowed with the sup-norm denoted by $\|.\|_\infty$.

For every $\zeta \in \mathcal{C}_b(\mathcal{I}^2)$, the GBS operator associated with the operators defined in (4.1) is defined as:

$$\mathcal{S}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\zeta;x_1,x_2) = \sum_{k=0}^{n_1}\sum_{j=0}^{n_2}\mathfrak{P}^{\alpha_1,\alpha_2}_{n_1,n_2,k,j}(x_1,x_2)\Bigg[\zeta\left(\frac{[k]_{q_{n_1}}}{[n_1]_{q_{n_1}}},x_2\right) + \zeta\left(x_1,\frac{[j]_{q_{n_2}}}{[n_2]_{q_{n_2}}}\right) - \zeta\left(\frac{[k]_{q_{n_1}}}{[n_1]_{q_{n_1}}},\frac{[j]_{q_{n_2}}}{[n_2]_{q_{n_2}}}\right)\Bigg]. \tag{6.1}$$

The operator $\mathcal{S}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}$ is well defined on the space $\mathcal{C}_b(\mathcal{I}^2)$ into $\mathcal{C}(\mathcal{I}^2)$. We shall analyze the order of approximation of $\mathcal{S}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\zeta)$ to ζ, for all $\zeta \in \mathcal{C}_b(\mathcal{I}^2)$, using mixed modulus of smoothness.

First, we show the uniform convergence of the operators (6.1) to ζ, where $\zeta \in \mathcal{C}_b(\mathcal{I}^2)$ by using the following result:

LEMMA 6.1 ([7]). *Let $\mathcal{K}_{m,n} : C_b(X\times Y) \to B(X\times Y), m,n \in \mathbb{N}$ be a sequence of bivariate positive linear operators. Further, let $G_{m,n}$ be the associated GBS operators and the following identities hold:*

(1) $\mathcal{K}_{m,n}(e_{00};x_1,x_2) = 1$;
(2) $\mathcal{K}_{m,n}(e_{10};x_1,x_2) = x_1 + \alpha_{m,n}(x_1,x_2)$;
(3) $\mathcal{K}_{m,n}(e_{01};x_1,x_2) = x_2 + \beta_{m,n}(x_1,x_2)$
(4) $\mathcal{K}_{m,n}(e_{20}+e_{02};x_1,x_2) = x_1^2 + x_2^2 + \gamma_{m,n}(x_1,x_2)$

for all $(x_1,x_2) \in X\times Y$. If the sequences $\alpha_{m,n}, \beta_{m,n}$, and $\gamma_{m,n}$ converge to zero, as $m,n\to\infty$, uniformly on $X\times Y$, then the sequence $(G_{m,n}(\zeta))$ converges to ζ, as $m,n\to\infty$, uniformly on $X\times Y$ for all $\zeta \in C_b(X\times Y)$.

THEOREM 6.2. *For $\zeta \in \mathcal{C}_b(\mathcal{I}^2)$, we have*

$$\lim_{n\to\infty}\mathcal{S}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\zeta;x_1,x_2) = \zeta(x_1,x_2),$$

uniformly in $(x_1,x_2) \in \mathcal{I}^2$.

Proof. From Lemma 4.1, it follows that

$$\lim_{n\to\infty}\|\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(e_{ij}) - e_{ij}\|_{\mathcal{C}(\mathcal{I}^2)} = 0,\ \forall\,(i,j) \in \{(0,0),(1,0),(0,1)\}$$

and

$$\lim_{n\to\infty}\|\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(e_{20}+e_{02}) - (e_{20}+e_{02})\|_{\mathcal{C}(\mathcal{I}^2)} = 0,$$

hence applying Lemma 6.1, we obtain the desired conclusion. □

In the next result, we examine convergence estimates of $\mathcal{S}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\zeta)$ to ζ by virtue of mixed modulus of smoothness.

For (x_1,x_2), $(\hbar_1,\hbar_2) \in \mathcal{I}^2$, the mixed modulus of smoothness of $\zeta \in \mathcal{C}_b(\mathcal{I}^2)$ is defined by

$$\widetilde{\omega_B}(\zeta,\rho_1,\rho_2) = \sup\{|\Delta_{(x_1,x_2)}\zeta[\hbar_1,\hbar_2;x_1,x_2]| : |\hbar_1 - x_1| < \rho_1, |\hbar_2 - x_2| < \rho_2\}$$

for any $(\rho_1,\rho_2) \in (0,\infty)\times(0,\infty)$. From definition of $\widetilde{\omega_B}(\zeta,\rho_1,\rho_2)$, it follows that

$$\widetilde{\omega_B}(\zeta,c_1\rho_1,c_2\rho_2) \le (1+c_1)(1+c_2)\widetilde{\omega_B}(\zeta,\rho_1,\rho_2),$$

for any $c_1, c_2 > 0$.

THEOREM 6.3. *For every $\zeta \in \mathcal{C}_b(\mathcal{I}^2)$, and sufficiently large n_1 and n_2, the operator defined by (6.1), verifies the following result:*

$$\|\mathcal{S}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\zeta) - \zeta\|_{\mathcal{C}(\mathcal{I}^2)} \leq K^{\alpha_2}_{\alpha_1}\widetilde{\omega_B}(\zeta; [n_1]^{-\frac{1}{2}}_{q_{n_1}}, [n_2]^{-\frac{1}{2}}_{q_{n_2}}),$$

where $K^{\alpha_2}_{\alpha_1}$ is some positive constant depending on α_1 and α_2.

Proof. Considering the properties of the function $\widetilde{\omega_B}$, we get

$$|\Delta_{(x_1,x_2)}\zeta[(\hbar_1,\hbar_2);(x_1,x_2)]| \leq \widetilde{\omega_B}(\zeta; |\hbar_1 - x_1|, |\hbar_2 - x_2|)$$

$$\leq \left(1 + \frac{|\hbar_1 - x_1|}{\rho_1}\right)\left(1 + \frac{|\hbar_2 - x_2|}{\rho_2}\right)\widetilde{\omega_B}(\zeta; \rho_1, \rho_2), \tag{6.2}$$

for every $(x_1, x_2), (\hbar_1, \hbar_2) \in \mathcal{I}^2$ and for any $\rho_1, \rho_2 > 0$. Using the definition of $\Delta_{(x_1,x_2)}\zeta[(\hbar_1,\hbar_2);(x_1,x_2)]$, we may write

$$\zeta(x_1,\hbar_2) + \zeta(\hbar_1,x_2) - \zeta(\hbar_1,\hbar_2) = \zeta(x_1,x_2) - \Delta_{(x_1,x_2)}\zeta[(\hbar_1,\hbar_2);(x_1,x_2)].$$

Applying the operator $\mathcal{S}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(.; x_1, x_2)$ on both sides of the above equality, we get

$$\begin{aligned}\mathcal{S}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\zeta; x_1, x_2) =& \zeta(x_1,x_2)\,\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(e_{00}; x_1, x_2)\\ &- \mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}\left(\Delta_{(x_1,x_2)}\zeta[(\hbar_1,\hbar_2);(x_1,x_2)]; x_1, x_2\right).\end{aligned} \tag{6.3}$$

Hence using (6.2), Lemma 4.1 and applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}&|\mathcal{S}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\zeta; x_1, x_2) - \zeta(x_1,x_2)| \leq\\ &\leq \mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}\left(|\Delta_{(x_1,x_2)}\zeta[(\hbar_1,\hbar_2);(x_1,x_2)]|; x_1, x_2\right)\\ &\leq \Bigg(1 + \rho_1^{-1}\sqrt{\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}((\hbar_1 - x_1)^2; x_1, x_2)}\\ &\quad + \rho_2^{-1}\sqrt{\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}((\hbar_2 - x_2)^2; x_1, x_2)}\\ &\quad + \rho_1^{-1}\rho_2^{-1}\sqrt{\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}((\hbar_1 - x_1)^2(\hbar_2 - x_2)^2; x_1, x_2)}\Bigg)\,\widetilde{\omega_B}(\zeta; \rho_1, \rho_2).\end{aligned} \tag{6.4}$$

Now, choosing $\rho_i = [n_i]^{\frac{-1}{2}}_{q_{n_i}}, i = 1, 2$, and applying Lemma 4.3, the required assertion is proved. □

The following result is concerned with the error in the approximation of the B-differentiable functions by the operators (6.1).

THEOREM 6.4. *If $\zeta \in D_b(\mathcal{I}^2)$ and $D_B\zeta \in B(\mathcal{I}^2)$, then*

$$\|\mathcal{S}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\zeta; x_1, x_2) - \zeta(x_1,x_2)\|_{\mathcal{C}(\mathcal{I}^2)} \leq$$

$$\leq \frac{C}{[n_1]^{\frac{1}{2}}_{q_{n_1}}[n_2]^{\frac{1}{2}}_{q_{n_2}}}\left\{\|D_B\zeta\|_\infty + \widetilde{\omega_B}(D_b\zeta; [n_1]^{-\frac{1}{2}}_{q_{n_1}}, [n_2]^{-\frac{1}{2}}_{q_{n_2}})\right\},$$

where $C > 0$ *is some constant.*

Proof. Since $\zeta \in D_b(\mathcal{I}^2)$, by mean value theorem, we have
$\Delta_{(x_1,x_2)}\zeta[(\hbar_1,\hbar_2);(x_1,x_2)] = (\hbar_1 - x_1)(\hbar_2 - x_2)D_B\zeta(\alpha,\beta),\ with\, x_1 < \alpha < \hbar_1,$ and $x_2 < \beta < \hbar_2$. Clearly,
$D_B\zeta(\alpha,\beta) = \Delta_{(x_1,x_2)}D_B\zeta[(\alpha,\beta);(x_1,x_2)] + D_B\zeta(\alpha,x_2) + D_B\zeta(x_1,\beta) - D_B\zeta(x_1,x_2).$
Since $D_B\zeta \in B(\mathcal{I}^2)$, from the above equalities, we have

$$
\begin{aligned}
(6.5)\quad & |\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\Delta_{(x_1,x_2)}\zeta[(\hbar_1,\hbar_2);(x_1,x_2)];x_1,x_2)| = \\
& = |\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}((\hbar_1 - x_1)(\hbar_2 - x_2)D_B\zeta(\alpha,\beta);x_1,x_2)]| \\
& \leq \mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(|\hbar_1 - x_1||\hbar_2 - x_2||\Delta_{(x_1,x_2)}D_B\zeta[(\alpha,\beta);(x_1,x_2)]|;x_1,x_2) \\
& \quad + \mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}\Big(|\hbar_1 - x_1||\hbar_2 - x_2|\Big(|D_B\zeta(\alpha,x_2)| \\
& \qquad\qquad + |D_B\zeta(x_1,\beta)| + |D_B\zeta(x_1,x_2)|\Big);x_1,x_2\Big) \\
& \leq \mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(|\hbar_1 - x_1||\hbar_2 - x_2|\widetilde{\omega_B}(D_B\zeta;|\alpha - x_1|,|\beta - x_2|);x_1,x_2) \\
& \quad + 3|||D_B\zeta||_\infty\ \mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(|\hbar_1 - x_1||\hbar_2 - x_2|;x_1,x_2).
\end{aligned}
$$

Considering the properties of mixed modulus of smoothness $\widetilde{\omega_B}$, for any ρ_1, $\rho_2 > 0$, we have

$$
\begin{aligned}
& \widetilde{\omega_B}(D_B\zeta;|\alpha - x_1|,|\beta - x_2|) \leq \widetilde{\omega_B}(D_B\zeta;|\hbar_1 - x_1|,|\hbar_2 - x_2|) \\
(6.6)\quad & \qquad \leq \prod_{i=1}^{2}(1 + \rho_i^{-1}|\hbar_i - x_i|)\ \widetilde{\omega_B}(D_B\zeta;\rho_1,\rho_2).
\end{aligned}
$$

Hence taking into account (6.5), (6.6) and applying the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
(6.7)\quad & |\mathcal{S}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\zeta;x_1,x_2) - \zeta(x_1,x_2)| = \\
& = |\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\Delta\zeta[(\hbar_1,\hbar_2);(x_1,x_2)];x_1,x_2)| \leq \\
& \leq 3||D_B\zeta||_\infty\sqrt{\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}((\hbar_1 - x_1)^2(\hbar_2 - x_2)^2;x_1,x_2)} \\
& \quad + \Big(\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(|\hbar_1 - x_1||\hbar_2 - x_2|;x_1,x_2) \\
& \quad + \rho_1^{-1}\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}((\hbar_1 - x_1)^2|\hbar_2 - x_2|;x_1,x_2) \\
& \quad + \rho_2^{-1}\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(|\hbar_1 - x_1|(\hbar_2 - x_2)^2;x_1,x_2) \\
& \quad + \rho_1^{-1}\rho_2^{-1}\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}((\hbar_1 - x_1)^2(\hbar_2 - x_2)^2;x_1,x_2)\Big)\widetilde{\omega_B}(D_B\zeta;\rho_1,\rho_2) \\
& \leq 3||D_B\zeta||_\infty\sqrt{\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}((\hbar_1 - x_1)^2(\hbar_2 - x_2)^2;x_1,x_2)} \\
& \quad + \Big(\sqrt{\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}((\hbar_1 - x_1)^2(\hbar_2 - x_2)^2;x_1,x_2)}
\end{aligned}
$$

$$+\rho_1^{-1}\sqrt{\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}((\hbar_1-x_1)^4(\hbar_2-x_2)^2;x_1,x_2)}$$
$$+\rho_2^{-1}\sqrt{\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}((\hbar_1-x_1)^2(\hbar_2-x_2)^4;x_1,x_2)}$$
$$+\rho_1^{-1}\rho_2^{-1}\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}((\hbar_1-x_1)^2(\hbar_2-x_2)^2;x_1,x_2)\Big)\widetilde{\omega_B}(D_B\zeta;\rho_1,\rho_2).$$

Since, for $i,j\in\{1,2\}$, we have

$$\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}((\hbar_1-x_1)^{2i}(\hbar_2-x_2)^{2j};x_1,x_2)=$$
$$=\mathcal{O}(\tfrac{1}{[n_1]^i_{q_{n_1}}})\mathcal{O}(\tfrac{1}{[n_2]^j_{q_{n_2}}}),\qquad as\, n_1,n_2\to\infty, \tag{6.8}$$

uniformly in $(x_1,x_2)\in\mathcal{I}^2$, combining (6.7)–(6.8) and choosing $\rho_i=[n_i]^{\frac{-1}{2}}_{q_{n_i}}, i=1,2$, we reach the desired result. □

Now, we discuss the approximation degree of the operators (6.1) for Lipschitz class of B-continuous functions.

The Lipschitz class $\mathrm{Lip}^\theta_{\mathcal{M},b}$ with $\theta\in(0,1]$, for B-continuous functions is defined by

$$\mathrm{Lip}^\theta_{\mathcal{M},b}=\Big\{\zeta\in\mathcal{C}_b(\mathcal{I}^2):|\Delta_{(x_1,x_2)}\zeta[(\hbar_1,\hbar_2);(x_1,x_2)]|\le\mathcal{M}\{(\hbar_1-x_1)^2+(\hbar_2-x_2)^2\}^{\frac{\theta}{2}}\Big\}$$

for every $(\hbar_1,\hbar_2)$, $(x_1,x_2)\in\mathcal{I}^2$.

THEOREM 6.5. *For* $\zeta\in\mathrm{Lip}^\theta_{\mathcal{M},b}$, $\theta\in(0,1]$, *we have*

$$\|\mathcal{S}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\zeta)-\zeta\|_{\mathcal{C}(\mathcal{I}^2)}\le\mathcal{M}(\theta^{q_{n_1}}_{n_1,\alpha_1}+\gamma^{q_{n_2}}_{n_2,\alpha_2})^{\frac{\theta}{2}}.$$

Proof. Considering Lemma 4.1 and (6.3), by our hypothesis we get

$$\Big|\mathcal{S}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\zeta;x_1,x_2)-\zeta(x_1,x_2)\Big|\le$$
$$\le\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}\left(\Big|\Delta_{(\hbar_1,\hbar_2)}\zeta[(x_1,x_2);(x_1,x_2)]\Big|;x_1,x_2\right)$$
$$\le\mathcal{M}\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}\left(((\hbar_1-x_1)^2+(\hbar_2-x_2)^2)^{\frac{\theta}{2}};x_1,x_2\right).$$

Now, applying the Hölder's inequality with $(p_1,q_1)=\left(\frac{2}{\theta},\frac{2}{(2-\theta)}\right)$ and using Lemma 4.1, we get

$$|\mathcal{S}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(\zeta;x_1,x_2)-\zeta(x_1,x_2)|\le$$
$$\le\{\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}(e_{00};x_1,x_2)\}^{\frac{2}{(2-\theta)}}$$
$$\times\mathcal{M}\{\mathcal{G}^{q_{n_1},q_{n_2}}_{n_1,n_2,\alpha_1,\alpha_2}\left(((\hbar_1-x_1)^2+(\hbar_2-x_2)^2);x_1,x_2\right)\}^{\theta/2}$$
$$\le\mathcal{M}(\theta^{q_{n_1}}_{n_1,\alpha_1}+\gamma^{q_{n_2}}_{n_2,\alpha_2})^{\frac{\theta}{2}},$$

which yields us the required result. This completes the proof. □

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