

GENERAL MULTIVARIATE ARCTANGENT FUNCTION ACTIVATED
NEURAL NETWORK APPROXIMATIONS

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Abstract. Here we expose multivariate quantitative approximations of Banach space valued continuous multivariate functions on a box or \mathbb{R}^N , $N \in \mathbb{N}$, by the multivariate normalized, quasi-interpolation, Kantorovich type and quadrature type neural network operators. We treat also the case of approximation by iterated operators of the last four types. These approximations are derived by establishing multidimensional Jackson type inequalities involving the multivariate modulus of continuity of the engaged function or its high order Fréchet derivatives. Our multivariate operators are defined by using a multidimensional density function induced by the arctangent function. The approximations are pointwise and uniform. The related feed-forward neural network is with one hidden layer.

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1. INTRODUCTION

The author in [2] and [3], see chapters 2–5, was the first to establish neural network approximations to continuous functions with rates by very specifically defined neural network operators of Cardaliagnet-Euvrard and “Squashing” types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators “bell-shaped” and “squashing” functions are assumed to be of compact support. Also in [3] he gives the N th order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see chapters 4–5 there.

For this article the author is motivated by the article [13] of Z. Chen and F. Cao, also by [4], [5], [6], [7], [8], [9], [10], [11], [14], [15].

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The author here performs multivariate arctangent function based neural network approximations to continuous functions over boxes or over the whole \mathbb{R}^N , $N \in \mathbb{N}$. Also he does iterated approximation. All convergences here are with rates expressed via the multivariate modulus of continuity of the involved function or its high order Fréchet derivative and given by very tight multidimensional Jackson type inequalities.

The author here comes up with the “right” precisely defined multivariate normalized, quasi-interpolation neural network operators related to boxes or \mathbb{R}^N , as well as Kantorovich type and quadrature type related operators on \mathbb{R}^N . Our boxes are not necessarily symmetric to the origin. In preparation to prove our results we establish important properties of the basic multivariate density function induced by arctangent function and defining our operators.

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this chapter, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^n c_j \sigma(\langle a_j \cdot x \rangle + b_j), \quad x \in \mathbb{R}^s, \quad s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{R}$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of a_j and x , and σ is the activation function of the network. In many fundamental network models, the activation function is the arctangent function. About neural networks read [16], [17], [18].

2. AUXILIARY NOTIONS

We consider the

$$(1) \quad \arctan x = \int_0^x \frac{dz}{1+z^2}, \quad x \in \mathbb{R}.$$

We will be using

$$(2) \quad h(x) := \frac{2}{\pi} \arctan\left(\frac{\pi x}{2}\right) = \frac{2}{\pi} \int_0^{\frac{\pi x}{2}} \frac{dz}{1+z^2}, \quad x \in \mathbb{R},$$

which is a sigmoid type function and it is strictly increasing. We have that

$$h(0) = 0, \quad h(-x) = -h(x), \quad h(+\infty) = 1, \quad h(-\infty) = -1,$$

and

$$(3) \quad h'(x) = \frac{4}{4+\pi^2 x^2} > 0, \quad \text{all } x \in \mathbb{R}.$$

We consider the activation function

$$(4) \quad \psi(x) := \frac{1}{4} (h(x+1) - h(x-1)), \quad x \in \mathbb{R},$$

and we notice that

$$(5) \quad \psi(-x) = \psi(x),$$

it is an even function.

Since $x+1 > x-1$, then $h(x+1) > h(x-1)$, and $\psi(x) > 0$, all $x \in \mathbb{R}$.

We see that

$$(6) \quad \psi(0) = \frac{1}{\pi} \arctan \frac{\pi}{2} \cong 0.319.$$

Let $x > 0$, we have that

$$(7) \quad \psi'(x) = \frac{1}{4} (h'(x+1) - h'(x-1)) = \frac{-4\pi^2 x}{(4+\pi^2(x+1)^2)(4+\pi^2(x-1)^2)} < 0.$$

That is

$$(8) \quad \psi'(x) < 0, \text{ for } x > 0.$$

That is ψ is strictly decreasing on $[0, \infty)$ and clearly is strictly increasing on $(-\infty, 0]$, and $\psi'(0) = 0$.

Observe that

$$(9) \quad \begin{aligned} \lim_{x \rightarrow +\infty} \psi(x) &= \frac{1}{4} (h(+\infty) - h(+\infty)) = 0, \\ \text{and} \\ \lim_{x \rightarrow -\infty} \psi(x) &= \frac{1}{4} (h(-\infty) - h(-\infty)) = 0. \end{aligned}$$

That is the x -axis is the horizontal asymptote on ψ .

All in all, ψ is a bell symmetric function with maximum $\psi(0) \cong 0.319$.

We need

THEOREM 1 ([11, p. 286]). *We have that*

$$(10) \quad \sum_{i=-\infty}^{\infty} \psi(x-i) = 1, \quad \forall x \in \mathbb{R}.$$

THEOREM 2 ([11, p. 287]). *It holds*

$$(11) \quad \int_{-\infty}^{\infty} \psi(x) dx = 1.$$

So that $\psi(x)$ is a density function on \mathbb{R} .

We mention

THEOREM 3 ([11, p. 288]). *Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds*

$$(12) \quad \sum_{\substack{k=-\infty \\ |nx-k| \geq n^{1-\alpha}}}^{\infty} \psi(nx-k) < \frac{2}{\pi^2(n^{1-\alpha}-2)}.$$

Denote by $[\cdot]$ the integral part of the number and by $\lceil \cdot \rceil$ the ceiling of the number.

We need

THEOREM 4 ([11, p. 289]). *Let $x \in [a, b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lceil nb \rceil$. It holds*

$$(13) \quad \frac{1}{\sum_{k=\lceil na \rceil}^{\lceil nb \rceil} \psi(nx-k)} < \frac{1}{\psi(1)} \cong 4.9737, \quad \forall x \in [a, b].$$

NOTE 1 ([11, pp. 290–291]). i) We have that

$$(14) \quad \lim_{n \rightarrow \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx - k) \neq 1,$$

for at least some $x \in [a, b]$.

ii) For large enough $n \in \mathbb{N}$ we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$.

In general, by [Theorem 1](#), it holds

$$(15) \quad \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx - k) \leq 1.$$

We introduce

$$(16) \quad Z(x_1, \dots, x_N) := Z(x) := \prod_{i=1}^N \psi(x_i), \quad x = (x_1, \dots, x_N) \in \mathbb{R}^N, \quad N \in \mathbb{N}.$$

It has the properties:

(i) $Z(x) > 0, \forall x \in \mathbb{R}^N$,

(ii)

$$(17) \quad \sum_{k=-\infty}^{\infty} Z(x - k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \dots \sum_{k_N=-\infty}^{\infty} Z(x_1 - k_1, \dots, x_N - k_N) = 1,$$

where $k := (k_1, \dots, k_n) \in \mathbb{Z}^N, \forall x \in \mathbb{R}^N$,

hence

(iii)

$$(18) \quad \sum_{k=-\infty}^{\infty} Z(nx - k) = 1,$$

$\forall x \in \mathbb{R}^N; n \in \mathbb{N}$,

and

(iv)

$$(19) \quad \int_{\mathbb{R}^N} Z(x) dx = 1,$$

that is Z is a multivariate density function.

Here denote $\|x\|_{\infty} := \max\{|x_1|, \dots, |x_N|\}, x \in \mathbb{R}^N$, also set $\infty := (\infty, \dots, \infty)$, $-\infty := (-\infty, \dots, -\infty)$ upon the multivariate context, and

$$(20) \quad \lceil na \rceil := (\lceil na_1 \rceil, \dots, \lceil na_N \rceil),$$

$$\lfloor nb \rfloor := (\lfloor nb_1 \rfloor, \dots, \lfloor nb_N \rfloor),$$

where $a := (a_1, \dots, a_N), b := (b_1, \dots, b_N)$.

We obviously see that

$$\begin{aligned}
(21) \quad \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k) &= \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(\prod_{i=1}^N \psi(nx_i - k_i) \right) \\
&= \sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \dots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} \left(\prod_{i=1}^N \psi(nx_i - k_i) \right) \\
&= \prod_{i=1}^N \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \psi(nx_i - k_i) \right).
\end{aligned}$$

For $0 < \beta < 1$ and $n \in \mathbb{N}$, a fixed $x \in \mathbb{R}^N$, we have that

$$(22) \quad \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx - k) = \sum_{\substack{k=\lceil na \rceil \\ \|\frac{k}{n} - x\|_\infty \leq \frac{1}{n^\beta}}}}^{\lfloor nb \rfloor} \psi(nx - k) + \sum_{\substack{k=\lceil na \rceil \\ \|\frac{k}{n} - x\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} \psi(nx - k).$$

In the last two sums the counting is over disjoint vector sets of k 's, because the condition $\|\frac{k}{n} - x\|_\infty > \frac{1}{n^\beta}$ implies that there exists at least one $|\frac{k_r}{n} - x_r| > \frac{1}{n^\beta}$, where $r \in \{1, \dots, N\}$.

(v) As in [10, pp. 379–380], we derive that

$$(23) \quad \sum_{\substack{k=\lceil na \rceil \\ \|\frac{k}{n} - x\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} Z(nx - k) \stackrel{(12)}{<} \frac{2}{\pi^2(n^{1-\beta} - 2)}, \quad 0 < \beta < 1,$$

with $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \prod_{i=1}^N [a_i, b_i]$.

(vi) By Theorem 4 we get that

$$(24) \quad 0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)} < \frac{1}{(\psi(1))^N} \cong (4.9737)^N,$$

$\forall x \in \left(\prod_{i=1}^N [a_i, b_i] \right)$, $n \in \mathbb{N}$.

It is also clear that

(vii)

$$(25) \quad \sum_{\substack{k=-\infty \\ \|\frac{k}{n} - x\|_\infty > \frac{1}{n^\beta}}}^{\infty} Z(nx - k) < \frac{2}{\pi^2(n^{1-\beta} - 2)},$$

$0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \mathbb{R}^N$.

Furthermore it holds

$$(26) \quad \lim_{n \rightarrow \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k) \neq 1,$$

for at least some $x \in \left(\prod_{i=1}^N [a_i, b_i] \right)$.

Here $(X, \|\cdot\|_\gamma)$ is a Banach space.

Let $f \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$, $x = (x_1, \dots, x_N) \in \prod_{i=1}^N [a_i, b_i]$, $n \in \mathbb{N}$ such that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, \dots, N$.

We introduce and define the following multivariate linear normalized neural network operator ($x := (x_1, \dots, x_N) \in \left(\prod_{i=1}^N [a_i, b_i]\right)$):

$$\begin{aligned}
(27) \quad A_n(f, x_1, \dots, x_N) &:= \\
&:= A_n(f, x) \\
&:= \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} \\
&= \frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} \dots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, \dots, \frac{k_N}{n}\right) \left(\prod_{i=1}^N \psi(nx_i - k_i)\right)}{\prod_{i=1}^N \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \psi(nx_i - k_i)\right)}.
\end{aligned}$$

For large enough $n \in \mathbb{N}$ we always obtain $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, \dots, N$. Also $a_i \leq \frac{k_i}{n} \leq b_i$, iff $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor$, $i = 1, \dots, N$.

When $g \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$ we define the companion operator

$$(28) \quad \tilde{A}_n(g, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} g\left(\frac{k}{n}\right) Z(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)}.$$

Clearly \tilde{A}_n is a positive linear operator. We have that

$$\tilde{A}_n(1, x) = 1, \quad \forall x \in \left(\prod_{i=1}^N [a_i, b_i]\right).$$

Notice that $A_n(f) \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$ and $\tilde{A}_n(g) \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$.

Furthermore it holds

$$(29) \quad \|A_n(f, x)\|_\gamma \leq \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \|f\left(\frac{k}{n}\right)\|_\gamma Z(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} = \tilde{A}_n(\|f\|_\gamma, x),$$

$\forall x \in \prod_{i=1}^N [a_i, b_i]$.

Clearly $\|f\|_\gamma \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$.

So, we have that

$$(30) \quad \|A_n(f, x)\|_\gamma \leq \tilde{A}_n(\|f\|_\gamma, x),$$

$$\forall x \in \prod_{i=1}^N [a_i, b_i], \forall n \in \mathbb{N}, \forall f \in C\left(\prod_{i=1}^N [a_i, b_i], X\right).$$

Let $c \in X$ and $g \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$, then $cg \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$.

Furthermore it holds

$$(31) \quad A_n(cg, x) = c\tilde{A}_n(g, x), \quad \forall x \in \prod_{i=1}^N [a_i, b_i].$$

Since $\tilde{A}_n(1) = 1$, we get that

$$(32) \quad A_n(c) = c, \quad \forall c \in X.$$

We call \tilde{A}_n the companion operator of A_n .

For convinience we call

$$(33) \quad \begin{aligned} A_n^*(f, x) &:= \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z(nx - k) = \\ &= \sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} \dots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, \dots, \frac{k_N}{n}\right) \left(\prod_{i=1}^N \psi(nx_i - k_i)\right), \end{aligned}$$

$$\forall x \in \left(\prod_{i=1}^N [a_i, b_i]\right).$$

That is

$$(34) \quad A_n(f, x) := \frac{A_n^*(f, x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)},$$

$$\forall x \in \left(\prod_{i=1}^N [a_i, b_i]\right), n \in \mathbb{N}.$$

Hence

$$(35) \quad A_n(f, x) - f(x) = \frac{A_n^*(f, x) - f(x) \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)}.$$

Consequently we derive

$$(36) \quad \|A_n(f, x) - f(x)\|_\gamma \stackrel{(24)}{\leq} (4.9737)^N \left\| A_n^*(f, x) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k) \right\|_\gamma,$$

$$\forall x \in \left(\prod_{i=1}^N [a_i, b_i]\right).$$

We will estimate the right hand side of (36).

For the last and others we need

DEFINITION 1 ([11, p. 274]). Let M be a convex and compact subset of $(\mathbb{R}^N, \|\cdot\|_p)$, $p \in [1, \infty]$, and $(X, \|\cdot\|_\gamma)$ be a Banach space. Let $f \in C(M, X)$. We define the first modulus of continuity of f as

$$(37) \quad \omega_1(f, \delta) := \sup_{\substack{x, y \in M \\ \|x - y\|_p \leq \delta}} \|f(x) - f(y)\|_\gamma, \quad 0 < \delta \leq \text{diam}(M).$$

If $\delta > \text{diam}(M)$, then

$$(38) \quad \omega_1(f, \delta) = \omega_1(f, \text{diam}(M)).$$

Notice $\omega_1(f, \delta)$ is increasing in $\delta > 0$. For $f \in C_B(M, X)$ (continuous and bounded functions) $\omega_1(f, \delta)$ is defined similarly.

LEMMA 1 ([11, p. 274]). We have $\omega_1(f, \delta) \rightarrow 0$ as $\delta \downarrow 0$, iff $f \in C(M, X)$, where M is a convex compact subset of $(\mathbb{R}^N, \|\cdot\|_p)$, $p \in [1, \infty]$.

Clearly we have also: $f \in C_U(\mathbb{R}^N, X)$ (uniformly continuous functions), iff $\omega_1(f, \delta) \rightarrow 0$ as $\delta \downarrow 0$, where ω_1 is defined similarly to (37). The space $C_B(\mathbb{R}^N, X)$ denotes the continuous and bounded functions on \mathbb{R}^N .

When $f \in C_B(\mathbb{R}^N, X)$ we define,

(39)

$$\begin{aligned} B_n(f, x) &:= B_n(f, x_1, \dots, x_N) \\ &:= \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx - k) \\ &:= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \dots \sum_{k_N=-\infty}^{\infty} f\left(\frac{k_1}{n}, \frac{k_2}{n}, \dots, \frac{k_N}{n}\right) \left(\prod_{i=1}^N \psi(nx_i - k_i)\right), \end{aligned}$$

$n \in \mathbb{N}, \forall x \in \mathbb{R}^N, N \in \mathbb{N}$, the multivariate quasi-interpolation neural network operator.

Also for $f \in C_B(\mathbb{R}^N, X)$ we define the multivariate Kantorovich type neural network operator

(40)

$$\begin{aligned} C_n(f, x) &:= C_n(f, x_1, \dots, x_N) \\ &:= \sum_{k=-\infty}^{\infty} \left(n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) dt \right) Z(nx - k) \\ &= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \dots \sum_{k_N=-\infty}^{\infty} \left(n^N \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}} \dots \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1, \dots, t_N) dt_1 \dots dt_N \right) \end{aligned}$$

$$\cdot \left(\prod_{i=1}^N \psi(nx_i - k_i) \right),$$

$n \in \mathbb{N}, \forall x \in \mathbb{R}^N$.

Again for $f \in C_B(\mathbb{R}^N, X)$, $N \in \mathbb{N}$, we define the multivariate neural network operator of quadrature type $D_n(f, x)$, $n \in \mathbb{N}$, as follows.

Let $\theta = (\theta_1, \dots, \theta_N) \in \mathbb{N}^N$, $r = (r_1, \dots, r_N) \in \mathbb{Z}_+^N$, $w_r = w_{r_1, r_2, \dots, r_N} \geq 0$, such that $\sum_{r=0}^{\theta} w_r = \sum_{r_1=0}^{\theta_1} \sum_{r_2=0}^{\theta_2} \dots \sum_{r_N=0}^{\theta_N} w_{r_1, r_2, \dots, r_N} = 1$; $k \in \mathbb{Z}^N$ and

(41)

$$\begin{aligned} \delta_{nk}(f) &:= \delta_{n, k_1, k_2, \dots, k_N}(f) := \sum_{r=0}^{\theta} w_r f\left(\frac{k}{n} + \frac{r}{n\theta}\right) \\ &= \sum_{r_1=0}^{\theta_1} \sum_{r_2=0}^{\theta_2} \dots \sum_{r_N=0}^{\theta_N} w_{r_1, r_2, \dots, r_N} f\left(\frac{k_1}{n} + \frac{r_1}{n\theta_1}, \frac{k_2}{n} + \frac{r_2}{n\theta_2}, \dots, \frac{k_N}{n} + \frac{r_N}{n\theta_N}\right), \end{aligned}$$

where $\frac{r}{\theta} := \left(\frac{r_1}{\theta_1}, \frac{r_2}{\theta_2}, \dots, \frac{r_N}{\theta_N}\right)$.

We set

$$\begin{aligned} (42) \quad D_n(f, x) &:= D_n(f, x_1, \dots, x_N) := \sum_{k=-\infty}^{\infty} \delta_{nk}(f) Z(nx - k) \\ &= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \dots \sum_{k_N=-\infty}^{\infty} \delta_{n, k_1, k_2, \dots, k_N}(f) \left(\prod_{i=1}^N \psi(nx_i - k_i) \right), \end{aligned}$$

$\forall x \in \mathbb{R}^N$.

In this article we study the approximation properties of A_n, B_n, C_n, D_n neural network operators and as well of their iterates. That is, the quantitative pointwise and uniform convergence of these operators to the unit operator I .

3. MULTIVARIATE GENERAL NEURAL NETWORK APPROXIMATIONS

Here we present several vectorial neural network approximations to Banach space valued functions given with rates.

We give

THEOREM 5. *Let $f \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$, $0 < \beta < 1$, $x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$, $N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$. Then*

1)

$$(43) \quad \|A_n(f, x) - f(x)\|_{\gamma} \leq (4.9737)^N \left[\omega_1\left(f, \frac{1}{n^{\beta}}\right) + \frac{4\|f\|_{\gamma}}{\pi^2(n^{1-\beta}-2)} \right] =: \lambda_1(n),$$

and

2)

$$(44) \quad \left\| \|A_n(f) - f\|_\gamma \right\|_\infty \leq \lambda_1(n).$$

We notice that $\lim_{n \rightarrow \infty} A_n(f) \stackrel{\|\cdot\|_\gamma}{=} f$, pointwise and uniformly.
Above ω_1 is with respect to $p = \infty$.

Proof. We observe that

$$(45) \quad \begin{aligned} \Delta(x) &:= A_n^*(f, x) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k) \\ &= \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z(nx - k) - \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f(x) Z(nx - k) \\ &= \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f\left(\frac{k}{n}\right) - f(x)\right) Z(nx - k). \end{aligned}$$

Thus

$$(46) \quad \begin{aligned} \|\Delta(x)\|_\gamma &\leq \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_\gamma Z(nx - k) \\ &= \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_\gamma Z(nx - k) \\ &\quad \left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^\beta} \\ &\quad + \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_\gamma Z(nx - k) \\ &\quad \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \\ &\stackrel{(18)}{\leq} \omega_1\left(f, \frac{1}{n^\beta}\right) + 2 \left\| \|f\|_\gamma \right\|_\infty \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k) \\ &\quad \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \\ &\stackrel{(23)}{\leq} \omega_1\left(f, \frac{1}{n^\beta}\right) + \frac{4 \left\| \|f\|_\gamma \right\|_\infty}{\pi^2(n^{1-\beta} - 2)}. \end{aligned}$$

So that

$$(47) \quad \|\Delta(x)\|_\gamma \leq \omega_1\left(f, \frac{1}{n^\beta}\right) + \frac{4 \left\| \|f\|_\gamma \right\|_\infty}{\pi^2(n^{1-\beta} - 2)}.$$

Now using (36) we finish the proof. \square

We make

REMARK 1 ([11, pp. 263–266]). Let $(\mathbb{R}^N, \|\cdot\|_p)$, $N \in \mathbb{N}$; where $\|\cdot\|_p$ is the L_p -norm, $1 \leq p \leq \infty$. \mathbb{R}^N is a Banach space, and $(\mathbb{R}^N)^j$ denotes the j -fold product space $\mathbb{R}^N \times \dots \times \mathbb{R}^N$ endowed with the max-norm $\|x\|_{(\mathbb{R}^N)^j} := \max_{1 \leq \lambda \leq j} \|x_\lambda\|_p$, where $x := (x_1, \dots, x_j) \in (\mathbb{R}^N)^j$.

Let $(X, \|\cdot\|_\gamma)$ be a general Banach space. Then the space $L_j := L_j((\mathbb{R}^N)^j; X)$ of all j -multilinear continuous maps $g : (\mathbb{R}^N)^j \rightarrow X$, $j = 1, \dots, m$, is a Banach space with norm

$$(48) \quad \|g\| := \|g\|_{L_j} := \sup_{(\|x\|_{(\mathbb{R}^N)^j}=1)} \|g(x)\|_\gamma = \sup \frac{\|g(x)\|_\gamma}{\|x_1\|_p \dots \|x_j\|_p}.$$

Let M be a non-empty convex and compact subset of \mathbb{R}^k and $x_0 \in M$ is fixed.

Let O be an open subset of $\mathbb{R}^N : M \subset O$. Let $f : O \rightarrow X$ be a continuous function, whose Fréchet derivatives (see [19]) $f^{(j)} : O \rightarrow L_j = L_j((\mathbb{R}^N)^j; X)$ exist and are continuous for $1 \leq j \leq m$, $m \in \mathbb{N}$.

Call $(x - x_0)^j := (x - x_0, \dots, x - x_0) \in (\mathbb{R}^N)^j$, $x \in M$.

We will work with $f|_M$.

Then, by Taylor's formula [12], [19, p. 124], we get

$$(49) \quad f(x) = \sum_{j=0}^m \frac{1}{j!} f^{(j)}(x_0) (x - x_0)^j + R_m(x, x_0), \quad \text{all } x \in M,$$

where the remainder is the Riemann integral

$$(50) \quad R_m(x, x_0) := \int_0^1 \frac{(1-u)^{m-1}}{(m-1)!} \left(f^{(m)}(x_0 + u(x - x_0)) - f^{(m)}(x_0) \right) (x - x_0)^m du,$$

here we set $f^{(0)}(x_0) (x - x_0)^0 = f(x_0)$.

We consider

$$(51) \quad w := \omega_1(f^{(m)}, h) := \sup_{\substack{x, y \in M: \\ \|x-y\|_p \leq h}} \|f^{(m)}(x) - f^{(m)}(y)\|,$$

$h > 0$.

We obtain

$$(52) \quad \begin{aligned} & \left\| \left(f^{(m)}(x_0 + u(x - x_0)) - f^{(m)}(x_0) \right) (x - x_0)^m \right\|_\gamma \leq \\ & \leq \left\| f^{(m)}(x_0 + u(x - x_0)) - f^{(m)}(x_0) \right\| \cdot \|x - x_0\|_p^m \\ & \leq w \|x - x_0\|_p^m \left\lceil \frac{u \|x - x_0\|_p}{h} \right\rceil, \end{aligned}$$

by Lemma 7.1.1, [1, p. 208], where $\lceil \cdot \rceil$ is the ceiling.

Therefore for all $x \in M$ (see [1, pp. 121–122]):

$$(53) \quad \begin{aligned} \|R_m(x, x_0)\|_\gamma &\leq w \|x - x_0\|_p^m \int_0^1 \left[\frac{u\|x-x_0\|_p}{h} \right] \frac{(1-u)^{m-1}}{(m-1)!} du \\ &= w\Phi_m(\|x - x_0\|_p) \end{aligned}$$

by a change of variable, where

$$(54) \quad \Phi_m(t) := \int_0^{|t|} \left[\frac{s}{h} \right] \frac{(|t|-s)^{m-1}}{(m-1)!} ds = \frac{1}{m!} \left(\sum_{j=0}^{\infty} (|t| - jh)_+^m \right), \quad \forall t \in \mathbb{R},$$

is a (polynomial) spline function, see [1, p. 210–211].

Also from there we get

$$(55) \quad \Phi_m(t) \leq \left(\frac{|t|^{m+1}}{(m+1)!h} + \frac{|t|^m}{2m!} + \frac{h|t|^{m-1}}{8(m-1)!} \right), \quad \forall t \in \mathbb{R},$$

with equality true only at $t = 0$.

Therefore it holds

$$(56) \quad \|R_m(x, x_0)\|_\gamma \leq w \left(\frac{\|x-x_0\|_p^{m+1}}{(m+1)!h} + \frac{\|x-x_0\|_p^m}{2m!} + \frac{h\|x-x_0\|_p^{m-1}}{8(m-1)!} \right), \quad \forall x \in M.$$

We have found that

$$(57) \quad \begin{aligned} \left\| f(x) - \sum_{j=0}^m \frac{1}{j!} f^{(j)}(x_0) (x - x_0)^j \right\|_\gamma &\leq \\ &\leq \omega_1(f^{(m)}, h) \left(\frac{\|x-x_0\|_p^{m+1}}{(m+1)!h} + \frac{\|x-x_0\|_p^m}{2m!} + \frac{h\|x-x_0\|_p^{m-1}}{8(m-1)!} \right) < \infty, \end{aligned}$$

$\forall x, x_0 \in M$.

Here $0 < \omega_1(f^{(m)}, h) < \infty$, by M being compact and $f^{(m)}$ being continuous on M .

One can rewrite (57) as follows:

$$(58) \quad \begin{aligned} \left\| f(\cdot) - \sum_{j=0}^m \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!} \right\|_\gamma &\leq \\ &\leq \omega_1(f^{(m)}, h) \left(\frac{\|\cdot - x_0\|_p^{m+1}}{(m+1)!h} + \frac{\|\cdot - x_0\|_p^m}{2m!} + \frac{h\|\cdot - x_0\|_p^{m-1}}{8(m-1)!} \right), \quad \forall x_0 \in M, \end{aligned}$$

a pointwise functional inequality on M .

Here $(\cdot - x_0)^j$ maps M into $(\mathbb{R}^N)^j$ and it is continuous, also $f^{(j)}(x_0)$ maps $(\mathbb{R}^N)^j$ into X and it is continuous. Hence their composition $f^{(j)}(x_0)(\cdot - x_0)^j$ is continuous from M into X .

Clearly $f(\cdot) - \sum_{j=0}^m \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!} \in C(M, X)$, hence $\left\| f(\cdot) - \sum_{j=0}^m \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!} \right\|_\gamma \in C(M)$.

Let $\{\tilde{L}_N\}_{N \in \mathbb{N}}$ be a sequence of positive linear operators mapping $C(M)$ into $C(M)$.

Therefore we obtain

$$(59) \quad \left(\tilde{L}_N \left(\left\| f(\cdot) - \sum_{j=0}^m \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!} \right\|_{\gamma} \right) \right) (x_0) \leq \\ \leq \omega_1 \left(f^{(m)}, h \right) \left[\frac{(\tilde{L}_N(\|\cdot - x_0\|_p^{m+1}))(x_0)}{(m+1)!h} + \frac{(\tilde{L}_N(\|\cdot - x_0\|_p^m))(x_0)}{2m!} \right. \\ \left. + \frac{h(\tilde{L}_N(\|\cdot - x_0\|_p^{m-1}))(x_0)}{8(m-1)!} \right],$$

$\forall N \in \mathbb{N}, \forall x_0 \in M$. □

Clearly (59) is valid when $M = \prod_{i=1}^N [a_i, b_i]$ and $\tilde{L}_n = \tilde{A}_n$, see (28).

All the above is preparation for the following theorem, where we assume Fréchet differentiability of functions.

This will be a direct application of Theorem 10.2, [11, pp. 268–270]. The operators A_n, \tilde{A}_n fulfill its assumptions, see (27), (28), (30), (31) and (32).

We present the following high order approximation results.

THEOREM 6. *Let O open subset of $(\mathbb{R}^N, \|\cdot\|_p)$, $p \in [1, \infty]$, such that $\prod_{i=1}^N [a_i, b_i] \subset O \subseteq \mathbb{R}^N$, and let $(X, \|\cdot\|_{\gamma})$ be a general Banach space. Let $m \in \mathbb{N}$ and $f \in C^m(O, X)$, the space of m -times continuously Fréchet differentiable functions from O into X . We study the approximation of $f|_{\prod_{i=1}^N [a_i, b_i]}$. Let $x_0 \in$*

$\left(\prod_{i=1}^N [a_i, b_i] \right)$ and $r > 0$. Then

$$(60) \quad \left\| (A_n(f))(x_0) - \sum_{j=0}^m \frac{1}{j!} \left(A_n \left(f^{(j)}(x_0)(\cdot - x_0)^j \right) \right) (x_0) \right\|_{\gamma} \leq \\ \leq \frac{1}{rm!} \omega_1 \left(f^{(m)}, r \left(\left(\tilde{A}_n \left(\|\cdot - x_0\|_p^{m+1} \right) \right) (x_0) \right)^{\frac{1}{m+1}} \right) \\ \cdot \left(\left(\tilde{A}_n \left(\|\cdot - x_0\|_p^{m+1} \right) \right) (x_0) \right)^{\left(\frac{m}{m+1} \right)} \left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8} \right],$$

2) additionally if $f^{(j)}(x_0) = 0$, $j = 1, \dots, m$, we have

$$(61) \quad \|(A_n(f))(x_0) - f(x_0)\|_{\gamma} \leq \\ \leq \frac{1}{rm!} \omega_1 \left(f^{(m)}, r \left(\left(\tilde{A}_n \left(\|\cdot - x_0\|_p^{m+1} \right) \right) (x_0) \right)^{\frac{1}{m+1}} \right)$$

$$\begin{aligned}
& \cdot \left(\left(\tilde{A}_n \left(\|\cdot - x_0\|_p^{m+1} \right) \right) (x_0) \right)^{\left(\frac{m}{m+1} \right)} \left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8} \right], \\
& 3) \\
(62) \quad & \left\| (A_n(f))(x_0) - f(x_0) \right\|_\gamma \leq \\
& \leq \sum_{j=1}^m \frac{1}{j!} \left\| \left(A_n \left(f^{(j)}(x_0) (\cdot - x_0)^j \right) \right) (x_0) \right\|_\gamma \\
& \quad + \frac{1}{rm!} \omega_1 \left(f^{(m)}, r \left(\left(\tilde{A}_n \left(\|\cdot - x_0\|_p^{m+1} \right) \right) (x_0) \right)^{\frac{1}{m+1}} \right) \\
& \quad \cdot \left(\left(\tilde{A}_n \left(\|\cdot - x_0\|_p^{m+1} \right) \right) (x_0) \right)^{\left(\frac{m}{m+1} \right)} \left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8} \right],
\end{aligned}$$

and

$$\begin{aligned}
(63) \quad & \left\| \|A_n(f) - f\|_\gamma \right\|_{\infty, \prod_{i=1}^N [a_i, b_i]} \leq \\
& \leq \sum_{j=1}^m \frac{1}{j!} \left\| \left\| \left(A_n \left(f^{(j)}(x_0) (\cdot - x_0)^j \right) \right) (x_0) \right\|_\gamma \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i, b_i]} \\
& \quad + \frac{1}{rm!} \omega_1 \left(f^{(m)}, r \left\| \left(\tilde{A}_n \left(\|\cdot - x_0\|_p^{m+1} \right) \right) (x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i, b_i]}^{\frac{1}{m+1}} \right) \\
& \quad \cdot \left\| \left(\tilde{A}_n \left(\|\cdot - x_0\|_p^{m+1} \right) \right) (x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i, b_i]}^{\left(\frac{m}{m+1} \right)} \left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8} \right].
\end{aligned}$$

We need

LEMMA 2. *The function $\left(\tilde{A}_n \left(\|\cdot - x_0\|_p^m \right) \right) (x_0)$ is continuous in $x_0 \in \left(\prod_{i=1}^N [a_i, b_i] \right)$, $m \in \mathbb{N}$.*

Proof. By Lemma 10.3, [11, p. 272]. □

We make

REMARK 2. By Remark 10.4, [11, p. 273], we get that

$$\begin{aligned}
(64) \quad & \left\| \left(\tilde{A}_n \left(\|\cdot - x_0\|_p^k \right) \right) (x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i, b_i]} \leq \\
& \leq \left\| \left(\tilde{A}_n \left(\|\cdot - x_0\|_p^{m+1} \right) \right) (x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i, b_i]}^{\left(\frac{k}{m+1} \right)},
\end{aligned}$$

for all $k = 1, \dots, m$. □

We give

COROLLARY 1. (to [Theorem 6](#), case of $m = 1$) Then
1)

$$(65) \quad \begin{aligned} & \| (A_n(f))(x_0) - f(x_0) \|_\gamma \leq \\ & \leq \left\| \left(A_n \left(f^{(1)}(x_0)(\cdot - x_0) \right) \right) (x_0) \right\|_\gamma \\ & \quad + \frac{1}{2r} \omega_1 \left(f^{(1)}, r \left(\left(\tilde{A}_n \left(\|\cdot - x_0\|_p^2 \right) \right) (x_0) \right)^{\frac{1}{2}} \right) \\ & \quad \cdot \left(\left(\tilde{A}_n \left(\|\cdot - x_0\|_p^2 \right) \right) (x_0) \right)^{\frac{1}{2}} \left[1 + r + \frac{r^2}{4} \right], \end{aligned}$$

and
2)

$$(66) \quad \begin{aligned} & \left\| \| (A_n(f)) - f \|_\gamma \right\|_{\infty, \prod_{i=1}^N [a_i, b_i]} \leq \\ & \leq \left\| \left\| \left(A_n \left(f^{(1)}(x_0)(\cdot - x_0) \right) \right) (x_0) \right\|_\gamma \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i, b_i]} \\ & \quad + \frac{1}{2r} \omega_1 \left(f^{(1)}, r \left\| \left(\tilde{A}_n \left(\|\cdot - x_0\|_p^2 \right) \right) (x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i, b_i]}^{\frac{1}{2}} \right) \\ & \quad \cdot \left\| \left(\tilde{A}_n \left(\|\cdot - x_0\|_p^2 \right) \right) (x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i, b_i]}^{\frac{1}{2}} \left[1 + r + \frac{r^2}{4} \right], \end{aligned}$$

$r > 0$.

We make

REMARK 3. We estimate $0 < \alpha < 1$, $m, n \in \mathbb{N} : n^{1-\alpha} > 2$,

$$(67) \quad \begin{aligned} & \tilde{A}_n \left(\|\cdot - x_0\|_\infty^{m+1} \right) (x_0) = \\ & = \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^{m+1} Z(nx_0 - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx_0 - k)} \\ & \stackrel{(24)}{<} (4.9737)^N \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^{m+1} Z(nx_0 - k) \end{aligned}$$

$$\begin{aligned}
&= (4.9737)^N \left\{ \sum_{\substack{k=\lceil na \rceil \\ \|\frac{k}{n} - x_0\|_\infty \leq \frac{1}{n^\alpha}}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^{m+1} Z(nx_0 - k) \right. \\
&\quad \left. + \sum_{\substack{k=\lceil na \rceil \\ \|\frac{k}{n} - x_0\|_\infty > \frac{1}{n^\alpha}}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^{m+1} Z(nx_0 - k) \right\} \\
(68) \quad &\stackrel{(25)}{\leq} (4.9737)^N \left\{ \frac{1}{n^{\alpha(m+1)}} + \frac{2\|b-a\|_\infty^{m+1}}{\pi^2(n^{1-\alpha}-2)} \right\},
\end{aligned}$$

(where $b - a = (b_1 - a_1, \dots, b_N - a_N)$).

We have proved that $(\forall x_0 \in \prod_{i=1}^N [a_i, b_i])$

$$(69) \quad \tilde{A}_n \left(\|\cdot - x_0\|_\infty^{m+1} \right) (x_0) < (4.9737)^N \left\{ \frac{1}{n^{\alpha(m+1)}} + \frac{2\|b-a\|_\infty^{m+1}}{\pi^2(n^{1-\alpha}-2)} \right\} =: \varphi_1(n)$$

($0 < \alpha < 1$, $m, n \in \mathbb{N} : n^{1-\alpha} > 2$).

And, consequently it holds

$$\begin{aligned}
(70) \quad &\left\| \tilde{A}_n \left(\|\cdot - x_0\|_\infty^{m+1} \right) (x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i, b_i]} < \\
&< (4.9737)^N \left\{ \frac{1}{n^{\alpha(m+1)}} + \frac{2\|b-a\|_\infty^{m+1}}{\pi^2(n^{1-\alpha}-2)} \right\} = \varphi_1(n) \rightarrow 0, \text{ as } n \rightarrow +\infty.
\end{aligned}$$

So, we have that $\varphi_1(n) \rightarrow 0$, as $n \rightarrow +\infty$. Thus, when $p \in [1, \infty]$, from [Theorem 6](#) we have the convergence to zero in the right hand sides of parts (1), (2).

Next we estimate $\left\| \left(\tilde{A}_n \left(f^{(j)}(x_0) (\cdot - x_0)^j \right) \right) (x_0) \right\|_\gamma$.

We have that

$$(71) \quad \left(\tilde{A}_n \left(f^{(j)}(x_0) (\cdot - x_0)^j \right) \right) (x_0) = \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f^{(j)}(x_0) \left(\frac{k}{n} - x_0 \right)^j Z(nx_0 - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx_0 - k)}.$$

When $p = \infty$, $j = 1, \dots, m$, we obtain

$$(72) \quad \left\| f^{(j)}(x_0) \left(\frac{k}{n} - x_0 \right)^j \right\|_\gamma \leq \left\| f^{(j)}(x_0) \right\| \left\| \frac{k}{n} - x_0 \right\|_\infty^j.$$

We further have that

$$(73) \quad \left\| \left(\tilde{A}_n \left(f^{(j)}(x_0) (\cdot - x_0)^j \right) \right) (x_0) \right\|_\gamma \stackrel{(24)}{<} <$$

$$\begin{aligned}
& \stackrel{(24)}{<} (4.9737)^N \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| f^{(j)}(x_0) \left(\frac{k}{n} - x_0 \right)^j \right\|_{\gamma} Z(nx_0 - k) \right) \\
& \leq (4.9737)^N \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| f^{(j)}(x_0) \right\| \left\| \frac{k}{n} - x_0 \right\|_{\infty}^j Z(nx_0 - k) \right) \\
& = (4.9737)^N \left\| f^{(j)}(x_0) \right\| \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_{\infty}^j Z(nx_0 - k) \right) \\
& = (4.9737)^N \left\| f^{(j)}(x_0) \right\| \left\{ \begin{array}{l} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_{\infty}^j Z(nx_0 - k) \\ : \left\| \frac{k}{n} - x_0 \right\|_{\infty} \leq \frac{1}{n^{\alpha}} \end{array} \right. \\
(74) \quad & + \left. \sum_{\substack{k=\lceil na \rceil \\ : \left\| \frac{k}{n} - x_0 \right\|_{\infty} > \frac{1}{n^{\alpha}}} }^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_{\infty}^j Z(nx_0 - k) \right\} \\
& \stackrel{(25)}{\leq} (4.9737)^N \left\| f^{(j)}(x_0) \right\| \left\{ \frac{1}{n^{\alpha j}} + \frac{2\|b-a\|_{\infty}^j}{\pi^2(n^{1-\alpha}-2)} \right\} \rightarrow 0, \text{ as } n \rightarrow \infty.
\end{aligned}$$

That is

$$\left\| \left(\tilde{A}_n \left(f^{(j)}(x_0) (\cdot - x_0)^j \right) \right) (x_0) \right\|_{\gamma} \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Therefore when $p = \infty$, for $j = 1, \dots, m$, we have proved:

$$\begin{aligned}
(75) \quad & \left\| \left(\tilde{A}_n \left(f^{(j)}(x_0) (\cdot - x_0)^j \right) \right) (x_0) \right\|_{\gamma} < \\
& < (4.9737)^N \left\| f^{(j)}(x_0) \right\| \left\{ \frac{1}{n^{\alpha j}} + \frac{2\|b-a\|_{\infty}^j}{\pi^2(n^{1-\alpha}-2)} \right\} \\
& \leq (4.9737)^N \left\| f^{(j)} \right\|_{\infty} \left\{ \frac{1}{n^{\alpha j}} + \frac{2\|b-a\|_{\infty}^j}{\pi^2(n^{1-\alpha}-2)} \right\} =: \varphi_{2j}(n) < \infty,
\end{aligned}$$

and converges to zero, as $n \rightarrow \infty$. \square

We conclude:

In [Theorem 6](#), the right hand sides of (62) and (63) converge to zero as $n \rightarrow \infty$, for any $p \in [1, \infty]$.

Also in [Corollary 1](#), the right hand sides of (65) and (66) converge to zero as $n \rightarrow \infty$, for any $p \in [1, \infty]$.

CONCLUSION 1. *We have proved that the left hand sides of (60), (61), (62), (63) and (65), (66) converge to zero as $n \rightarrow \infty$, for $p \in [1, \infty]$. Consequently $A_n \rightarrow I$ (unit operator) pointwise and uniformly, as $n \rightarrow \infty$, where $p \in [1, \infty]$. In the presence of initial conditions we achieve a higher speed of convergence,*

see (61). Higher speed of convergence happens also to the left hand side of (60).

We give

COROLLARY 2. (to (6)) Let O open subset of $(\mathbb{R}^N, \|\cdot\|_\infty)$, such that $\prod_{i=1}^N [a_i, b_i] \subset O \subseteq \mathbb{R}^N$, and let $(X, \|\cdot\|_\gamma)$ be a general Banach space. Let $m \in \mathbb{N}$ and $f \in C^m(O, X)$, the space of m -times continuously Fréchet differentiable functions from O into X . We study the approximation of $f|_{\prod_{i=1}^N [a_i, b_i]}$.

Let $x_0 \in \left(\prod_{i=1}^N [a_i, b_i]\right)$ and $r > 0$. Here $\varphi_1(n)$ as in (69) and $\varphi_{2j}(n)$ as in (75), where $n \in \mathbb{N} : n^{1-\alpha} > 2, 0 < \alpha < 1, j = 1, \dots, m$. Then

1)

$$(76) \quad \left\| (A_n(f))(x_0) - \sum_{j=0}^m \frac{1}{j!} \left(A_n(f^{(j)}(x_0)(\cdot - x_0)^j) \right)(x_0) \right\|_\gamma \leq \\ \leq \frac{1}{rm!} \omega_1 \left(f^{(m)}, r(\varphi_1(n))^{\frac{1}{m+1}} \right) (\varphi_1(n))^{\left(\frac{m}{m+1}\right)} \left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8} \right],$$

2) additionally, if $f^{(j)}(x_0) = 0, j = 1, \dots, m$, we have

$$(77) \quad \|(A_n(f))(x_0) - f(x_0)\|_\gamma \leq \\ \leq \frac{1}{rm!} \omega_1 \left(f^{(m)}, r(\varphi_1(n))^{\frac{1}{m+1}} \right) (\varphi_1(n))^{\left(\frac{m}{m+1}\right)} \left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8} \right],$$

3)

(78)

$$\left\| \|A_n(f) - f\|_\gamma \right\|_{\infty, \prod_{i=1}^N [a_i, b_i]} \leq \\ \leq \sum_{j=1}^m \frac{\varphi_{2j}(n)}{j!} + \frac{1}{rm!} \omega_1 \left(f^{(m)}, r(\varphi_1(n))^{\frac{1}{m+1}} \right) (\varphi_1(n))^{\left(\frac{m}{m+1}\right)} \left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8} \right] \\ =: \varphi_3(n) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

We continue with

THEOREM 7. Let $f \in C_B(\mathbb{R}^N, X), 0 < \beta < 1, x \in \mathbb{R}^N, N, n \in \mathbb{N}$ with $n^{1-\beta} > 2, \omega_1$ is for $p = \infty$. Then

1)

$$(79) \quad \|B_n(f, x) - f(x)\|_\gamma \leq \omega_1 \left(f, \frac{1}{n^\beta} \right) + \frac{4\|f\|_\gamma}{\pi^2(n^{1-\beta}-2)} =: \lambda_2(n),$$

2)

$$(80) \quad \left\| \|B_n(f) - f\|_\gamma \right\|_\infty \leq \lambda_2(n).$$

Given that $f \in (C_U(\mathbb{R}^N, X) \cap C_B(\mathbb{R}^N, X))$, we obtain $\lim_{n \rightarrow \infty} B_n(f) = f$, uniformly.

Proof. We have that

$$(81) \quad \begin{aligned} B_n(f, x) - f(x) &\stackrel{(18)}{=} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx - k) - f(x) \sum_{k=-\infty}^{\infty} Z(nx - k) \\ &= \sum_{k=-\infty}^{\infty} \left(f\left(\frac{k}{n}\right) - f(x)\right) Z(nx - k). \end{aligned}$$

Hence

$$(82) \quad \begin{aligned} \|B_n(f, x) - f(x)\|_\gamma &\leq \sum_{k=-\infty}^{\infty} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_\gamma Z(nx - k) \\ &= \sum_{\substack{k=-\infty \\ \|\frac{k}{n} - x\|_\infty \leq \frac{1}{n^\beta}}^{\infty}} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_\gamma Z(nx - k) \\ &\quad + \sum_{\substack{k=-\infty \\ \|\frac{k}{n} - x\|_\infty > \frac{1}{n^\beta}}^{\infty}} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_\gamma Z(nx - k) \\ &\stackrel{(18)}{\leq} \omega_1\left(f, \frac{1}{n^\beta}\right) + 2 \left\| \|f\|_\gamma \right\|_\infty \sum_{\substack{k=-\infty \\ \|\frac{k}{n} - x\|_\infty > \frac{1}{n^\beta}}^{\infty}} Z(nx - k) \\ &\stackrel{(25)}{\leq} \omega_1\left(f, \frac{1}{n^\beta}\right) + \frac{4 \left\| \|f\|_\gamma \right\|_\infty}{\pi^2(n^{1-\beta} - 2)}, \end{aligned}$$

proving the claim. \square

We give

THEOREM 8. Let $f \in C_B(\mathbb{R}^N, X)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$, ω_1 is for $p = \infty$. Then

1)

$$(83) \quad \|C_n(f, x) - f(x)\|_\gamma \leq \omega_1\left(f, \frac{1}{n} + \frac{1}{n^\beta}\right) + \frac{4 \left\| \|f\|_\gamma \right\|_\infty}{\pi^2(n^{1-\beta} - 2)} =: \lambda_3(n),$$

2)

$$(84) \quad \left\| \|C_n(f) - f\|_\gamma \right\|_\infty \leq \lambda_3(n).$$

Given that $f \in (C_U(\mathbb{R}^N, X) \cap C_B(\mathbb{R}^N, X))$, we obtain $\lim_{n \rightarrow \infty} C_n(f) = f$, uniformly.

Proof. We notice that

$$\begin{aligned} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) dt &= \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}} \dots \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1, t_2, \dots, t_N) dt_1 dt_2 \dots dt_N = \\ (85) \quad &\int_0^{\frac{1}{n}} \int_0^{\frac{1}{n}} \dots \int_0^{\frac{1}{n}} f\left(t_1 + \frac{k_1}{n}, t_2 + \frac{k_2}{n}, \dots, t_N + \frac{k_N}{n}\right) dt_1 \dots dt_N = \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt. \end{aligned}$$

Thus it holds (by (40))

$$(86) \quad C_n(f, x) = \sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt \right) Z(nx - k).$$

We observe that

$$\begin{aligned} &\|C_n(f, x) - f(x)\|_{\gamma} = \\ &= \left\| \sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt \right) Z(nx - k) - \sum_{k=-\infty}^{\infty} f(x) Z(nx - k) \right\|_{\gamma} \\ &= \left\| \sum_{k=-\infty}^{\infty} \left(\left(n^N \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt \right) - f(x) \right) Z(nx - k) \right\|_{\gamma} \\ (87) \quad &= \left\| \sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left(f\left(t + \frac{k}{n}\right) - f(x) \right) dt \right) Z(nx - k) \right\|_{\gamma} \\ &\leq \sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \|f\left(t + \frac{k}{n}\right) - f(x)\|_{\gamma} dt \right) Z(nx - k) \\ &= \sum_{\substack{k=-\infty \\ \|\frac{k}{n} - x\|_{\infty} \leq \frac{1}{n^{\beta}}} }^{\infty} \left(n^N \int_0^{\frac{1}{n}} \|f\left(t + \frac{k}{n}\right) - f(x)\|_{\gamma} dt \right) Z(nx - k) \\ &\quad + \sum_{\substack{k=-\infty \\ \|\frac{k}{n} - x\|_{\infty} > \frac{1}{n^{\beta}}} }^{\infty} \left(n^N \int_0^{\frac{1}{n}} \|f\left(t + \frac{k}{n}\right) - f(x)\|_{\gamma} dt \right) Z(nx - k) \\ &\leq \sum_{\substack{k=-\infty \\ \|\frac{k}{n} - x\|_{\infty} \leq \frac{1}{n^{\beta}}} }^{\infty} \left(n^N \int_0^{\frac{1}{n}} \omega_1\left(f, \|t\|_{\infty} + \left\| \frac{k}{n} - x \right\|_{\infty} \right) dt \right) Z(nx - k) \end{aligned}$$

$$\begin{aligned}
& + 2 \left\| \|f\|_\gamma \right\|_\infty \left(\sum_{\substack{k=-\infty \\ \|\frac{k}{n}-x\|_\infty > \frac{1}{n^\beta}}^{\infty}} Z(|nx-k|) \right) \\
(88) \quad & \leq \omega_1 \left(f, \frac{1}{n} + \frac{1}{n^\beta} \right) + \frac{4 \left\| \|f\|_\gamma \right\|_\infty}{\pi^2 (n^{1-\beta} - 2)},
\end{aligned}$$

proving the claim. \square

We also present

THEOREM 9. *Let $f \in C_B(\mathbb{R}^N, X)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$, ω_1 is for $p = \infty$. Then*

1)

$$(89) \quad \|D_n(f, x) - f(x)\|_\gamma \leq \omega_1 \left(f, \frac{1}{n} + \frac{1}{n^\beta} \right) + \frac{4 \left\| \|f\|_\gamma \right\|_\infty}{\pi^2 (n^{1-\beta} - 2)} = \lambda_4(n),$$

2)

$$(90) \quad \left\| \|D_n(f) - f\|_\gamma \right\|_\infty \leq \lambda_4(n).$$

Given that $f \in (C_U(\mathbb{R}^N, X) \cap C_B(\mathbb{R}^N, X))$, we obtain $\lim_{n \rightarrow \infty} D_n(f) = f$, uniformly.

Proof. We have that (by (42))

$$\begin{aligned}
(91) \quad & \|D_n(f, x) - f(x)\|_\gamma = \\
& = \left\| \sum_{k=-\infty}^{\infty} \delta_{nk}(f) Z(nx-k) - \sum_{k=-\infty}^{\infty} f(x) Z(nx-k) \right\|_\gamma \\
& = \left\| \sum_{k=-\infty}^{\infty} (\delta_{nk}(f) - f(x)) Z(nx-k) \right\|_\gamma \\
& = \left\| \sum_{k=-\infty}^{\infty} \left(\sum_{r=0}^{\theta} w_r \left(f\left(\frac{k}{n} + \frac{r}{n\theta}\right) - f(x) \right) \right) Z(nx-k) \right\|_\gamma \\
& \leq \sum_{k=-\infty}^{\infty} \left(\sum_{r=0}^{\theta} w_r \left\| f\left(\frac{k}{n} + \frac{r}{n\theta}\right) - f(x) \right\|_\gamma \right) Z(nx-k) \\
& = \sum_{\substack{k=-\infty \\ \|\frac{k}{n}-x\|_\infty \leq \frac{1}{n^\beta}}^{\infty} \left(\sum_{r=0}^{\theta} w_r \left\| f\left(\frac{k}{n} + \frac{r}{n\theta}\right) - f(x) \right\|_\gamma \right) Z(nx-k)
\end{aligned}$$

$$\begin{aligned}
& + \sum_{\substack{k=-\infty \\ \|\frac{k}{n}-x\|_\infty > \frac{1}{n^\beta}}^{\infty} \left(\sum_{r=0}^{\theta} w_r \left\| f\left(\frac{k}{n} + \frac{r}{n\theta}\right) - f(x) \right\|_\gamma \right) Z(nx - k) \\
& \leq \sum_{\substack{k=-\infty \\ \|\frac{k}{n}-x\|_\infty \leq \frac{1}{n^\beta}}^{\infty} \left(\sum_{r=0}^{\theta} w_r \omega_1 \left(f, \left\| \frac{k}{n} - x \right\|_\infty + \left\| \frac{r}{n\theta} \right\|_\infty \right) \right) Z(nx - k) \\
& \quad + 2 \left\| \|f\|_\gamma \right\|_\infty \left(\sum_{\substack{k=-\infty \\ \|\frac{k}{n}-x\|_\infty > \frac{1}{n^\beta}}^{\infty} Z(nx - k) \right) \\
(92) \quad & \leq \omega_1 \left(f, \frac{1}{n} + \frac{1}{n^\beta} \right) + \frac{4 \left\| \|f\|_\gamma \right\|_\infty}{\pi^2 (n^{1-\beta} - 2)},
\end{aligned}$$

proving the claim. \square

We make

DEFINITION 2. Let $f \in C_B(\mathbb{R}^N, X)$, $N \in \mathbb{N}$, where $(X, \|\cdot\|_\gamma)$ is a Banach space. We define the general neural network operator

(93)

$$F_n(f, x) := \sum_{k=-\infty}^{\infty} l_{nk}(f) Z(nx - k) = \begin{cases} B_n(f, x), & \text{if } l_{nk}(f) = f\left(\frac{k}{n}\right), \\ C_n(f, x), & \text{if } l_{nk}(f) = n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t) dt, \\ D_n(f, x), & \text{if } l_{nk}(f) = \delta_{nk}(f). \end{cases}$$

Clearly $l_{nk}(f)$ is an X -valued bounded linear functional such that $\|l_{nk}(f)\|_\gamma \leq \left\| \|f\|_\gamma \right\|_\infty$.

Hence $F_n(f)$ is a bounded linear operator with $\left\| \|F_n(f)\|_\gamma \right\|_\infty \leq \left\| \|f\|_\gamma \right\|_\infty$.

We need

THEOREM 10. Let $f \in C_B(\mathbb{R}^N, X)$, $N \geq 1$. Then $F_n(f) \in C_B(\mathbb{R}^N, X)$.

Proof. Clearly $F_n(f)$ is a bounded function.

Next we prove the continuity of $F_n(f)$. Notice for $N = 1$, $Z = \psi$ by (16).

We will use the generalized Weierstrass M test: If a sequence of positive constants M_1, M_2, M_3, \dots , can be found such that in some interval

(a) $\|u_n(x)\|_\gamma \leq M_n$, $n = 1, 2, 3, \dots$

(b) $\sum M_n$ converges,

then $\sum u_n(x)$ is uniformly and absolutely convergent in the interval.

Also we will use:

If $\{u_n(x)\}$, $n = 1, 2, 3, \dots$ are continuous in $[a, b]$ and if $\sum u_n(x)$ converges uniformly to the sum $S(x)$ in $[a, b]$, then $S(x)$ is continuous in $[a, b]$. I.e. a

uniformly convergent series of continuous functions is a continuous function. First we prove claim for $N = 1$.

We will prove that $\sum_{k=-\infty}^{\infty} l_{nk}(f) \psi(nx - k)$ is continuous in $x \in \mathbb{R}$.

There always exists $\lambda \in \mathbb{N}$ such that $nx \in [-\lambda, \lambda]$.

Since $nx \leq \lambda$, then $-nx \geq -\lambda$ and $k - nx \geq k - \lambda \geq 0$, when $k \geq \lambda$.

Therefore

$$(94) \quad \sum_{k=\lambda}^{\infty} \psi(nx - k) = \sum_{k=\lambda}^{\infty} \psi(k - nx) \leq \sum_{k=\lambda}^{\infty} \psi(k - \lambda) = \sum_{k'=0}^{\infty} \psi(k') \leq 1.$$

So for $k \geq \lambda$ we get

$$\|l_{nk}(f)\|_{\gamma} \psi(nx - k) \leq \| \|f\|_{\gamma} \|_{\infty} \psi(k - \lambda),$$

and

$$\| \|f\|_{\gamma} \|_{\infty} \sum_{k=\lambda}^{\infty} \psi(k - \lambda) \leq \| \|f\|_{\gamma} \|_{\infty}.$$

Hence by the generalized Weierstrass M test we obtain that $\sum_{k=\lambda}^{\infty} l_{nk}(f) \psi(nx - k)$ is uniformly and absolutely convergent on $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$.

Since $l_{nk}(f) \psi(nx - k)$ is continuous in x , then $\sum_{k=\lambda}^{\infty} l_{nk}(f) \psi(nx - k)$ is continuous on $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$.

Because $nx \geq -\lambda$, then $-nx \leq \lambda$, and $k - nx \leq k + \lambda \leq 0$, when $k \leq -\lambda$. Therefore

$$\sum_{k=-\infty}^{-\lambda} \psi(nx - k) = \sum_{k=-\infty}^{-\lambda} \psi(k - nx) \leq \sum_{k=-\infty}^{-\lambda} \psi(k + \lambda) = \sum_{k'=-\infty}^0 \psi(k') \leq 1.$$

So for $k \leq -\lambda$ we get

$$(95) \quad \|l_{nk}(f)\|_{\gamma} \psi(nx - k) \leq \| \|f\|_{\gamma} \|_{\infty} \psi(k + \lambda),$$

and

$$\| \|f\|_{\gamma} \|_{\infty} \sum_{k=-\infty}^{-\lambda} \psi(k + \lambda) \leq \| \|f\|_{\gamma} \|_{\infty}.$$

Hence by Weierstrass M test we obtain that $\sum_{k=-\infty}^{-\lambda} l_{nk}(f) \psi(nx - k)$ is uniformly and absolutely convergent on $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$.

Since $l_{nk}(f) \psi(nx - k)$ is continuous in x , then $\sum_{k=-\infty}^{-\lambda} l_{nk}(f) \psi(nx - k)$ is continuous on $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$.

So we proved that $\sum_{k=\lambda}^{\infty} l_{nk}(f) \psi(nx - k)$ and $\sum_{k=-\infty}^{-\lambda} l_{nk}(f) \psi(nx - k)$ are continuous on \mathbb{R} . Since $\sum_{k=-\lambda+1}^{\lambda-1} l_{nk}(f) \psi(nx - k)$ is a finite sum of continuous functions on \mathbb{R} , it is also a continuous function on \mathbb{R} .

Writing

$$(96) \quad \sum_{k=-\infty}^{\infty} l_{nk}(f) \psi(nx-k) = \sum_{k=-\infty}^{-\lambda} l_{nk}(f) \psi(nx-k) \\ + \sum_{k=-\lambda+1}^{\lambda-1} l_{nk}(f) \psi(nx-k) + \sum_{k=\lambda}^{\infty} l_{nk}(f) \psi(nx-k)$$

we have it as a continuous function on \mathbb{R} . Therefore $F_n(f)$, when $N = 1$, is a continuous function on \mathbb{R} .

When $N = 2$ we have

$$F_n(f, x_1, x_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} l_{nk}(f) \psi(nx_1 - k_1) \psi(nx_2 - k_2) \\ = \sum_{k_1=-\infty}^{\infty} \psi(nx_1 - k_1) \left(\sum_{k_2=-\infty}^{\infty} l_{nk}(f) \psi(nx_2 - k_2) \right)$$

(there always exist $\lambda_1, \lambda_2 \in \mathbb{N}$ such that $nx_1 \in [-\lambda_1, \lambda_1]$ and $nx_2 \in [-\lambda_2, \lambda_2]$)

$$= \sum_{k_1=-\infty}^{\infty} \psi(nx_1 - k_1) \left[\sum_{k_2=-\infty}^{-\lambda_2} l_{nk}(f) \psi(nx_2 - k_2) \right. \\ \left. + \sum_{k_2=-\lambda_2+1}^{\lambda_2-1} l_{nk}(f) \psi(nx_2 - k_2) + \sum_{k_2=\lambda_2}^{\infty} l_{nk}(f) \psi(nx_2 - k_2) \right] \\ = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{-\lambda_2} l_{nk}(f) \psi(nx_1 - k_1) \psi(nx_2 - k_2) \\ + \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\lambda_2+1}^{\lambda_2-1} l_{nk}(f) \psi(nx_1 - k_1) \psi(nx_2 - k_2) \\ + \sum_{k_1=-\infty}^{\infty} \sum_{k_2=\lambda_2}^{\infty} l_{nk}(f) \psi(nx_1 - k_1) \psi(nx_2 - k_2) =: (*).$$

(For convenience call

$$F(k_1, k_2, x_1, x_2) := l_{nk}(f) \psi(nx_1 - k_1) \psi(nx_2 - k_2).)$$

Thus

$$(*) = \sum_{k_1=-\infty}^{-\lambda_1} \sum_{k_2=-\infty}^{-\lambda_2} F(k_1, k_2, x_1, x_2) + \sum_{k_1=-\lambda_1+1}^{\lambda_1-1} \sum_{k_2=-\infty}^{-\lambda_2} F(k_1, k_2, x_1, x_2) \\ + \sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=-\infty}^{-\lambda_2} F(k_1, k_2, x_1, x_2) + \sum_{k_1=-\infty}^{-\lambda_1} \sum_{k_2=-\lambda_2+1}^{\lambda_2-1} F(k_1, k_2, x_1, x_2)$$

$$\begin{aligned}
& + \sum_{k_1=-\lambda_1+1}^{\lambda_1-1} \sum_{k_2=-\lambda_2+1}^{\lambda_2-1} F(k_1, k_2, x_1, x_2) + \sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=-\lambda_2+1}^{\lambda_2-1} F(k_1, k_2, x_1, x_2) \\
(97) \quad & + \sum_{k_1=-\infty}^{-\lambda_1} \sum_{k_2=\lambda_2}^{\infty} F(k_1, k_2, x_1, x_2) + \sum_{k_1=-\lambda_1+1}^{\lambda_1-1} \sum_{k_2=\lambda_2}^{\infty} F(k_1, k_2, x_1, x_2) \\
& + \sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=\lambda_2}^{\infty} F(k_1, k_2, x_1, x_2).
\end{aligned}$$

Notice that the finite sum of continuous functions $F(k_1, k_2, x_1, x_2)$, $\sum_{k_1=-\lambda_1+1}^{\lambda_1-1} \sum_{k_2=-\lambda_2+1}^{\lambda_2-1} F(k_1, k_2, x_1, x_2)$ is a continuous function.

The rest of the summands of $F_n(f, x_1, x_2)$ are treated all the same way and similarly to the case of $N = 1$. The method is demonstrated as follows.

We will prove that $\sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=-\infty}^{-\lambda_2} l_{nk}(f) \psi(nx_1 - k_1) \psi(nx_2 - k_2)$ is continuous in $(x_1, x_2) \in \mathbb{R}^2$.

The continuous function

$$\|l_{nk}(f)\|_{\gamma} \psi(nx_1 - k_1) \psi(nx_2 - k_2) \leq \| \|f\|_{\gamma} \|_{\infty} \psi(k_1 - \lambda_1) \psi(k_2 + \lambda_2),$$

and

$$\begin{aligned}
& \| \|f\|_{\gamma} \|_{\infty} \sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=-\infty}^{-\lambda_2} \psi(k_1 - \lambda_1) \psi(k_2 + \lambda_2) = \\
& = \| \|f\|_{\gamma} \|_{\infty} \left(\sum_{k_1=\lambda_1}^{\infty} \psi(k_1 - \lambda_1) \right) \left(\sum_{k_2=-\infty}^{-\lambda_2} \psi(k_2 + \lambda_2) \right) \\
& \leq \| \|f\|_{\gamma} \|_{\infty} \left(\sum_{k'_1=0}^{\infty} \psi(k'_1) \right) \left(\sum_{k'_2=-\infty}^0 \psi(k'_2) \right) \leq \| \|f\|_{\gamma} \|_{\infty}.
\end{aligned}$$

So by the Weierstrass M test we get that

$\sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=-\infty}^{-\lambda_2} l_{nk}(f) \psi(nx_1 - k_1) \psi(nx_2 - k_2)$ is uniformly and absolutely convergent. Therefore it is continuous on \mathbb{R}^2 .

Next we prove continuity on \mathbb{R}^2 of

$$\sum_{k_1=-\lambda_1+1}^{\lambda_1-1} \sum_{k_2=-\infty}^{-\lambda_2} l_{nk}(f) \psi(nx_1 - k_1) \psi(nx_2 - k_2).$$

Notice here that

$$\begin{aligned}
& \|l_{nk}(f)\|_{\gamma} \psi(nx_1 - k_1) \psi(nx_2 - k_2) \leq \\
& \leq \| \|f\|_{\gamma} \|_{\infty} \psi(nx_1 - k_1) \psi(k_2 + \lambda_2) \\
& \leq \| \|f\|_{\gamma} \|_{\infty} \psi(0) \psi(k_2 + \lambda_2) = 0.319 \cdot \| \|f\|_{\gamma} \|_{\infty} \psi(k_2 + \lambda_2),
\end{aligned}$$

and

$$\begin{aligned}
& 0.319 \cdot \left\| \|f\|_\gamma \right\|_\infty \left(\sum_{k_1=-\lambda_1+1}^{\lambda_1-1} 1 \right) \left(\sum_{k_2=-\infty}^{-\lambda_2} \psi(k_2 + \lambda_2) \right) = \\
& = 0.319 \cdot \left\| \|f\|_\gamma \right\|_\infty (2\lambda_1 - 1) \left(\sum_{k'_2=-\infty}^0 \psi(k'_2) \right) \\
(98) \quad & \leq 0.319 \cdot (2\lambda_1 - 1) \left\| \|f\|_\gamma \right\|_\infty.
\end{aligned}$$

So the double series under consideration is uniformly convergent and continuous. Clearly $F_n(f, x_1, x_2)$ is proved to be continuous on \mathbb{R}^2 .

Similarly reasoning one can prove easily now, but with more tedious work, that $F_n(f, x_1, \dots, x_N)$ is continuous on \mathbb{R}^N , for any $N \geq 1$. We choose to omit this similar extra work. \square

REMARK 4. By (27) it is obvious that $\left\| \|A_n(f)\|_\gamma \right\|_\infty \leq \left\| \|f\|_\gamma \right\|_\infty < \infty$, and $A_n(f) \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$, given that $f \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$.

Call L_n any of the operators A_n, B_n, C_n, D_n .

Clearly then

$$(99) \quad \left\| \|L_n^2(f)\|_\gamma \right\|_\infty = \left\| \|L_n(L_n(f))\|_\gamma \right\|_\infty \leq \left\| \|L_n(f)\|_\gamma \right\|_\infty \leq \left\| \|f\|_\gamma \right\|_\infty,$$

etc.

Therefore we get

$$(100) \quad \left\| \|L_n^k(f)\|_\gamma \right\|_\infty \leq \left\| \|f\|_\gamma \right\|_\infty, \quad \forall k \in \mathbb{N},$$

the contraction property.

Also we see that

$$(101) \quad \left\| \|L_n^k(f)\|_\gamma \right\|_\infty \leq \left\| \|L_n^{k-1}(f)\|_\gamma \right\|_\infty \leq \dots \leq \left\| \|L_n(f)\|_\gamma \right\|_\infty \leq \left\| \|f\|_\gamma \right\|_\infty.$$

Here L_n^k are bounded linear operators. \square

NOTATION 1. Here $N \in \mathbb{N}$, $0 < \beta < 1$. Denote by

$$(102) \quad c_N := \begin{cases} (4.9737)^N, & \text{if } L_n = A_n, \\ 1, & \text{if } L_n = B_n, C_n, D_n, \end{cases}$$

$$(103) \quad \varphi(n) := \begin{cases} \frac{1}{n^\beta}, & \text{if } L_n = A_n, B_n, \\ \frac{1}{n} + \frac{1}{n^\beta}, & \text{if } L_n = C_n, D_n, \end{cases}$$

$$(104) \quad \Omega := \begin{cases} C\left(\prod_{i=1}^N [a_i, b_i], X\right), & \text{if } L_n = A_n, \\ C_B(\mathbb{R}^N, X), & \text{if } L_n = B_n, C_n, D_n, \end{cases} \quad \text{and}$$

$$(105) \quad Y := \begin{cases} \prod_{i=1}^N [a_i, b_i], & \text{if } L_n = A_n, \\ \mathbb{R}^N, & \text{if } L_n = B_n, C_n, D_n. \end{cases}$$

We give the condensed

THEOREM 11. *Let $f \in \Omega$, $0 < \beta < 1$, $x \in Y$; $n, N \in \mathbb{N}$ with $n^{1-\beta} > 2$. Then*

(i)

$$(106) \quad \|L_n(f, x) - f(x)\|_\gamma \leq c_N \left[\omega_1(f, \varphi(n)) + \frac{4\|f\|_\gamma}{\pi^2(n^{1-\beta}-2)} \right] =: \tau(n),$$

where ω_1 is for $p = \infty$,

and

(ii)

$$(107) \quad \left\| \|L_n(f) - f\|_\gamma \right\|_\infty \leq \tau(n) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

For f uniformly continuous and in Ω we obtain

$$\lim_{n \rightarrow \infty} L_n(f) = f,$$

pointwise and uniformly.

Proof. By [Theorems 5](#) and [7](#) to [9](#). □

Next we do iterated neural network approximation (see also [\[9\]](#)).

We make

REMARK 5. Let $r \in \mathbb{N}$ and L_n as above. We observe that

$$\begin{aligned} L_n^r f - f &= (L_n^r f - L_n^{r-1} f) + (L_n^{r-1} f - L_n^{r-2} f) \\ &\quad + (L_n^{r-2} f - L_n^{r-3} f) + \dots + (L_n^2 f - L_n f) + (L_n f - f). \end{aligned}$$

Then

$$\begin{aligned} &\left\| \|L_n^r f - f\|_\gamma \right\|_\infty \leq \\ &\leq \left\| \|L_n^r f - L_n^{r-1} f\|_\gamma \right\|_\infty + \left\| \|L_n^{r-1} f - L_n^{r-2} f\|_\gamma \right\|_\infty \\ &\quad + \left\| \|L_n^{r-2} f - L_n^{r-3} f\|_\gamma \right\|_\infty + \dots + \left\| \|L_n^2 f - L_n f\|_\gamma \right\|_\infty + \left\| \|L_n f - f\|_\gamma \right\|_\infty \\ &= \left\| \|L_n^{r-1} (L_n f - f)\|_\gamma \right\|_\infty + \left\| \|L_n^{r-2} (L_n f - f)\|_\gamma \right\|_\infty + \left\| \|L_n^{r-3} (L_n f - f)\|_\gamma \right\|_\infty \\ (108) \quad &+ \dots + \left\| \|L_n (L_n f - f)\|_\gamma \right\|_\infty + \left\| \|L_n f - f\|_\gamma \right\|_\infty \leq r \left\| \|L_n f - f\|_\gamma \right\|_\infty. \end{aligned}$$

That is

$$(109) \quad \left\| \|L_n^r f - f\|_\gamma \right\|_\infty \leq r \left\| \|L_n f - f\|_\gamma \right\|_\infty. \quad \square$$

We give

THEOREM 12. All here as in Theorem 11 and $r \in \mathbb{N}$, $\tau(n)$ as in (106). Then

$$(110) \quad \left\| \|L_n^r f - f\|_\gamma \right\|_\infty \leq r\tau(n).$$

So that the speed of convergence to the unit operator of L_n^r is not worse than of L_n .

Proof. By (109) and (107). \square

We make

REMARK 6. Let $m_1, \dots, m_r \in \mathbb{N} : m_1 \leq m_2 \leq \dots \leq m_r$, $0 < \beta < 1$, $f \in \Omega$. Then $\varphi(m_1) \geq \varphi(m_2) \geq \dots \geq \varphi(m_r)$, φ as in (103).

Therefore

$$(111) \quad \omega_1(f, \varphi(m_1)) \geq \omega_1(f, \varphi(m_2)) \geq \dots \geq \omega_1(f, \varphi(m_r)).$$

Assume further that $m_i^{1-\beta} > 2$, $i = 1, \dots, r$. Then

$$(112) \quad \frac{2}{\pi^2(m_1^{1-\beta}-2)} \geq \frac{2}{\pi^2(m_2^{1-\beta}-2)} \geq \dots \geq \frac{2}{\pi^2(m_r^{1-\beta}-2)}.$$

Let L_{m_i} as above, $i = 1, \dots, r$, all of the same kind.

We write

$$(113) \quad \begin{aligned} & L_{m_r}(L_{m_{r-1}}(\dots L_{m_2}(L_{m_1}f))) - f = \\ & = L_{m_r}(L_{m_{r-1}}(\dots L_{m_2}(L_{m_1}f))) - L_{m_r}(L_{m_{r-1}}(\dots L_{m_2}f)) \\ & \quad + L_{m_r}(L_{m_{r-1}}(\dots L_{m_2}f)) - L_{m_r}(L_{m_{r-1}}(\dots L_{m_3}f)) \\ & \quad + L_{m_r}(L_{m_{r-1}}(\dots L_{m_3}f)) - L_{m_r}(L_{m_{r-1}}(\dots L_{m_4}f)) + \dots \\ & \quad + L_{m_r}(L_{m_{r-1}}f) - L_{m_r}f + L_{m_r}f - f \\ & = L_{m_r}(L_{m_{r-1}}(\dots L_{m_2})) (L_{m_1}f - f) + L_{m_r}(L_{m_{r-1}}(\dots L_{m_3})) (L_{m_2}f - f) \\ & \quad + L_{m_r}(L_{m_{r-1}}(\dots L_{m_4})) (L_{m_3}f - f) + \dots + L_{m_r}(L_{m_{r-1}}f - f) + L_{m_r}f - f. \end{aligned}$$

Hence by the triangle inequality property of $\|\|\cdot\|_\gamma\|_\infty$ we get

$$\begin{aligned} & \left\| \|L_{m_r}(L_{m_{r-1}}(\dots L_{m_2}(L_{m_1}f))) - f\|_\gamma \right\|_\infty \leq \\ & \leq \left\| \|L_{m_r}(L_{m_{r-1}}(\dots L_{m_2})) (L_{m_1}f - f)\|_\gamma \right\|_\infty \\ & \quad + \left\| \|L_{m_r}(L_{m_{r-1}}(\dots L_{m_3})) (L_{m_2}f - f)\|_\gamma \right\|_\infty \\ & \quad + \left\| \|L_{m_r}(L_{m_{r-1}}(\dots L_{m_4})) (L_{m_3}f - f)\|_\gamma \right\|_\infty + \dots \\ & \quad + \left\| \|L_{m_r}(L_{m_{r-1}}f - f)\|_\gamma \right\|_\infty + \left\| \|L_{m_r}f - f\|_\gamma \right\|_\infty \end{aligned}$$

(repeatedly applying (99))

$$(114) \quad \leq \left\| \|L_{m_1}f - f\|_\gamma \right\|_\infty + \left\| \|L_{m_2}f - f\|_\gamma \right\|_\infty + \left\| \|L_{m_3}f - f\|_\gamma \right\|_\infty + \dots \\ + \left\| \|L_{m_{r-1}}f - f\|_\gamma \right\|_\infty + \left\| \|L_{m_r}f - f\|_\gamma \right\|_\infty$$

$$(115) \quad = \sum_{i=1}^r \left\| \|L_{m_i}f - f\|_\gamma \right\|_\infty.$$

That is, we proved

$$(116) \quad \left\| \|L_{m_r}(L_{m_{r-1}}(\dots L_{m_2}(L_{m_1}f))) - f\|_\gamma \right\|_\infty \leq \sum_{i=1}^r \left\| \|L_{m_i}f - f\|_\gamma \right\|_\infty. \quad \square$$

We give

THEOREM 13. *Let $f \in \Omega$; $N, m_1, m_2, \dots, m_r \in \mathbb{N} : m_1 \leq m_2 \leq \dots \leq m_r, 0 < \beta < 1; m_i^{1-\beta} > 2, i = 1, \dots, r, x \in Y$, and let $(L_{m_1}, \dots, L_{m_r})$ as $(A_{m_1}, \dots, A_{m_r})$ or $(B_{m_1}, \dots, B_{m_r})$ or $(C_{m_1}, \dots, C_{m_r})$ or $(D_{m_1}, \dots, D_{m_r})$, $p = \infty$. Then*

$$(117) \quad \begin{aligned} & \left\| \|L_{m_r}(L_{m_{r-1}}(\dots L_{m_2}(L_{m_1}f))) - f\|_\gamma \right\|_\infty \leq \\ & \leq \left\| \|L_{m_r}(L_{m_{r-1}}(\dots L_{m_2}(L_{m_1}f))) - f\|_\gamma \right\|_\infty \\ & \leq \sum_{i=1}^r \left\| \|L_{m_i}f - f\|_\gamma \right\|_\infty \\ & \leq c_N \sum_{i=1}^r \left[\omega_1(f, \varphi(m_i)) + \frac{4\|f\|_\gamma}{\pi^2(m_i^{1-\beta}-2)} \right] \\ & \leq rc_N \left[\omega_1(f, \varphi(m_1)) + \frac{4\|f\|_\gamma}{\pi^2(m_1^{1-\beta}-2)} \right]. \end{aligned}$$

Clearly, we notice that the speed of convergence to the unit operator of the multiply iterated operator is not worse than the speed of L_{m_1} .

Proof. Using (116), (111), (112) and (106), (107). \square

We continue with










THEOREM 14. *Let all as in Corollary 2, and $r \in \mathbb{N}$. Here $\varphi_3(n)$ is as in (78). Then*

$$(118) \quad \left\| \|A_n^r f - f\|_\gamma \right\|_\infty \leq r \left\| \|A_n f - f\|_\gamma \right\|_\infty \leq r\varphi_3(n).$$

Proof. By (109) and (78). \square

APPLICATION 1. *A typical application of all of our results is when $(X, \|\cdot\|_\gamma) = (\mathbb{C}, |\cdot|)$, where \mathbb{C} are the complex numbers.*

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