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QUANTITATIVE INHERITANCE PROPERTIES FOR SIMULTANEOUS APPROXIMATION BY TENSOR PRODUCT OPERATORS II: APPLICATIONS*

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Dedicated to Professor Gabriel V. Orman

Abstract. We summarize several general results concerning quantitative inheritance properties for simultaneous approximation by tensor product operators and apply these to various situations. All inequalities are given in terms of moduli of continuity of higher order.

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1. INTRODUCTORY REMARKS AND NOTATIONS

Products of parametric extensions of d univariate operators are appropriate tools to approximate functions defined on the product of d compact metric spaces. We will deal in this paper with twodimensional tensor products over compact real intervals. These are used in *Computer Aided Geometric Design* (CAGD) to construct approximating or interpolating free form surfaces (see [21], [42], [7] and the references therein). According to Schumakers classification (see [66]) the tensor product of two univariate operators can be viewed as a simple two-stage process. For the case of d dimensions see, *e.g.*, [29], [33] where, however, no assertions concerning simultaneous approximation were made. This paper is a continuation of our recent paper [3] and focusses on applications. A more detailed presentation including full proofs was given in our technical report [2].

The tensor product method is a restrictive one, especially because of the requirement that the information must be given in tensor product form, *e.g.*,

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in the case of discretely defined operators on a Cartesian product grid. We also have to recall that some directions are preferred on a tensor product surface, and if these directions are not properly chosen, then the obtained surface might not be "good enough".

Despite of these limitations one should use the tensor product method as often as it makes sense; as de Boor mentions in his book [16], this method is extremely efficient when it comes to implement it in comparison with other surface approximation techniques. Furthermore, two recent preprints by Höllig, Reif and Wipper [43], [44] show in an impressive way that the tensor product method is not dead at all. The method also remains useful in fitting functions given on the sphere. This was demonstrated in a talk given by Sbibih [64] at the 2001 International Conference on Numerical Algorithms (Marrakesh, Morocco).

The method of using the product of parametric extensions of univariate operators is quite a classic one. It was first used to a larger extent in the context of multivariate polynomial interpolation. Much historical information and many early references on the subject can be found in an article by Gasca and Sauer [23]. Here we mention in particular the 1926 paper by Neder [58]. To our knowledge the tensor product of the (noninterpolatory) Bernstein polynomial operators was first considered in 1933 by Hildebrandt and Schoenberg [41]. An early use of the tensor product method in combination with spline interpolation was made in de Boors 1962 article [14] on bicubic splines and–almost simultaneously–by Ferguson [22]. Both authors were guided by the work of Birkhoff and Garabedian [6].

The tensor product of two linear operators was considered for example by Stancu [74] in 1964. For further references and results concerning the method (and other multivariate ones) see, *e.g.*, a paper by Coatmelec [11], the dissertation of Haussmann [39], an article of Haussmann and Pottinger [40], and the ones by Mastroianni [54] and Lancaster [49]. Many other notes treated the method as well. Our bibliography lists several of these, although our paper does not refer to them explicitely; there is no claim for completeness at all.

A recent deep result concerning d-variate tensor product Bernstein operators was given by Xinlong Zhou who proved a strong converse inequality (see [87]). Similar results for d-variate tensor product Jackson operators were proved by Knoop and Zhou [48].

For the bivariate case the *method of parametric extensions* can be described as follows (see [37], [17], [19]); de Boor calls this the naive approach. We emphasize that this approach is the only one which is of interest in this note.

Let I and J be nontrivial compact intervals of the real axis $\mathbb{R}$. For $i = 1, 2$ let G_i be linear subspaces of $\mathbb{R}^I = \{g : I \ni x \ \to g(x) \in \mathbb{R}\}$, and H_i be such of $\mathbb{R}^J = \{h : J \ni y \to h(y) \in \mathbb{R}\}$. We consider linear operators

$$L : G_1 \to G_2 \text{ and } M : H_1 \to H_2.$$

For $f : I \to \mathbb{R}$ we define the partial functions $f_x : J \to \mathbb{R}$ and $f_y : I \to \mathbb{R}$ by

$$f_x\left(y\right) := f\left(x,y\right) =: f_y\left(x\right), \text{ for all } \left(x,y\right) \in I \times J.$$

Furthermore, putting

$$F := \{f \in \mathbb{R}^{I\times J} : f_y \in G_1, \forall y \in J\} \cap \{f \in \mathbb{R}^{I\times J} : f_x \in H_1, \forall x \in I\},$$

the parametric extension of L is given by ${}_xL : F \to \mathbb{R}^{I\times J}$,

$${}_xL\left(f;x;y\right) := L\left(f_y;x\right), \text{ for all } \left(x,y\right) \in I \times J.$$

Symmetrically, the parametric extension of M has the form ${}_yM : F \to \mathbb{R}^{I\times J}$,

$${}_yM\left(f;x;y\right) := M\left(f_x;y\right), \text{ for all } \left(x,y\right) \in I \times J.$$

We assume, furthermore, that ${}_xL : F \to F$ and

$${}_yM : F \to F$$

so that the linear operators

$${}_xL \circ {}_yM \text{ and } {}_yM \circ {}_xL$$

are both defined, mapping F into itself. The product of these parametric extensions is called the tensor product of L and M (M and L, respectively).

An answer to the question under which conditions the parametric extensions commute is known in certain cases. For example, Gordon and Cheney (see Theorem 9 in [37]) and, independently Potapov and Jimenez Pozo (see Theorem 1 in [61]) observed that this is so if $G_1 = G_2 = C(X)$, $H_1 = H_2 = C(Y)$, $F = C(X \times Y)$, where X and Y are compact metric spaces, and if $L : C(X) \to C(X)$, $M : C(Y) \to C(Y)$ are continuous linear operators with respect to the sup norms. Other conditions are discussed by de Boor [17, pp. 50–52], and by Lancaster [49, Th. 3], for example.

The situation relevant to us is described as follows: For $k, l \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, let the symbol $D^{(k,l)}$ denote the partial differential operator $\partial^{k+l}/\partial x^k \partial y^l$; occasionally we will write $f^{(k,l)}$ instead of $D^{(k,l)}f$. We define

$$\begin{aligned} &C^{p,q}\left(I \times J\right) := \\ &:= \{f : I \times J \to \mathbb{R} : D^{(k,l)}f \text{ is continuous for } \left(0,0\right) \le \left(k,l\right) \le \left(p,q\right)\}. \end{aligned}$$

The notation $(k,l) \le (p,q)$ indicates that $k \le p$ and $l \le q$. The corresponding symbols used for the univariate case will be $C^p\left(I\right), D^k, d^k/dx^k$ and $f^{(k)}$. For $p = q = 0$ we obtain the spaces $C(I), C(J)$ and $C(I \times J)$ of real-valued functions which are continuous on I, J and $I \times J$, respectively. Similarly, D^0 and $D^{(0,0)}$ mean the one and two dimensional identity operators.

Although we will not restrict our attention to these, many of our univariate building blocks will be *discretely defined*, a notion to be explained next.

Let $I = [a,b]$ and $\triangle_n; a \le x_0 < x_1 < \ldots < x_n \le b$ be a partition of $[a,b]$. With this sequence of points we associate an incidence matrix

$$E = \left(e_{i,j}\right)_{i=0,\ldots,n;j=0,\ldots,p}$$

where p is a nonnegative integer. Such matrices have as entries $|E| \geq n+1$ ones and $(n+1)(p+1)-|E|$ zeros and are such that in each row there is at least one entry equal to one. We also assume that the last column contains at least one entry equal to one.

Now suppose that $L_{\triangle_n,E} = L : C^p(I) \to \mathbb{R}$ is of the form

$$L(g;x) = \sum_{e_{i,,j}=l} g^{(i)}(x_i) \cdot A_{i,j}(x), \quad x \in I,$$

where $A_{i,j} \in \mathbb{R}^I$ are the fundamental functions of L. If, analogously, $M_{\triangle_m,F} = M : C^q(J) \to \mathbb{R}^J$ is given by

$$M(h;y) = \sum_{f_{i',,j'}=1} h^{(j')}(y_{i'}) \cdot B_{i',j'}(y), \quad y \in J,$$

then for $f \in C^{p,q}(I \times J)$ one has

$$\begin{aligned}
({}_xL \circ {}_yM)(f;x,y) &= \sum_{e_{i,,j}=1} \sum_{f_{i',,j'}=1} \left(\frac{\partial}{\partial x}\right)^j \left(\frac{\partial}{\partial y}\right)^{j'} f(x_i, y_{i'}) \cdot A_{i,j}(x) \cdot B_{i',j'}(y) \\
&= \sum_{f_{i',,j'}=1} \sum_{e_{i,,j}=1} \left(\frac{\partial}{\partial y}\right)^{j'} \left(\frac{\partial}{\partial x}\right)^j f(x_i, y_{i'}) \cdot B_{i',j'}(y) \cdot A_{i,j}(x) \\
&= ({}_yM \circ {}_xL)(f;x,y).
\end{aligned}$$

In our applications the univariate building blocks will be mostly of the form of L and M, respectively, so that in these cases it will be justified to speak about the tensor product of $\{L, M\}$.

For discretely defined operators on $C^{p,q}(I \times J)$ there also hold the following commutativity properties:

$$D^{(0,q)} \circ {}_xL = {}_xL \circ D^{(0,q)} \text{ and } D^{(p,0)} \circ {}_yM = {}_yM \circ D^{(p,0)}. \tag{1}$$

We note here that the functionals $g \to g^{(j)}(x_i)$, $e_{i,j}=1$, $g \in C^p(I)$, figuring in the representation of $L_{\triangle,E}$ and occuring, for example, in Birkhoff interpolation problems, can be replaced by more general linear functionals. This is, for instance, the case when dealing with the tensor product of two linear interpolation problems. See, *e.g.*, [15], [16], [17] or [49] for details.

The space $C^p(I)$ will be equipped with the norm

$$\|f\|_{C^p(I)} := \max\left\{\|f^{(k)}\|_\infty ; 0 \leq k \leq p\right\};$$

here $\|\cdot\|_\infty$ denotes the Čebyčëv norm on I. Clearly, $\|\cdot\|_{C(I)} = \|\cdot\|_\infty$. If L is a continuous linear operator mapping $\left(C^p(I), \|\cdot\|_{C^p(I)}\right)$ into $\left(C^q(J), \|\cdot\|_{C^q(J)}\right)$, its operator norm will be denoted by $\|\cdot\|_{C^p(I)\to C^q(J)}$. This norm is defined as follows:

$$\|L\|_{C^p(I)\to C^q(J)} = \sup_{\substack{f \in C^p(I) \\ \|f\|_{C^p(I)} \leq 1}} \max_{0 \leq k \leq q} \max_{x \in J} \left|(Lf)^{(k)}(x)\right|.$$

For $p = q = 0$ we simply write $\|L\|$.

The main results of this paper and some applications will be given in terms of so-called partial moduli of smoothness of order r, given for the compact intervals $I, J \subset \mathbb{R}$, for $f \in C(I \times J)$, $r \in \mathbb{N}_0$ and $\delta \in \mathbb{R}_+$ by

$$\omega_r(f;\delta,0) :=$$
$$:= \sup\left\{\left|\sum_{v=0}^{r}(-1)^{r-v}\binom{r}{v}\cdot f(x+vh,y)\right| : (x,y),(x+rh,y)\in I\times J, |h|\leq\delta\right\}$$

and symmetrically by

$$\omega_r(f;0,\delta) :=$$
$$:= \sup\left\{\left|\sum_{v=0}^{r}(-1)^{r-v}\binom{r}{v}\cdot f(x,y+vh)\right| : (x,y),(x,y+rh)\in I\times J, |h|\leq\delta\right\}.$$

Some other applications will be formulated in terms of total moduli of smoothness of order r, defined by

$$\omega_r(f;\delta_1,\delta_2) := \sup\left\{\left|\sum_{v=0}^{r}(-1)^{r-v}\binom{r}{v}\cdot f(x+vh_1,y+vh_2)\right| : \right.$$
$$\left. (x,y),(x+rh_1,y+rh_2)\in I\times J, |h_1|\leq\delta_1, h_2\leq\delta_2\right\},$$

for the compact intervals $I \times J \subset \mathbb{R}$,for $f \in C(I \times J)$, $r \in \mathbb{N}_0$ and $\delta_1, \delta_2 \in \mathbb{R}_+$.

The third type of moduli figuring in this note will be the mixed moduli of smoothness, given for $r, s \in \mathbb{N}_0$ by

$$\omega_{r,s}(f;\delta_1,\delta_2) := \sup\left\{\left|\sum_{v=0}^{r}\sum_{\mu=0}^{s}(-1)^{r+s-v-\mu}\binom{r}{v}\binom{s}{\mu}\cdot f(x+vh_1,y+\mu h_2)\right| : \right.$$
$$\left. (x,y),(x+rh_1,y+sh_2)\in I\times J, |h_i|\leq\delta_i, i=1,2\right\}.$$

Several properties of these moduli can be found in L.L. Schumakers book [67] and in [29]. Further notations will be introduced below when needed.

In this note we will first summarize how quantitative properties of certain univariate operators are inherited by the product of their parametric extensions, and this in a form as simple as possible. Details can be found in [2] and [3]. In order to illustrate what is meant by this we cite the following special case (for $n = 1$) of Theorem 4.1 in the article [31] on generalized n-th order blending as

Theorem 1. *Let I and J be nontrivial compact intervals of the real axis $\mathbb{R}$. For $(p,q) \geq (p',q') \geq (0,0)$ let linear operators*

$$L : \left(C^p(I), \|\cdot\|_{C^p(I)}\right) \to \left(C^{p'}(I), \|\cdot\|_{C^{p'}(I)}\right),$$

$$M:\left(C^{q}\left(J\right),\left\|\cdot\right\|_{C^{q}\left(J\right)}\right)\rightarrow\left(C^{q'}\left(J\right),\left\|\cdot\right\|_{C^{q'}\left(J\right)}\right),$$

by given such that for fixed $r,s\in\mathbb{N}_0$

$$\left|\left(g-L_g\right)^{(k)}\left(x\right)\right|\leq\Gamma_{r,k,L}\left(x\right)\cdot\omega_r\left(g^{(p)};\Lambda_{r,L}\left(x\right)\right)$$
$$\text{for } 0\leq k\leq p'\leq p,\ \text{for any } x\in I \text{ and any } g\in C^{p}\left(I\right),$$

and

$$\left|\left(h-Mh\right)^{(l)}\left(y\right)\right|\leq\Gamma_{s,l,M}\left(y\right)\cdot\omega_s(h^{(q)};\Lambda_{s,M}\left(y\right)),$$
$$\text{for } 0\leq l\leq q'\leq q,\ \text{for any } y\in J \text{ and any } h\in C^{q}\left(J\right).$$

Here, Γ *and* Λ *are bounded functions.*

Then for any $(x,y)\in I\times J$*, for all* $f\in C^{p,q}\left(I\times J\right)$ *and for* $(0,0)\leq(k,l)\leq(p',q')\leq(p,q)$ *the following pointwise inequalities simultaneously hold:*

$$\begin{aligned}&\left|\left[f-\left({}_xL\circ{}_yM\right)f\right]^{(k,l)}\left(x,y\right)\right|\leq\\&\leq\Gamma_{r,k,L}\left(x\right)\cdot\omega_r\left(f^{(p,l)};\Lambda_{r,L}\left(x\right),0\right)\\&+\Gamma_{s,l,M}\left(y\right)\cdot\omega_s\left(f^{(k,q)};0,\Lambda_{s,M}\left(y\right)\right)\\&+\Gamma_{r,k,L}\left(x\right)\cdot\Gamma_{s,l,M}\left(y\right)\cdot\omega_{r,s}\left(f^{(p,q)};\Lambda_{r,L}\left(x\right),\Lambda_{s,M}\left(y\right)\right).\end{aligned}$$

The quantities p' and q' were introduced mainly for two reasons. On the one hand we wanted a theorem applicable to a cubic spline operator defined on $C^4(I)$, say. In this case $p=4$, $p'=2$ are the proper choices. On the other hand, we also want to be able to cover such methods which map $C^p(I)$ into itself, but where good inequalities are only available for $0\leq k\leq p'<p$.

As was indicated before, generalized n-th order blending is one possible generalization of the tensor product method. For further extensions and modifications which have been of some interest in the past see our concluding remark. We emphasize here the fact that this is not intended to be a paper on scattered data interpolation (Where would the univariate building blocks be?).

Tensor product operators totally depend upon the quality of their building blocks, that is, they live on good or the best possible univariate results available by inheriting many of them, sometimes in a modified fashion. This is why in our sections on applications we will mention in detail some rather recent and very good results from the theory of functions of one variable which-due to the permanence principles below - more or less immediately carry over to tensor product operators.

The organization of the remainder of this report is as follows: In Section 2 we give a summary of recent theoretical results and conclude the section with several remarks on the differences between the theorems presented. Sections 3 and 4 contain applications of Theorems 2 to 4, respectively. In this way we arrive at new results for tensor products of several univariate operators which have been attracting the interest of many researchers for many years. We have chosen to add a rather extensive bibliography, also in order to draw the reader's attention to some less-known articles dealing with the tensor product construct.

2. DEGREE OF SIMULTANEOUS APPROXIMATION BY PRODUCTS OF PARAMETRIC EXTENSIONS. RECENT THEORETICAL RESULTS.

In this section let $I,\ J,\ I'\ J'$ be non-trivial compact intervals of the real axis $\mathbb{R}$, such that $I' \subseteq I$ and $J' \subseteq J$. For $(p,q) \geq (0,0)$ let linear operators

$$L : (C^p(I),)\ \|\cdot\|_{C^p(I)} \to \left(C^{p'}(I').\ \|\cdot\|_{C^{p'}(I')}\right),$$
$$M : (C^q\left(J), \|\cdot\|_{C^q(J)}\right) \to (C^{q'}(J'), \|\cdot\|_{C^{q'}(J')})$$

be given. Γ and Λ are non-negative, bounded functions and $R, S \subset \mathbb{N}_0$ are finite, nonempty sets (preferably with few elements).

Theorem 2 (see Theorem 1 in [3]). *If the operators L and M satisfy the conditions*

$$|(g-Lg)^{(k)}(x)| \leq \sum_{r\in R} \Gamma_{r,k,L}\left(x)\right) \cdot \omega_r\left(g^{(p)}; \Lambda_{r,L}(x)\right), \tag{2}$$
$$\textit{for } 0 \leq l \leq p' \leq p, \textit{for any } x \in I' \textit{ and any } g \in C^p(I)$$

and

$$\left|(h-Mh)^{(l)}(y)\right| \leq \sum_{s\in S} \Gamma_{s,l,M}(y) \cdot \omega_s\left(h^{(q)}; \Lambda_{s,M}(y)\right)$$
$$\textit{for } 0 \leq l \leq q',\ \textit{for any } y \in J' \textit{ and any } h \in C^q(J)$$

then we have for any $(x,y) \in I' \times J'$, for all $f \in C^{p,q}(I \times J)$ and for $(0,0) \leq (k,l) \leq (p',q') \leq (p,q)$ the following pointwise inequalities:

$$\begin{aligned} &\left|\left[f - ({}_xL \circ {}_yM)\, f\right]^{(k,l)}(x,y)\right| \leq \\ &\leq \sum_{r\in R} \Gamma_{r,k,L}(x) \cdot \omega_r\left(f^{(p,l)}; \Lambda_{r,L}(x), 0\right) \\ &\quad + \left\|D^k \circ L\right\|_{C^p(I)\to C(I')} \cdot \sup_{0\leq i\leq p} \sum_{s\in S} \Gamma_{s,l,M}(y) \cdot \omega_s\left(f^{(i,q)}; 0, \Lambda_{s,M}(y)\right) \end{aligned} \tag{3}$$

and

$$\begin{aligned} &\left|\left[f - ({}_xL \circ {}_yM)\, f\right]^{(k,l)}(x,y)\right| \leq \\ &\leq \sum_{s\in S} \Gamma_{s,l,M}(y) \cdot \omega_s\left(f^{(k,q)}; 0, \Lambda_{s,M}(y)\right) \\ &+ \left\|D^l \circ M\right\|_{C^q(J)\to C(J')} \cdot \sup_{0\leq j\leq q} \sum_{r\in R} \Gamma_{r,k,L}(x) \cdot \omega_r\left(f^{(p,j)}; \Lambda_{r,L}(x), 0\right). \end{aligned} \tag{4}$$

Proof. Can be found in [2], [3]. In the case of *discretely defined operators* we are in a much more convenient situation: it suffices to only consider one fixed partial of order (p,q) and its univariate predecessors. This is due to the fact that the equalities (1) are trivially satisfied for such operators. □

THEOREM 3 (see Theorem 31 in [4]). *For $p=p', q=q'$ let L and M be discretely defined operators as given above such that*

$$\left|(g-Lg)^{(p)}\right| \leq \sum_{r\in R} \Gamma_{r,p,L}(x)\cdot\omega_r\left(g^{(p)};\Lambda_{r,p,L}(x)\right), \qquad x\in I',\ g\in C^p(I),$$

and

$$\left|(h-Mh)^{(q)}(y)\right| \leq \sum_{s\in S} \Gamma_{s,q,M}(y)\cdot\omega_s\left(h^{(q)};\Lambda_{s,q,M}(y)\right), \quad h\in C^q(J).$$

(i) *Then for $(x,y)\in I'\times J'$ and $f\in C^{p,q}(I\times J)$ the following hold:*

$$\begin{aligned}&\left|\left[f-\left({}_xL\circ {}_yM\right)f\right]^{(p,q)}(x,y)\right| \leq \\ &\leq \sum_{r\in R}\Gamma_{r,p,L}(x)\cdot\omega_r\left(f^{(p,q)};\Lambda_{r,p,L}(x),0\right)\\ &\quad+\|D^p\circ L\|^*\cdot\sum_{s\in S}\Gamma_{s,q,M}(y)\cdot\omega_s\left(f^{(p,q)};0,\Lambda_{s,q,M}(y)\right).\end{aligned}$$

Here

$$\|D^p\circ L\|^* := \inf\{c : \|(D^p\circ L)\,g\|_{\infty,I'} \leq c\cdot\left\|g^{(p)}\right\|_{\infty,I}, \forall g\in C^p(I)\}.$$

(ii) *A symmetric upper bound is given by*

$$\begin{aligned}&\sum_{s\in S}\Gamma_{s,q,M}(y)\cdot\omega_s\left(f^{(p,q)};0,\Lambda_{s,q,M}(y)\right)\\ &+\|D^q\circ M\|^*\cdot\sum_{r\in R}\Gamma_{r,p,L}(x)\cdot\omega_r\left(f^{(p,q)};\Lambda_{r,p,L}(x),0\right).\end{aligned}$$

Observe that in the upper bound of Theorem 3–as compared to Theorem 2–there is no sup any more.

The following result is similar to Theorem 2, but makes different assumptions for the univariate building blocks. Before discussing these further we state it as

THEOREM 4 (see Theorem 6 in [3]). *Let the operators L and M be given such that*

$$\left|(g-Lg)^{(k)}(x)\right| \leq \sum_{r\in R}\Gamma_{r,k,L}(x)\cdot\omega_r\left(g^{(k)};\Lambda_{r,k,L}(x)\right),$$

$$\text{for } 0\leq k\leq p'\leq p, \text{ for any } x\in I' \text{ and any } g\in C^p(I),$$

and

$$\left|(h-Mh)^{(l)}(y)\right| \leq \sum_{s\in S}\Gamma_{s,l,M}(y)\cdot\omega_s\left(h^{(l)};\Lambda_{s,l,M}(y)\right),$$

for $0 \leq l \leq q' \leq q,$ *for any* $y \in J'$ *and any* $h \in C^q(J)$.

Then we have for any $(x, y) \in I' \times J'$, *for all* $f \in C^{p,q}(I \times J\ )$ *and for* $(0,0) \leq (k,l) \leq (p',q') \leq (p,q)$ *the following pointwise inequalities:*

$$\begin{aligned}
&\left|\left[f - ({}_xL \circ {}_yM)\, f\right]^{(k,l)}(x,y)\right| \leq \\
&\leq \sum_{r \in R} \Gamma_{r,k,L}(x) \cdot \omega_e\left(f^{(k,l)}; \Lambda_{r,k,L}(x), 0\right) \\
&\quad + \left\|D^k \circ L\right\|_{C^p(I) \to C(I')} \cdot \sup_{0 \leq i \leq p} \sum_{s \in S} \Gamma_{s,l,M}(y) \cdot \omega_s\left(f^{(i,l)}; 0, \Lambda_{s,l,M}(y)\right)
\end{aligned}$$

and

$$\begin{aligned}
&\left|\left[f - ({}_xL \circ {}_yM)\, f\right]^{(k,l)}(x,y)\right| \leq \\
&\leq \sum_{s \in S} \Gamma_{s,l,M}(y) \cdot \omega_s\left(f^{(k,l)}; 0, \Lambda_{s,l,M}(y)\right) \\
&\quad + \left\|D^l \circ M\right\|_{C^q(J) \to C(J')} \cdot \sup_{0 \leq j \leq q} \sum_{r \in R} \Gamma_{r,k,L}(x) \cdot \omega_r\left(f^{(k,j)}; \Lambda_{r,k,L}(x), 0\right).
\end{aligned}$$

Proof. Can be found in [2]. □

Remark 1. A word is in order concerning the difference between Theorems 2 and 4. In Theorem 2 we assumed that the order of the derivatives (in the univariate cases) on the left varies, while the order on the right hand side is fixed. It can, for example, be applied to tensor products of certain interpolatory spline operators (two special instances will be discussed below) or of algebraic polynomial operators satisfying inequalities of the Brudnyĭ-Gopengauz-type. See, *e.g.*, the 1985 paper [30] for the use of such inequalities in the framework of approximation by Boolean sums of parametric extensions. Much more is known today about univariate operators satisfying such interpolatory pointwise estimates; see [35] for details.

In contrast to that, in Theorem 4 we assume that the order of the derivatives on the left and on the right are the same. This is the appropriate assumption for tensor products of the Bernstein operators or of Bernstein-Durrmeyer operators, just to mention two examples. Theorem 2 is not a suitable tool to cover these situations. □

Remark 2. (i) For $p = q = 0$ Theorems 2 and 4 coincide. Interesting for us are the cases where this does not occur, because we are mainly interested in simultaneous approximation.

(ii) The assumptions in both Theorems 2 and 4 are not artificial ones. Indeed, there exist interesting univariate building blocks L for which one-term upper bounds such as $\Gamma_{r,k,L}(x) \cdot \omega_r(g^{(p)}; \Lambda_{r,L}(x))$ are not quite appropriate in order to describe their approximation behavior properly. For example, if L is a univariate positive linear operator satisfying $Le_0 = e_0$, but $Le_1 \neq e_1(e_i(x) :=$

$x^i, i \in \mathbb{N}_0$), then the appropriate upper bound is given in

$$\begin{aligned}|L(f,x)-f(x)| \leq & h^{-1}\cdot|L(e_1-x;x)|\cdot\omega_1(f;h)\\ & +\left[1+\tfrac{1}{2}\cdot h^{-2}\cdot L\left((e_1-x)^2\right);x\right]\cdot\omega_2(f;h)\end{aligned}$$

for $0 < h \leq \frac{1}{2}(b-a)$. See [59, Corollary 1] for this result.

One further related situation in which moduli of smoothness of several orders occur in the right hand side of the inequality is that of "approximation of derivatives by derivatives". Here we refer to a recent paper by D. Kacsó [45]; Theorem 3 there contains a three term expression which is the appropriate quantity in the context of Kacsó's paper. That this is so was first discovered in [28]. Details are given below.

(iii) Both Theorems 2 and 4 are proved under the assumptions that information is available for the derivatives of order $0 \leq k \leq p'$ and $0 \leq l \leq q'$. This is needed in order to apply Lemma 2 in [3]. If the operators L and M are discretely defined (like, for example, the Bernstein operators), then Theorem 3 is also applicable and a more efficient tool. Below we list several results from our paper [4] where Theorem 3 was applied to variation-diminishing Schoenberg spline operators.

(iv) For $p = q = 0$, $|R| = |S| = 1, r = s$ we obtain from Theorem 4 the same upper bounds as in Corollary 3 in [3]. $\square$

3. APPLICATIONS OF THEOREM 2

3.1. Brudnyĭ-Gopengauz operators $\mathbf{Q}_n$. The univariate operators Q_n we use here are the result of many years of research which culminated in papers by Gonska and Hinnemann [34] and by Dahlhaus [13]. Guided by early work (in chronological order) of Timan, Dzjadyk, Trigub, Brudnyĭ, Teljakovskiĭ, Gopengauz and DeVore on the subject, the authors mentioned investigated linear polynomial operators satisfying certain interpolation conditions at the endpoints of $I = [-1, 1]$. We decided to call the Q_n below Brudnyĭ-Gopengauz operators because–according to our knowledge–Brudnyĭ was the first to observe the possibility of using moduli of smoothness of arbitrary order in pointwise Jackson-type estimates for algebraic polynomial approximation. Furthermore, Gopengauz first used higher order moduli for estimates on simultaneous approximation and observed (correctly!) that in inequalities with the first modulus the quantity $\frac{\sqrt{1-x^2}}{n}$ can be used in simultaneous estimates. For the sake of completeness we mention here one further paper by Gonska *et al.* [35] in which the problem of pointwise interpolatory inequalities was treated in combination with questions concerning shape preservation.

We describe the work of Gonska, Hinnemann and Dahlhaus briefly in order to make the subsequent statements comprehensible. The first named two authors constructed in [34] a new linear operator Q_n (not necessarily being discretely defined), namely

$$Q_n : \left(C^p(I), \|\cdot\|_{C^p(I)}\right) \to \Pi_n(I)\,;\ \text{here } Q_n : R_n \oplus \bar{L}_n := R_n + \bar{L}_n - R_n \circ \bar{L}_n.$$

For details regarding this definition see [34, p. 247]. The linearity of Q_n follows straightforward from the linearity of its building blocks R_n and L_n.

Extending the work in [34], Dahlhaus showed in [13] that for $p, r \in \mathbb{N}_0$ fixed, $n \geq \max\{4(p+1), p+r\}, 0 \leq k \leq \min\{p-r+2, p\}, f \in C^p(I)$ and $x \in I$ there holds the inequality

$$\left|(f - Q_n f)^{(k)}(x)\right| \leq c_{p,r} \cdot \left(\frac{\sqrt{1-x^2}}{n}\right)^{p-k} \cdot \omega_r\left(f^{(p)}, \frac{\sqrt{1-x^2}}{n}\right). \tag{5}$$

Furthermore, Dahlhaus proved that his assertion is best possible in a most meaningful sense. The above inequality is of type (2) with bounded $\Gamma_{r,k,Q_n}(x) = c_{p,r} \cdot \left(\frac{\sqrt{1-x^2}}{n}\right)^{p-k}, \Lambda_{r,Q_n}(x) = \frac{\sqrt{1-x^2}}{n}$ and $p' := \min\{p-r+2, p\} \leq p$.

Choosing a second copy Q_m, say, mapping $C^q(I)$ into $\Pi_m(I)$ and satisfying

$$\left|(f - Q_m f)^{(l)}(y)\right| \leq c_{q,s} \cdot \left(\frac{\sqrt{1-y^2}}{m}\right)^{q-1} \cdot \omega_s\left(f^{(q)}, \frac{\sqrt{1-y^2}}{m}\right),$$

for $0 \leq l \leq q' := \min\{q-s+2, q\}$, all the assumptions of Theorem 2 are fulfilled. Thus we have, for example, for $(x, y) \in I^2, f \in C^{p,q}(I^2)$ and for $(0,0) \leq (k,l) \leq (p', q')$ the inequality

$$\begin{aligned}
&\left|\left[f - ({}_xQ_n \circ {}_yQ_m) f\right]^{(k,l)}(x,y)\right| \leq \\
&\leq c_{p,r} \cdot \left(\frac{\sqrt{1-x^2}}{n}\right)^{p-k} \cdot \omega_r\left(f^{(p,l)}; \frac{\sqrt{1-x^2}}{n}, 0\right) \\
&\quad + \left\|D^k \circ Q_n\right\|_{C^p(I)\to C(I)} \cdot c_{q,s}\left(\frac{\sqrt{1-y^2}}{m}\right)^{q-1} \cdot \sup_{0\leq i\leq p} \omega_s\left(f^{(i,q)}; 0, \frac{\sqrt{1-y^2}}{m}\right).
\end{aligned}$$

It follows from (5) that there is a constant $d_{p,r}$ which does not depend on n such that

$$\left\|D^k \circ Q_n\right\|_{C^p(I)\to C(I)} \leq d_{p,r}.$$

Hence the above upper bound of

$$\left|\left[f - ({}_xQ_n \circ {}_yQ_m) f\right]^{(k,l)}(x,y)\right|$$

simplifies accordingly.

3.2. Cubic interpolatory splines. As a second application of Theorem 2 we deal with two types of cubic interpolatory splines. Part of the results presented here can also be found in [3].

Let $\triangle_n$ be a partition of the real interval $I = [a,b]$. By $S_{\triangle_n}$ we denote the Type I cubic spline operator which attaches to each function $f \in C^1 I$ the corresponding clamped cubic spline interpolant $s_f \in C^2(I)$. Further, let $T_{\triangle_n}$ be the cubic spline operator which attaches to each $f \in C(I)$ the corresponding natural cubic spline interpolant $t_f \in C^2(I)$. For more detailed definitions see, *e.g.*, [32].

In order to give two-dimensional estimates we use one-dimensional ones. These are, in the case of the Type I cubic spline operator, Theorem 3.4 in [32], and Theorem 1 in [38] for the natural cubic spline operator. We recall them briefly.

Theorem 5. *Let $S_{\triangle_n}$ be given as above, and let $p = 1, 2, 3$, or 4. Then for any $f \in C^p(I)$ the following inequalities hold:*

$$\left\|(S_{\triangle_n} f - f)^{(k)}\right\|_\infty \leq c(p,k) \cdot \delta^{p-k} \cdot \omega_{4-p}\left(f^{(p)}, \delta\right), \qquad 0 \leq k \leq \min\{p, 2\}.$$

Here, the constants $c(p,k)$ depend only on p and k.

Theorem 6. *If $T_{\triangle_n}$ is given as above, then for all $f \in C^2(I)$ and $k = 0, 1$ there holds*

$$\left\|(T_{\triangle_n} f - f)^{(k)}\right\|_\infty \leq 15 \cdot \delta^{2-k} \cdot \omega_0\left(f'', \delta\right).$$

In both theorems $\delta = \delta_n$ is the mesh gauge of $\triangle_n$.

Let now be $S_{\triangle_n}$ and $S_{\triangle_m}$ two Type I cubic spline operators. By Theorem 5 we have

$$\left\|(S_{\triangle_n} f - f)^{(k)}\right\|_\infty \leq \Gamma_{4-p,k,S_{\triangle_n}} \cdot \omega_{4-p}\left(f^{(p)}, \Lambda_{4-p,S_{\triangle_n}}\right)$$

for $p = 1, 2, 3$, or $4, f \in C^p(I), 0 \leq k \leq \min\{p, 2\}, \Gamma_{4-p,k,S_{\triangle_n}} := c(p,k) \cdot \delta_1^{p-k}, \Lambda_{4-p,S_{\triangle_n}} := \delta_1$ (the mesh gauge of $\triangle_n$), and

$$\left\|(S_{\triangle_m} f - f)^{(l)}\right\|_\infty \leq \Gamma_{4-q,l,S_{\triangle_m}} \cdot \omega_{4-q}\left(f^{(q)}, \Lambda_{4-q,S_{\triangle_m}}\right)$$

for $q = 1, 2, 3$, or $4, f \in C^q(J), 0 \leq l \leq \min\{q, 2\}, \Gamma_{4-q,l,S_{\triangle_m}} := c(q,l) \cdot \delta_2^{q-1}, \Lambda_{4-q,S_{\triangle_m}} : \delta_2$ (the mesh gauge of $\triangle_m$). Applying now Theorem 2 we obtain the following

Theorem 7 (see Theorem 5 in [3]). *With the above notation,*

$$\begin{aligned}
&\left\|[f - ({}_xS_{\triangle_n} \circ {}_yS_{\triangle_m}) f]^{k,l}\right\|_\infty \leq \\
&\leq c(p,k) \cdot \delta_1^{p-k} \cdot \omega_{4-p}\left(f^{(p,l)}; \delta_1, 0\right) \\
&\quad + d(p,k,\delta_1) \cdot c(q,l) \cdot \delta_2^{q-1} \cdot \sup_{0 \leq i \leq p}\left\{\omega_{4-q}(f^{(i,q)}; 0, \delta_2)\right\},
\end{aligned}$$

for $p, q \in \{1,2,3,4\}$, $f \in C^{p,q}(I \times J), 0 \leq k \leq \min\{p, 2\}$ and $0 \leq 1 \leq \min\{q, 2\}$. Here, $d(p,k,\delta_1) := 1 + c(p,k) \cdot \delta_1^{p-k} \cdot 2^{4-p}$.

In order to arrive at the latter inequality it is only necessary to observe that

$$\|D^k \circ S_{\triangle_n}\|_{C^p(I) \to C(I)} \leq 1 + c(p,k) \cdot \delta_1^{p-k} \cdot 2^{4-p}.$$

Further let $T_{\triangle_n}$ and $T_{\triangle_m}$ be two natural cubic spline operators. According to Theorem 6 we have the one-dimensional estimates

$$\left\|(T_{\triangle_n} f - f)^{(k)}\right\|_\infty \le \Gamma_{0,k,T_{\triangle_n}} \cdot \omega_0\left(f'', \Lambda_{0,T_{\triangle_n}}\right)$$

for $f \in C^2(I)$, $k \in \{0,1\}$, $\Gamma_{0,k,T_{\triangle_n}} := c \cdot \delta_1^{2-k}$, $\Lambda_{0,T_{\triangle_n}} := \delta_1$ (the mesh gauge of $\triangle_n$), and

$$\left\|(T_{\triangle_m} f - f)^{(l)}\right\|_\infty \le \Gamma_{0,l,T_{\triangle_m}} \cdot \omega_0\left(f'', \Lambda_{0,T_{\triangle_m}}\right)$$

for $f \in C^2(J)$, $l \in \{0,1\}$, $\Gamma_{0,l,T_{\triangle_m}} := c \cdot \delta_2^{2-l}$, $\Lambda_{0,T_{\triangle_m}} := \delta_2$ (the mesh gauge of $\triangle_m$). Applying again Theorem 2 there follows

Theorem 8. *With the notation from above we have*

$$\begin{aligned}
&\left\|[f - ({}_xT_{\triangle_n} \circ {}_yT_{\triangle_m}) f]^{(k,l)}\right\|_\infty \le \\
&\le 15 \cdot \delta_1^{2-k} \cdot \omega_0(f^{(2,l)}; \delta_1, 0) \\
&\quad + \left(1 + 15 \cdot \delta_1^{2-k}\right) \cdot 15 \cdot \delta_2^{2-l} \cdot \sup_{0\le i\le 2} \left\{\omega_0(f^{(i,2)}; 0, \delta_2)\right\},
\end{aligned}$$

for $f \in C^{2,2}(I \times J)$ *and* $k, l \in \{0,1\}$.

3.3. Applications of Theorem 3. We now turn to three classes of discretely defined operators and thus to applications of Theorem 3.

3.3.1. *Bivariate operators of binomial type.* A sequence $(p_n)_{n\in\mathbb{N}}$ where for all n the polynomial p_n is exactly of degree n is called a *polynomial sequence*. A polynomial sequence is said to be *of binomial type*, iff for all x, y and all $n \in \mathbb{N}$ it satisfies the identity

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) \cdot p_{n-k}(y).$$

For a survey on this issue see [53]. Three further recent contributions on the subject are [78], [80] and [79].

We consider now a linear approximation operator of degree n associated with the sequence $(p_n)_{n\in\mathbb{N}}$ of binomial type by

$$T_n(f;x) := \tfrac{1}{p_n(1)} \sum_{k=0}^{n} \binom{n}{k} p_k(x) \cdot p_{n-k}(1-x) \cdot f\left(\tfrac{k}{n}\right) \tag{6}$$

for all $f \in C[0,1]$, provided that $p_n(1) \neq 0$ for all $n \in \mathbb{N}_0$. These operators are called by D.D. Stancu and A.D. Vernescu in [81] *Popoviciu type operators* (see [60]) and by P. Sablonnière in [63] *Bernstein-Sheffer operators*. See these and the further paper [78] for additional references. The well-known Bernstein-Stancu operator S_n^α (introduced in [75]) corresponds to the sequence of factorial powers

$$p_n(x) = x(x+\alpha)\dots(x+(n-1)\alpha)$$

and the classical Bernstein operator B_n to the sequence of monomials $p_n(x) = x^n$.

Theorem 2 (ii) in [63] states that $T_n e_i = e_i$ for $i = 0, 1$, and $T_n e_2 = e_2 + \frac{b_n}{n}(e_1 - e_2)$, where

$$b_n = 1 + (n-1) \cdot \frac{r_{n-2}(1)}{p_n(1)}.$$

For the specification of the sequence $(r_n(x))_{n \in \mathbb{N}_0}$ see [63]. If for all $n \in \mathbb{N}_0$ and for $x \in [0,1]$ one has $p_n(x) \geq 0$, then the operator T_n is positive and we can apply Păltănea's Corollary 1 in [59]. With $p = 1$, $b = 0$, $\lambda = 0$ and $h = \sqrt{x(1-x)\frac{b_n}{n}}$ it yields

$$|T_n(f;x) - f(x)| \leq \tfrac{3}{2} \cdot \omega_2\left(f; \sqrt{x(1-x)\tfrac{b_n}{n}}\right),$$

for all $f \in C[0,1]$. If $\frac{r_{n-2}(1)}{p_n(1)} = o(1)$, then $\frac{b_n}{n} \to 0$, and the latter inequality implies uniform convergence of $T_n f$ towards f for all $f \in C[0,1]$, at the same time constituing a significant refinement of Theorem 3 by Sablonnière [63] (see also [53, Th. 2.11]).

Further we consider a second operator of binomial type T_m with the corresponding estimate, namely

$$|T_m(f;y) - f(y)| \leq \tfrac{3}{2} \cdot \omega_2\left(f; \sqrt{y(1-y)\tfrac{b_m}{m}}\right),$$

for all $f \in C[0,1]^2$. Applying Theorem 3 with $I = I' = J = J' = [0,1]$, $p = q = 0$, $R = S = \{2\}$, $\Gamma_{2,0,T_n}(x) = \Gamma_{2,0,T_m}(y) = \frac{3}{2}$, $\Lambda_{2,0,T_n}(x) = \sqrt{x(1-x)\frac{b_n}{n}}$, $\Lambda_{2,0,T_m}(y) = \sqrt{y(1-y)\frac{b_m}{m}}$ it follows that

$$\begin{aligned}
&\left|[f - ({}_xT_n \circ {}_yT_m) f](x,y)\right| \leq \\
&\leq \tfrac{3}{2}\left\{\omega_2\left(f; \sqrt{x(1-x)\tfrac{b_n}{n}}, 0\right) + \omega_2\left(f; 0, \sqrt{y(1-y)\tfrac{b_m}{m}}\right)\right\} \qquad (7)\\
&\leq 3 \cdot \omega_2\left(f; \sqrt{x(1-x)\tfrac{b_n}{n}}, \sqrt{y(1-y)\tfrac{b_m}{m}}\right),
\end{aligned}$$

for all $f \in C[0,1]^2$. This improves the estimate of the error given in [81] in the sense that now a second order modulus instead of a first order one is used.

For the Bernstein-Stancu operators S_n^α we have $b_n = \frac{1+n\alpha}{1+\alpha}$ and thus

$$\begin{aligned}
&\left|[f - \left({}_xS_n^\alpha \circ {}_yS_m^\beta\right) f](x,y)\right| \leq \qquad (8)\\
&\leq 3 \cdot \omega_2\left(f; \sqrt{\tfrac{x(1-x)}{n} \cdot \tfrac{1+n\alpha}{1+\alpha}}, \sqrt{\tfrac{y(1-y)}{m} \cdot \sqrt{\tfrac{1+m\beta}{1+\beta}}}\right).
\end{aligned}$$

This inequality can be viewed as a pointwise version of Theorems 3 and 4 in Stancu's paper [76].

The best-known examples of tensor product Bernstein-Stancu operators are the tensor product Bernstein operators themselves. As mentioned earlier, these were investigated as early as 1933 by Hildebrandt and Schoenberg in order to prove the Weierstraß approximation theorem in $C[0,1]^2$. Subsequently their research was supplemented by a number of authors who, among other results, proved quantitative assertions of different kinds. See [29, sect. 6.2.1] for references and a brief discussion of those published until the mid 80's.

If we put $\alpha = \beta = 0$ in (8) we arrive at

$$\left|\left[f - ({}_xB_n \circ {}_yB_m)\, f\right](x,y)\right| \leq 3 \cdot \omega_2\left(f; \sqrt{\tfrac{x(1-x)}{n}}, \sqrt{\tfrac{y(1-y)}{m}}\right)$$

which improves Corollary 9.2 in [29] for the case $d = 2$. The latter estimate holds for any $f \in C[0,1]^2$.

Remark 3. From (7) we also have

$$\begin{aligned} &\left|\left[f - ({}_xB_n \circ {}_yB_m)\, f\right]\right|(x,y) \leq \\ &\leq \tfrac{3}{2} \cdot \left(\omega_2\left(f; \sqrt{\tfrac{x(1-x)}{n}}, 0\right) + \omega_2\left(f; 0, \sqrt{\tfrac{y(1-y)}{m}}\right)\right) \\ &\leq \tfrac{3}{2} \cdot \left(\bar{\omega}_2\left(f; \sqrt{\tfrac{x(1-x)}{n}}\right) + \bar{\omega}_2\left(f; \sqrt{\tfrac{y(1-y)}{m}}\right)\right) =: A, \end{aligned}$$

where

$$\bar{\omega}_2\left(f;\delta\right) := \sup_{\substack{h \in \mathbb{R}^2, \|h\|_{2 \leq \delta} \\ x \pm h \in [0,1]^2}} \left|f\left(x+h\right) - 2f\left(x\right) + f\left(x-\delta\right)\right|$$

and $\|\cdot\|_2$ is the Euclidian norm in $\mathbb{R}^2$.

Since

$$\begin{aligned} A &\leq 3 \cdot \bar{\omega}_2\left(f; \max\left\{\sqrt{\tfrac{x(1-x)}{n}}, \sqrt{\tfrac{y(1-y)}{m}}\right\}\right) \\ &\leq 3 \cdot \bar{\omega}_2\left(f; \tfrac{1}{\sqrt{2 \cdot \min\{n,m\}}}\right) \end{aligned}$$

we have thus improved a recent result of López-Moreno and Muñoz-Delgado [51]. □

Remark 4. We also use the Bernstein operators in order to emphasize the differences between Theorem 2 and the three theorems from Section 2. To that end we only consider the case $p = q = 0$. From Theorem 1 we arrive at the 3-term upper bound

$$\begin{aligned} &\tfrac{3}{2} \cdot \omega_2\left(f; \sqrt{\tfrac{x(1-x)}{n}}, 0\right) + \tfrac{3}{2} \cdot \omega_2\left(f; 0, \sqrt{\tfrac{y(1-y)}{m}}\right) + \\ &+ \tfrac{9}{4} \cdot \omega_{2,2}\left(f; \sqrt{\tfrac{x(1-x)}{n}}, \sqrt{\tfrac{y(1-y)}{m}}\right). \end{aligned}$$

The three assertions in Section 2 lead to the 2-term sum

$$\frac{3}{2}\cdot\omega_2\left(f;\sqrt{\frac{x(1-x)}{n}},0\right)+\frac{3}{2}\cdot\omega_2\left(f;0,\sqrt{\frac{y(1-y)}{m}}\right)$$

showing that the term involving the mixed modulus $\omega_{2,2}$ is superfluous. This observation sheds a light on the principal differences between the two approaches. The second one uses the decomposition

$$Id - {}_xL\circ {}_yM = Id - {}_xL + {}_xL\circ\left(Id - {}_yM\right),$$

or the symmetric one, namely

$$Id - {}_xL\circ {}_yM = Id - {}_yM + \left(Id - {}_xL\right)\circ {}_yM.$$

Both representations lead to 2-term upper bounds. In the proof of Theorem 1 we decomposed the error a priori into

$$\begin{aligned} Id - {}_xL\circ {}_yM &= \left(Id - {}_xL\right) + \left(Id - {}_yM\right) - \left(Id - {}_xL\right)\circ\left(Id - {}_yM\right) \\ &= \left(Id - {}_xL\right)\oplus\left(Id - {}_yM\right), \end{aligned}$$

that is, the Boolean sum of the individual errors. This creates the 3-term bound. □

3.4. Tensor Product Schoenberg Splines. As a second application of Theorem 3 we summarize several results on the degree of simultaneous approximation by the tensor product of variation-diminishing Schoenberg spline operators. For details the reader is refered to [4]. We briefly recall some of the basic definitions and of the fundamental univariate results.

Consider the knot sequence $\triangle_n = \{x_i\}_{-k}^{n+k}$ $(n>0, k>0)$ with

$$x_{-k} = x_{-k+1} = \ldots = x_0 = 0 < x_1 < \ldots < x_n = \ldots = x_{n+k} = 1.$$

For a function $f\in\mathbb{R}^{[0,1]}$, the variation-diminishing spline of degree k w.r.t. $\triangle_n$ is given

$$\begin{aligned} S_{\triangle_{n,k}}f\left(x\right) &:= \sum_{j=-k}^{n-1} f\left(\xi_{j;k}\right)\cdot N_{j,k}\left(x\right) \text{ for } 0\le x<1 \text{ and} \\ S_{\triangle_{n,k}}f\left(1\right) &:= \lim_{\substack{y\to 1\\ y<1}} S_{\triangle_{n,k}}f\left(y\right), \end{aligned}$$

with the nodes (Greville abscissas) $\xi_{j,k} := \frac{x_{j+1}+\ldots+x_{j+k}}{k}$, $-k\le j\le n-1$, and the normalized B-splines as fundamental functions

$$N_{j,k}\left(x\right) := \left(x_{j+k+1}-x_j\right)\left[x_j,x_{j+1},\ldots,x_{j+k+1}\right]\left(\cdot - x\right)_+^k.$$

If the knots are equidistant, *i.e.*, $x_j = \frac{j}{n}, 0\le j\le n$, the k-th degree Schoenberg spline operator will be denoted simply by $S_{n,k}$.

The following inequalities will be given predominantly in terms of the so-called mesh gauge $\|\triangle_n\| := \max_j\left(x_{j+1}-x_j\right)$.

Proposition 1 (see Corollary 7 in [4]). *For all $f \in C[0,1]$ one has the following uniform estimates*

$$\left\| S_{\triangle_{n,k}} f - f \right\|_{\infty} \leq \left(1 + \tfrac{k+1}{24}\right) \cdot \omega_2\left(f; \|\triangle_n\|\right).$$

Proposition 2 (see Corollary 23 in [4]). *Let $f \in C^1[0,1$. Then, for $n \geq 1$, $k \geq 2$, one has*

$$\left| DS_{\triangle_{n,k}} f(x) - Df(x) \right| \leq \omega_1\left(Df; \|\triangle_n\|\right) + \tfrac{3}{2}\left(1 + \sqrt{\tfrac{k}{12}}\right)^2 \cdot \omega_2\left(Df; \|\triangle_n\|\right).$$

For splines with $x_j = \frac{j}{n}$, $0 \leq j \leq n$, and second order derivatives one has uniform convergence on compact subsets of $(0,1)$ only. In this case we have, for example,

Proposition 3 (see Corollary 26 in [4]). *Let $f \in C^2[0,1]$, $x \in [\frac{k-1}{n}, 1-\frac{k-1}{n}]$ and $3 \leq k \leq \frac{n}{2}+1$. Then there holds:*

$$\left| D^2 S_{n,k} f(x) - D^2 f(x) \right| \leq \omega_1\left(D^2 f; \tfrac{1}{n}\right) + \tfrac{3}{2}\left(1 + \sqrt{\tfrac{k-1}{12}}\right)^2 \cdot \omega_2\left(D^2 f; \tfrac{1}{n}\right).$$

For $m, l \geq 1$, we now consider a second operator $S_{\triangle_{m,l}}$ defined for functions on $[0,1]$ and satisfying inequalities analogous to those in Proposition 1–Proposition 3. Hence all the univariate inequalities have the appropriate form to apply Theorem 3.

For the tensor product of two Schoenberg spline operators we first state

Theorem 9 (see Theorem 38 in [4]). *For $n, m \geq 1$ and $k, l \geq 1$ we have*

$$\begin{aligned} &\left\| f - \left({}_x S_{\triangle_{n,k}} \circ {}_y S_{\triangle_{m,k}} \right) f \right\|_{\infty, I \times J} \leq \\ &\leq \left(1 + \tfrac{k+1}{24}\right) \cdot \omega_2\left(f; \|\triangle_n\|, 0\right) + \left(1 + \tfrac{l+1}{24}\right) \cdot \omega_2\left(f; 0, \|\triangle_m\|\right) \\ &\leq \left(2 + \tfrac{k+l+2}{24}\right) \cdot \omega_2\left(f; \|\triangle_n\|, \|\triangle_m\|\right). \end{aligned}$$

Proof. This is the case $p = q = 0$, $R = S = \{2\}$. With $\Gamma_{2,0,S_{\triangle_{n,k}}}(x) = 1 + \frac{k+1}{24}$, $\Lambda_{2,0,S_{\triangle_{n,k}}}(x) = \|\triangle_n\|$ and analogous choices with respect to the variable y we arrive at the above upper bound, also observing that $\left\| D^0 \circ S_{\triangle_{n,k}} \right\|^* = 1$. $\square$

For the partial derivatives up to order $(1,1)$ we obtain

Theorem 10 (see Theorem 39 in [4]). *For $n, m \geq 1$ and $k, l \geq 2$ we have the following inequalities for any $f \in C^{2,2}[0,1]^2$.*

(*i*) $\left\| f - \left({}_x S_{\triangle_{n,k}} \circ {}_y S_{\triangle_{m,l}} \right) f \right\|_{\infty} = \mathcal{O}\left(\|\triangle_n\|^2 + \|\triangle_m\|^2 \right)$;

(*ii*) $\left\| \left(f - \left({}_x S_{\triangle_{n,k}} \circ {}_y S_{\triangle_{m,l}} \right) \right)^{(1,0)} \right\|_{\infty} = \mathcal{O}\left(\|\triangle_n\| + \|\triangle_m\|^2 \right)$;

(*iii*) $\left\| \left(f - \left({}_x S_{\triangle_{n,k}} \circ {}_y S_{\triangle_{m,l}} \right) \right)^{(0,1)} \right\|_{\infty} = \mathcal{O}\left(\|\triangle_n\|^2 + \|\triangle_m\| \right)$;

(iv) $\left\|\left(f-\left({}_xS_{\triangle_{n,k}}\circ{}_yS_{\triangle_{m,l}}\right)\right)^{(1,1)}\right\|_\infty=\mathcal{O}\left(\|\triangle_n\|+\|\triangle_m\|\right).$

In all four cases $\mathcal{O}$ depends on k and l, and the sup norms are those over $[0,1]^2$.

Proof. It is mainly a consequence of properties of the partial moduli of smoothness; for details see [4]. □

For the remaining partials up to order 2 again we consider only the case of equidistant knots and the smaller intervals $\left[\frac{k-1}{n},1-\frac{k-1}{n}\right]\times\left[\frac{l-1}{m},1-\frac{l-1}{m}\right]$. We now have

Theorem 11 (see Theorem 40 in [4]). *For $3\le k\le\frac{n}{2}+1, 3\le l\le\frac{m}{2}+1$ the following are true for $f\in C^{3,3}[0,1]^2$.*

(i) $\left\|\left(f-\left({}_xS_{n,k}\circ{}_yS_{m,l}\right)f\right)^{(2,0)}\right\|_\infty=\mathcal{O}\left(\frac{1}{n}+\frac{1}{m^2}\right);$

(ii) $\left\|\left(f-\left({}_xS_{n,k}\circ{}_yS_{m,l}\right)f\right)^{(2,1)}\right\|_\infty=\mathcal{O}\left(\frac{1}{n}+\frac{1}{m^2}\right);$

(iii) $\left\|\left(f-\left({}_xS_{n,k}\circ{}_yS_{m,l}\right)f\right)^{(2,2)}\right\|_\infty=\mathcal{O}\left(\frac{1}{n}+\frac{1}{m}\right).$

Analogous statements hold for the partials of orders $(0,2)$ and 2; $\mathcal{O}$ depends on k and l in all cases and the sup norms are those over the smaller subinterval given above.

Proof. Proof. See [4] for details. □

3.5. Discretely defined $\mathcal{H}$-operators. We finish this section with certain operators which are of particular interest in the theory of positive linear operators. All three theorems from above are applicable in this case.

In order to solve the strong form of Butzer's Problem, Gavrea, Gonska and Kacsó introduced in [25] a new class of discrete, linear and positive operators of the following form

$$\begin{aligned}\mathcal{H}^*_{n+s+2}(f;x)&:=(1-x)^2\cdot f(0)\cdot\int_0^1 P^*_{n+s}(t(1-x))\,dt\\ &+x^2\cdot f(1)\cdot\int_0^1 P^*_{n+s}(xt)\,dt+\sum_{k=1}^{n+s}A_k\cdot K_{n+s}(x,x_k)\cdot f(x_k).\end{aligned}\tag{9}$$

For details regarding the quantities appearing in the above definition see [25], [26], [24].

Theorem 7 in [26] states that the operater $\mathcal{H}^*_{n+s+2}:C[0,1]\to\Pi_{n+s+2}$ satisfies the DeVore-Gopengauz inequality

$$\left|\mathcal{H}^*_{n+s+2}(f;x)-f(x)\right|\le c(s)\cdot\omega_2\left(f;\frac{\sqrt{x(1-x)}}{n}\right).$$

In particular, $\left\|\mathcal{H}^*_{n+s+2}\right\|=1$. We consider a second operator $\mathcal{H}^*_{m+t+2}$ which satisfies an analogous inequality in the variable y. Proceeding as in the case of binomial-type operators we get for $f \in C[0,1]^2$ the following:

$$\left| \left[f - \left({}_x\mathcal{H}^*_{n+s+2} \circ {}_y\mathcal{H}^*_{m+t+2} \right) f \right](x,y) \right| \leq$$
$$\leq [c(s) + c(t)] \cdot \omega_2 \left(f; \tfrac{\sqrt{x(1-x)}}{n}, \tfrac{\sqrt{y(1-y)}}{m} \right).$$

4. APPLICATIONS OF THEOREM 4

4.1. Simultaneous Approximation by Bernstein Operators. We return to the classical Bernstein operators which we already considered in Section 4.1 as special operators of binomial type. Here we supplement the results from there by deriving assertions for the approximation of derivatives by derivatives.

Uniform convergence of certain mixed partial derivatives $[f - ({}_xB_n \circ {}_yB_m)f]^{(k,l)}$ to 0 was considered in the past by many authors. As a more advanced paper we mention one by I. Badea and C. Badea [1] in which the following was shown.

Theorem 12. *If $f \in C^{p,q}[0,1]^2$ and $n > p, m > q$, then*

$$\left\| \left[f - \left({}_xB_n \circ {}_yB_m \right) f \right]^{(p,q)} \right\|_\infty \leq$$
$$\leq t(p,q) \cdot \omega_1 \left(f^{(p,q)}; \tfrac{1}{\sqrt{n-p}}, \tfrac{1}{\sqrt{m-q}} \right) + M^{p,q}_{n,m}(f).$$

Here $t(p,q)$is a certain real-valued function providing small constants, and

$$M^{p,q}_{n,m}(f) := \max\left\{ \tfrac{p(p-1)}{n}, \tfrac{q(q-1)}{m} \right\} \cdot \left\| f^{(p,q)} \right\|_\infty.$$

The article by the two Badeas contains numerous references to earlier papers dealing with simultaneous approximation by tensor product Bernstein operators. In order to modify the above inequality we first cite a result of Kacsó, namely Theorem 5 in [46]:

$$|[D^p(f - B_nf)](x)| \leq \tfrac{p(p-1)}{2n} \cdot |D^pf(x)| + \tfrac{1}{h} \cdot \tfrac{p}{2n} \cdot \omega_1(D^pf;h)$$
$$+ \left(1 + \tfrac{1}{2h^2} \cdot \tfrac{3n-2p}{12n^2}\right) \cdot \omega_2(D^pf;h).$$

The above holds for $p \in \mathbb{N}_0, n \geq \max\{p+2, p(p+1)\}, f \in C^p[0,1],\ x \in [0,1]$ and $h > 0$.

Taking $h = \frac{1}{\sqrt{n}}$ leads to

$$\begin{aligned}\|D^p(f - B_nf)\|_\infty \leq& \tfrac{p(p-1)}{2n} \|D^pf\|_\infty + \tfrac{p}{2\sqrt{n}} \cdot \omega_1\left(D^pf; \tfrac{1}{\sqrt{n}}\right) \\ &+ \tfrac{27n-2p}{24n} \cdot \omega_2\left(D^pf; \tfrac{1}{\sqrt{n}}\right) \\ \leq& c_p \left[\tfrac{1}{n}\omega_0\left(D^p; f\tfrac{1}{\sqrt{n}}\right) + \tfrac{1}{\sqrt{n}}\omega_1\left(D^pf; \tfrac{1}{\sqrt{n}}\right) + \omega_2\left(D^pf; \tfrac{1}{\sqrt{n}}\right) \right],\end{aligned}$$

for $p \geq 0$. This shows that Theorem 4 is applicable if we consider B_n as an operator from $C^p[0,1]$ into itself and make an analogous assumption for B_m.

For simplicity we only discuss the derivative of fixed order (p, q) and derive

$$\begin{aligned}
&\left\| f - ({}_xB_n \circ {}_yB_m)\, f]^{(p,q)} \right\| \leq \\
&\leq \sum_{r=0}^{2} c_p \cdot n^{\frac{r}{2}-1} \cdot \omega_r \left(f^{(p,q)}; \tfrac{1}{\sqrt{n}}, 0 \right) \\
&\quad + \| D^p \circ B_n \|_{C^p[0,1] \to C[0,1]} \cdot \sup_{0 \leq i \leq p} \sum_{s=0}^{2} c_q \cdot m^{\frac{s}{2}-1} \cdot \omega_s \left(f^{(i,q)}; 0, \tfrac{1}{\sqrt{m}} \right).
\end{aligned}$$

Due to the representation

$$\left(D^p B_n f \right)(x) = \frac{(n)_p}{n^p} \cdot p! \sum_{v=0}^{n-p} \left[\frac{v}{n}, \ldots, \frac{v+p}{n}; f \right] \cdot \binom{n-p}{v} x^v (1-x)^{n-p-v}$$

we obtain $\| D^p B_n f \|_\infty \leq \left\| f^{(p)} \right\|_\infty$, so that $\| D^p \circ B_n \|_{C^p[0,1] \to C[0,1]} \leq 1$. Thus

$$\begin{aligned}
&\left| [f - ({}_xB_n \circ {}_yB_m)\, f]^{(p,q)}(x, y) \right| \leq \\
&\leq \max\{c_p, c_q\} \cdot \left\{ \sum_{r=0}^{2} n^{\frac{r}{2}-1} \cdot \omega_r \left(f^{(p,q)}; \tfrac{1}{\sqrt{n}}, 0 \right) \right. \\
&\quad \left. + \sup_{0 \leq i \leq p} \sum_{s=0}^{2} m^{\frac{s}{2}-1} \cdot \omega_s \left(f^{(i,q)}; 0, \tfrac{1}{\sqrt{m}} \right) \right\}.
\end{aligned}$$

Clearly the latter inequality shows that

$$\left\| [f - ({}_xB_n \circ {}_yB_m)\, f]^{(p,q)} \right\|_\infty = o(1) \text{ for all } f \in C^{p,q}[0,1]^2 \text{ when } n, m \to \infty.$$

Moreover, assuming that $f \in C^{p+2,q+2}[0,1]^2$ the inequality implies

$$\left\| [f - ({}_xB_n \circ {}_yB_m)\, f]^{(p,q)} \right\|_\infty = \mathcal{O}\left(\tfrac{1}{n} + \tfrac{1}{m} \right), \quad n, m \to \infty.$$

This order cannot be derived from Theorem 12.

Remark 5. For the discretely defined Bernstein operators it would also have been possible to apply Theorem 3. This would have avoided the appearence of the sup in the upper bound. However, the $o(1)$ and the $\mathcal{O}\left(\frac{1}{n} + \frac{1}{m} \right)$ orders would have remained unchanged. □

4.2. Simultaneous approximation by Bernstein–Durrmeyer operators with Jacobi weights. In this subsection we consider certain operators which are not discretely defined so that, according to our above presentation, only Theorem 4 will be applicable.

The univariate building blocks were recently investigated again by Kacsó [47] and we briefly recall here the basic definition and the result which we will use. For details, historical remarks and useful references the reader should consult [47].

Consider the positive linear operators

$$M_n^{\langle\alpha,\beta\rangle} : C[0,1] \ni f \longmapsto \sum_{k=0}^{n} p_{n,k}(\cdot) \cdot \frac{\int_0^1 \omega^{(\alpha,\beta)}(t) \cdot p_{n,k}(t) f(t)\, dt}{\int_0^1 \omega^{(\alpha,\beta)}(t) \cdot p_{n,k}(t)\, dt} \in \Pi_n,$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \omega^{(\alpha,\beta)}(x) = x^\beta (1-x)^\alpha$ and $\alpha, \beta > -1$. These are the Bernstein-Durrmeyer operators w.r.t. the Jacobi weight $\omega^{(\alpha,\beta)}$. Kacsó [47, Th. 2.2] showed that

$$\left|\left[D^p\left(f - M_n^{\langle\alpha,\beta\rangle}\right)\right](x)\right| \le \tfrac{p\cdot(\alpha+\beta+p+1)}{n} \cdot |D^p f(x)| \\ + \tfrac{\sqrt{n}(\max\{\alpha,\beta\}+p+1)}{b+\alpha+\beta+p+2}\omega_1\left(D^p f; \tfrac{1}{\sqrt{n}}\right) + \tfrac{5}{4}\omega_2\left(D^p f; \tfrac{1}{\sqrt{n}}\right),$$

for $p \in \mathbb{N}\cup\{0\}$, any $f \in C^p[0,1]$, $x \in [0,1]$ and n sufficiently large.

We consider a second copy of the operators in question, namely $M_m^{\langle\bar\alpha,\bar\beta\rangle}$ acting on functions in $C^q[0,1]$, and the variable is $y \in [0,1]$.

For brevity again we only consider the derivative of order (p,q). Theorem 4 implies in this case

$$\left\|\left[f - \left({}_xM_n^{\langle\bar\alpha,\bar\beta\rangle} \circ {}_yM_m^{\langle\bar\alpha,\bar\beta\rangle}\right) f\right]^{(p,q)}\right\|_\infty \le \\ \le c(p,\alpha,\beta) \cdot \sum_{r=0}^{2} n^{\frac{r}{2}-1} \cdot \omega_r\left(f^{(p,q)}; \tfrac{1}{\sqrt{n}}, 0\right) \\ + \left\|D^p \circ M_n^{\langle\alpha,\beta\rangle}\right\|_{C^p[0,1]\to C[0,1]} \cdot c\left(q,\bar\alpha,\bar\beta\right) \sup_{0\le i\le p} \sum_{s=0}^{2} m^{\frac{s}{2}-1} \cdot \omega_s\left(f^{(i,q)}; 0, \tfrac{1}{\sqrt{m}}\right).$$

Recall that n and m have to be sufficiently large in the above.

In order to turn the latter inequality into a more compact and instructive assertion, we consider the quantity

$$\left\|D^p \circ M_n^{\langle\alpha,\beta\rangle}\right\|_{C^p[0,1]\to C[0,1]}.$$

The representation we need here can also be found in [47].

$$D^p \circ M_n^{\langle\alpha,\beta\rangle} f(x) = \tfrac{(n)_p}{(n+\alpha+\beta+p+1)_p} \cdot M_{n-p}^{\langle\alpha,p,\beta+p\rangle} f^{(p)}(x),$$

where the Pochhammer symbol $(a)_b$ is defined as

$$(a)_0 : 1, \qquad (a)_b := \prod_{k=0}^{b-1}(a-k), \qquad a \in \mathbb{R},\ b \in \mathbb{N}.$$

From this representation it follows that

$$\left\|D^p \circ M_n^{\langle\alpha,\beta\rangle}\right\|_{C^p[0,1]\to C[0,1]} \le 1,$$

hence

$$\left\|\left[f - \left({}_xM_n^{\langle\alpha,\beta\rangle} \circ {}_yM_m^{\langle\bar\alpha,\bar\beta\rangle}\right) f\right]^{(p,q)}\right\|_\infty \le$$

$$\leq c\left(p,\alpha,\beta\right)\cdot\sum_{r=0}^{2}n^{\frac{r}{2}-1}\cdot\omega_r\left(f^{(p,q)};\tfrac{1}{\sqrt{n}},0\right)$$
$$+\,c\left(q,\bar{\alpha},\bar{\beta}\right)\sup_{0\leq i\leq p}\sum_{s=0}^{2}m^{\frac{s}{2}-1}\cdot\omega_s\left(f^{(i,q)};0,\tfrac{1}{\sqrt{m}}\right).$$

Again we have

$$\left\|\left[f-\left({}_xM_n^{\langle\alpha,\beta\rangle}\circ\ {}_yM_m^{\langle\bar{\alpha},\bar{\beta}\rangle}\right)f\right]^{(p,q)}\right\|_\infty=o\left(1\right),\ \text{when } n,m\to\infty,$$

and for $f\in C^{p+2,q+2}\left[0,1\right]^2$ the inequality implies

$$\left\|\left[f-\left({}_xM_n^{\langle\alpha,\beta\rangle}\circ\ {}_yM_m^{\langle\bar{\alpha},\bar{\beta}\rangle}\right)f\right]^{(p,q)}\right\|_\infty=\mathcal{O}\left(\tfrac{1}{n}+\tfrac{1}{m}\right),\ n,m\to\infty.$$

5. CONCLUDING REMARK

(i) Tensor product operators (products of parametric extensions) as considered in this note were modified in the past in several ways in order to also handle less regularly spaced data. We mention here in chronological order the work on interpolation by Biermann [5], Steffensen [83], Stancu [70], [74], Delvos and Posdorf [18], and that of Coman *et al.* [12], among others. For example, Stancu considers bivariate Lagrange interpolation at an array of points $\{(x_{ij},y_{ij}):0\leq i\leq m,\ 0\leq j\leq n\}$, thus generalizing Steffensen interpolation.

(ii) Furthermore, also generalizing the notion of a tensor product, Bos *et al.* [8] introduced pseudo-tensor products of univariate schemes. In this context "pseudo" refers to the fact that ${}_xL\left(f;x,y\right)=L\left(f_y;x\right)$ may change with y and that ${}_yM\left(f;x,y\right)=M\left(f_x;y\right)$ may change with x. This method was also investigated in the diploma thesis of Dyllong [20] to some extent.

For neither of the above methods significant quantitative inheritance statements seem to have been stated according to the authors knowledge until [2] and [3] were written.

REFERENCES

[1] I. Badea, C. Badea, *On the order of simultaneous approximation of bivariate functions by Bernstein operators*, Anal. Numér. Théor. Approx., **16** (1987), pp. 11–17, `https://ictp.acad.ro/jnaat/journal/article/view/1987-vol16-no1-art2`.

[2] L. Beutel, H. Gonska, *Simultaneous Approximation by Tensor Product Operators*, Duisburg: Schriftenreihe des Fachbereichs Mathematik der Gerhard–Mercator–Universität SM-DU-504 (2001).

[3] L. Beutel, H. Gonska, *Quantitative inheritance properties for simultaneous approximation by tensor product operators*, Numer. Algor., **33** (2003), 83–92, `https://doi.org/10.1023/A:1025539316792`.

[4] L. Beutel, H.H. Gonska, D. Kacsó , G. Tachev, *On variation-diminishing Schoenberg operators: new quantitative statements. In: Multivariate Approximation and Interpolation with Applications* (ed. by M. Gasca), Monogr. Academia Ciencias de Zaragoza 20 (2002), pp. 9–58.

[5] O. Biermann, *Über näherungsweise Kubaturen*, Monatsh. Math. Physik, **14** (1903), pp. 211–225, https://doi.org/10.1007/bf01706869.
[6] G. Birkhoff, H. Garabedian, *Smooth surface interpolation*, J. Math. Phys., **39** (1960), pp. 258–268, https://doi.org/10.1002/sapm1960391258.
[7] W. Böhm, G. Farin, J. Kahmann, *A survey of curve and surface methods in CAGD*, Comput. Aided Geom. Design, **1** (1984), pp. 1–60, https://doi.org/10.1016/0167-8396(84)90003-7.
[8] L.B. Bos, J.E. Grabenstetter, K. Salkauskas, *Pseudo–tensor product interpolation and blending with families of univariate schemes*, Comput. Aided Geom. Design, **13** (1996), pp. 429–440, https://doi.org/10.1016/0167-8396(95)00036-4.
[9] E.W. Cheney, *Best approximation in tensor product spaces*. In: Numerical Analysis, Proc. 8th Biennial Conf., Dundee 1979 (ed. by G.A. Watson), pp. 25–32.
[10] C. Cismaşiu, D. Acu, *An integral representation of the remainder in an approximation formula by a bivariate operator of Cheney-Sharma*, in: "RoGer 2000 – Braşov", Proc. 4th Romanian–German Seminar on Approximation Theory and its Applications, Braşov 2000 (ed. by H.H. Gonska, D. Kacsó, L. Beutel), 34–39. Duisburg: Schriftenreihe des Fachbereichs Mathematik der Gerhard-Mercator-Universität SM-DU-485 (2000).
[11] Ch. Coatmélec, *Approximation et interpolation des fonctions différentiables de plusieurs variables*, Ann. Sci. Ecole Norm. Sup., **83** (1966) 3, pp. 271–341, https://doi.org/10.24033/asens.1157.
[12] Gh. Coman, L. Ţâmbulea, I. Gânscă, *Multivariate approximation, Babeş-Bolyai University, Cluj-Napoca*, Faculty of Mathematics and Informatics, Research Seminars, Seminar of Numerical and Statistical Calculus, Preprint no. 1 (1996), pp. 29–60.
[13] R. Dahlhaus, *Pointwise approximation by algebraic polynomials*, J. Approx. Theory, **57** (1989), pp. 274–277, https://doi.org/10.1016/0021-9045(89)90042-7.
[14] C. de Boor, *Bicubic spline interpolation*, J. Math. Phys., **41** (1962), pp. 212–218, https://doi.org/10.1002/sapm1962411212.
[15] C. de Boor, Appendix to *"Splines and histograms"* by I.J. Schoenberg. In: Spline Functions and Approximation Theory, Proc. Sympos., Univ. Alberta, Edmonton, Alta., 1972 (ed. by A. Meir, A. Sharma), pp. 329–358, Birkhäuser, Basel, 1973.
[16] C. de Boor, *A Practical Guide to Splines*, Springer, New York, 1978.
[17] C. de Boor, *Topics in multivariate approximation theory*, In: Topics in Numerical Analysis, Proceedings of the S.E.R.C. Summer School, Lancaster 1981 (ed. by P.R. Turner). Lecture Notes in Mathematics, **965**, pp. 39–78, Springer, Berlin, 1982.
[18] F.J. Delvos, H. Posdorf, *Generalized Biermann interpolation*, Result. Math., **5** (1982), pp. 6–18, https://doi.org/10.1007/BF03323297.
[19] F.J. Delvos, W. Schempp, *The method of parametric extension applied to right invertible operators*, Numer. Funct. Anal. Optim., **6** (1983), pp. 135–148, https://doi.org/10.1080/01630568308816158.
[20] E. Dyllong, *Mehrstufenverfahren und ihre Anwendungen im Flächendesign*. Diplomarbeit, Universität Duisburg, 1997.
[21] G. Farin, *Curves and Surfaces for Computer–Aided Geometric Design, A Practical Guide*, Academic Press, San Diego, 1997.
[22] J. Ferguson, Multiv*ariable curve interpolation*, J. ACM, II/**2** (1964), pp. 221–228, https://doi.org/10.1145/321217.321225.
[23] M. Gasca, Th. Sauer, *On the history of multivariate polynomial interpolation*, in: Numerical Analysis 2000, Vol. II: Interpolation and Extrapolation (ed. by C. Brezinski). J. Comput. Appl. Math., **122** (2000) 1–2, pp. 23–35, https://doi.org/10.1016/s0377-0427(00)00353-8.

[24] I. Gavrea, H.H. Gonska, D.P. Kacsó, *Variation on Butzer's problem: characterization of the solutions*, Comput. Math. Appl., **34** (1997) 9, pp. 51–64, https://doi.org/10.1016/s0898-1221(97)00188-0.

[25] I. Gavrea, H.H. Gonska, D.P. Kacsó, *A class of discretely defined positive linear operators satisfying DeVore-Gopengauz inequalities*, Rev. Anal. Numér. Théor. Approx., **27** (1998) 2, pp. 263–275, https://ictp.acad.ro/jnaat/journal/issue/view/1998-vol27-no2.

[26] I. Gavrea, H.H. Gonska, D.P. Kacsó, *On discretely defined positive linear polynomial operators giving optimal degrees of approximation.* In: Proc. 3rd Int. Conf. on Functional Analysis and Approximation Theory, Acquafredda di Maratea 1996. Rend. Circ. Mat. Palermo Serie II No. 52 Vol. II (1998), pp. 455–473.

[27] I. Gânscă, Gh. Coman, L. Ţâmbulea, *Contributions to rational Bézier curves and surfaces. In: Approximation and Optimization* (Vol. II). Proc. ICAOR Cluj-Napoca 1996 (ed. by D.D. Stancu, Gh. Coman, W.W. Breckner, P. Blaga), 107–116, Transilvania Press, Cluj-Napoca, 1997.

[28] H.H. Gonska, *Quantitative Korovkin–type theorems on simultaneous approximation*, Math. Z., **186** (1984), pp. 419–433, https://doi.org/10.1007/bf01174895.

[29] H.H.Gonska, *Quantitative Approximation in C(X)*, Habilitationsschrift, Universität Duisburg 1985. Duisburg: Schriftenreihe des Fachbereichs Mathematik der Universität Duisburg SM-DU-94 (1986).

[30] H.H. Gonska, *Simultaneous approximation by algebraic blending functions*, in: Proc. Alfred Haar Memorial Conf., Budapest 1985 (ed. by J. Szabados, K. Tandori), Colloq. Soc. János Bolyai 49, 363–382, North-Holland, Amsterdam, 1987.

[31] H.H. Gonska, *Simultaneous approximation by generalized n-th order blending operators*, In: Multivariate Approximation Theory IV, Proc. Conf. Oberwolfach 1989 (ed. by C.K. Chui, W. Schempp, K. Zeller), 173–180, Birkhäuser, Basel, 1989.

[32] H.H. Gonska, *Degree of simultaneous approximation of bivariate functions by Gordon operators*, J. Approx. Theory, **62** (1990), pp. 170–191, https://doi.org/10.1016/0021-9045(90)90030-t.

[33] H.H. Gonska, *Products of parametric extensions: refined estimates*, In: Proc. 2nd Int. Conf. on "Symmetry and Antisymmetry", in Mathematics, Formal Languages and Computer Science, Braşov 2000 (ed. by G.V. Orman, D. Bocu), 1–15, Editura Universităţii Transilvania, Braşov, 2000.

[34] H.H. Gonska, E. Hinnemann, *Punktweise Abschätzungen zur Approximation durch algebraische Polynome*, Acta Math. Hung., **46** (1985) 3-4, pp. 243-254.

[35] H.H. Gonska, D. Leviatan, I.A. Shevchuk, H.-J. Wenz, *Interpolatory pointwise estimates for polynomial approximation*, Constr. Approx., **16** (2000), pp. 603–629, https://doi.org/10.1007/s003650010008.

[36] T.N.T. Goodman, *Shape preserving representations*, In: Mathematical Methods in Computer Aided Geometric Design (ed. by T. Lyche, L.L. Schumaker), 333–351, Academic Press, Boston, 1989.

[37] W.J. Gordon, E.W. Cheney, *Bivariate and multivariate interpolation with noncommutative projectors.* In: Linear Spaces and Approximation (ed. by P.L. Butzer, B. Sz.-Nagy), 381–387, Birkhäuser, Basel, 1978.

[38] C.A. Hall, *Natural cubic and bicubic spline interpolation*, SIAM J. Numer. Anal., **10** (1973), pp. 1055–1060, https://doi.org/10.1137/0710088.

[39] W. Haussmann, *Hermite-Interpolation in mehreren Veränderlichen*, Dissertation, Ruhr-Universität Bochum, 1969.

[40] W. Haussmann, P. Pottinger, *On the construction and convergence of multivariate interpolation operators*, J. Approx. Theory, **19** (1977), pp. 205–221, https://doi.org/10.1016/0021-9045(77)90052-1.

[41] T.H. Hildebrandt, I.J. Schoenberg, *On linear functional operations and the moment problem*, Ann. Math., **34** (1933) 2, pp. 317–328, https://doi.org/10.2307/1968205.

[42] J. Hoschek, D. Lasser, *Fundamentals of Computer Aided Geometric Design*, A.K. Peters, Wellesley, MA, 1993.

[43] K. Höllig, U. Reif, J. Wipper, *Error estimates for the web-method*, Preprint Nr. 2126, Technische Universität Darmstadt, 2000.

[44] K. Höllig, U. Reif, J. Wipper, *Weighted extended b-spline approximation of Dirichlet problems*, Preprint 2000–8, Universität Stuttgart, 2000.

[45] D. Kacsó, *On the degree of simultaneous approximation.* In: "RoGer 2000 – Braşov", Proc. 4th Romanian–German Seminar on Approximation Theory and its Applications, Braşov 2000 (ed. by H.H. Gonska, D. Kacsóo, L. Beutel), 62–69. Duisburg: Schriftenreihe des Fachbereichs Mathematik der Gerhard-Mercator-Universität SM-DU-485 (2000).

[46] D. Kacsó, *Simultaneous approximation by almost convex operators*, Rend. Circ. Mat. Palermo (2) Suppl., **68** (2002) 2, pp. 523–538.

[47] D. Kacsó, *Quantitative statements for the Bernstein–Durrmeyer operators with Jacobi weights.* In: Mathematical Analysis and Approximation Theory, Proc. 5th Romanian-German Seminar on Approximation Theory and its Applications (ed. by A. Lupaş et al.), 135–144, Burg-Verlag, Sibiu, 2002.

[48] H.-B. Knoop, X.L. Zhou, *On the Jackson integral operator*, Rend. Circ. Mat. Palermo (2) Suppl., **68** (2002) 2, pp. 555–567.

[49] P. Lancaster, *Composite methods for generating surfaces. In: Polynomial and Spline Approximation.* Theory and Applications, Proc. NATO Adv. Study Inst. Univ. Calgary, Alta. 1978 (ed. by B.N. Sahney), 91–102. Dordrecht-Boston, MA: D. Reidel 1979.

[50] W.A. Light, E.W. Cheney, *Approximation Theory in Tensor Product Spaces*, Springer, Berlin, 1985.

[51] A.J. López Moreno, F.J. Muñoz Delgado, *Grado de aproximación de operadores lineales positivos*, Actas de las V Jornadas Zaragoza-Pau de Matemática Aplicada y Estadística, Jaca 1997, Publicaciones del Seminario Matemático, Universidad de Zaragoza.

[52] R.A. Lorentz, *Multivariate Birkhoff Interpolation*, Lecture Notes in Mathematics 1516, Springer, Berlin, 1992.

[53] A. Lupaş, *Approximation operators of binomial type.* In: New Developments in Approximation Theory. Papers 2nd Int. Dortmund Meeting on Approximation Theory 1998 (ed. by M.W. Müller, M.D. Buhmann, D.H. Mache, M. Felten). Internat. Ser. Numer. Math. 132, 175–198, Birkhäuser, Basel, 1999.

[54] G. Mastroianni, *Sull' approssimazione di funzioni continue mediante operatori lineari*, Calcolo, **14** (1977), pp. 343–368, http://dx.doi.org/10.1007/BF02575991.

[55] H. Mettke, *Biquadratische Interpolations-und Volumenabgleichssplines*, Beitr. Num. Math., **9** (1981), pp. 135–149.

[56] H. Mettke, *Fehlerabschätzungen zur zweidimensionalen Splineinterpolation*, Beitr. Num. Math., **11** (1983), pp. 81–91 https://doi.org/10.1515/9783486992977-007.

[57] G. Moldovan, *L'evaluation de l'erreur de l'approximation d'une fonction continue par certains opérateurs linéaires positifs*, Mathematica (Cluj), **22** (45) (1980) no. 1, pp. 85–95.

[58] L. Neder, *Interpolationsformeln für Funktionen mehrerer Argumente*, Scandinavisk Aktuarietidskrift, **9** (1926), pp. 59–69.

[59] R. Păltănea, *New types of estimates with moduli of continuity.* In: "RoGer 2000 – Braşov", Proc. 4th Romanian-German Seminar on Approximation Theory and its Applications, Braşov 2000 (ed. by H.H. Gonska, D. Kacsó, L. Beutel), 110–114. Duisburg:

Schriftenreihe des Fachbereichs Mathematik der Gerhard-Mercator-Universität SM-DU-485 (2000).

[60] T. Popoviciu, *Remarques sur les polynômes binomiaux*, [Bul. Soc. Sci. Cluj 6 (1931), 146-148]= [Mathematica (Cluj), 6 (1932), pp. 8–10].

[61] M.K. Potapov, M.A. Jiménez Pozo, *Sobre la aproximación mediante ángulos en espacios de funciones continuas*, Cienc. Mat. (Havana), **2** (1981), pp. 101–110.

[62] H. Prautzsch, T. Gallagher, *Is there a geometric variation diminishing property for B-spline or Bézier surfaces?*, Comput. Aided Geom. Design, **9** (1992), pp. 119–124, `https://doi.org/10.1016/0167-8396(92)90011-d`.

[63] P. Sablonnière, *Positive Bernstein-Sheffer operators*, J. Approx. Theory, **83** (1995), pp. 330–341, `https://doi.org/10.1006/jath.1995.1124`.

[64] D. Sbibih, E.B. Ameur, P. Sablonnière, *A general multiresolution method for fitting functions on the sphere*, Slides presented at the "Int. Conf. on Num. Algorithms", Marrakesh (Morocco), 2001.

[65] H.-J. Schelske, *Lokale Glättung segmentierter Bézierkurven und Bézierflächen, Dissertation*, Darmstadt: Fachbereich Mathematik der Technischen Hochschule Darmstadt, 1984.

[66] L.L. Schumaker, *Fitting surfaces to scattered data*, In: Approximation Theory II (ed. by G.G. Lorentz), 203–268. Academic Press, New York, 1976.

[67] L.L. Schumaker, *Spline Functions*: Basic Theory, Wiley, New York, 1981.

[68] F. Schurer, F.W. Steutel, *On the exact degree of approximation of Bernstein operators on $C([0,1]^2)$*, In: Multivariate Approximation Theory. Proc. Conf. Math. Res. Inst. Oberwolfach 1979 (ed. by W. Schempp, K. Zeller), 413–435, Birkhäuser, Basel, 1979.

[69] F. Schurer, F.W. Steutel, *Exact degrees of approximation for two-dimensional Bernstein operators on C1*, In: Approximation Theory III. Proc. Int. Sympos. Austin 1980 (ed. by E.W. Cheney), 823–828, Academic Press, New York, 1980.

[70] D.D. Stancu, *Generalizarea unor formule de interpolare pentru funcţiile de mai multe variabile şi unele consideraţii asupra formulei de integrare numerică a lui Gauss*, Acad. R.P. Române, Bul. Şti. Secţ. Şti. Mat. Fiz., **9** (1957), pp. 287–313.

[71] D.D. Stancu, *Generalizarea unor polinoame de interpolare pentru funcţiile de mai multe variabile*, Bul. Inst. Politehn. Iaşi. Serie nouă, tomul III (VII), fasc. 1-2 (1957), pp. 31–38.

[72] D.D. Stancu, *Sur l'approximation des dérivées des fonctions par les dérivées correspondantes de certains polynômes du type Bernstein*, Mathematica (Cluj), 2 (**25**) (1960), pp. 335–348.

[73] D.D. Stancu, *Evaluation of the remainder term in approximation formulas by Bernstein polynomials*, Math. Comp., **17** (1963), pp. 270-278, `https://doi.org/10.1090/s0025-5718-1963-0179524-6`.

[74] D.D. Stancu, *The remainder of certain linear approximation formulas in two variables*, SIAM J. Numer. Anal. Ser. B, **1** (1964), pp. 137–163, `https://www.jstor.org/stable/2949772`.

[75] D.D. Stancu, *Approximation of functions by a new class of linear polynomial operators*, Rev. Roumaine Math. Pures Appl., **13** (1968), 1173–1194.

[76] D.D. Stancu, *Aproximarea funcţiilor de două şi mai multe variabile printr-o clasă de polinoame de tip Bernstein*, Stud. Cerc. Mat., **22** (1970), pp. 335–345.

[77] D.D. Stancu, *Approximation of bivariate functions by means of some Bernstein-type operators*, in: Multivariate Approximation. Proc. Sympos. Durham 1977 (ed. by D.C. Handscomb), 189–208, Academic Press, London, 1978.

[78] D.D. Stancu, *Representation of remainders in approximation formulae by some discrete type linear positive operators*, Rend. Circ. Mat. Palermo, Ser. II, Suppl., **52** (1998), pp. 781–791.

[79] D.D. Stancu, *Remainder in approximation by means of certain linear operators*. In: Mathematical Analysis and Approximation Theory, Proc. 5th Romanian-German Seminar on Approximation Theory and its Applications (ed. by A. Lupaş et al.), 259–271, Burg-Verlag, Sibiu, 2002.

[80] D.D. Stancu, M.R. Occorsio, *On approximation by binomial operators of Tiberiu Popoviciu type*, Rev. Anal. Numér. Théor. Approx., **27** (1998), pp. 167–181. `https://ictp.acad.ro/jnaat/journal/issue/view/1998-vol27-no1`

[81] D.D. Stancu, A.D. Vernescu, *Approximation of bivariate functions by means of a class of operators of Tiberiu Popoviciu type*, Mathematical Reports, 1 (**51**) (1999), pp. 411–419.

[82] D.D. Stancu, *Aproximarea funcţiilor de două variabile cu ajutorul unor noi clase de operatori liniari pozitivi*, An. Univ. Timişoara, Ser. Şti. Mat., 9 (2) (1973), pp. 159–168.

[83] J.F. Steffensen, *Interpolation*, Chelsea, New York, 1927 (second edition reprinted 1965).

[84] A. Vernescu, *Contributions to the application of umbral calculus in the theory of the approximation of functions*, Abstract of the doctoral thesis, Universitatea Babeş-Bolyai, Cluj-Napoca, Romania, 2000.

[85] A. Vernescu, *Some generalizations of a Lupaş operator*, In: "RoGer 2000 – Braşov", Proc. 4th Romanian-German Seminar on Approximation Theory and its Applications, Braşov 2000 (ed. by H.H. Gonska, D. Kacsó, L. Beutel), 166–169. Duisburg: Schriftenreihe des Fachbereichs Mathematik der Gerhard-Mercator-Universität SM-DU-485 (2000).

[86] V.I. Zabutnaya, S.B. Vakarchuk, *The smoothing operators and the problems of the approximation recovery of curves and surfaces*, manuscript, 1996.

[87] X.L. Zhou, *Approximation by multivariate Bernstein operators*, Result. Math., **25** (1994), pp. 166–191, `https://doi.org/10.1007/BF03323150`.

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