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BASKAKOV-KANTOROVICH OPERATORS REPRODUCING AFFINE FUNCTIONS: INVERSE RESULTS

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Abstract. In a previous paper the author presented a Kantorovich modification of Baskakov operators which reproduce affine functions and he provided an upper estimate for the rate of convergence in polynomial weighted spaces. In this paper, for the same family of operators, a strong inverse inequality is given for the case of approximation in norm.

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1. INTRODUCTION

Let $C[0,\infty)$ be the family of all real continuous functions on the semiaxis. Denote $e_k(t)=t^k$, $k\geq 0$.

Throughout the paper, we fix $m\in\mathbb{N}$, $m\geq 2$, and set

$$\varrho(x)=\tfrac{1}{(1+x)^m}\quad\text{and}\quad \varphi(x)=\sqrt{x(1+x)}.$$

Moreover

$$C_\varrho[0,\infty)=\{f\in C[0,\infty):\ \|f\|_\varrho<\infty\},$$

where $\|f\|_\varrho=\sup_{x\geq 0}|\varrho(x)f(x)|$.

For a real $\lambda>1$, $f:[0,\infty)\to\mathbb{R}$ and $x\geq 0$, the Baskakov operator is defined by

$$V_\lambda(f,x)=\sum_{k=0}^{\infty} f\big(\tfrac{k}{\lambda}\big)v_{\lambda,k}(x),\qquad v_{\lambda,k}(x)=\tbinom{\lambda+k-1}{k}\tfrac{x^k}{(1+x)^{\lambda+k}},$$

whenever the series converges absolutely.

For some functions $f\in C[0,\infty)$, a family of Kantorovich-Baskakov type operators reproducing affine functions was introduced in [1] by setting

$$M_\lambda(f,x)=\lambda\sum_{k=0}^{\infty}Q_{\lambda,k}(f)v_{\lambda,k}(x),\qquad Q_{\lambda,k}(f)=\int_{I_{\lambda,k}}f(a_kt)dt,$$

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where

$$I_{\lambda,k}=\left[\tfrac{k}{\lambda},\tfrac{k+1}{\lambda}\right] \qquad \text{and} \qquad a_k=\tfrac{2k}{2k+1}, \quad k\in\mathbb{N}_0.$$

Approximation properties of the operators M_λ in some weighted spaces were presented in [1]. The following notations are needed: for $0\le\beta\le 1$, set

$$C_{\varrho,\beta}[0,\infty)=\Big\{h\in C[0,\infty):\ h(0)=0,\ \ \|h\|_{\varrho,\beta}=\|\varphi^{2\beta}h\|_\varrho<\infty\Big\}$$

and

$$K(f,t)_{\varrho,\beta}=\inf\Big\{\|f-g\|_{\varrho,\beta}+t\|\varphi^2g''\|_{\varrho,\beta}:\ g\in D(\varrho,\beta)\Big\}, \tag{1}$$

where

$$D(\varrho,\beta)=\Big\{g\in C_{\varrho,\beta}[0,\infty)\ :\ g,g'\in AC_{loc}:\ \|\varphi^2g''\|_{\varrho,\beta}<\infty\Big\}.$$

The following result was proved in [1].

Theorem 1. *If $\beta\in[0,1]$ and $m\ge 2$, then there exists a constant C such that, for all $\lambda>2(1+m)$ and every $f\in C_{\varrho,\beta}[0,\infty)$,*

$$\|M_\lambda(f)-f\|_{\varrho,\beta}\le CK\big(f,\tfrac{1}{\lambda}\big)_{\varrho,\beta}.$$

In this paper we present a strong inverse result related with Theorem 1.

The work is organized as follows. In Section 2 we present notations that will be used throughout the paper, as well as some identities related with the operators M_λ and their derivatives. Section 3 is devoted to prove inequalities related with the moments of the operators M_λ. In Section 4 we collect several inequalities related with the weight ϱ. In Section 5 we include some Bernstein type inequalities. A Voronovskaya type theorem is given in Section 6. Finally the main result is proved in Section 7.

In what follows C and C_i $(i\in\mathbb{N})$ will denote absolute constants. They may be different on each occurrence. We remark that our arguments allow to obtain bounds for the constants, but not the best.

2. NOTATIONS AND IDENTITIES

We will use the notations

$$M'_\lambda(f,x)=\frac{d}{dx}M_\lambda(f,x), \quad M''_\lambda(f,x)=\frac{d^2}{dx^2}M_\lambda(f,x)$$

and

$$M'''_\lambda(f,x)=\frac{d^3}{dx^3}M_\lambda(f,x).$$

For $\lambda>1$, $k\in\mathbb{N}_0$ and $x\ge 0$, we define

$$R_{\lambda,k}(x)=\Big(\tfrac{k}{\lambda}-x\Big)^2-\tfrac{(1+2x)}{\lambda}\Big(\tfrac{k}{\lambda}-x\Big)-\tfrac{\varphi^2(x)}{\lambda}, \tag{2}$$

$$I_{\lambda,1}(x)=\sum_{k=1}^{\infty}\frac{v_{\lambda,k}(x)}{\varrho(\frac{k}{\lambda})\varphi^{2\beta}(\frac{k}{n})}\Big(\tfrac{k}{\lambda}-x\Big)^2, \tag{3}$$

$$I_{\lambda,2}(x) = \tfrac{1+2x}{\lambda} \sum_{k=1}^{\infty} \Big| \tfrac{k}{\lambda} - x \Big| \tfrac{v_{\lambda,k}(x)}{\varrho(\frac{k}{\lambda})\varphi^{2\beta}(\frac{k}{n})}, \tag{4}$$

and

$$I_{\lambda,3}(x) = \tfrac{\varphi^2(x)}{\lambda} \sum_{k=1}^{\infty} \tfrac{v_{\lambda,k}(x)}{\varrho(\frac{k}{\lambda})\varphi^{2\beta}(\frac{k}{\lambda})}. \tag{5}$$

For $p, q \geq 0$, set

$$A_{\lambda,p,q}(x) = \sum_{k=1}^{\infty} \tfrac{v_{\lambda,k}(x)}{\varrho^p(\frac{k}{\lambda})\varphi^q(\frac{k}{\lambda})}. \tag{6}$$

For $f \in C[0,\infty)$ and $k, j \in \mathbb{N}$, we use the notation

$$J_{\lambda,k,j}(f) = \int_{I_{\lambda,k}} \int_{k/\lambda}^{a_{k+j}(t+j/\lambda)} f(u) \cdot \Big((a_{k+j}(t + \tfrac{j}{\lambda}) - u \Big)\, du\, dt.$$

Proposition 1. ([1, Prop. 2.4]) *For each $f \in C_\varrho[0,\infty)$, $\lambda > 1$ and $x > 0$, one has*

$$M'_\lambda(f,x) = \lambda^2 \sum_{k=0}^{\infty} \Big(Q_{\lambda,k+1}(f) - Q_{\lambda,k}(f) \Big) v_{\lambda+1,k}(x)$$

$$= \tfrac{\lambda^2}{\varphi^2(x)} \sum_{k=0}^{\infty} Q_{\lambda,k}(f) \Big(\tfrac{k}{\lambda} - x \Big) v_{\lambda,k}(x).$$

If $f(0) = 0$, the term corresponding to $k = 0$ should be omitted.

Remark 1. The proof of Proposition 1 (see [1]) is a consequence of the identities ($v_{\lambda,-1} = 0$)

$$v'_{\lambda,k}(x) = \tfrac{k-\lambda x}{x(1+x)} v_{\lambda,k}(x) = \lambda\big(v_{\lambda+1,k-1}(x) - v_{\lambda+1,k}(x)\big)\big). \tag{7}$$ □

Here need the analogous of Proposition 1 for the second and the third derivatives.

Proposition 2. *For each $\lambda > 1$, $f \in C_\varrho[0,\infty)$, and $x > 0$, one has*

$$M''_\lambda(f,x) = \lambda^2(\lambda+1) \sum_{k=0}^{\infty} \Big(Q_{\lambda,k+2}(f) - 2Q_{\lambda,k+1}(f) + Q_{\lambda,k}(f) \Big) v_{\lambda+2,k}(x) \tag{8}$$

$$= \tfrac{\lambda^3}{\varphi^4(x)} \sum_{k=0}^{\infty} Q_{\lambda,k}(f) R_{\lambda,k}(x) v_{\lambda,k}(x), \tag{9}$$

where $R_{\lambda,k}$ was defined in (2).

Moreover, if $g \in C^2_\varrho(0,\infty)$ and $h \in C^3_\varrho(0,\infty)$, then

$$M''_\lambda(g,x) = \lambda^2(\lambda+1) \sum_{k=0}^{\infty} \Big(\sum_{j=0}^{2} \tbinom{2}{j} (-1)^j J_{\lambda,k,j}(g'') \Big) v_{\lambda+2,k}(x) \tag{10}$$

and

$$\frac{\varphi^2(x)M_\lambda'''(g,x)}{\lambda^2(\lambda+1)(\lambda+2)}=\sum_{k=0}^{\infty}\Big(\sum_{j=0}^{2}\tbinom{2}{j}(-1)^jJ_{\lambda,k,j}(g'')\Big)\Big(\tfrac{k}{\lambda+2}-x\Big)v_{\lambda+2,k}(x).$$

If $f(0)=0$ or $g(0)=0$ the term corresponding to $k=0$ should be omitted.

Proof. It follows from Proposition 1 and (7) that

$$\begin{aligned}M_\lambda''(f,x)=&\lambda^2\sum_{k=0}^{\infty}\Big(Q_{\lambda,k+1}(f)-Q_{\lambda,k}(f)\Big)v'_{\lambda+1,k}(x)\\=&\lambda^2(\lambda+1)\sum_{k=0}^{\infty}\Big(Q_{\lambda,k+1}(f)-Q_{\lambda,k}(f)\Big)(v_{\lambda+2,k-1}(x)-v_{\lambda+2,k}(x)))\\=&\lambda^2(\lambda+1)\sum_{k=0}^{\infty}\Big(Q_{\lambda,k+2}(f)-2Q_{\lambda,k+1}(f)+Q_{\lambda,k}(f)\Big)v_{\lambda+2,k}(x).\end{aligned}$$

On the other hand,

$$\begin{aligned}\tfrac{\varphi^4(x)}{\lambda^2}M_\lambda''(f,x)=&\tfrac{\varphi^2(x)}{\lambda}\Big(\tfrac{\varphi^2(x)}{\lambda}M_\lambda'(f,x)\Big)'-\tfrac{1+2x}{\lambda}\tfrac{\varphi^2(x)}{\lambda}M_\lambda'(f,x)\\=&\tfrac{\varphi^2(x)}{\lambda}\Big(\lambda\sum_{k=0}^{\infty}Q_{\lambda,k}(f)\Big(\tfrac{k}{\lambda}-x\Big)v'_{\lambda,k}(x)-\lambda\sum_{k=0}^{\infty}Q_{\lambda,k}(f)v_{\lambda,k}(x)\Big)\\&-(1+2x)\sum_{k=0}^{\infty}Q_{\lambda,k}(f)\Big(\tfrac{k}{\lambda}-x\Big)v_{\lambda,k}(x)\\=&\lambda\sum_{k=0}^{\infty}Q_{\lambda,k}(f)\Big(\Big(\tfrac{k}{\lambda}-x\Big)^2-\tfrac{(1+2x)}{\lambda}\Big(\tfrac{k}{\lambda}-x\Big)-\tfrac{\varphi^2(x)}{\lambda}\Big)v_{\lambda,k}(x).\end{aligned}$$

Notice that, if

$$T_\lambda(t,k)=\Big(a_{k+2}(t+\tfrac{2}{\lambda})-\tfrac{k}{\lambda}\Big)-2\Big(a_{k+1}(t+\tfrac{1}{\lambda})-\tfrac{k}{\lambda}\Big)+\Big(a_kt-\tfrac{k}{\lambda}\Big),$$

then

$$\begin{aligned}&2\int_{\frac{k}{\lambda}}^{\frac{k+1}{\lambda}}T_\lambda(t,k)dt=\\&=a_{k+2}\Big((\tfrac{k+3}{\lambda})^2-(\tfrac{k+2}{\lambda})^2\Big)-2a_{k+1}\Big((\tfrac{k+2}{\lambda})^2-(\tfrac{k+1}{\lambda})^2\Big)+a_k\Big((\tfrac{k+1}{\lambda})^2-(\tfrac{k}{\lambda})^2\Big)\\&=\tfrac{1}{\lambda^2}\Big(a_{k+2}(2k+5)-2a_{k+1}(2k+3)+a_k(2k+1)\Big)\\&=\tfrac{1}{\lambda^2}(2(k+2)-4(k+1)+2k)=0.\end{aligned}$$

Hence, using the representation

$$g(y)=g(\tfrac{k}{\lambda})+g'(\tfrac{k}{\lambda})(y-\tfrac{k}{\lambda})+\int_{k/\lambda}^{y}g''(u)(y-u)du,$$

one has

$$Q_{\lambda,k+2}(g)-2Q_{\lambda,k+1}(g)+Q_{\lambda,k}(g)=$$

$$= \int_{k/\lambda}^{(k+1)/\lambda} \Big(g\Big(a_{k+2}(t+2/\lambda)\Big) - 2g\Big(a_{k+1}(t+\tfrac{1}{\lambda})\Big) + g(a_k t)\Big) dt$$
$$= \int_{k/\lambda}^{(k+1)/\lambda} \Big(\int_{k/\lambda}^{a_{k+2}(t+2/\lambda)} g''(u)(a_{k+2}(t+\tfrac{2}{\lambda}) - u)$$
$$- 2\int_{k/\lambda}^{a_{k+1}(t+2/\lambda)} g''(u)(a_{k+1}(t+\tfrac{1}{\lambda}) - u) + \int_{k/\lambda}^{a_k t} g''(u)(a_k t - u)\Big) du$$
$$= \sum_{j=0}^{2} \tbinom{2}{j}(-1)^j J_{\lambda,k,j}(g'').$$

On the other hand

$$\varphi^2(x) M_\lambda'''(g,x) =$$
$$= \varphi^2(x)\lambda^2(\lambda+1)\sum_{k=0}^{\infty}\Big(\sum_{j=0}^{2}\tbinom{2}{j}(-1)^j J_{\lambda,k,j}(g'')\Big) v'_{\lambda+2,k}(x)$$
$$= \varphi^2(x)\lambda^2(\lambda+1)\sum_{k=0}^{\infty}\Big(\sum_{j=0}^{2}\tbinom{2}{j}(-1)^j J_{n,k,j}(g'')\Big) \tfrac{k-(\lambda+2)x}{x(1+x)} v_{\lambda+2,k}(x)$$
$$= \lambda^2(\lambda+1)(\lambda+2)\sum_{k=0}^{\infty}\Big(\sum_{j=0}^{2}\tbinom{2}{j}(-1)^j J_{n,k,j}(g'')\Big)\Big(\tfrac{k}{\lambda+2} - x\Big) v_{\lambda+2,k}(x). \qquad \square$$

3. ESTIMATES FOR THE MOMENTS

Here, for $q \in \mathbb{N}$, we should consider the absolute moments

$$V_{\lambda,q}(x) = V_\lambda(|e_1 - x|^q, x) \quad \text{and} \quad M_{\lambda,q}(x) = M_\lambda(|e_1 - x|^q, x).$$

PROPOSITION 3. *For each fixed $q \in \mathbb{N}$, there exists a constant C_q such that, if $\lambda > 1$, $x > 0$ and $\lambda x \geq \frac{2}{3}$, then*

$$V_{\lambda,q}(x) \leq C_q \frac{\varphi^q(x)}{\lambda^{\frac{q}{2}}}.$$

Proof. It was proved in [3, Prop. 3.2] that, if $q = 2j$, $j \in \mathbb{N}$, the assertion holds whenever $\lambda x \geq 1$. But the proof can be modified to include the case $\lambda x \geq \frac{2}{3}$. Of course, with a different constant.

If $q = 2j - 1$, $j \in \mathbb{N}$, then

$$V_\lambda(|\, e_1 - x\,|^{2j-1}, x) \leq \Big(V_\lambda((e_1 - x)^{2(j-1)}, x) V_\lambda((e_1 - x)^{2j}, x)\Big)^{1/2}$$
$$\leq C\Big(\tfrac{\varphi^{2(j-1)}(x)}{\lambda^{j-1}}\tfrac{\varphi^{2j}(x)}{\lambda^j}\Big)^{1/2} = C\tfrac{\varphi^{2j-1}(x)}{\lambda^{j-1/2}}. \qquad \square$$

We need an extension of Proposition 3 to the case of the operators M_λ, but only for $1 \leq q \leq 6$.

Proposition 4. *For each fixed $q\in\mathbb{N}$, $1\leq q\leq 6$, there exists a constant C_q such that, if $\lambda>1$, $x>0$ and $\lambda x\geq \frac{2}{3}$, then*

$$M_{\lambda,q}(x)\leq C_q\frac{\varphi^q(x)}{\lambda^{\frac{q}{2}}}.$$

Proof. It was proved in [1, Cor. 2.3] that, for each $\lambda>1$ and $x\geq 0$, one has

$$M_{\lambda,2}(x)\leq \tfrac{13}{12}\tfrac{\varphi^2(x)}{\lambda}\qquad\text{and}\qquad M_{\lambda,1}(x)\leq\sqrt{\tfrac{13}{12}}\tfrac{\varphi(x)}{\sqrt{\lambda}}.$$

Moreover, if $x\geq\frac{1}{2(\lambda+1)}$, then $M_{\lambda,4}(x)\leq\frac{16\varphi^4(x)}{\lambda^2}$. Since $\frac{2}{3\lambda}\geq\frac{1}{2(\lambda+1)}$, the inequality holds under the conditions assumed above.

Notice that

$$M_{\lambda,3}(x)\leq\sqrt{M_{\lambda,2}(x)M_{\lambda,4}(x)}\leq 4\sqrt{2}\tfrac{\varphi^3(x)}{\lambda^{3/2}}.$$

We should present a proof for $M_{\lambda,6}(x)$. For $k\in\mathbb{N}$,

$$\begin{aligned}
0&\leq\lambda\int_{I_{\lambda,k}}(a_kt-x)^6dt=\tfrac{\lambda}{7a_k}\Big(\Big(\tfrac{a_kk}{\lambda}-x+\tfrac{a_k}{\lambda}\Big)^7-\Big(\tfrac{a_kk}{\lambda}-x\Big)^7\Big)\\
&=\tfrac{\lambda}{7a_k}\sum_{i=0}^{6}\tbinom{7}{i}\Big(\tfrac{a_k}{\lambda}\Big)^{7-i}\Big(\tfrac{a_kk}{\lambda}-x\Big)^i=\tfrac{1}{7}\sum_{i=0}^{6}\tbinom{7}{i}\Big(\tfrac{a_k}{\lambda}\Big)^{6-i}\Big(\tfrac{a_kk}{\lambda}-x\Big)^i\\
&=\tfrac{1}{7}\sum_{i=0}^{6}\tbinom{7}{i}\Big(\tfrac{a_k}{\lambda}\Big)^{6-i}\Big(\tfrac{k}{\lambda}-x-\tfrac{k}{(2k+1)\lambda}\Big)^i\\
&\leq\tfrac{1}{7}\sum_{i=0}^{6}\tbinom{7}{i}2^i\Big(\tfrac{1}{\lambda}\Big)^{6-i}\Big(\Big|\tfrac{k}{\lambda}-x\Big|^i+\Big(\tfrac{k}{(2k+1)\lambda}\Big)^i\Big)\\
&\leq\tfrac{1}{7\lambda^6}\sum_{i=0}^{6}\tbinom{7}{i}+\tfrac{1}{7}\sum_{i=0}^{6}\tbinom{7}{i}2^i\Big(\tfrac{1}{\lambda}\Big)^{6-i}\Big|\tfrac{k}{\lambda}-x\Big|^i.
\end{aligned}$$

Therefore

$$\begin{aligned}
\lambda\sum_{k=1}^{\infty}Q_{\lambda,k}((e_1-x)^6)v_{\lambda,k}(x)&\leq\tfrac{2^7}{7\lambda^6}+\sum_{i=1}^{6}\tbinom{7}{i}\tfrac{2^i}{\lambda^{6-i}}V_\lambda(\mid e_1-x\mid^i,x)\\
&\leq\tfrac{2^7}{7\lambda^6}+\sum_{i=1}^{6}\tbinom{7}{i}\tfrac{2^iC_i}{\lambda^{6-i}}\tfrac{\varphi^i(x)}{\lambda^{i/2}}\\
&=\tfrac{2^7}{7\lambda^6}+\sum_{i=1}^{6}\tbinom{7}{i}\tfrac{2^iC_i\varphi^i(x)}{\lambda^{3+(6-i)/2}}\\
&\leq\tfrac{3^3x^3}{2^37\lambda^3}+\sum_{i=1}^{6}\tbinom{7}{i}\tfrac{2^i3^{(6-i)/2}C_i\varphi^i(x)x^{(6-i)/2}}{2^{(6-i)/2}\lambda^3}\\
&\leq\tfrac{\varphi^6(x)}{\lambda^3}+C_1\sum_{i=1}^{6}\tbinom{7}{i}\tfrac{\varphi^i(x)\varphi^{6-i}(x)}{\lambda^3}\leq C_2\tfrac{\varphi^6(x)}{\lambda^3}.
\end{aligned}$$

Moreover, taking into account Proposition 3, one has

$$0 \le \lambda \int_{I_{\lambda,0}} (a_0 t - x)^6 dt v_{\lambda,0} = x^6 v_{\lambda,0} \le V_\lambda((e_1 - x)^6, x) \le C_6 \frac{\varphi^6(x)}{\lambda^3}.$$

This yields the inequality for $m = 6$.

Finally the proof in the case $m = 5$ is obtained by using Hölder inequality. □

4. PREPARATORY COMPUTATIONS

Proposition 5. *Suppose that $m \ge 2$, $\gamma \in [0,2)$, $f \in C[0,\infty)$, $f(0) = 0$, and $\|\varphi^{2\gamma} f\|_\varrho < \infty$. If $\lambda > 1$ and $k \in \mathbb{N}$, then*

$$| Q_{\lambda,k}(f) | \le \frac{2^m}{\lambda} \frac{\|\varphi^{2\gamma} f\|_\varrho}{\varrho(\frac{k}{\lambda})\varphi^{2\gamma}(\frac{k}{\lambda})}.$$

Proof. It was proved in [1, Prop. 3.4] that, if $\gamma \in [0,2)$, $m \ge \gamma$, $\lambda > 1$ and $k > 0$, then

$$\int_{k/\lambda}^{(k+1)/\lambda} \frac{dt}{\varrho(a_k t)\varphi^{2\gamma}(a_k t)} \le \frac{2^m}{\lambda} \frac{1}{\varrho(\frac{k}{\lambda})\varphi^{2\gamma}(\frac{k}{\lambda})}.$$ □

Proposition 6. *Assume $m \in \mathbb{N}$, $p, q \ge 0$, $mp \ge q \ge 0$ and set $s = 2(1 + 2mp - q)$. There exists a constant $C(p,q)$ such that, if $\lambda \ge s$ and $x > 0$, then*

$$A_{\lambda,p,q}(x) \le C(p,q) \frac{(1+x)^{mp}}{\varphi^q(x)},$$

where $A_{\lambda,p,q}(x)$ is defined in (6).

Proof. It is known that (see [3, Prop. 3.8]), if $a > -1$, $q \in \mathbb{R}$, and $r = \max\left\{2a, 2|c|, \frac{|c|}{1+a}\right\}$, then

$$\sum_{k=1}^{\infty} \left(\frac{\lambda}{k}\right)^a \left(1 + \frac{k}{\lambda}\right)^c v_{\lambda,k}(x) \le C \frac{(1+x)^c}{x^a}, \qquad \lambda \ge 2(1+r).$$

We apply this result with $a = \frac{q}{2}$ and $c = mp - \frac{q}{2}$. Notice that $q \le 2mp - q$. Hence, if $\lambda \ge 2(1 + 2mp - q)$,

$$\sum_{k=1}^{\infty} \frac{v_{\lambda,k}(x)}{\varrho^p(\frac{k}{\lambda})\varphi^q(\frac{k}{\lambda})} = \sum_{k=1}^{\infty} \left(\frac{\lambda}{k}\right)^{\frac{q}{2}} \left(1 + \frac{k}{\lambda}\right)^{mp - \frac{q}{2}} v_{\lambda,k}(x)$$

$$\le C \frac{(1+x)^{mp - \frac{q}{2}}}{x^{\frac{q}{2}}} = C \frac{(1+x)^{mp}}{\varphi^q(x)}.$$ □

Proposition 7. *Assume $\beta \in [0,1]$ and $m \ge 2$. There exists a constant C such that, if $k > 0$, $j \in \{0,1,2\}$, $\lambda > 2$, and $g \in D(\varrho, \beta)$, then*

$$| J_{\lambda,k,j}(g'') | \le \frac{C}{\varrho(\frac{k}{\lambda+2})\varphi^{2+2\beta}(\frac{k}{\lambda+2})} \frac{\|\varphi^2 g''\|_{\varrho,\beta}}{\lambda^3}.$$

Proof. It is sufficient to consider the case $\|\varphi^2 g''\|_{\varrho,\beta}=1$.

If we set $g(x)=\varrho(x)\varphi^{2+2\beta}(x)$, then

$$|J_{\lambda,k,j}(g'')|\le \int_{I_{\lambda,k}}\int_{k/\lambda}^{a_{k+j}(t+\frac{j}{\lambda})}\frac{((a_{k+j}(t+\frac{j}{\lambda})-u)}{g(u)}\,du\,dt.$$

(A) Let us first consider the case $j\in\{1,2\}$. Since

$$k(1+2(k+j))\le 2k(k+j)+2j(k+j)=2(k+j)^2,$$

if $t\ge \frac{k}{\lambda}$, then

$$\tfrac{k}{\lambda}\le \Big(\tfrac{2(k+j)}{1+2(k+j)}\Big(\tfrac{k}{\lambda}+\tfrac{j}{\lambda}\Big)\le a_{k+j}(t+\tfrac{j}{\lambda}).$$

Moreover, if $u\le a_{k+j}(t+j/\lambda)$ and $t\le (k+1)/\lambda$, then

$$\begin{aligned}\frac{1}{\varrho(u)}&=(1+u)^m\le (1+(a_{k+j}(\tfrac{k+1+j}{\lambda}))^m\le (1+(\tfrac{2k+2}{\lambda}))^m\\ &\le (1+(\tfrac{4k}{\lambda}))^m\le 4^m(1+\tfrac{k}{\lambda})^m\le \frac{4^m\cdot 2^m}{\varrho(\frac{k}{\lambda+2})}.\end{aligned}$$

On the other hand, since $\varphi^{2+2\beta}(x)$ increases, if $\frac{k}{\lambda}\le u$, then

$$\frac{1}{\varphi^{2+2\beta}(u)}\le\frac{1}{\varphi^{2+2\beta}(\frac{k}{\lambda})}\le\frac{1}{\varphi^{4\beta}(\frac{k}{\lambda+2})}.$$

Therefore,

$$\begin{aligned}|J_{\lambda,k,j}(g'')|\le&\tfrac{2^{3m}}{g(\frac{k}{\lambda+2})}\int_{I_{\lambda,k}}\int_{k/\lambda}^{a_{k+j}(t+j/\lambda)}(a_{k+j}(t+\tfrac{j}{\lambda})-u)dudt\\ =&\tfrac{2^{3m}}{2\,g(\frac{k}{\lambda+2})}\int_{I_{\lambda,k}}(a_{k+j}(t+\tfrac{j}{\lambda})-\tfrac{k}{\lambda})^2dt\\ \le&\tfrac{2^{3m}}{2\,g(\frac{k}{\lambda+2})}\int_{k/\lambda}^{(k+1)/\lambda}((t+\tfrac{j}{\lambda})-\tfrac{k}{\lambda})^2dt\\ \le&\tfrac{2^{3m}}{2\,g(\frac{k}{\lambda+2})}\int_{k/\lambda}^{(k+1)/\lambda}(\tfrac{1+j}{\lambda})^2dt\le\tfrac{9\cdot 2^{3m}}{2\,g(\frac{k}{\lambda+2})}\tfrac{1}{\lambda^3}.\end{aligned}$$

(B) Now assume $j=0$. First notice that, if $k\in\mathbb{N}$, then

$$\tfrac{2}{3}\le a_k<1. \tag{11}$$

In this case, taking into account (11), for $t\ge k/\lambda$,

$$\varphi^2(a_kt)=a_kt(1+a_kt)\ge\tfrac{2}{3}t(1+\tfrac{2t}{3})\ge\tfrac{4}{9}\varphi(\tfrac{k}{\lambda})\ge\tfrac{4}{9}\varphi(\tfrac{k}{\lambda+2}).$$

On the other hand, since

$$1+\tfrac{k}{\lambda}\le 2+\tfrac{k}{\lambda+2}\tfrac{\lambda+2}{\lambda}\le 2\Big(1+\tfrac{k}{\lambda+2}\Big),$$

one has

$$\frac{1}{\varrho(\frac{k}{\lambda})}\le\frac{2^m}{\varrho(\frac{k}{\lambda+2})}. \tag{12}$$

Moreover

$$(k+1)a_k = \tfrac{2k(k+1)}{2k+1} \geq \tfrac{k(2k+1)}{2k+1} = k.$$

Therefore

$$\begin{aligned}
&\int_{I_{\lambda,k}} \left| \int_{k/\lambda}^{a_k t} \tfrac{a_k t-u}{g(u)} du \right| dt = \\
&= \int_{k/\lambda}^{k/(\lambda a_k)} \int_{a_k t}^{k/\lambda} \tfrac{u-a_k t}{g(u)} du dt + \int_{k/(\lambda a_k)}^{(k+1)/\lambda} \int_{k/\lambda}^{a_k t} \tfrac{a_k t-u}{g(u)} du dt \\
&\leq \tfrac{C_1}{g(k/(\lambda+2))} \Big(\int_{k/n}^{k/(\lambda a_k)} \int_{a_k t}^{k/\lambda} (u-a_k t) du dt + \int_{k/(n a_k)}^{(k+1)/\lambda} \int_{k/\lambda}^{a_k t} (a_k t - u) du dt \Big) \\
&= \tfrac{C_1}{2g(\frac{k}{\lambda+2})} \Big(\int_{k/\lambda}^{k/(\lambda a_k)} (\tfrac{k}{\lambda} - a_k t)^2 dt + \int_{k/(\lambda a_k)}^{(k+1)/\lambda} (a_k t - \tfrac{k}{\lambda})^2 dt \Big) \\
&= \tfrac{C_1}{6a_k\, g(\frac{k}{\lambda+2})} \Big(\tfrac{k^3}{\lambda^3}(1-a_k)^3 + \Big(\tfrac{a_k(k+1)}{\lambda} - \tfrac{k}{\lambda} \Big)^3 \Big) \\
&\leq \tfrac{C_1}{4\, g(\frac{k}{\lambda+2})} \Big(\tfrac{k^3}{\lambda^3} \tfrac{1}{(2k+1)^3} + \Big(\tfrac{k+1}{\lambda} - \tfrac{k}{\lambda} \Big)^3 \Big) \leq \tfrac{C_2}{g(\frac{k}{\lambda+2})} \tfrac{1}{\lambda^3},
\end{aligned}$$

where we use (12). □

Proposition 8. *If $\lambda \geq 2$, $\beta \in [0,1)$, there exists a constant C such that, if $x > 0$, $k \in \{1,2\}$, and $\lambda x < 2/3$, then*

$$\varphi^{2\beta}(x) v_{\lambda,k}(x) \lambda \int_{k/\lambda}^{(k+1)/\lambda} \left| \int_x^{a_k t} \frac{(a_k t - s)^2 ds}{(\varrho \varphi^{3+2\beta}(s)} \right| dt \leq \tfrac{C(k+1)^2}{\lambda^2}.$$

Proof. Since $x < 2/(3\lambda) \leq a_k/\lambda \leq a_k k/\lambda$ and $0 < x < 1$,

$$\begin{aligned}
&\varphi^{2\beta}(x) v_{\lambda,k}(x) \lambda \int_{k/\lambda}^{(k+1)/\lambda} \int_x^{a_k t} \tfrac{(a_k t - s)^2 ds}{(\varrho \varphi^{3+2\beta}(s)} dt \leq \\
&\leq \tfrac{x^k}{\varphi^2(x)\sqrt{1+x}} \lambda \int_{k/\lambda}^{(k+1)/\lambda} (1+t)^m (t-x)^2 \int_x^t \tfrac{ds}{s^{1/2}} dt \\
&= \tfrac{2x^k}{\varphi^2(x)\sqrt{1+x}} \lambda \int_{k/\lambda}^{(k+1)/\lambda} (1+t)^m (t-x)^2 (\sqrt{t} - \sqrt{x}) dt \\
&\leq C_1 x^{k-1} \Big(1 + \tfrac{k+1}{\lambda}\Big)^m \lambda \int_{k/\lambda}^{(k+1)/\lambda} (t-x)^2 dt \leq C_2 \Big(\tfrac{k+1}{\lambda}\Big)^2.
\end{aligned}$$

□

Proposition 9. *If $m \geq 2$ and $\beta \in [0,1)$, there exists a constant C such that for $i \in \{1,2,3\}$, $\lambda \geq 2(1+2m)$, and $\lambda x \geq 2/3$,*

$$I_{\lambda,i}(x) \leq C \frac{(1+x)^m}{\varphi^{2\beta}(x)} \frac{\varphi^2(x)}{\lambda}.$$

where we use the notations in (3)–(5).

Proof. We will use the notation in (6). Notice that $4\beta < 4 \le 2m$, thus we can apply Proposition 6 with $\lambda \ge 2(1+2m) \ge 2(1+2m-4\beta)$.

Taking into account Hölder's inequality and using Proposition 6 and Proposition 3, we have

$$I_1(\lambda,x) = \sum_{k=1}^{\infty} \frac{v_{\lambda,k}(x)}{\varrho(\frac{k}{\lambda})\varphi^{2\beta}(\frac{k}{n})}\Big(\tfrac{k}{\lambda}-x\Big)^2$$
$$\le \sqrt{A_{\lambda,2,4\beta}(x)}\Big(\sum_{k=1}^{\infty} v_{\lambda,k}(x)\Big(\tfrac{k}{\lambda}-x\Big)^4\Big)^{1/2} \le C_1\frac{(1+x)^m}{\varphi^{2\beta}(x)}\frac{\varphi^2(x)}{\lambda}.$$

On the other hand

$$I_2(\lambda,x) = \tfrac{1+2x}{\lambda}\sum_{k=1}^{\infty}\Big|\tfrac{k}{\lambda}-x\Big|\tfrac{v_{\lambda,k}(x)}{\varrho(\frac{k}{\lambda})\varphi^{2\beta}(\frac{k}{n})}$$
$$\le \tfrac{1+2x}{\lambda}\sqrt{A_{\lambda,2,4\beta}(x)}\Big(\sum_{k=1}^{\infty}v_{\lambda,k}(x)(\tfrac{k}{n}-x)^2\Big)^{1/2}$$
$$\le C_2\tfrac{(1+x)^m}{\varphi^{2\beta}(x)}\tfrac{(1+x)}{\lambda}\sqrt{\tfrac{x(1+x)}{\lambda}} = C_2\tfrac{(1+x)^m}{\varphi^{2\beta}(x)}\tfrac{x(1+x)}{\lambda}\sqrt{\Big(1+\tfrac{1}{x}\Big)\tfrac{1}{\lambda}}$$
$$\le C_2\tfrac{(1+x)^m}{\varphi^{2\beta}(x)}\tfrac{\varphi^2(x)}{\lambda}\sqrt{\tfrac{1+\lambda}{\lambda}} \le \sqrt{2}C_2\tfrac{(1+x)^m}{\varphi^{2\beta}(x)}\tfrac{\varphi^2(x)}{\lambda}.$$

Finally

$$\frac{\lambda}{\varphi^2(x)}I_{\lambda,3}(x) = \sum_{k=1}^{\infty}\frac{v_{\lambda,k}(x)}{\varrho(\frac{k}{\lambda})\varphi^{2\beta}(\frac{k}{\lambda})} = A_{\lambda,1,2\beta}(x) \le C_3\frac{(1+x)^m}{\varphi^{2\beta}(x)}. \qquad \square$$

5. BERNSTEIN TYPE INEQUALITIES

THEOREM 2. *Suppose $\beta \in [0,1)$ and $m \ge 2$. There exists a constant C such that, if $\lambda \ge 2(1+2m)$ and $f \in C_{\varrho,\beta}[0,\infty)$, then*

$$\|\varphi^2 M''_\lambda(f)\|_{\varrho,\beta} \le C\,\lambda\,\|f\|_{\varrho,\beta}.$$

Proof. Taking into account the notations (3)–(5), it follows from (9), Proposition 5, and Proposition 9 that

$$\varrho(x)\varphi^{2+2\beta}(x)|M''_\lambda(f,x)| = \frac{\varrho(x)\lambda^3}{\varphi^{2(1-\beta)}(x)}\Big|\sum_{k=0}^{\infty}Q_{\lambda,k}(f)R_{\lambda,k}(x)v_{\lambda,k}(x)\Big|$$
$$\le \frac{\varrho(x)\lambda^2\|f\|_{\varrho,\beta}}{\varphi^{2(1-\beta)}(x)}\Big(I_1(\lambda,x)+I_2(\lambda,x)+I_3(\lambda,x)\Big)$$
$$\le C\lambda\,\|f\|_{\varrho,\beta}. \qquad \square$$

THEOREM 3. *Suppose $\beta \in [0,1)$ and $m \ge 2$. There exist a constant Λ_1 such that, if $\lambda \ge 2(1+2m)$ and $g \in D(\varrho,\beta)$, then*

$$\|\varphi^3 M'''_\lambda(g)\|_{\varrho,\beta} \le \Lambda_1\sqrt{\lambda}\,\|\varphi^2 g''\|_{\varrho,\beta}.$$

Proof. Set $Z(f) = \|\varphi^2 g''\|_{\varrho,\beta}$. From Propositions 2, 6 and 7 we obtain

$$
\begin{aligned}
&(\varrho\varphi^{3+2\beta})(x)|M_\lambda'''(g,x)| \le \\
&\le C_1\, Z(f)\, (\varrho\varphi^{1+2\beta})(x)\lambda^4 \sum_{k=1}^{\infty} \Big(\sum_{j=0}^{2} \tbinom{2}{j} \mid J_{n,k,j} \mid \Big| \tfrac{k}{\lambda+2} - x \Big| v_{\lambda+2,k}(x) \\
&\le C_2 Z(f)(\varrho\varphi^{1+2\beta})(x)\, \lambda \sum_{k=1}^{\infty} \Big| \tfrac{k}{\lambda+2} - x \Big| \tfrac{v_{\lambda+2,k}(x)}{\varrho(\frac{k}{\lambda+2})\varphi^{2+2\beta}(\frac{k}{\lambda+2})} \\
&\le C_2\, Z(f)\, (\varrho\varphi^{1+2\beta})(x)\lambda \sqrt{A_{\lambda+2,2,2(1+\beta)}(x)} \sqrt{V_{\lambda+2}((e_1-x)^2,x)} \\
&\le C_3\, Z(f)\, (\varrho\varphi^{1+2\beta})(x) \tfrac{(1+x)^m}{\varphi^{2+2\beta}(x)} \lambda \tfrac{\varphi(x)}{\sqrt{\lambda+2}} \le C_4\, Z(f)\, \sqrt{\lambda}.
\end{aligned}
$$

□

6. A VORONOVSKAYA TYPE THEOREM

We need a result given in Theorem 5.1 of [1].

THEOREM 4. *If $\beta \in [0,1]$ and $m \ge 2$, there exists a constant Λ_2 such that, for all $\lambda > 2(1+m)$ and every $f \in C_{\varrho,\beta}[0,\infty)$, one has*

$$\|M_\lambda(f)\|_{\varrho,\beta} \le \Lambda_2 \|f\|_{\varrho,\beta}.$$

Let

$$C^3_{\varrho,\beta}[0,\infty) = \Big\{ g \in C_{\varrho,\beta}[0,\infty) \,:\, g, g', g'' \in AC_{loc},\ \ \|\varphi^3 g'''\|_{\varrho,\beta} < \infty \Big\}.$$

THEOREM 5. *Suppose $\beta \in [0,1)$ and $m \ge 3$ (or $m = 2$ and $\beta \in [0,1/2]$). There exists a constant Λ_3 such that, if $g \in C^3_{\varrho,\beta}[0,\infty)$ and $\lambda \ge 2(1+2m)$, then*

$$\Big\| M_\lambda(g) - g - \tfrac{g''}{2} M_{\lambda,2} \Big\|_{\varrho,\beta} \le \tfrac{\Lambda_3}{\lambda^{3/2}} \|\varphi^3 g'''\|_{\varrho.\beta}\,.$$

Proof. Let us denote

$$R_\lambda(g,x) = M_\lambda(g,x) - g(x) - \tfrac{g''(x)}{2} M_\lambda((t-x)^2,x).$$

By Taylor's expansion

$$g(t) = g(x) + g'(x)(x-t) + \tfrac{1}{2} g''(x)(t-x)^2 + \tfrac{1}{2}\int_x^t g'''(s)(t-s)^2 ds,$$

one has

$$
\begin{aligned}
&\varphi^{2\beta}(x)\varrho(x) \mid R_\lambda(g,x) \mid = \\
&= \varphi^{2\beta}(x)\varrho(x) \Big| \tfrac{\lambda}{2} \sum_{k=0}^{\infty} v_{\lambda,k}(x) \Big(\int_{k/\lambda}^{(k+1)/\lambda} \int_x^{a_k t} (a_k t - s)^2\, g'''(s) ds \Big) dt \Big| \\
&\le\ \varphi^{2\beta}(x)\varrho(x) \|\varphi^3 g'''\|_{\varrho,\beta} \tfrac{\lambda}{2} \sum_{k=0}^{\infty} v_{\lambda,k}(x) \int_{k/\lambda}^{(k+1)/\lambda} \Big| \int_x^{a_k t} \tfrac{(a_k t - s)^2 ds}{(\varrho\varphi^{3+2\beta}(s))} \Big| dt \\
&= \tfrac{1}{2} \|\varphi^3 g'''\|_{\varrho,\beta}\, F_n(x),
\end{aligned}
$$

where

$$F_n(x)=\varphi^{2\beta}(x)\varrho(x)\lambda\sum_{k=0}^{\infty}v_{\lambda,k}(x)\int_{k/\lambda}^{(k+1)/\lambda}\left|\int_x^{a_kt}\frac{(a_kt-s)^2ds}{(\varrho\varphi^{3+2\beta}(s)}\right|dt.$$

Case 1. Assume $\lambda x<2/3$.
First we estimate the terms corresponding to $k=0,1,2$.
Since $a_0=0$, one has

$$\begin{aligned}\frac{(\varrho\varphi^{2\beta})(x)}{(1+x)^{\lambda}}\lambda\int_0^{1/\lambda}\int_0^x\frac{s^2ds}{(\varrho\varphi^{3+2\beta}(s)}dt &\le \varrho(x)x^{\beta}\int_0^x s^{(1-2\beta)/2}(1+s)^m ds\\ &\le \varrho(x)x^{\beta}(1+x)^m\int_0^x s^{(1-2\beta)/2}ds\\ &=\frac{2x^{\beta}x^{3/2-\beta}}{(3-2\beta)}\le\frac{2}{(3-2\beta)}\frac{1}{\lambda^{3/2}}.\end{aligned}$$

On the other hand, it follows from Proposition 8 that

$$(\varrho\varphi^{2\beta})(x)\lambda\sum_{k=1}^{2}v_{\lambda,k}(x)\int_{k/\lambda}^{(k+1)/\lambda}\left|\int_x^{a_kt}\frac{(a_kt-s)^2ds}{(\varrho\varphi^{3+2\beta}(s)}\right|dt\le\frac{C_1(2^2+3^2)}{\lambda^2}.$$

Now we consider the tail of the series. It is known that (see [3])

$$\frac{k}{\lambda}v_{\lambda,k}(x)=xv_{\lambda+1,k-1}(x).$$

In particular

$$\frac{k}{\lambda}\frac{k-1}{\lambda+1}\frac{k-2}{\lambda+2}v_{\lambda,k}(x)=x^3v_{\lambda+3,k-3}(x).$$

From this we obtain

$$\begin{aligned}&(\varrho\varphi^{2\beta})(x)\,\lambda\sum_{k=3}^{\infty}v_{\lambda,k}(x)\int_{k/\lambda}^{(k+1)/\lambda}\left|\int_x^{a_kt}\frac{(a_kt-s)^2ds}{(\varrho\varphi^{3+2\beta}(s)}\right|dt=\\ &=(\varrho\varphi^{2\beta})(x)\,\lambda\sum_{k=3}^{\infty}v_{\lambda,k}(x)\int_{k/\lambda}^{(k+1)/\lambda}\int_x^{a_kt}\frac{(a_kt-s)^2(1+s)^m\,ds}{\varphi^{3+2\beta}(s)}dt\\ &\le\frac{\lambda\,\varrho(x)}{\varphi^3(x)}\sum_{k=3}^{\infty}v_{\lambda,k}(x)\int_{k/\lambda}^{(k+1)/\lambda}\int_x^{t}(t-s)^2(1+s)^m\,ds\,dt\\ &\le\frac{\lambda\,\varrho(x)}{\varphi^3(x)}\sum_{k=1}^{\infty}\left(1+\frac{k+1}{\lambda}\right)^m v_{\lambda,k}(x)\int_{k/\lambda}^{(k+1)/\lambda}t^3dt\\ &\le\frac{C_2\,\varrho(x)}{\varphi^3(x)}\sum_{k=3}^{\infty}\left(1+\frac{k}{\lambda}\right)^m\frac{k^3}{\lambda^3}v_{\lambda,k}(x)\\ &\le\frac{C_3\,\varrho(x)}{\varphi^3(x)}\sum_{k=3}^{\infty}v_{\lambda,k}(x)\left(1+\frac{k}{\lambda+3}\right)^m\frac{k}{\lambda}\frac{k-1}{\lambda+1}\frac{k-2}{\lambda+2}\\ &\le C_4\,\varrho(x)\,x^{3/2}\sum_{k=3}^{\infty}v_{\lambda+3,k-3}(x)\left(1+\frac{k-3}{\lambda+3}\right)^m\le C_5\,x^{3/2}\le\frac{C_6}{\lambda^{3/2}},\end{aligned}$$

where we use Proposition 6, with $q=0$ and $p=1$.

Case 2. Assume $\lambda x \geq 2/3$ and set $c = m - 3/2 - \beta$.
From Proposition 3.3 of [3] we know that

$$\left|\int_x^{a_k t} \frac{(a_k t - u)^2 du}{(\varrho\varphi^{3+2\beta}(u)}\right| = \left|\int_x^{a_k t} \frac{(a_k t-u)^2(1+u)^c du}{u^{(3+2\beta)/2}}\right|$$
$$\leq \frac{|a_k t - x|^3}{(3-(1+2\beta)/2)x^{(3+2\beta)/2}}\Big((1+x)^c + (1+t)^c\Big)$$

If $\lambda x \geq 2/3$, since $3/2 + \beta < 5/2 < 3$, one has

$$\begin{aligned}
F_n(x) =&(\varrho\varphi^{2\beta})(x)\,\lambda \sum_{k=0}^{\infty} v_{\lambda,k}(x) \int_{k/\lambda}^{(k+1)/\lambda} \left|\int_x^{a_k t} \frac{(a_k t-u)^2 du}{(\varrho\varphi^{3+2\beta}(u)}\right| dt \\
\leq& \frac{C_1\varphi^{2\beta}(x)\varrho(x)\lambda}{x^{(3+2\beta)/2}} \sum_{k=0}^{\infty} v_{\lambda,k}(x) \int_{k/\lambda}^{(k+1)/\lambda} \mid a_k t - x \mid^3 \Big((1+x)^c + (1+t)^c\Big) dt \\
=& \frac{C_1\varphi^{2\beta}(x)\varrho(x)(1+x)^c}{x^{(3+2\beta)/2}} M_{\lambda,3}(x) \\
&+ \frac{C_1\varphi^{2\beta}(x)\varrho(x)}{x^{(3+2\beta)/2}}\,\lambda \sum_{k=0}^{\infty} v_{\lambda,k}(x) \int_{k/\lambda}^{(k+1)/\lambda} |a_k t - x|^3 (1+t)^c dt \\
=& \frac{C_1}{\varphi^3(x)} M_{\lambda,3}(x) \\
&+ \frac{C_1(1+x)^\beta}{x^{3/2}(1+x)^m}\,\lambda \sum_{k=0}^{\infty} v_{\lambda,k}(x) \int_{k/\lambda}^{(k+1)/\lambda} |a_k t - x|^3 (1+t)^c dt.
\end{aligned}$$

It follows from Proposition 4 that

$$\frac{1}{\varphi^3(x)} M_\lambda(\mid t - x \mid^3, x) \leq \frac{C_2}{\lambda^{3/2}}.$$

For the other terms we first estimate the case $k = 0$. Notice that, for $t \in (0, \frac{1}{\lambda})$,

$$(1+t)^c \leq (1 + \tfrac{1}{\lambda})^c \leq (1 + \tfrac{3x}{2})^c \leq 2^c(1+x)^c.$$

Here the condition $m \geq 3$ was used. Therefore

$$\begin{aligned}
\frac{(1+x)^\beta}{x^{3/2}(1+x)^m}\,\lambda v_{\lambda,0}(x) \int_0^{1/\lambda} x^3 (1+t)^c dt &\leq \frac{2^c(1+x)^\beta x^3}{x^{3/2}(1+x)^m}\, v_{\lambda,0}(x)(1+x)^c \\
&= \frac{2^c x^3}{x^{3/2}(1+x)^{3/2}}\, v_{\lambda,0}(x) \\
&\leq \frac{2^c}{\varphi^3(x)} V_\lambda(\mid t - x \mid^3, x) \leq \frac{C_3}{\lambda^{3/2}}.
\end{aligned}$$

On the other hand, since

$$\begin{aligned}
&\int_{k/\lambda}^{(k+1)/\lambda} \mid a_k t - x \mid^3 (1+t)^c dt \leq \\
&\leq \Big(\int_{k/\lambda}^{(k+1)/\lambda} (a_k t - x)^6 dt\Big)^{1/2} \Big(\int_{k/\lambda}^{(k+1)/\lambda} (1+t)^{2c} dt\Big)^{1/2},
\end{aligned}$$

from Theorem 4 and Proposition 4 we obtain

$$\begin{aligned}
&\tfrac{(1+x)^\beta}{x^{3/2}(1+x)^m}\,\lambda\sum_{k=1}^{\infty}v_{\lambda,k}(x)\int_{k/\lambda}^{(k+1)/\lambda}\mid a_k t-x\mid^3(1+t)^c dt\leq\\
&\leq\tfrac{(1+x)^\beta}{x^{3/2}(1+x)^m}\sqrt{M_\lambda((1+t)^{2c},x)}\sqrt{M_\lambda((e_1-x)^6,x)}\\
&\leq\tfrac{C_4(1+x)^\beta}{x^{3/2}(1+x)^m}\,(1+x)^{m-3/2-\beta}\tfrac{\varphi^3(x)}{\lambda^{3/2}}=\tfrac{C_4}{\lambda^{3/2}}.
\end{aligned}$$

This completes the proof. □

Remark 2. We do not know if Theorem 5 holds $m=2$ and $\beta\in(\frac{1}{2},1)$.

7. INVERSE RESULT

Theorem 6. *Suppose $\beta\in[0,1)$ and $m\geq 3$ (or $m=2$ and $\beta\in[0,1/2]$). There exist positive constants κ and Λ_4 such that, if $f\in C_\varrho[0,\infty)$ and $\lambda\geq 2(1+2m)$, then*

$$\tfrac{1}{\lambda}\left\|\varphi^2(M^2_\lambda(f))''\right\|_{\varrho,\beta}\leq\Lambda_3\Big(\|M_\lambda(f)-f\|_{\varrho,\beta}+\|M_{\kappa\,\lambda}(f)-f\|_{\varrho,\beta}\Big),$$

where $M^2_\lambda(f)=M_\lambda(M_\lambda(f))$.

Proof. Let Λ_1 and Λ_3 be constants such that the inequalities in Theorem 3 and Theorem 5 hold for $\lambda\geq 2(1+2m)$. Set

$$\mu=16(\Lambda_1\Lambda_3)^2\,\lambda.$$

If $\lambda\geq 2(1+2m)$ and $g=M^2_\lambda(f)$, from Theorems 2 to 5 we know that

$$\begin{aligned}
&\tfrac{1}{2\mu}\left\|\varphi^2(M^2_\lambda)''f\right\|_{\varrho,\beta}\leq\\
&\leq\tfrac{1}{2}\|M_{\mu,2}(M^2_\lambda)''f\|_{\varrho,\beta}\\
&\leq\left\|M_\mu(M^2_\lambda f)-M^2_\lambda f\right\|_{\varrho,\beta}+\left\|M_\mu(M^2_\lambda f)-M^2_\lambda f-\tfrac{M_{\mu,2}}{2}(M^2_\lambda f)''\right\|_{\varrho,\beta}\\
&\leq\|M_\mu(M^2_\lambda f-M_\lambda f)+M_\mu(M_\lambda f-f)+M_\mu f-f+f-M_\lambda f+M_\lambda f-M^2_\lambda f\|_{\varrho,\beta}\\
&\quad+\tfrac{\Lambda_3}{\mu^{3/2}}\,\|\varphi^3(M^2_\lambda f)'''\|_{\varrho,\beta}\\
&\leq C_1\Big(\|M_\mu f-f\|_{\varrho,\beta}+\|M_\lambda f-f\|_{\varrho,\beta}\Big)+\tfrac{\Lambda_1\Lambda_3\sqrt{\lambda}}{\mu^{3/2}}\,\|\varphi^2 M''_\lambda(f)\|_{\varrho,\beta}\\
&\leq C_1\Big(\|M_\mu f-f\|_{\varrho,\beta}+\|M_\lambda f-f\|_{\varrho,\beta}\Big)\\
&\quad+\tfrac{\Lambda_1\Lambda_3\sqrt{\lambda}}{\mu^{3/2}}\,\|\varphi^2 M''_\lambda(f-M_\lambda(f))\|_{\varrho,\beta}+\tfrac{\Lambda_1\Lambda_3\sqrt{\lambda}}{\mu^{3/2}}\,\|\varphi^2 M''_\lambda(M_\lambda(f))\|_{\varrho,\beta}\\
&\leq C_1\Big(\|M_\mu f-f\|_{\varrho,\beta}+\|M_\lambda f-f\|_{\varrho,\beta}\Big)\\
&\quad+\tfrac{C\Lambda_1\Lambda_3\,\lambda^{3/2}}{\mu^{3/2}}\,\|f-M_\lambda(f)\|_{\varrho,\beta}+\tfrac{1}{4\,\mu}\,\|\varphi^2 M''_\lambda(M_\lambda(f))\|_{\varrho,\beta}.
\end{aligned}$$

Therefore

$$\frac{1}{4\mu}\|\varphi^2(M_\lambda^2)''f\|_{\varrho,\beta} \le C_2\Big(\|M_\mu f - f\|_{\varrho,\beta} + \|M_\lambda f - f\|_{\varrho,\beta}\Big),$$

and it is sufficient to prove the result, because

$$\frac{1}{\lambda}\left\|\varphi^2(M_\lambda^2(f))''\right\|_{\varrho,\beta} = \frac{2^6(\Lambda_1\Lambda_2)^2}{4\mu}\left\|\varphi^2(M_\lambda^2(f))''\right\|_{\varrho,\beta}. \qquad \square$$

THEOREM 7. *Suppose* $\beta \in [0,1)$, $m \ge 3$ (*or* $m = 2$ *and* $\beta \in [0, \frac{1}{2}]$), *and* κ *is given as in Theorem 6. There exists a constant* C *such that, if* $f \in C_\varrho[0,\infty)$ *and* $\lambda \ge 2(1+2m)$, *then*

$$K_\beta\left(f, \frac{1}{\lambda}\right)_\varrho \le C\Big(\|M_\lambda(f) - f\|_{\varrho,\beta} + \|M_{\kappa\lambda}(f) - f\|_{\varrho,\beta}\Big),$$

where $K_\beta(f,t)_\varrho$ *is defined as in* (1).

Proof. Fix $f \in C_\varrho[0,\infty)$ and denote $g = M_\lambda^2(f) = M_\lambda(M_\lambda(f))$. From Theorem 4 we know that $g \in C_{\varrho,b}[0,\infty)$.

From the definition of the K-functional $K_\beta(f,t)_\varrho$, Theorem 4 and Theorem 6 we know that

$$\begin{aligned}
K_\beta\left(f, \frac{1}{\lambda}\right)_\varrho \le& \|f - M_\lambda^2(f)\|_{\varrho,\beta} + \frac{1}{\lambda}\|\varphi^2(M_\lambda^2(f))''\|_{\varrho,\beta}\\
\le& \|f - M_\lambda(f)\|_{\varrho,\beta} + \|M_\lambda(f - M_\lambda(f))\|_{\varrho,\beta} + \frac{1}{\lambda}\left\|\varphi^2(M_\lambda^2(f))''\right\|_{\varrho,\beta}\\
\le& (1+\Lambda_2)\big\|f - M_\lambda(f)\big\|_{\varrho,\beta} + \frac{1}{\lambda}\left\|\varphi^2(M_\lambda^2(f))''\right\|_{\varrho,\beta}\\
\le& C\Big(\|M_\lambda(f) - f\|_{\varrho,\beta} + \|M_{\kappa,\lambda}(f) - f\|_{\varrho,\beta}\Big). \qquad \square
\end{aligned}$$

REMARK 3. In the case $\beta = 0$ the K-functional can be replaced by a weighted modulus of smoothness as in [4]. For the classical Baskakov operators V_n related results were given [6] and [5]. □

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