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A NOTE ON THE UNIQUE SOLVABILITY CONDITION FOR GENERALIZED ABSOLUTE VALUE MATRIX EQUATION

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Abstract. We present the spectral radius condition $\rho\big(|A^{-1}| \cdot |B|\big) < 1$ for the unique solvability of the *generalized absolute value matrix equation* (GAVME) $AX + B|X| = D$. For some instances, our condition is superior to the earlier published singular values conditions $\sigma_{\max}(|B|) < \sigma_{\min}(A)$ [3] and $\sigma_{\max}(B) < \sigma_{\min}(A)$ [12]. For the validity of our condition, we provided some examples.

MSC. 15A06, 15A18, 90C30.

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1. INTRODUCTION

In this note, we consider the following GAVME

$$AX + B|X| = D, \tag{1}$$

where $A, B, D \in \mathbb{R}^{n\times n}$ and $X \in \mathbb{R}^{n\times n}$ is unknown.

The GAVME is a generalization form of the following *generalized absolute value equation* (GAVE)

$$Ax + B|x| = d, \tag{2}$$

where $A, B \in \mathbb{R}^{n\times n}$, $d \in \mathbb{R}^n$ are known and $x \in \mathbb{R}^n$ is unknown.

The GAVE was first introduced by Rohn [7] and studied more detail in [1, 2, 5, 6], where authors provided its unique solvability conditions and discussed its numerical solution. The importance of *absolute value equations* (AVEs) is due to their broad applications in many mathematics and applied sciences domains. For instance, the linear complementarity problem, bimatrix games, mixed-integer programming, system of linear interval matrix and convex quadratic optimization can be formulated as AVEs. Because of that reason, AVEs attract the attention of researchers in this field. The GAVME is a generalization of the

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GAVE. There are many different types of conditions for the unique solution of GAVE (2) studied (see [8, 9, 10, 11] and references therein), but GAVME (1) has few conditions for detecting the unique solution. The study of GAVE and GAVME is challenging and exciting because both the equations contain the non-differentiable term.

The GAVME was first considered by Dehghan *et al.* [3] and provided a multi-splitting Picard-iterative method for its solution. Kai [12] extended the unique solvability result and improved the convergence condition of Dehghan *et al.* [3]. For more details about the GAVME one can see ([3, 12] and references therein).

We will prove that if the following spectral radius condition

$$\rho(|A^{-1}| \cdot |B|) < 1,$$

satisfy for nonsingular matrix A, then the GAVME (1) has a unique solution for any matrix D, where $\rho(A)$ denote the spectral radius of the matrix $A \in \mathbb{R}^{n\times n}$. For some instances, our result is superior to the following two singular value conditions

$$\sigma_{\max}(|B|) < \sigma_{\min}(A), \tag{3}$$

$$\sigma_{\max}(B) < \sigma_{\min}(A), \tag{4}$$

where $\sigma_{\max}(A)$ denote the maximum and $\sigma_{\min}(A)$ denote the minimum singular values of matrix $A \in \mathbb{R}^{n\times n}$. In papers by Dehghan *et al.* [3] and Kai [12], the conditions (3) and (4) are provided, respectively. These conditions (3) and (4) are also used to determine whether the GAVME is uniquely solvable, but occasionally they are insufficient to determine whether the GAVME (1) is uniquely solvable. For example, see Example 1 and Example 2, where our condition is sufficient to determine the unique solution of the GAVME (1), while conditions (3) and (4) are invalid to judge the unique solution of GAVME (1).

2. MAIN RESULTS

In this section, for our main result, the following Lemma is required.

Lemma 1 ([4]). *If nonsingular matrix A and matrix B is satisfy the conditions*

$$\rho(|A^{-1}| \cdot |B|) < 1,$$

then GAVE (2) has a unique solution for any d.

Under the same unique solvability condition of the GAVE in the previous Lemma 1, we prove that GAVME (1) also has a unique solution. So based on Lemma 1, we give our main result.

Theorem 1. *If matrices A and B satisfy the condition*

$$\rho(|A^{-1}| \cdot |B|) < 1, \tag{5}$$

then GAVME (1) has a unique solution for any matrix D, where A is invertible.

Proof. Suppose $X = (x_1, x_2, ..., x_n)$ and $D = (d_1, d_2, ..., d_n)$, where x_k and d_k are the kth column of the matrices X and D, respectively. By $|X| = (|x_1|, |x_2|, ..., |x_n|)$, the GAVME (1) can be rewritten as $A(x_1, x_2, ..., x_n) + B(|x_1|, |x_2|, ..., |x_n|) = (d_1, d_2, ..., d_n)$, or equivalently,

$$Ax_k + B|x_k| = d_k, \tag{6}$$

where $k = 1, 2, 3, ..., n$. Then with the aid of Lemma 1, GAVE (6) has a unique solution for any k when the condition $\rho(|A^{-1}| \cdot |B|) < 1$ holds. From here we can calculate all x_k separately. This completes the proof. □

In Theorem 1, we have a new sufficient condition for the GAVME (1). To check our condition's validity we are considering the following examples, see Example 1 and Example 2, in which our condition (5) is holds, while both the conditions (3) and (4) are not holds.

Example 1. *Consider the matrices A and B given below*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -0.4 & -0.9 \\ 0.48 & 0.04 \end{bmatrix}$$

Here $\sigma_{\min}(A) = 1$, $\sigma_{\max}(B) = \sigma_{\max}(|B|) = 1.0172$, and $\rho(|A^{-1}| \cdot |B|) = 0.9015 < 1$. Clearly, both the conditions (3) and (4) are not satisfying, while our condition (5) is satisfying. Moreover, GAVME (1) has a unique solution according to Theorem 1.

Example 2. *Consider the following GAVME (1)*

$$\begin{bmatrix} 2 & -4 & 0 \\ 0 & 1.2 & 1.1 \\ -2 & 0.8 & 0 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} |x_1| & |x_2| & |x_3| \\ |x_4| & |x_5| & |x_6| \\ |x_7| & |x_8| & |x_9| \end{bmatrix} =$$
$$= \begin{bmatrix} -9 & -17 & -45 \\ 8.7 & 8.3 & 21.8 \\ -3.6 & 0.2 & 11.4 \end{bmatrix}$$

Clearly, $\rho(|A^{-1}| \cdot |B|) = 0.9091 < 1$. Further, $\sigma_{\max}(|B|) = 1.8019 \nless \sigma_{\min}(A) = 0.9038$ and $\sigma_{\max}(B) = 1.8019 \nless \sigma_{\min}(A) = 0.9038$. This GAVME has a unique solution

$$X = \begin{bmatrix} 2 & 1 & -5 \\ 3 & 4 & 8 \\ 1 & -5 & 2 \end{bmatrix}.$$

Example 3. *Consider the matrix $A = I_{n\times n}$ and matrix*

$$B = b_{ij} = \begin{cases} 0.97 & \text{if } i = j \\ 0 & \text{if } i > j \\ 0.8 & \text{if } i < j \end{cases}$$

We take n=25, to satisfy the condition of Theorem 1. We get $\rho(|A^{-1}|\cdot|B|) = 0.97 < 1$*, so the GAVME* (1) *has a unique solution.*

From Theorem 1, we get the Corollary that follows.

Corollary 1. *If A is invertible matrix and satisfy the following spectral radius condition*

$$\rho(|A^{-1}|) < 1, \tag{7}$$

then absolute value matrix equation $AX + |X| = D$ *has a unique solution for any matrix* D*.*

3. CONCLUSIONS

In this note, a new condition is stated for the unique solvability of the generalized absolute value matrix equation. Since the generalized absolute value equation and generalized absolute value matrix equation both have the same known terms on the left-hand side, namely matrices A and B. So in the future, we can also use the unique solvability conditions of the GAVE to detect the unique solvability of GAVME. This needs further investigation. The numerical results for the GAVME are also an interesting topic in the future.

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