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BALL CONVERGENCE OF POTRA-PTAK-TYPE METHOD WITH OPTIMAL FOURTH ORDER OF CONVERGENCE

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Abstract. We present a local convergence analysis Potra-Ptak-type method with optimal fourth order of convergence in order to approximate a solution of a nonlinear equation. In earlier studies such as [1], [5]–[28] hypotheses up to the fourth derivative are used. In this paper we use hypotheses up to the first derivative only, so that the applicability of these methods is extended under weaker hypotheses. Moreover the radius of convergence and computable error bounds on the distances involved are also given in this study. Numerical examples are also presented in this study.

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1. INTRODUCTION

Let $F : D \subseteq S \to S$ is a nonlinear function, D is a convex subset of S and S is $\mathbb{R}$ or $\mathbb{C}$. Consider the problem of approximating a locally unique solution x^* of equation

$$F(x) = 0. \tag{1}$$

Newton-like methods are famous for finding solution of (1), these methods are usually studied based on: semi-local and local convergence [3, 4, 20, 21, 22, 24, 26].

Third order methods such as Euler's, Halley's, super Halley's, Chebyshev's [1]–[28] require the evaluation of the second derivative F'' at each step, which in general is very expensive. That is why many authors have used higher order multipoint methods [1]–[28]. In this paper, we study the local convergence of fourth order method defined for each $n = 0, 1, 2, \dots$ by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1} F(x_n) \\ z_n &= x_n - F'(x_n)^{-1} (F(x_n) + F(y_n)) \end{aligned}$$

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$$(2) \qquad x_{n+1} = x_n - F(x_n)^{-2}F'(x_n)^{-1}F(y_n)^2(2F(x_n)+F(y_n)),$$

where x_0 is an initial point. Method (2) was studied by Cordero et.al. in [13]. In particular the fourth order of convergence was shown under hypotheses reaching up to the fourth derivative of function F. Notice that method (2) involves three functional evaluations. Therefore the efficiencyindex $EI = p^{\frac{1}{m}}$ where p is theorder of convergence and m is the number of functional evaluations per step gives $EI = 4^{\frac{1}{3}} = 1.5874$. Kung and Traub conjecture [28] that the order of convergence of any multipoint method without memory cannot exceed the bound 2^{m-1} (called the optimal order). Thus, the optimal order for a method with three function evaluations per step should be four.

Other single and multi-point methods can be found in [2, 3, 20, 25] and the references therein. The local convergence of the preceding methods has been shown under hypotheses up to the fourth derivative (or even higher). These hypotheses restrict the applicability of these methods. As a motivational example, let us define function f on $D = [-\frac{1}{2}, \frac{5}{2}]$ by

$$f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Choose $x^* = 1$. We have that

$$\begin{aligned} f'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \ f'(1) = 3, \\ f''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x \\ f'''(x) &= 6 \ln x^2 + 60x^2 - 24x + 22. \end{aligned}$$

Then, obviously, function f''' is unbounded on D. In the present paper we only use hypotheses on the first Fréchet derivative. This way we expand the applicability of method (2).

The rest of the paper is organized as follows: Section 2 contains the local convergence analysis of methods (2). The numerical examples are presented in the concluding Section 3.

2. LOCAL CONVERGENCE ANALYSIS

We present the local convergence analysis of method (2) in this section. Let $L_0 > 0, L > 0$ and $M \geq 1$. It is convenient for the local convergence analysis of method (2) to introduce some functions and parameters. Define functions g_1, g_2, h_2, g_3, h_3 on the interval $[0, \frac{1}{L_0})$ by

$$\begin{aligned} g_1(t) &= \tfrac{Lt}{2(1-L_0t)}, \\ g_2(t) &= \tfrac{1}{2(1-L_0t)}[Lt + 2Mg_1(t)], \\ h_2(t) &= g_2(t) - 1 \\ g_3(t) &= g_2(t) + \frac{M^3 g_1^2(t)(2+g_1(t))}{(1-\frac{L_0}{2}t)^2(1-L_0t)}, \\ h_3(t) &= g_3(t) - 1 \end{aligned}$$

and parameter

$$r_A = \frac{2}{2L_0+L}.$$

We have that $h_2(0) = -1 < 0$ and $h_2(r_A) = \frac{M}{1-L_0r_A} > 0$, since $\frac{Lr_A}{2(1-L_0r_A)} = 1$ and $1 - L_0r_A > 0$. It then follows from the intermediate value theorem that function h_2 has zeros in the interval $(0, r_A)$. Denote by r_2 the smallest such zero. Similarly, we have that $h_3(0) = -1 < 0$ and $h_3(r_2) > 0$. Denote by r the smallest zero of function h_3 in the interval $(0, r_2)$. Then, for each $t \in [0, r)$ we have

$$0 \leq g_1(t) < 1, \tag{3}$$

$$0 \leq g_2(t) < 1, \tag{4}$$

and

$$0 \leq g_3(t) < 1. \tag{5}$$

Denote by $U(v,\rho), \bar{U}(v,\rho)$ the open and closed balls in S, respectively, with center $v \in S$ and of radius $\rho > 0$. Next, we show the following local convergence result for method (2) using the preceding notation.

Theorem 1. *Let $F : D \subseteq S \to S$ be a differentiable function. Suppose that there exist $x^* \in D$, $L_0 > 0, L > 0$ and $M \geq 1$ such that for each $x, y \in D$*

$$F(x^*) = 0, \ F'(x^*) \neq 0, \tag{6}$$

$$|F'(x^*)^{-1}(F'(x) - F'(x^*))| \leq L_0|x - x^*|, \tag{7}$$

$$|F'(x^*)^{-1}(F'(x) - F'(y))| \leq L|x - y|, \tag{8}$$

$$|F'(x^*)^{-1}F'(x)| \leq M, \tag{9}$$

and

$$\bar{U}(x^*, (1 + M_0)r) \subseteq D, \tag{10}$$

where r is defined above Theorem 1. Then, the sequence $\{x_n\}$ generated by method (2) for $x_0 \in U(x^*, r) - \{x^*\}$ is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \cdots$ and converges to x^*. Moreover, the following estimates hold

$$|y_n - x^*| \leq g_1(|x_n - x^*|)|x_n - x^*| < |x_n - x^*| < r, \tag{11}$$

$$|z_n - x^*| \leq g_2(|x_n - x^*|)|x_n - x^*| < |x_n - x^*| \tag{12}$$

and

$$|x_{n+1} - x^*| \leq g_3(|x_n - x^*|)|x_n - x^*| < |x_n - x^*|, \tag{13}$$

where the "g" functions are defined above in Theorem 1. Furthermore, if that there exists $T \in [r, \frac{2}{L_0})$ such that $\bar{U}(x^*, T) \subset D$, then the limit point x^* is the only solution of equation $F(x) = 0$ in $\bar{U}(x^*, T)$.

Proof. We shall use induction to show estimates (11)–(13). Using the hypothesis $x_0 \in U(x^*, r) - \{x^*\}$, (7) and the definition of r, we have that

$$|F'(x^*)^{-1}(F'(x_0) - F'(x^*))| \leq L_0|x_0 - x^*| < L_0 r < 1. \tag{14}$$

It follows from (14) and the Banach Lemma on invertible functions [3, 4, 19, 20, 22, 23] that $F'(x_0) \neq 0$ and

$$|F'(x_0)^{-1}F'(x^*)| \leq \frac{1}{1-L_0|x_0-x^*|} < \frac{1}{1-L_0 r}. \tag{15}$$

Hence, y_0 and x_0 are well defined. We also get from (2), (3), (8) and (15) that

$$|y_0 - x^*| \leq |F'(x_0)^{-1}F'(x^*)|| \int_0^1 F'(x^*)^{-1}(F'(x^* + \theta(x_0 - x^*)) - F'(x_0))(x_0 - x^*)d\theta$$

$$\leq \frac{L|x_0-x^*|^2}{2(1-L_0|x_0-x^*|)} \tag{16}$$

$$= g_1(|x_0 - x^*|)|x_0 - x^*| < |x_0 - x^*| < r, \tag{17}$$

which shows (11) for $n = 0$ and $y_0 \in U(x^*, r)$. We can write

$$F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta. \tag{18}$$

Then, by (9), (18) we obtain that

$$\begin{aligned} |F'(x^*)^{-1}F(x_0)| &\leq |\int_0^1 F'(x^*)^{-1}F'(x^* + \theta(x_0 - x^*))(x_0 - x^*)d\theta| \\ &\leq M|x_0 - x^*|, \end{aligned} \tag{19}$$

where we also used $|x^* + \theta(x_0 - x^*) - x^*| = \theta|x_0 - x^*| < r$. That is $x^* + \theta(x_0 - x^*) - x^* \in U(x^*, r)$. We also have that

$$|F'(x^*)^{-1}F(y_0)| \leq M|y_0 - x^*| \leq Mg_1(|x_0 - x^*|)|x_0 - x^*|. \tag{20}$$

Then, by the second substep of method (2) for $n = 0$, (4), (6) and (20) we get that

$$\begin{aligned} |z_0 - x^*| &\leq |x_0 - x^* - F'(x_0)^{-1}F(x_0)| \\ &\quad + |F'(x_0)^{-1}F'(x^*)||F'(x^*)^{-1}F(y_0)| \\ &\leq \frac{L|x_0-x^*|^2}{2(1-L_0|x_0-x^*|} + \frac{M|y_0-x^*|}{1-L_0|x_0-x^*|} \\ &\leq \frac{(L|x_0-x^*|+2Mg_1(|x_0-x^*|))|x_0-x^*|}{2(1-L_0|x_0-x^*|)} \\ &= g_2(|x_0 - x^*|)|x_0 - x^*| < |x_0 - x^*| < r, \end{aligned} \tag{21}$$

which shows (12) for $n = 0$ and $z_0 \in U(x^*, r)$. Next, we show that $F(x_0) \neq 0$. Using (7), we get that

$$\begin{aligned} &|(F'(x^*)(x_0 - x^*))^{-1}[F(x_0) - F(x^*) - F'(x^*)(x_0 - x^*)]| \\ &\leq |x_0 - x^*|^{-1}| \int_0^1 F'(x^*)^{-1}(F'(x^* + \theta(x_0 - x^*)) - F'(x^*))(x_0 - x^*)d\theta| \\ &\leq |x_0 - x^*|^{-1} \frac{L_0|x_0-x^*|^2}{2} \end{aligned}$$

$$(22)\quad \le \tfrac{L_0|x_0-x^*|}{2} < \tfrac{L_0 r}{2} < 1.$$

it follows from (22) that $F(x_0) \neq 0$ and

$$|F(x_0)^{-1}F'(x^*)| \le \tfrac{1}{|x_0-x^*|(1-\frac{L_0}{2}|x_0-x^*|)}. \tag{23}$$

Hence x_1 is well defined. Then, using the last substep of method (2) for $n=0$, (5), (15), (19), (20), (21) and (23) we get in turn that

$$\begin{aligned}
|x_1 - x^*| &\le |z_0 - x^*| + |F'(x^*)^{-1}F(y_0)|^2|F'(x^*)^{-1}F(x_0)|^{-2}\\
&\quad \times |F'(x^*)^{-1}F'(x_0)|^{-1}(2|F'(x^*)^{-1}F(x_0)|\\
&\quad + |F'(x^*)^{-1}F(y_0)|)\\
&\le g_2(|x_0-x^*|)|x_0-x^*|\\
&\quad + \tfrac{M^3|y_0-x^*|^2(2|x_0-x^*|+|y_0-x^*|)}{|x_0-x^*|^2(1-\frac{L_0}{2}|x_0-x^*|)^2(1-L_0|x_0-x^*|)}\\
&\le [g_2(|x_0-x^*|)\\
&\quad + \tfrac{M^3 g_1^2(|x_0-x^*|)(2+g_1(|x_0-x^*|)}{(1-\frac{L_0}{2}|x_0-x^*|)^2(1-L_0|x_0-x^*|)}]|x_0-x^*|\\
&= g_3(|x_0-x^*|)|x_0-x^*| < |x_0-x^*| < r,
\end{aligned}$$

which shows (13) for $n=0$ and $x_1 \in U(x^*, r)$. By simply replacing x_0, y_0, z_0, x_1 by x_k, y_k, z_k, x_{k+1} in the preceding estimates we arrive at estimates (11)–(13). Using the estimate $|x_{k+1}-x^*| < |x_k - x^*| < r$, we deduce that $x_{k+1} \in U(x^*, r)$ and $\lim_{k\to\infty} x_k = x^*$. To show the uniqueness part, let $Q = \int_0^1 F'(y^* + \theta(x^* - y^*)d\theta$ for some $y^* \in \bar{U}(x^*, T)$ with $F(y^*) = 0$. Using (7) we get that

$$\begin{aligned}
|F'(x^*)^{-1}(Q - F'(x^*))| &\le \int_0^1 L_0|y^* + \theta(x^*-y^*) - x^*|d\theta\\
&\le \int_0^1 (1-\theta)|x^*-y^*|d\theta \le \frac{L_0}{2}R < 1.
\end{aligned} \tag{24}$$

It follows from (24) and the Banach Lemma on invertible functions that Q is invertible. Finally, from the identity $0 = F(x^*) - F(y^*) = Q(x^* - y^*)$, we deduce that $x^* = y^*$. □

Remark 2. 1. In view of (7) and the estimate

$$\begin{aligned}
\|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\|\\
&\le 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le 1 + L_0\|x - x^*\|
\end{aligned}$$

condition (9) can be dropped and M can be replaced by

$$M(t) = 1 + L_0 t.$$

2. The results obtained here can be used for operators F satisfying autonomous differential equations [3] of the form

$$F'(x) = P(F(x))$$

where P is a continuous operator. Then, since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then, we can choose: $P(x) = x + 1$.

3. The radius r_A was shown by us to be the convergence radius of Newton's method [2]–[4]

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \text{ for each } n = 0, 1, 2, \cdots \tag{25}$$

under the conditions (7) and (8). It follows from the definition of r that the convergence radius r of the method (2) cannot be larger than the convergence radius r_A of the second order Newton's method (25). As already noted in [3, 4] r_A is at least as large as the convergence ball given by Rheinboldt [27]

$$r_R = \tfrac{2}{3L}. \tag{26}$$

In particular, for $L_0 < L$ we have that

$$r_R < r$$

and

$$\tfrac{r_R}{r_A} \to \tfrac{1}{3} \;\; as \;\; \tfrac{L_0}{L} \to 0.$$

That is our convergence ball r_A is at most three times larger than Rheinboldt's. The same value for r_R was given by Traub [28].

4. It is worth noticing that method (2) is not changing when we use the conditions of Theorem 1 instead of the stronger conditions used in [1, 5, 12]–[28]. Moreover, we can compute the computational order of convergence (COC) defined by

$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) / \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

This way we obtain in practice the order of convergence in a way that avoids the bounds involving estimates using estimates higher than the first Fréchet derivative of operator F.

3. NUMERICAL EXAMPLES

We present numerical examples in this section.

EXAMPLE 3. Let $D = [-\infty, +\infty]$. Define function f of D by

$$f(x) = \sin(x). \tag{27}$$

Then we have for $x^* = 0$ that $L_0 = L = M = 1, \alpha = 1$. The parameters are $r_A = 0.6667$, $r_2 = 0.6667$, $r = 0.3991$ and $\xi_1 = 5.1010$.

EXAMPLE 4. Let $D = [-1, 1]$. Define function f of D by

$$f(x) = e^x - 1. \tag{28}$$

Using (28) and $x^* = 0$, we get that $L_0 = e - 1 < L = M = e, \alpha = 1$. The parameters are $r_A = 0.3249$, $r_2 = 0.1458$, $r = 0.0699$ and $\xi_1 = 3.9088$.

□

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