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WAVELET BI-FRAMES ON LOCAL FIELDS

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Abstract. In this paper, we introduce the notion of periodic wavelet bi-frames on local fields and establish the theory for the construction of periodic Bessel sequences and periodic wavelet bi-frames on local fields.

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1. INTRODUCTION

Duffin and Schaeffer [14] introduced the concept of frame in separable Hilbert space while dealing with some deep problems in non-harmonic Fourier series. Frames are basis-like systems that span a vector space but allow for linear dependency, which can be used to reduce noise, find sparse representations, or obtain other desirable features unavailable with orthonormal bases.

During the last two decades, there is a substantial body of work that has been concerned with the construction of wavelets on local fields. Even though the structures and metrics of local fields of zero and positive characteristics are similar, their wavelet and MRA (multiresolution analysis) theory are quite different. For example, R. L. Benedetto and J. J. Benedetto [12] developed a wavelet theory for local fields and related groups. They did not develop the multiresolution analysis (MRA) approach, their method is based on the theory of wavelet sets and only allows the construction of wavelet functions whose Fourier transforms are characteristic functions of some sets. Khrennikov, Shelkovich and Skopina [21] constructed a number of scaling functions generating an MRA of $L^2(\mathbb{Q}_p)$. But later on in [10], Albeverio, Evdokimov and Skopina proved that all these scaling functions lead to the same Haar MRA and that there exist no other orthogonal test scaling functions generating an MRA except those described in [21]. Some wavelet bases for $L^2(\mathbb{Q}_p)$ different from the Haar system were constructed in [9, 15]. These wavelet bases

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were obtained by relaxing the basis condition in the definition of an MRA and form Riesz bases without any dual wavelet systems. For some related works on wavelets and frames on $\mathbb{Q}_p$, we refer to [11, 20, 22, 23]. On the other hand, Lang [24, 25, 26] constructed several examples of compactly supported wavelets for the Cantor dyadic group. Farkov [16, 17] has constructed many examples of wavelets for the Vilenkin p-groups. Jiang *et al.* [18] pointed out a method for constructing orthogonal wavelets on local field $\mathbb{K}$ with a constant generating sequence and derived necessary and sufficient conditions for a solution of the refinement equation to generate a multiresolution analysis of $L^2(\mathbb{K})$. In the series of papers [1, 2, 3, 4, 5, 6, 7, 8, 30, 31, 32, 33], we have obtained various results related to wavelet and Gabor frames on local fields.

The study of periodic bi-frames was carried by Li and Jia [35] but the parallel development on local fields is not reported yet. In this paper, we introduce the notion of periodic wavelet bi-frames on local field of positive characteristic and establish the theory for the construction of periodic wavelet bi-frames on local fields.

The rest of the article is structured as follows. In Section 2, we discuss the preliminaries of local fields and some basic definitions which plays vital role in the rest of the paper. In Section 3, we establish some results related to periodic Bessel sequences on local fields of positive characteristic. Section 4 is devoted to the construction of periodic wavelet bi-frames on local fields.

2. PRELIMINARIES ON LOCAL FIELDS

Let K be a field and a topological space. Then K is called a *local field* if both K^+ and K^* are locally compact Abelian groups, where K^+ and K^* denote the additive and multiplicative groups of K, respectively. If K is any field and is endowed with the discrete topology, then K is a local field. Further, if K is connected, then K is either $\mathbb{R}$ or $\mathbb{C}$. If K is not connected, then it is totally disconnected. Hence by a local field, we mean a field K which is locally compact, non-discrete and totally disconnected. The p-adic fields are examples of local fields. In the rest of this paper, we use the symbols $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural, non-negative integers and integers, respectively.

Let K be a local field. Let dx be the Haar measure on the locally compact Abelian group K^+. If $\alpha \in K$ and $\alpha \neq 0$, then $d(\alpha x)$ is also a Haar measure. Let $d(\alpha x) = |\alpha| dx$. We call $|\alpha|$ the *absolute value* of α. Moreover, the map $x \to |x|$ has the following properties: (a) $|x| = 0$ if and only if $x = 0$; (b) $|xy| = |x|\cdot|y|$ for all $x, y \in K$; and (c) $|x+y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$. Property (c) is called the *ultrametric inequality.* The set $\mathfrak{D} = \{x \in K : |x| \leq 1\}$ is called the *ring of integers* in K. Define $\mathfrak{B} = \{x \in K : |x| < 1\}$. The set $\mathfrak{B}$ is called the *prime ideal* in K. The prime ideal in K is the unique maximal ideal in $\mathfrak{D}$ and hence as result $\mathfrak{B}$ is both principal and prime. Since the local field K is totally disconnected, so there exist an element of $\mathfrak{B}$ of maximal absolute value. Let $\mathfrak{p}$ be a fixed element of maximum absolute value in $\mathfrak{B}$. Such an element is called a *prime element* of K. Therefore, for such an ideal $\mathfrak{B}$ in $\mathfrak{D}$, we have $\mathfrak{B} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$. As it was proved in [34], the set $\mathfrak{D}$ is compact and open. Hence, $\mathfrak{B}$ is compact and open. Therefore, the residue space $\mathfrak{D}/\mathfrak{B}$ is isomorphic to a finite field $GF(q)$, where $q = p^k$ for some prime p and $k \in \mathbb{N}$.

Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{B} = \{x \in K : |x| = 1\}$. Then, it can be proved that $\mathfrak{D}^*$ is a group of units in K^* and if $x \neq 0$, then we may write $x = \mathfrak{p}^k x', x' \in \mathfrak{D}^*$. For a proof of this fact we refer to [18]. Moreover, each $\mathfrak{B}^k = \mathfrak{p}^k\mathfrak{D} = \{x \in K : |x| < q^{-k}\}$ is a compact subgroup of K^+ and usually known as the *fractional ideals* of K^+. Let $\mathcal{U} = \{c_i\}_{i=0}^{q-1}$ be any fixed full set of coset representatives of $\mathfrak{B}$ in $\mathfrak{D}$, then every element $x \in K$ can be expressed uniquely as $x = \sum_{\ell=k}^{\infty} c_\ell \mathfrak{p}^\ell$ with $c_\ell \in \mathcal{U}$. Let χ be a fixed character on K^+ that is trivial on $\mathfrak{D}$ but is non-trivial on $\mathfrak{B}^{-1}$. Therefore, χ is constant on cosets of $\mathfrak{D}$ so if $y \in \mathfrak{B}^k$, then $\chi_y(x) = \chi(yx), x \in K$. Suppose that χ_u is any character on K^+, then clearly the restriction $\chi_u|\mathfrak{D}$ is also a character on $\mathfrak{D}$. Therefore, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathfrak{D}$ in K^+, then, as it was proved in [34], the set $\{\chi_{u(n)} : n \in \mathbb{N}_0\}$ of distinct characters on $\mathfrak{D}$ is a complete orthonormal system on $\mathfrak{D}$.

The Fourier transform $\hat{f}$ of a function $f \in L^1(K) \cap L^2(\mathbb{K})$ is defined by

$$\hat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}\, dx. \tag{2.1}$$

It is noted that

$$\hat{f}(\xi) = \int_K f(x)\, \overline{\chi_\xi(x)} dx = \int_K f(x)\chi(-\xi x)\, dx.$$

Furthermore, the properties of Fourier transform on local field K are much similar to those of on the real line. In particular Fourier transform is unitary on $L^2(\mathbb{K})$. Also, if $f \in L^2(\mathfrak{D})$, then we define the Fourier coefficients of f as

$$\hat{f}(u(n)) = \int_{\mathfrak{D}} f(x)\overline{\chi_{u(n)}(x)}\, dx. \tag{2.2}$$

The series $\sum_{n\in\mathbb{N}_0} \hat{f}(u(n))\chi_{u(n)}(x)$ is called the Fourier series of f. From the standard L^2-theory for compact Abelian groups, we conclude that the Fourier series of f converges to f in $L^2(\mathfrak{D})$ and Parseval's identity holds:

$$\|f\|_2^2 = \int_{\mathfrak{D}} |f(x)|^2 dx = \sum_{n\in\mathbb{N}_0} \left|\hat{f}(u(n))\right|^2. \tag{2.3}$$

We now impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$. We have $\mathfrak{D}/\mathfrak{B} \cong GF(q)$ where $GF(q)$ is a c-dimensional vector space over the field $GF(p)$. We choose a set $\{1 = \zeta_0, \zeta_1, \zeta_2, \dots, \zeta_{c-1}\} \subset \mathfrak{D}^*$ such that span $\{\zeta_j\}_{j=0}^{c-1} \cong GF(q)$. For $n \in \mathbb{N}_0$ satisfying

$$0 \le n < q,\ n = a_0 + a_1 p + \cdots + a_{c-1}p^{c-1},\ 0 \le a_k < p,\ \text{and } k = 0, 1, \dots, c-1,$$

we define

$$u(n) = (a_0 + a_1\zeta_1 + \cdots + a_{c-1}\zeta_{c-1})\, \mathfrak{p}^{-1}. \tag{2.4}$$

Also, for $n=b_0+b_1q+b_2q^2+\cdots+b_sq^s,\ n\in\mathbb{N}_0,\ 0\le b_k<q, k=0,1,2,\ldots,s$, we set

$$u(n)=u(b_0)+u(b_1)\mathfrak{p}^{-1}+\cdots+u(b_s)\mathfrak{p}^{-s}. \tag{2.5}$$

This defines $u(n)$ for all $n\in\mathbb{N}_0$. In general, it is not true that $u(m+n)=u(m)+u(n)$. But, if $r,k\in\mathbb{N}_0$ and $0\le s<q^k$, then $u(rq^k+s)=u(r)\mathfrak{p}^{-k}+u(s)$. Further, it is also easy to verify that $u(n)=0$ if and only if $n=0$ and $\{u(\ell)+u(k):k\in\mathbb{N}_0\}=\{u(k):k\in\mathbb{N}_0\}$ for a fixed $\ell\in\mathbb{N}_0$. Hereafter we use the notation $\chi_n=\chi_{u(n)},\ n\ge 0$.

Let the local field K be of characteristic $p>0$ and $\zeta_0,\zeta_1,\zeta_2,\ldots,\zeta_{c-1}$ be as above. We define a character χ on K as follows:

$$\chi(\zeta_\mu\mathfrak{p}^{-j})=\begin{cases}\exp(2\pi i/p), & \mu=0 \text{ and } j=1,\\ 1, & \mu=1,\ldots,c-1 \text{ or } j\ne 1.\end{cases} \tag{2.6}$$

We also denote the test function space on K by $\Omega(K)$, that is, each function f in $\Omega(K)$ is a finite linear combination of functions of the form $\mathbf{1}_k(x-h),\ h\in K,\ k\in\mathbb{Z}$, where $\mathbf{1}_k$ is the characteristic function of $\mathfrak{B}^k$. This class of functions can also be described in the following way. A function $g\in\Omega(K)$ if and only if there exist integers k,ℓ such that g is constant on cosets of $\mathfrak{B}^k$ and is supported on $\mathfrak{B}^\ell$. It follows that Ω is closed under Fourier transform and is an algebra of continuous functions with compact support, which is dense in $\mathcal{C}_0(K)$ as well as in $L^p(K),\ 1\le p<\infty$.

For $j\in\mathbb{N}_0$, let $\mathcal{N}_j$ denote a full collection of coset representatives of $\mathbb{N}_0/q^j\mathbb{N}_0$, *i.e.*,

$$\mathcal{N}_j=\left\{0,1,2,\ldots,q^j-1\right\},\quad j\ge 0.$$

Then, $\mathbb{N}_0=\bigcup_{n\in\mathcal{N}_j}\left(n+q^j\mathbb{N}_0\right)$, and for any distinct $n_1,n_2\in\mathcal{N}_j$, we have $\left(n_1+q^j\mathbb{N}_0\right)\cap\left(n_2+q^j\mathbb{N}_0\right)=\emptyset$. Thus, every non-negative integer k can uniquely be written as $k=rq^j+s$, where $r\in\mathbb{N}_0,\ s\in\mathcal{N}_j$. Further, a bounded function $W:\mathbb{K}\to\mathbb{K}$ is said to be a radial decreasing L^1-majorant of $f(x)\in L^2(\mathbb{K})$ if $|f(x)|\le W(x)$, $W\in L^1(\mathbb{K})$, and $W(0)<\infty$.

For $j\in\mathbb{Z}$ and $y\in\mathbb{K}$, we define the dilation δ_j and the translation operators T_y as follows:

$$D_jf(x)=q^{j/2}f\left(\mathfrak{p}^{-j}x\right)\quad\text{and}\quad T_yf(x)=f(x-y),\quad f\in L^2(\mathbb{K}).$$

For an arbitrary measurable function f, we define

$$f^{\mathrm{per}}(x)=\sum_{k\in\mathbb{N}_0}f(x+u(k))$$

and

$$f_{j,k}(x)=q^{j/2}f\left(\mathfrak{p}^{-j}x-u(k)\right)\ \text{ for } j\in\mathbb{Z},\ k\in\mathbb{N}_0.$$

In particular, we define

$$f_{j,k}^{\mathrm{per}}(x)=\sum_{s\in\mathbb{N}_0}f_{j,k}\left(x+u(s)\right)\ \text{ for } j\in\mathbb{Z} \text{ and } k\in\mathbb{N}_0.$$

For a finite subset E of $L^2(\mathbb{K})$, we write

$$X(E) = \{f_{j,k} : f \in E,\ j \in \mathbb{Z},\ k \in \mathbb{N}_0\}, \tag{2.7}$$

$$X^{\mathrm{per}}(E) = \{1, f^{\mathrm{per}}_{j,k} : f \in E,\ j \in \mathbb{N}_0,\ k \in \mathcal{N}_j\}. \tag{2.8}$$

we require that k in (2.8) belongs to $\mathcal{N}_j$ instead of $\mathbb{N}_0$. Otherwise, every $f^{\mathrm{per}}_{j,k}$ with $k \in \mathcal{N}_j$ will repeat infinitely many times since $f^{\mathrm{per}}_{j,k} = f_{j,k+\mathfrak{p}^{-j}u(s)}$ for $s \in \mathbb{N}_0$, and thus we cannot create a new frame. The restrictions on j and k in (2.8) are also related to our method for the construction of frames.

Let $\Psi = \{\psi_\ell : 1 \le \ell \le L\}$ and $\widetilde{\Psi} = \left\{\widetilde{\psi}_\ell : 1 \le \ell \le L\right\}$ be two finite subsets of $L^2(\mathbb{K})$ with the same cardinality. A *bi-frame* for $L^2(\mathbb{K})$ of the form $(X(\Psi), X(\widetilde{\Psi}))$ is called a *wavelet bi-frame* for $L^2(\mathbb{K})$, *i.e.*, $X(\Psi)$ and $X(\widetilde{\Psi})$ are two frames for $L^2(\mathbb{K})$ satisfying

$$f = \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0} \left\langle f, \widetilde{\psi}_{\ell,j,k}\right\rangle \psi_{\ell,j,k} \text{ for } f \in L^2(\mathbb{K}).$$

In the similar manner, a bi-frame for $L^2(\mathfrak{D})$ of the form $\left(X^{\mathrm{per}}(\Psi), X^{\mathrm{per}}(\widetilde{\Psi})\right)$ is called a *periodic wavelet bi-frame*.

A function $f \in L^1(K) \cap L^2(\mathbb{K})$ is said to be $\mathfrak{p}$-refinable if there exists a periodic measurable function m_f on K such that

$$\widehat{f}(\mathfrak{p}^{-1}\xi) = m_f(\xi)\widehat{f}(\xi),$$

where m_f is called a *symbol* of f.

Proposition 2.1. *Suppose ψ_0 and $\widetilde{\psi}_0$ are two $\mathfrak{p}$-refinable functions with symbols $H_0, \widetilde{H}_0 \in L^\infty(\mathfrak{D})$, $\widehat{\psi}_0$ and $\widehat{\widetilde{\psi}}_0$ are continuous at the origin with $\widehat{\psi}_0(0) = \widehat{\widetilde{\psi}}_0(0) = 1$ and*

$$\sum_{s\in\mathbb{N}_0}\left|\widehat{\psi_0}(\xi + u(s))\right|^2, \sum_{s\in\mathbb{N}_0}\left|\widehat{\widetilde{\psi}_0}(\xi + u(s))\right|^2 \in L^\infty(\mathfrak{D}).$$

If $\Psi = \{\psi_\ell : 1 \le \ell \le L\}$ and $\widetilde{\Psi} = \left\{\widetilde{\psi}_\ell : 1 \le \ell \le L\right\}$ satisfies

(a)

$$\widehat{\psi}_\ell(\mathfrak{p}^{-1}\xi) = H_\ell(\xi)\widehat{\psi_0}(\xi),\ \widehat{\widetilde{\psi}}_\ell(\mathfrak{p}^{-1}\xi) = \widetilde{H}_\ell\widehat{\widetilde{\psi}_0}(\xi) \tag{2.9}$$

with $H_\ell, \widetilde{H}_\ell \in L^\infty(\mathfrak{D})$,

(b) $X(\Psi)$ and $X(\widetilde{\Psi})$ are Bessel sequences in $L^2(\mathbb{K})$, and

(c)

$$\sum_{\ell=0}^{L} H_\ell(\xi)\overline{\widetilde{H}_\ell\left(\xi + \mathfrak{p}\gamma\right)} = \delta_{0,\gamma} \tag{2.10}$$

for a.e. $\xi \in K$ *and each* $\gamma \in \mathcal{N}_1$*, then* $\big(X(\Psi), X(\widetilde{\Psi})\big)$ *is a wavelet bi-frame for* $L^2(\mathbb{K})$*.*

PROPOSITION 2.2. *Suppose* ψ_0 *and* $\widetilde{\psi}_0$ *are two* $\mathfrak{p}$*-refinable functions with symbols* $H_0, \widetilde{H}_0 \in L^\infty(\mathfrak{D})$*,* $\widehat{\psi}_0$ *and* $\widehat{\widetilde{\psi}_0}$ *are continuous at the origin with* $\widehat{\psi}_0(0) = \widehat{\widetilde{\psi}_0}(0) = 1$ *and*

$$\sum_{s\in\mathbb{N}_0} \left|\widehat{\psi_0}(\xi + u(s))\right|^2, \sum_{s\in\mathbb{N}_0} \left|\widehat{\widetilde{\psi}_0}(\xi + u(s))\right|^2 \in L^\infty(\mathfrak{D}).$$

If $\Psi = \{\psi_\ell : 1 \le \ell \le L\}$ *and* $\widetilde{\Psi} = \left\{\widetilde{\psi}_\ell : 1 \le \ell \le L\right\}$ *satisfies*

(a)

$$\widehat{\psi}_\ell(\mathfrak{p}^{-1}\xi) = H_\ell(\xi)\widehat{\psi_0}(\xi),\ \ \widehat{\widetilde{\psi}}_\ell(\mathfrak{p}^{-1}\xi) = \widetilde{H_\ell}\widehat{\widetilde{\psi}_0}(\xi)$$

with $H_\ell, \widetilde{H}_\ell \in L^\infty(\mathfrak{D})$*,*

(b) $X(\Psi)$ *and* $X(\widetilde{\Psi})$ *are Bessel sequences in* $L^2(\mathbb{K})$*, and*

(c)

$$H_0(\xi)\overline{\widetilde{H}_0\,(\xi + \mathfrak{p}\gamma)}\varphi(\mathfrak{p}^{-1}\xi) + \sum_{\ell=1}^{L} H_\ell(\xi)\overline{\widetilde{H}_\ell\,(\xi + \mathfrak{p}\gamma)} = \varphi(\xi)\delta_{0,\gamma} \tag{2.11}$$

for a.e. $\xi \in \mathbb{K}$ *and each* $\gamma \in \mathcal{N}_1$*, where* φ *is a periodic measurable function which is positively bounded from below and above and continuous at the origin with* $\varphi(0) = 1$*,*
then $\big(X(\Psi), X(\widetilde{\Psi})\big)$ *is a wavelet bi-frame for* $L^2(\mathbb{K})$*.*

DEFINITION 2.3. *A function* $\rho : \mathbb{K} \to \mathbb{K}$ *is called a quasi-norm if the following conditions hold:*

(i) $\rho(x) = 0$ *if and only if* $x = 0$*;*

(ii) there exists $c_0 > 0$ *such that* $\rho(x + y) \le c_0\,(\rho(x) + \rho(y))$ *for* $x, y \in \mathbb{K}$*;*

(iii) $\rho(\mathfrak{p}^{-1}x) = q\rho(x)$*;*

(iv) ρ *is continuous on* $\mathbb{K}$ *and smooth on* $\mathbb{K}\backslash\{0\}$*;*

(v) there exist $c_1, \alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ *such that*

$$c_1{}^{-1}|x|^{\alpha_1} \le \rho(x) \le c_1|x|^{\beta_1} \text{ if } x \in \mathfrak{B},$$

$$c_1{}^{-1}|x|^{\alpha_2} \le \rho(x) \le c_1|x|^{\beta_2} \text{ if } x \notin \mathfrak{B}.$$

DEFINITION 2.4. *An atmost countable collection* $\{S_i : i \in \mathcal{I}\}$ *of measurable sets is called a partition of a measurable set* S *if* $S = \cup_{i\in\mathcal{I}} S_i$ *and* $S_i \cap S_{i'} = \phi$ *in* $\mathcal{I}$*. Two measurable sets* S *and* S' *in* $L^2(\mathbb{K})$ *are said to be* $q\mathbb{N}_0$*-congruent if there exists a partition* $\{S_k : k \in \mathbb{N}_0\}$ *of* S *such that* $\{S_k + qu(k) : k \in \mathbb{N}_0\}$ *is a partition of* S'*.*

3. PERIODIC BESSEL SEQUENCES ON LOCAL FIELDS

In this section, we establish a Bessel sequence in $L^2(\mathfrak{D})$ from a Bessel sequence in $L^2(\mathbb{K})$ of the form $\{g_{m,n} : m, n \in \mathbb{N}_0\}$. For that purpose, we first introduce a Banach space $\mathcal{L}^p(\mathbb{K})$ with $1 \leq p \leq \infty$. The space $\mathcal{L}^p$ was introduced on $\mathbb{R}^d$ by Jia and Micchelli [19].

For $1 \leq p \leq \infty$ and a measurable f on $\mathbb{K}$, we define

$$|f|_p = \left\| \sum_{k\in\mathbb{N}_0} |f(\cdot + u(k))| \right\|_{L^p(\mathfrak{D})},$$

and we write

$$\mathcal{L}^p(\mathbb{K}) = \{f : |f|_p < \infty\}.$$

Then $\mathcal{L}^p(\mathbb{K})$ is a Banach space. It is well known that $\mathcal{L}^{p_2}(\mathbb{K}) \subset \mathcal{L}^{p_1}(\mathbb{K})$ if $1 \leq p_1 \leq p_2 \leq \infty$.

LEMMA 3.1. *For $f \in L^2(\mathbb{K})$, we have $f \in \mathcal{L}^2(\mathbb{K})$ if there exists $\alpha > 1$ such that $|f(x)| = \mathcal{O}\left((1 + |x|)^{-\alpha}\right)$ as $|x| \to \infty$.*

Proof. Since $|f(x)| = \mathcal{O}\left((1 + |x|)^{-\alpha}\right)$ as $|x| \to \infty$, there exists a constant C and $N \in \mathbb{N}_0$ such that

$$\big|f(x + u(k))\big| \leq C\,(1 + |x + u(k)|)^{-\alpha} \text{ for } x \in \mathfrak{D} \text{ and } k \in \mathbb{N}_0.$$

It follows that

$$\begin{aligned}
&\left\{ \sum_{k\in\mathbb{N}_0} |f(x + u(k))| \right\}^2 = \\
&= \left\{ \sum_{k\in\mathbb{N}_0, |k|\leq N} |f(x + u(k))| + \sum_{k\in\mathbb{N}_0, |k|>N} |f(x + u(k))| \right\}^2 \\
&\leq 2(2N + 1) \sum_{k\in\mathbb{N}_0} |f(x + u(k))|^2 + 2\left\{ \sum_{k\in\mathbb{N}_0, |k|>N} |f(x + u(k))| \right\}^2 \\
&\leq 2(2N + 1) \sum_{k\in\mathbb{N}_0} |f(x + u(k))|^2 + 2C^2\left\{ \sum_{k\in\mathbb{N}_0} f(1 + |x + u(k)|)^{-\alpha} \right\}^2 \\
&\leq 2(2N + 1) \sum_{k\in\mathbb{N}_0} |f(x + u(k))|^2 + C'
\end{aligned}$$

for a.e. $x \in \mathfrak{D}$ and some C' independent of x. This implies that

$$\int_{\mathfrak{D}} \left\{ \sum_{k\in\mathbb{N}_0} |f(x + u(k))| \right\}^2 dx \leq 2(2N + 1)\|f\|^2_{L^2(\mathbb{K})} + C' < \infty. \qquad \square$$

LEMMA 3.2. *For any $f \in \mathcal{L}^2(\mathbb{K}), \{f(\cdot - u(k)) : k \in \mathbb{N}_0\}$ is a Bessel sequence in $L^2(\mathbb{K})$.*

LEMMA 3.3. *$\bigcup_{s\in\mathcal{N}_1}(\mathfrak{D}+u(s))$ is $q\mathbb{N}_0$- congruent to $\mathfrak{p}^{-1}\mathfrak{D}$.*

Proof. Since $\bigcup_{p\in\mathcal{N}_1}(\mathfrak{D}+u(p))$ and $\mathfrak{p}^{-1}\mathfrak{D}$ have the same measure, to finish the proof we only need to prove that $\bigcup_{p\in\mathcal{N}_1}(\mathfrak{D}+u(p))$ is congruent to a subset of $\mathfrak{p}^{-1}\mathfrak{D}$. Observe that $\{\mathfrak{p}^{-1}\mathfrak{D}+\mathfrak{p}^{-1}u(r) : r\in\mathbb{N}_0\}$ is a partition of K. If $\bigcup_{p\in\mathcal{N}_1}(\mathfrak{D}+u(p))$ is not $q\mathbb{N}_0$-congruent to any subset of $\mathfrak{p}^{-1}\mathfrak{D}$, then there exists $E\subset\mathfrak{D}$ with $|E|>0$ such that

$E+u(p_1)+\mathfrak{p}^{-1}u(r_1)\subset\mathfrak{D}+u(p_2)+\mathfrak{p}^{-1}u(r_2)$ for some $(p_1,r_1)\neq(p_2,r_2)\in\mathcal{N}_j\times\mathbb{N}_0$.

It follows that

$$E\subset\mathfrak{D}+(u(p_2)-u(p_1))+\mathfrak{p}^{-1}\left(u(r_2)-u(r_1)\right),$$
$$0\neq(u(p_2)-u(p_1))+\mathfrak{p}^{-1}\left(u(r_2)-u(r_1)\right)\in\mathbb{N}_0.$$

This is a contradiction due to the fact that $E\subset\mathfrak{D}$. This completes the proof. □

LEMMA 3.4. *Let $f\in\mathcal{L}^2(\mathbb{K})$. Then $f_{j,k}\in\mathcal{L}^2(\mathbb{K})$, and $|f_{j,k}|\leq q^j|f|_2$ for $j\in\mathbb{N}_0$ and $k\in\mathbb{N}_0$.*

Proof. Since $\mathfrak{p}^{-j}\mathbb{N}_0\subset\mathbb{N}_0$ and by Lemma 3.3 we have

$$\begin{aligned}\int_{\mathfrak{D}}\left\{\sum_{r\in\mathbb{N}_0}|f_{j,k}(x+u(r))|\right\}^2dx &= q^j\int_{\mathfrak{D}}\left\{\sum_{r\in\mathbb{N}_0}\left|f(\mathfrak{p}^{-j}(x+u(r))-u(k))\right|\right\}^2dx\\ &\leq q^j\int_{\mathfrak{D}}\left\{\sum_{r\in\mathbb{N}_0}\left|f(\mathfrak{p}^{-j}x+u(r)-u(k))\right|\right\}^2dx\\ &= q^j\int_{\mathfrak{D}}\left\{\sum_{r\in\mathbb{N}_0}\left|f(\mathfrak{p}^{-j}x+u(r))\right|\right\}^2dx\\ &= q^{-j}\ int_{\mathfrak{p}^{-j}\mathfrak{D}}\left\{\sum_{r\in\mathbb{N}_0}|f(x+u(r))|\right\}^2dx\\ &= q^{-j}\int_{\bigcup_{s\in\mathcal{N}_j}(\mathfrak{D}+u(s))}\left\{\sum_{r\in\mathbb{N}_0}|f(x+u(r))|\right\}^2dx,\end{aligned}\tag{3.1}$$

It follows that

$$\begin{aligned}\int_{\mathfrak{D}}\left\{\sum_{r\in\mathbb{N}_0}\left|f(\mathfrak{p}^{-j}x+u(r))\right|\right\}^2dx &= q^{-j}\sum_{s\in\mathcal{N}_j}\int_{\mathfrak{D}}\left\{\sum_{r\in\mathbb{N}_0}|f(x+u(r))|\right\}^2dx\\ &=|f|_2,\end{aligned}$$

and thus $f_{j,k}\in\mathcal{L}^2(\mathbb{K})$, and $|f_{j,k}|_2\leq q^j|f|_2$ by (3.1). This completes the proof. □

As an immediate consequences of Lemma 3.4, we have

Lemma 3.5. *Let* $f \in \mathcal{L}^2(\mathbb{K})$*. Then* $f^{\text{per}}_{j,k} \in L^2(\mathfrak{D})$ *for* $j \in \mathbb{N}_0$ *and* $k \in \mathbb{N}_0$*.*

The following two lemmas are very useful in the later sections.

Lemma 3.6. *Let* $g \in \mathcal{L}^2(\mathbb{K})$*. Then for an arbitrary* $f \in L^2(\mathfrak{D})$*, we have*

$$\langle f, g^{\text{per}} \rangle = \sum_{r \in \mathbb{N}_0} \langle f, g(\cdot + u(r)) \rangle ,$$

where the series converges absolutely.

Proof. The proof of the lemma follows from the observation

$$\int_{\mathfrak{D}} |f(x)| \sum_{r \in \mathbb{N}_0} |g(x + u(r))| \, dx \le |g|_2 \|f\|. \qquad \square$$

Lemma 3.7. *Let* $\psi, \widetilde{\psi} \in \mathcal{L}^2(\mathbb{K})$*. Then*

$$\sum_{k \in \mathcal{N}_j} \left\langle f, \psi^{\text{per}}_{j,k} \right\rangle \overline{\left\langle g, \widetilde{\psi}^{\text{per}}_{j,k} \right\rangle} = \sum_{s \in \mathbb{N}_0} \sum_{r \in \mathbb{N}_0} \langle F_0, \psi_{j,r} \rangle_{L^2(\mathbb{K})} \overline{\left\langle G_s, \widetilde{\psi}_{j,r} \right\rangle}_{L^2(\mathbb{K})}$$

for $f, g \in L^2(\mathfrak{D})$*, where* $F_0 = f\mathbf{1}_{\mathfrak{D}}$ *and* $G_s = g\mathbf{1}_{\mathfrak{D}-\mathbf{u(s)}}$*.*

Proof. By Lemma 3.4 and Lemma 3.6,

$$\begin{aligned}
\left\langle f, \psi^{\text{per}}_{j,k} \right\rangle \overline{\left\langle g, \widetilde{\psi}^{\text{per}}_{j,k} \right\rangle} &= \sum_{s \in \mathbb{N}_0} \sum_{r \in \mathbb{N}_0} \langle f, \psi_{j,k}(\cdot + u(r)) \rangle \overline{\left\langle g, \widetilde{\psi}_{j,k}(\cdot + u(s)) \right\rangle} \\
&= \sum_{s \in \mathbb{N}_0} \sum_{r \in \mathbb{N}_0} \langle f, \psi_{j,k}(\cdot + u(r)) \rangle \overline{\left\langle g, \widetilde{\psi}_{j,k}(\cdot + u(r) + u(s)) \right\rangle} \\
&= \sum_{s \in \mathbb{N}_0} \sum_{r \in \mathbb{N}_0} \left\langle F_0, \psi_{j,k-\mathfrak{p}^{-j}u(r)} \right\rangle_{L^2(\mathbb{K})} \overline{\left\langle G_s, \widetilde{\psi}_{j,k-\mathfrak{p}^{-j}u(r)} \right\rangle}_{L^2(\mathbb{K})}.
\end{aligned}$$

It leads to the lemma due to the fact that $\mathbb{N}_0 = \mathcal{N}_j + \mathfrak{p}^j \mathbb{N}_0$. Thus the proof is complete. $\square$

Lemma 3.8. *For* $j \in \mathbb{N}_0$*, there exists a constant* C *such that*

$$|\mathfrak{p}^{-j} x| \ge C|x| \text{ for } x \in K.$$

Proof. It is clear that $\lim\limits_{j \to \infty} \|\mathfrak{p}^{-j}\|^{1/j} < 1$. It follows that there exists $J \in \mathbb{N}$ such that $\|\mathfrak{p}^{-j}\| < 1$ for $j > J$. By setting $C = \frac{1}{\max\{\|\mathfrak{p}^{-j}\| : 0 \le j \le J\}}$, then $\|\mathfrak{p}^{-j}\| \le \frac{1}{C}$ for $j \in \mathbb{N}_0$ and thus $|\mathfrak{p}^{-j}x| \le \frac{1}{C}|x|$ for $j \in \mathbb{N}_0$ and $x \in K$. This implies that $|\mathfrak{p}^j x| \le C|x|$ for $j \in \mathbb{N}_0$ and $x \in K$. $\square$

Theorem 3.9. *Let* $\{g_{j,k} : j \in \mathbb{N}_0, k \in \mathbb{N}_0\}$ *be a Bessel sequence in* $L^2(\mathbb{K})$*. Assume that*

$$|g(x)| = \mathcal{O}\big((1 + |x|)^{-\tau}\big) \ (|x| \to \infty) \tag{3.2}$$

for some $\tau > \max\left\{\beta_2, \frac{\beta_2}{\alpha_2}\right\}$, *where* α_2, β_2 *are as in Definition 2.3. Then* $\left\{1, g_{j,k}^{\text{per}} : j \in \mathbb{N}_0, k \in \mathcal{N}_m\right\}$ *is a Bessel sequence in* $L^2(\mathfrak{D})$.

Proof. By invoking Lemma 3.1 and Lemma 3.4 and the fact that $\tau > 1$, it is clear that $g_{j,k}^{\text{per}} \in L^2(\mathfrak{D})$. By a direct computation, we have $g_{j,k}^{\text{per}} = g_{j,k+\mathfrak{p}^{-j}u(r)}^{\text{per}}$ for $j \in \mathbb{N}_0$, k, $r \in \mathbb{N}_0$. First we claim that if there exists a constant $C > 0$ such that

$$\sum_{j\in\mathbb{N}_0}\sum_{k\in\mathcal{N}_j}\left\{\sum_{r\in\mathbb{N}_0}|\langle F_0, g_{j,k}(\cdot + u(r))\rangle|\right\}^2 \le C\|f\|^2 \text{ for } f \in L^2(\mathfrak{D}) \tag{3.3}$$

holds, then $\sum_{r\in\mathbb{N}_0}\langle F_0, g_{j,k}(\cdot + u(r))\rangle$ is well defined and

$$\left\langle f, g_{j,k}^{\text{per}}\right\rangle = \sum_{r\in\mathbb{N}_0}\langle F_0, g_{j,k}(\cdot + u(r))\rangle.$$

It follows that

$$\sum_{j\in\mathbb{N}_0}\sum_{n\in\mathcal{N}_j}\left|\left\langle f, g_{j,k}^{\text{per}}\right\rangle\right|^2 \le \sum_{j\in\mathbb{N}_0}\sum_{k\in\mathcal{N}_j}\left\{\sum_{r\in\mathbb{N}_0}|\langle F_0, g_{j,k}(\cdot + u(r))\rangle|\right\}^2.$$

Therefore by (3.3), we have

$$\left\langle f, g_{j,k}^{\text{per}}\right\rangle \le C\|f\|^2 \text{ for } f \in L^2(\mathfrak{D})$$

which implies that $\left\{1, g_{j,k}^{\text{per}} : j \in \mathbb{N}_0, k \in \mathcal{N}_j\right\}$ is a Bessel sequence in $L^2(\mathfrak{D})$.

Now we proceed to prove (3.3). By Lemma 3.8, there exists $C_1 > 0$ such that

$$\left|\mathfrak{p}^{-j}x + \mathfrak{p}^{-j}u(r) - u(k)\right| \ge C_1\left|u(r) + x - \mathfrak{p}^j u(k)\right| \ge C_1(|r| - 1)$$

for $j \in \mathbb{N}_0, x \in \mathfrak{D}, k \in \mathcal{N}_j$ and $r \in \mathbb{N}_0$. We observe that $|k| \to \infty$ if and only if $\rho(k) \to \infty$. It follows that there exists $J \in \mathbb{N}$ such that, if $|r| > J$, then

$$\left|g(\mathfrak{p}^{-j}x + \mathfrak{p}^{-j}u(r) - u(k))\right| \ge C_2\left|\mathfrak{p}^{-j}x + \mathfrak{p}^{-j}u(r) - u(k)\right|^{-\tau}$$

and $\rho(r) \ge 2c_0 \max_{x\in\mathfrak{D}}\rho(x)$ for $j \in \mathbb{N}_0, x \in \mathfrak{D}$ and $k \in \mathcal{N}_j$, where c_0 is as in Definition 2.3. So

$$\begin{aligned}
\left|g(\mathfrak{p}^{-j}x + \mathfrak{p}^{-j}u(r) - u(k))\right| &\le c_1^{\frac{\tau}{\beta_2}} C_2 \mathfrak{p}^{\frac{j\tau d}{\beta_2}}\left\{\rho\left(u(r) + x - \mathfrak{p}^j u(k)\right)\right\}^{\frac{-\tau}{\beta_2}} \\
&\le c_1^{\frac{\tau d}{\beta_2}} C_2 \mathfrak{p}^{\frac{j\tau}{\beta_2}}\left\{\rho\left(\frac{u(r)}{c_0}\right) - \max_{x\in\mathfrak{D}}\rho(x)\right\}^{\frac{-\tau}{\beta_2}} \\
&\le (c_0c_1)^{\frac{\tau}{\beta_2}} C_2 \mathfrak{p}^{\frac{j\tau}{\beta_2}}\left\{\rho(u(r))\right\}^{\frac{-\tau}{\beta_2}} \\
&\le (c_0{c_1}^2)^{\frac{\tau}{\beta_2}} C_2 \mathfrak{p}^{\frac{j\tau}{\beta_2}}|u(r))|^{\frac{-\tau}{\beta_2}}
\end{aligned}$$

for $k \in \mathbb{N}_0$ with $|r| > J$, $j \in \mathbb{N}_0$, $x \in \mathfrak{D}$ and $k \in \mathcal{N}_j$, where c_1 is as in Definition 2.3. Therefore, it follows that

$$\begin{aligned}|\langle F_0, g_{j,k}(\cdot + u(r))\rangle| &\leq q^{j/2}\|f\|\left\{\int_{\mathfrak{D}}\left|g(\mathfrak{p}^{-j}x + \mathfrak{p}^{-j}u(r) - u(k))\right|^2 dx\right\}^{1/2} \\ &\leq C_3\|f\|q^{j(\frac{1}{2}-\frac{\tau}{\beta_2})}|u(r)|^{\frac{-\tau}{\beta_2}}\end{aligned}$$

for $j \in \mathbb{N}_0$, $k \in \mathcal{N}_j$ and $r \in \mathbb{N}_0$ with $|r| > J$, and thus

$$\sum_{j\in\mathbb{N}_0}\sum_{k\in\mathcal{N}_j}\left\{\sum_{r\in\mathbb{N}_0,|r|>J}|\langle F_0, g_{j,k}(\cdot + u(r))\rangle|\right\}^2 \leq C_4\|f\|^2 \text{ for } f \in L^2(\mathfrak{D}),$$

where C_4 is a constant independent of f. Further, we have

$$\begin{aligned}&\sum_{j\in\mathbb{N}}\sum_{k\in\mathcal{N}_j}\left\{\sum_{r\in\mathbb{N}_0,|r|\leq J}|\langle F_0, g_{j,k}(\cdot + u(r))\rangle|\right\}^2 \leq \\ &\leq (2J+1)\sum_{j\in\mathbb{N}}\sum_{k\in\mathcal{N}_j}\sum_{k\in\mathbb{N}_0}\left|\left\langle F_0, g_{j,k-\mathfrak{p}^{-j}u(r)}(\cdot + u(r))\right\rangle\right|^2 \\ &\leq (2J+1)\sum_{j\in\mathbb{N}_0}\sum_{k\in\mathbb{N}_0}|\langle F_0, g_{j,k}\rangle|^2 \\ &\leq (2J+1)\|F_0\|^2 \\ &= \|f\|^2.\end{aligned}$$

Therefore, it follows that

$$\sum_{j\in\mathbb{N}_0}\sum_{k\in\mathcal{N}_j}\left\{\sum_{r\in\mathbb{N}_0}|\langle F_0, g_{j,k}(\cdot + u(r))\rangle|\right\}^2 \leq 2\{C_4 + C_6\}\|f\|^2,$$

and thus we get (3.3). This completes the proof of the theorem. □

4. CONSTRUCTION OF WAVELET BI-FRAMES ON LOCAL FIELDS

We devote this section to the construction of periodic wavelet bi-frames on local fields. we start with some lemmas.

Lemma 4.1. *Let $f \in L^1(K)$ be a $\mathfrak{p}$- refinable function satisfying $\widehat{f}(0) \neq 0$. Then $\widehat{f}(u(\beta)) = 0$ for $\beta \in \mathbb{N}$.*

Proof. Suppose H is a symbol of f. Then

$$\widehat{f}(\mathfrak{p}^{-1}\xi) = H(\xi)\widehat{f}(\xi) \text{ a.e. on } K \tag{4.1}$$

Also observe that $\widehat{f}(0) \neq 0$ and $\widehat{f}$ is continuous by the fact that $f \in L^1(K)$. This implies that $H(\xi) = \frac{\widehat{f}(\mathfrak{p}^{-1}\xi)}{f(\xi)}$ and is continuous in some neighborhood of the origin, and thus $H(0) = 1$ by letting $\xi \to 0$. So, at an arbitrary $\beta \in \mathbb{N}_0, H(\xi)$ is continuous and equals 1 by its periodicity. Combined with (4.1), it follows that $\widehat{f}(\mathfrak{p}^{-r}u(\beta)) = \widehat{f}(u(\beta))$ for $r \in \mathbb{N}, \beta \in \mathbb{N}$. Letting $r \to \infty$, we have $\widehat{f}(u(\beta)) = 0$ since $\lim_{|\xi|\to\infty} \widehat{f}(\xi) = 0$ due to the fact $f \in L^1(K)$. This completes the proof. $\square$

LEMMA 4.2. *For any finite set $S \subset \mathbb{N}_0$, there exists $j_0 \in \mathbb{N}$ such that $u(r_1) - u(r_2) \notin \mathfrak{p}^{j_0}\mathbb{N}_0$ for $r_1 \neq r_2 \in S$.*

Proof. Set $J = \max\{|u(r_1) - u(r_2)| : r_1, r_2 \in S\}$. Since we have $\lim\limits_{j\to\infty} \|\mathfrak{p}^{-j}\| = 0$, there exists $j_0 \in \mathbb{N}$ such that $\|\mathfrak{p}^{-j_0}\| < \frac{1}{J}$. It follows that

$$\left|\mathfrak{p}^{-j_0}(u(r_1) - u(r_2)\right| \leq \|\mathfrak{p}^{-j_0}\| \cdot |u(r_1) - u(r_2)| < 1 \text{ for } r_1 \neq r_2 \in S,$$

and thus $\mathfrak{p}^{-j_0}(u(r_1) - u(r_2)) \notin \mathbb{N}_0$. This completes the proof. $\square$

LEMMA 4.3. *For $r_1 \neq r_2 \in \mathcal{N}_j, u(r_1) - u(r_2) \notin \mathfrak{p}^{j+1}\mathbb{N}_0$ for $j \in \mathbb{N}_0$.*

Proof. For $r_1 \neq r_2 \in \mathcal{N}_j, u(r_1) - u(r_2) \notin \mathfrak{p}^{j}\mathbb{N}_0$, which implies $r_1 \neq r_2 \in \mathcal{N}_j, u(r_1) - u(r_2) \notin \mathfrak{p}^{j+1}\mathbb{N}_0$. This completes the proof. $\square$

LEMMA 4.4. *$\left\{\mathfrak{p}^{1/2}\overline{\chi_r(\mathfrak{p}\xi)} : k \in \mathcal{N}_j\right\}$ is an orthonormal basis for $\ell^2(\mathcal{N}_j)$ is the Hilbert space of the functions defined on $\mathcal{N}_j$ endowed with the inner product $\langle f, g\rangle = \sum\limits_{r\in\mathcal{N}_j} f(r)\overline{g(r)}$.*

LEMMA 4.5. *For any $f \in L^1(K)$, we have*
(i) $\sum\limits_{r\in\mathbb{N}_0} f(x + u(r))$ converges absolutely a.e. on K,
(ii) $f^{\text{per}} \in L^1(\mathfrak{D})$,
(iii) $\widehat{f}(u(r)) = \displaystyle\int_{\mathfrak{D}} f^{\text{per}}(x)\overline{\chi_r(x)}dx$ for $r \in \mathbb{N}_0$.

LEMMA 4.6. *For any two Bessel sequences $\{h_i\}_{i\in\mathcal{I}}$ and $\{\widetilde{h}_i\}_{i\in\mathcal{I}}$ is a separable Hilbert space $\mathcal{H}$ $\left(\{h_i\}_{i\in\mathcal{I}}, \{\widetilde{h}_i\}_{i\in\mathcal{I}}\right)$ is a bi-frame if and only if*

$$\sum_{i\in\mathcal{I}} \langle f, h_i\rangle \overline{\left\langle g, \widetilde{h}_i\right\rangle} = \langle f, g\rangle$$

for f and g in some dense subset of $\mathcal{H}$.

THEOREM 4.7. *Suppose $\psi_0, \widetilde{\psi_0} \in L^2(\mathbb{K})$ are two $\mathfrak{p}$-refinable functions with symbols $H_0, \widetilde{H}_0 \in L^\infty(\mathfrak{D})$, and*

$$|\psi_0(x)| = \mathcal{O}\big((1+|x|)^{-\tau}\big),\ |\widetilde{\psi}_0(x)| = \mathcal{O}\big((1+|x|)^{-\tau}\big)\ (|x| \to \infty), \tag{4.2}$$

$$\widehat{\psi}_0(0) = \widehat{\widetilde{\psi}}_0(0) = 1 \tag{4.3}$$

for some $\tau > \max\left\{\beta_2, \frac{\beta_2}{\alpha_2}\right\}$, where α_2, β_2 are as in Definition 2.3. Let $\Psi = \{\psi_\ell : 1 \leq \ell \leq L\}$ and $\widetilde{\Psi} = \left\{\widetilde{\psi}_\ell : 1 \leq \ell \leq L\right\}$ be two finite subsets of $L^2(\mathbb{K})$ satisfying

$$\text{(i) } |\psi_\ell(x)| = \mathcal{O}\big((1+|x|)^{-\tau}\big),\ |\widetilde{\psi}_\ell(x)| = \mathcal{O}\big((1+|x|)^{-\tau}\big)\ (|x| \to \infty), \tag{4.4}$$

$$\text{(4.5)} \qquad (ii)\ \widehat{\psi}_\ell(\mathfrak{p}^{-1}\xi) = H_\ell(\xi)\widehat{\psi}_0(\xi),\ \ \widehat{\widetilde{\psi}}_\ell(\mathfrak{p}^{-1}\xi) = \widetilde{H}_\ell(\xi)\widehat{\widetilde{\psi}}_0(\xi)$$

with $H_\ell, \widetilde{H}_\ell \in L^\infty(\mathfrak{D})$ *for* $1 \le \ell \le L$.

(iii) $X(\Psi)$ *and* $X(\widetilde{\Psi})$ *are Bessel sequences in* $L^2(\mathbb{K})$,

and

$$\text{(4.6)} \quad (iv)\ \sum_{\ell=0}^{L} H_\ell(\xi)\overline{\widetilde{H}_\ell\left(\xi+\mathfrak{p}\gamma\right)} = \delta_{0,\gamma} \text{ for a.e. } \xi \in K \text{ and for each } \gamma \in \mathcal{N}_1.$$

Then $\left(X^{\rm per}(\Psi), X^{\rm per}(\widetilde{\Psi})\right)$ *is a periodic wavelet bi-frame for* $L^2(\mathbb{K})$.

To prove the above theorem, we first prove the following lemma:

Lemma 4.8. *Under the hypothesis of the above theorem, we have*

$$\langle f, \mathbf{1}\rangle\,\overline{\langle g, \mathbf{1}\rangle} + \sum_{\ell=0}^{L}\sum_{m=0}^{j}\sum_{k\in\mathcal{N}_m}\left\langle f, \psi^{\rm per}_{\ell,m,k}\right\rangle\overline{\left\langle f, \widetilde{\psi}^{\rm per}_{\ell,m,k}\right\rangle} =$$

$$\text{(4.7)} \qquad = \sum_{k\in\mathcal{N}_{j+1}}\left\langle f, \psi^{\rm per}_{0,j+1,k}\right\rangle\overline{\left\langle f, \widetilde{\psi}^{\rm per}_{0,j+1,k}\right\rangle}$$

for $j \in \mathbb{N}_0$ *and* $f, g \in L^2(\mathfrak{D})$, *where* $\mathbf{1}$ *is the function equal to* 1 *on* $\mathfrak{D}$.

Proof. By invoking Lemma 3.7 to $\psi_\ell, \widetilde{\psi}_\ell$ with $0 \le \ell \le L$, we have

$$\text{(4.8)} \quad \sum_{k\in\mathcal{N}_j}\left\langle f, \psi^{\rm per}_{\ell,j,k}\right\rangle\overline{\left\langle f, \widetilde{\psi}^{\rm per}_{\ell,j,k}\right\rangle} = \sum_{s\in\mathbb{N}_0}\sum_{r\in\mathbb{N}_0}\langle F_0, \psi_{\ell,j,r}\rangle_{L^2(\mathbb{K})}\overline{\left\langle G_s, \widetilde{\psi}_{\ell,j,r}\right\rangle}_{L^2(\mathbb{K})}$$

for each $0 \le \ell \le L$. By (4.5) and $\mathfrak{p}$-refinable property of ψ_0, we have

$$\begin{aligned}
\langle F_0, \psi_{\ell,j,r}\rangle_{L^2(\mathbb{K})} &= q^{-j/2}\int_K \widehat{F}_0(\xi)\overline{\widehat{\psi}_\ell(\mathfrak{p}^j\xi)}\chi_r(\mathfrak{p}^j\xi)\,d\xi \\
&= q^{-j/2}\int_K \widehat{F}_0(\xi)\overline{\widehat{\psi}_0(\mathfrak{p}^{j+1}\xi)H_\ell(\mathfrak{p}^{j+1}\xi)}\chi_r(\mathfrak{p}^j\xi)\,d\xi \\
&= q^{-j/2}\int_{\mathfrak{p}^{-j}\mathfrak{D}} \mathcal{R}_{0,\ell,j}(f)(\xi)\chi_r(\mathfrak{p}^j\xi)\,d\xi,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{R}_{0,\ell,j}(f)(\xi) = \sum_{\alpha\in\mathbb{N}_0}\sum_{p\in\mathcal{N}_1}&\widehat{F}_0\left(\xi+\mathfrak{p}^{-j}(\mathfrak{p}^{-1}(u(\alpha)+u(p)))\right) \\
&\times\overline{\widehat{\psi}_0\left(\mathfrak{p}^{j+1}\xi+u(\alpha)+\mathfrak{p}u(p)\right)H_\ell\left(\mathfrak{p}^{j+1}\xi+\mathfrak{p}u(p)\right)}.
\end{aligned}$$

In the similar manner, we have

$$\left\langle G_s, \widetilde{\psi}_{\ell,j,r}\right\rangle_{L^2(\mathbb{K})} = q^{-j/2}\int_{\mathfrak{p}^{-j}\mathfrak{D}}\widetilde{\mathcal{R}}_{s,\ell,j}(g)(\xi)\chi_r(\mathfrak{p}^j\xi)\,d\xi,$$

where

$$\widetilde{\mathcal{R}}_{s,\ell,j}(g)(\xi) = \sum_{\beta\in\mathbb{N}_0}\sum_{p\in\mathcal{N}_1}\widehat{G}_s\left(\xi+\mathfrak{p}^{-j}(\mathfrak{p}^{-1}(u(\beta)+u(p)))\right)$$

$$\times \overline{\widehat{\psi}_0\left(\mathfrak{p}^{j+1}\xi+u(\beta)+\mathfrak{p}u(p)\right)\widetilde{H}_\ell\left(\mathfrak{p}^{j+1}\xi+\mathfrak{p}u(p)\right)}.$$

Since $X(\Psi), X(\widetilde{\Psi})$ are both Bessel sequences, and

$$\left\{\langle F_0,\psi_{\ell,j,r}\rangle_{L^2(\mathbb{K})}\right\}_{r\in\mathbb{N}_0}, \left\{\langle G_s,\psi_{\ell,j,r}\rangle_{L^2(\mathbb{K})}\right\}_{r\in\mathbb{N}_0} \in \ell^2(\mathbb{N}_0)$$

We observe that $\left\{q^{-j/2}\chi_r(\mathfrak{p}^j\xi) : r\in\mathbb{N}_0\right\}$ is an orthonormal basis for $L^2(\mathfrak{p}^{-j}\mathfrak{D})$. It follows that

$$\int_{\mathfrak{p}^{-j}\mathfrak{D}} \left|\mathcal{R}_{0,\ell,j}(f)(\xi)\right|^2 d\xi = \int_{\mathfrak{p}^{-j}\mathfrak{D}} \left|\widetilde{\mathcal{R}}_{s,\ell,j}(g)(\xi)\right|^2 d\xi < \infty,$$

and

$$\sum_{r\in\mathbb{N}_0} \langle F_0,\psi_{\ell,j,r}\rangle_{L^2(\mathbb{K})} \overline{\left\langle G_s,\widetilde{\psi}_{\ell,j,r}\right\rangle}_{L^2(\mathbb{K})} = \int_{\mathfrak{p}^{-j}\mathfrak{D}} \mathcal{R}_{0,\ell,j}(f)(\xi)\overline{\widetilde{\mathcal{R}}_{s,\ell,j}(g)(\xi)}\, d\xi.$$

So by (4.8), we have

(4.9)
$$\sum_{\ell=0}^{L}\sum_{k\in\mathcal{N}_j}\left\langle f,\psi^{\text{per}}_{\ell,j,k}\right\rangle\overline{\left\langle f,\widetilde{\psi}^{\text{per}}_{\ell,j,k}\right\rangle} = \sum_{\ell=0}^{L}\sum_{s\in\mathbb{N}_0}\int_{\mathfrak{p}^{-j}\mathfrak{D}} \mathcal{R}_{0,\ell,j}(f)(\xi)\overline{\widetilde{\mathcal{R}}_{s,\ell,j}(g)(\xi)}\, d\xi.$$

Further, by (4.6) we observe that

$$\sum_{\ell=0}^{L}\overline{H_\ell\left(\mathfrak{p}^{j+1}\xi+\mathfrak{p}u(p)\right)}\widetilde{H}_\ell\left(\mathfrak{p}^{j+1}\xi+\mathfrak{p}u(p')\right) = \delta_{p,p'} \text{ for } p,p'\in\mathcal{N}_j.$$

Also we have

$$\begin{aligned}
&\mathcal{R}_{0,\ell,j}(f)(\xi)\overline{\widetilde{\mathcal{R}}_{s,\ell,j}(g)(\xi)} = \\
&= \sum_{p\in\mathcal{N}_j}\sum_{\alpha\in\mathbb{N}_0}\widehat{F}_0\left(\xi+\mathfrak{p}^{-j-1}u(\alpha)+\mathfrak{p}^{-j}u(p)\right)\overline{\widehat{\psi}_0\left(\mathfrak{p}^{j+1}\xi+u(\alpha)+\mathfrak{p}^{-j}u(p)\right)} \\
&\qquad \times \overline{\left\{\widehat{G}_s\left(\xi+\mathfrak{p}^{-j-1}u(\alpha)+\mathfrak{p}^{-j}u(p)\right)\overline{\widehat{\widetilde{\psi}}_0\left(\mathfrak{p}^{j+1}\xi+u(\alpha)+\mathfrak{p}^{-j}u(p)\right)}\right\}}.
\end{aligned}$$

Thus by (4.9), we have

$$\begin{aligned}
&\sum_{\ell=0}^{L}\sum_{k\in\mathcal{N}_j}\left\langle f,\psi^{\text{per}}_{\ell,j,k}\right\rangle\overline{\left\langle f,\widetilde{\psi}^{\text{per}}_{\ell,j,k}\right\rangle} = \\
&= \sum_{s\in\mathbb{N}_0}\sum_{p\in\mathcal{N}_j}\int_{\mathfrak{p}^{-j}\mathfrak{D}}\sum_{\alpha\in\mathbb{N}_0}\widehat{F}_0\left(\xi+\mathfrak{p}^{-j-1}u(\alpha)+\mathfrak{p}^{-j}u(p)\right)\overline{\widehat{\psi}_0\left(\mathfrak{p}^{j+1}\xi+u(\alpha)+\mathfrak{p}^{-j}u(p)\right)} \\
&\quad \times \overline{\left\{\widehat{G}_s\left(\xi+\mathfrak{p}^{-j-1}u(\alpha)+\mathfrak{p}^{-j}u(p)\right)\overline{\widehat{\widetilde{\psi}}_0\left(\mathfrak{p}^{j+1}\xi+u(\alpha)+\mathfrak{p}^{-j}u(p)\right)}\right\}}\, d\xi \\
&= \sum_{s\in\mathbb{N}_0}\sum_{p\in\mathcal{N}_j}\int_{\mathfrak{p}^{-j}(\mathfrak{D}+u(p))}\sum_{\alpha\in\mathbb{N}_0}\widehat{F}_0\left(\xi+\mathfrak{p}^{-j-1}u(\alpha)\right)\overline{\widehat{\psi}_0\left(\mathfrak{p}^{j+1}\xi+u(\alpha)\right)}
\end{aligned}$$

$$\times \overline{\left\{\widehat{G}_s\left(\xi+\mathfrak{p}^{-j-1}u(\alpha)\right)\overline{\widehat{\widetilde{\psi}}_0\left(\mathfrak{p}^{j+1}\xi+u(\alpha)\right)}\right\}}\,d\xi$$

$$= \sum_{s\in\mathbb{N}_0}\int_{\mathfrak{p}^{-j}\left(\bigcup_{p\in\mathcal{N}_j}(\mathfrak{D}+u(p))\right)}\sum_{\alpha\in\mathbb{N}_0}\widehat{F}_0\left(\xi+\mathfrak{p}^{-j-1}u(\alpha)\right)\overline{\widehat{\psi}_0\left(\mathfrak{p}^{j+1}\xi+u(\alpha)\right)}$$

$$\times \overline{\left\{\widehat{G}_s\left(\xi+\mathfrak{p}^{-j-1}u(\alpha)\right)\overline{\widehat{\widetilde{\psi}}_0\left(\mathfrak{p}^{j+1}\xi+u(\alpha)\right)}\right\}}\,d\xi. \tag{4.10}$$

By applying Lemma 3.3 to $\mathfrak{p}^{-1}$ leads to $\bigcup_{p\in\mathcal{N}_1}(\mathfrak{D}+u(p))$ being $\mathfrak{p}^{-1}\mathbb{N}_0$-congruent to $\mathfrak{p}^{-1}\mathfrak{D}$, therefore $\mathfrak{p}^{-j}\left(\bigcup_{p\in\mathcal{N}_1}(\mathfrak{D}+u(p))\right)$ is $\mathfrak{p}^{-j-1}\mathbb{N}_0$-congruent to $\mathfrak{p}^{-j-1}\mathfrak{D}$. Combined with (4.10), it follows that

$$\sum_{\ell=0}^{L}\sum_{k\in\mathcal{N}_j}\left\langle f,\psi^{\mathrm{per}}_{\ell,j,k}\right\rangle\overline{\left\langle g,\widetilde{\psi}^{\mathrm{per}}_{\ell,j,k}\right\rangle} =$$

$$= \sum_{s\in\mathbb{N}_0}\int_{\mathfrak{p}^{-j-1}\mathfrak{D}}\sum_{\alpha\in\mathbb{N}_0}\widehat{F}_0\left(\xi+\mathfrak{p}^{-j-1}u(\alpha)\right)\overline{\widehat{\psi}_0\left(\mathfrak{p}^{j+1}\xi+u(\alpha)\right)}\cdot$$

$$\cdot\overline{\left\{\widehat{G}_s\left(\xi+\mathfrak{p}^{-j-1}u(\alpha)\right)\overline{\widehat{\widetilde{\psi}}_0\left(\mathfrak{p}^{j+1}\xi+u(\alpha)\right)}\right\}}\,d\xi. \tag{4.11}$$

Further, we have

$$\langle F_0,\psi_{0,j+1,r}\rangle_{L^2(\mathbb{K})} = q^{-(j+1)/2}\int_K \widehat{F_0}(\xi)\overline{\widehat{\psi_0}(\mathfrak{p}^{j+1})}\chi_r(\mathfrak{p}^{-j-1}\xi)\,d\xi$$

$$= q^{-(j+1)/2}\int_{\mathfrak{p}^{-j-1}\mathfrak{D}}\sum_{\alpha\in\mathbb{N}_0}\widehat{F}_0\left(\xi+\mathfrak{p}^{-j-1}u(\alpha)\right)\overline{\widehat{\psi}_0\left(\mathfrak{p}^{j+1}\xi+u(\alpha)\right)}\chi_r(\mathfrak{p}^{-j-1}\xi)\,d\xi. \tag{4.12}$$

Similarly, we have

$$\langle G_s,\psi_{0,j+1,r}\rangle_{L^2(\mathbb{K})} =$$

$$= q^{-(j+1)/2}\int_{\mathfrak{p}^{-j-1}\mathfrak{D}}\sum_{\alpha\in\mathbb{N}_0}\widehat{G}_s\left(\xi+\mathfrak{p}^{-j-1}u(\alpha)\right)\overline{\widehat{\widetilde{\psi}}_0\left(\mathfrak{p}^{j+1}\xi+u(\alpha)\right)}\chi_r(\mathfrak{p}^{-j-1}\xi)\,d\xi, \tag{4.13}$$

for $s\in\mathbb{N}_0$.

By (4.2) and Lemma 3.1, we have $\psi_0\in\mathcal{L}^2(\mathbb{K})$. This implies that $\{\psi_0(\cdot-u(r)) : r\in\mathbb{N}_0\}$ is a Bessel sequence in $L^2(\mathbb{K})$ by Lemma 3.2, and thus $\{\psi_{0,j+1,r} : r\in\mathbb{N}_0\}$ is a Bessel sequence in $L^2(\mathbb{K})$. This implies that $\left\{\langle F_0,\psi_{0,j+1,r}\rangle_{L^2(\mathbb{K})}\right\}_{r\in\mathbb{N}_0}\in\ell^2(\mathbb{N}_0)$ and thus

$$\sum_{\alpha\in\mathbb{N}_0}\widehat{F}_0\left(\xi+\mathfrak{p}^{-j-1}u(\alpha)\right)\overline{\widehat{\psi}_0\left(\mathfrak{p}^{j+1}\xi+u(\alpha)\right)}\in L^2(\mathfrak{p}^{-j-1}\mathfrak{D}).$$

In the similar manner, we have

$$\sum_{\alpha\in\mathbb{N}_0} \widehat{G}_s\left(\xi+\mathfrak{p}^{-j-1}u(\alpha)\right)\overline{\widehat{\widetilde{\psi}}_0\left(\mathfrak{p}^{j+1}\xi+u(\alpha)\right)}\in L^2(\mathfrak{p}^{-j-1}\mathfrak{D}).$$

Therefore, by (4.11), (4.12) and (4.13), we have

$$\sum_{\ell=0}^{L}\sum_{k\in\mathcal{N}_j}\left\langle f,\psi^{\text{per}}_{\ell,j,k}\right\rangle\overline{\left\langle g,\widetilde{\psi}^{\text{per}}_{\ell,j,k}\right\rangle}=\sum_{s\in\mathbb{N}_0}\sum_{r\in\mathbb{N}_0}\langle F_0,\psi_{0,j+1,r}\rangle_{L^2(\mathbb{K})}\overline{\left\langle G_s,\widetilde{\psi}_{0,j+1,r}\right\rangle}_{L^2(\mathbb{K})} \tag{4.14}$$

However, applying (4.8) to the case of $j+1$ and $\ell=0$, we have

$$\sum_{k\in\mathcal{N}_{j+1}}\left\langle f,\psi^{\text{per}}_{\ell,j+1,k}\right\rangle\overline{\left\langle g,\widetilde{\psi}^{\text{per}}_{\ell,j+1,k}\right\rangle}=\sum_{s\in\mathbb{N}_0}\sum_{r\in\mathbb{N}_0}\langle F_0,\psi_{0,j+1,r}\rangle_{L^2(\mathbb{K})}\overline{\left\langle G_s,\widetilde{\psi}_{0,j+1,r}\right\rangle}_{L^2(\mathbb{K})}.$$

Hence

$$\begin{aligned}&\sum_{k\in\mathcal{N}_{j+1}}\left\langle f,\psi^{\text{per}}_{\ell,j+1,k}\right\rangle\overline{\left\langle g,\widetilde{\psi}^{\text{per}}_{\ell,j+1,k}\right\rangle}=\\ &=\sum_{k\in\mathcal{N}_j}\left\langle f,\psi^{\text{per}}_{0,j,k}\right\rangle\overline{\left\langle g,\widetilde{\psi}^{\text{per}}_{0,j,k}\right\rangle}+\sum_{\ell=0}^{L}\sum_{k\in\mathcal{N}_j}\left\langle f,\psi^{\text{per}}_{\ell,j,k}\right\rangle\overline{\left\langle g,\widetilde{\psi}^{\text{per}}_{\ell,j,k}\right\rangle},\end{aligned} \tag{4.15}$$

for $k\in\mathbb{N}_0$.

Now we need to check ψ^{per}_0 and $\widetilde{\psi}^{\text{per}}_0$. By (4.2), we have $\psi_0,\widetilde{\psi}_0\in L^2(\mathbb{K})$, thus $\widehat{\psi}(u(\alpha))=\widehat{\widetilde{\psi}}_0(u(\alpha))=0$ for $\alpha\in\mathbb{N}$ by Lemma 4.1. It follows that

$$\psi^{\text{per}}_0(\xi)=\sum_{\alpha\in\mathbb{N}}\widehat{\psi}(u(\alpha))\chi_\alpha(\xi)=1\text{ and }\widetilde{\psi}^{\text{per}}_0(\xi)=\sum_{\alpha\in\mathbb{N}}\widehat{\widetilde{\psi}}_0(u(\alpha))\chi_\alpha(\xi)=1. \tag{4.16}$$

Combining (4.15) and (4.16), (4.7) follows. This completes the proof of Lemma 4.8. □

Proof of Theorem 4.7. By calling Theorem 3.9, it is clear that $\left\{\psi^{\text{per}}_{\ell,j,k}:j\in\mathbb{N}_0,k\in\mathcal{N}_j\right\}$ and $\left\{\widetilde{\psi}^{\text{per}}_{\ell,j,k}:j\in\mathbb{N}_0,k\in\mathcal{N}_j\right\}$ are Bessel sequences in $L^2(\mathfrak{D})$ for each $1\le\ell\le L$. It follows that $X^{\text{per}}(\Psi)$ and $X^{\text{per}}(\widetilde{\Psi})$ are both Bessel sequences in $L^2(\mathfrak{D})$. Also observe that the set of trigonometric polynomials is dense in $L^2(\mathfrak{D})$. In order to finish the proof, we only need to show that

$$\langle f,\mathbf{1}\rangle\overline{\langle g,\mathbf{1}\rangle}+\sum_{\ell=0}^{L}\sum_{m=0}^{\infty}\sum_{k\in\mathcal{N}_m}\left\langle f,\psi^{\text{per}}_{\ell,m,k}\right\rangle\overline{\left\langle f,\widetilde{\psi}^{\text{per}}_{\ell,m,k}\right\rangle}=\langle f,g\rangle$$

for arbitrary trigonometric polynomials f and g by Lemma 4.6. Again by Lemma 4.8, it is equivalent to

$$\mathcal{S}_j(f,g):=\sum_{k\in\mathcal{N}_j}\left\langle f,\psi^{\text{per}}_{0,j,k}\right\rangle\overline{\left\langle f,\widetilde{\psi}^{\text{per}}_{0,j,k}\right\rangle}\to\langle f,g\rangle\ \text{ as } j\to\infty \tag{4.17}$$

for arbitrary trigonometric polynomials f and g. Let us fix the trigonometric polynomials f and g. Now we prove (4.17). Clearly there exists a finite set $S \subset \mathbb{N}_0$ such that f and g have the form

$$f(x) = \sum_{r\in\mathbb{N}_0} c_r(f)\chi_r(x),\ g(x) = \sum_{r\in\mathbb{N}_0} c_r(f)\chi_r(x) \tag{4.18}$$

where

$$c_r(f) = c_r(g) = 0 \text{ for } r \notin S. \tag{4.19}$$

By Lemma 4.2, we can extend S as a full set $\mathcal{N}_{j_0}$. Also by Lemma 4.3, we can extend $\mathcal{N}_{j_0}$ to $\mathcal{N}_{j_0+1}$. By repeating this procedure, we can obtain a sequence $\{\mathcal{N}_j\}_{j=j_0}^{\infty}$ satisfying $\mathcal{N}_j \subset \mathcal{N}_{j+1}$. It implies that

$$c_r(f) = c_r(g) = 0 \text{ for } r \notin \mathcal{N}_j \text{ and } j \geq j_0. \tag{4.20}$$

By using Lemma 3.1 and Lemma 3.5, we have $\psi^{\text{per}}_{0,j,k}, \widetilde{\psi}^{\text{per}}_{0,j,k} \in L^2(\mathfrak{D})$. It follows by Lemma 4.5 that

$$\psi^{\text{per}}_{0,j,k} = \sum_{r\in\mathbb{N}_0} d_r\chi_r(x),\ \widetilde{\psi}^{\text{per}}_{0,j,k} \sum_{r\in\mathbb{N}_0} \widetilde{d}_r\chi_r(x), \tag{4.21}$$

where

$$d_r = \widehat{\psi}_{0,j,k}(u(r)) + q^{-j/2}\widehat{\psi}_0(\mathfrak{p}^j u(r))\overline{\chi_r(\mathfrak{p}^j u(k))},$$
$$\widetilde{d}_r = \widehat{\widetilde{\psi}}_{0,j,k}(u(r)) = q^{-j/2}\widehat{\widetilde{\psi}}_0(\mathfrak{p}^j u(r))\overline{\chi_r(\mathfrak{p}^j u(k))},$$

Combining the above with (4.20), we obtain

$$\left\langle f, \psi^{\text{per}}_{0,j,k}\right\rangle = \sum_{r\in\mathcal{N}_j} c_r(f) q^{-j/2}\overline{\widehat{\psi}_0(\mathfrak{p}^j u(r))\chi_r(\mathfrak{p}^j u(k))},$$

$$\left\langle g, \widetilde{\psi}^{\text{per}}_{0,j,k}\right\rangle = \sum_{r\in\mathcal{N}_j} c_r(g) q^{-j/2}\overline{\widehat{\widetilde{\psi}}_0(\mathfrak{p}^j u(r))\chi_r(\mathfrak{p}^j u(k))}.$$

Applying Lemma 4.4 to $\mathfrak{p}^{-j}$, we obtain an orthonormal basis $\left\{q^{-j/2}\overline{\chi_k(\mathfrak{p}^{-j}\xi)} : k \in \mathcal{N}_j\right\}$ for $\ell^2(\mathcal{N}_j)$. So

$$\mathcal{S}_j(f,g) = \sum_{r\in\mathcal{N}_j} c_r(f)\overline{c_r(g)\widehat{\psi}_0(\mathfrak{p}^j u(r))}\widehat{\widetilde{\psi}}_0(\mathfrak{p}^j u(r)) \text{ for } j \geq j_0. \tag{4.22}$$

Also observe that $\psi_0, \widetilde{\psi}_0 \in \mathcal{L}^2(\mathbb{K}) \subset L^1(K)$ by Lemma 3.1, which implies that $\psi_0, \widetilde{\psi}_0$ are continuous, and by (4.13), we have

$$\lim_{j\to\infty} \widehat{\psi}_0(\xi) = \lim_{j\to\infty} \widehat{\widetilde{\psi}}_0(\xi) = 1.$$

By letting $j \to \infty$ in (4.22), we obtain that

$$\lim_{j\to\infty} \mathcal{S}_j(f,g) = \lim_{j\to\infty} \sum_{r\in\mathcal{N}_j} c_r(f)\overline{c_r(g)} = \sum_{r\in\mathbb{N}_0} c_r(f)\overline{c_r(g)} = \langle f, g\rangle\,.$$

This completes the proof of Theorem 4.7. $\square$

THEOREM 4.9. *Suppose* $\psi_0, \widetilde{\psi_0} \in L^2(\mathbb{K})$ *are two* $\mathfrak{p}$*-refinable functions with symbols* $H_0, \widetilde{H}_0 \in L^\infty(\mathfrak{D})$, *and*

$$|\psi_0(x)| = \mathcal{O}((1+|x|)^{-\tau}),\ |\widetilde{\psi}_0(x)| = \mathcal{O}((1+|x|)^{-\tau})\ (|x| \to \infty),$$

$$\widehat{\psi}_0(0) = \widehat{\widetilde{\psi}}_0(0) = 1$$

for some $\tau > \max\left\{\beta_2, \frac{\beta_2}{\alpha_2}\right\}$, *where* α_2, β_2 *are as in Definition 2.3. Let* $\Psi = \{\psi_\ell : 1 \le \ell \le L\}$ *and* $\widetilde{\Psi} = \left\{\widetilde{\psi}_\ell : 1 \le \ell \le L\right\}$ *be two finite subsets of* $L^2(\mathbb{K})$ *such that*

(*i*) $|\psi_\ell(x)| = \mathcal{O}((1+|x|)^{-\tau d}),\ |\widetilde{\psi}_\ell(x)| = \mathcal{O}((1+|x|)^{-\tau d})\ (|x| \to \infty)$,

(*ii*) $\widehat{\psi}_\ell(\mathfrak{p}^{-1}\xi) = H_\ell(\xi)\widehat{\psi}_0(\xi),\ \widehat{\widetilde{\psi}}_\ell(\mathfrak{p}^{-1}\xi) = \widetilde{H}_\ell(\xi)\widehat{\widetilde{\psi}}_0(\xi)$ *with* $H_\ell, \widetilde{H}_\ell \in L^\infty(\mathfrak{D})$ *for* $1 \le \ell \le L$.

(*iii*) $X(\Psi)$ *and* $X(\widetilde{\Psi})$ *are Bessel sequences in* $L^2(\mathbb{K})$, *and*

(*iv*) *there exists a periodic measurable function* φ *which is positively bounded from below and above and continuous at the origin with* $\varphi(0) = 1$ *such that*

$$H_0(\xi)\overline{\widetilde{H}_0\,(\xi+\mathfrak{p}\gamma)}\varphi(\mathfrak{p}^{-1}\xi) + \sum_{\ell=1}^{L} H_\ell(\xi)\overline{\widetilde{H}_\ell\,(\xi+\mathfrak{p}\gamma)} = \varphi(\xi)\delta_{0,\gamma} \tag{4.23}$$

for a.e. $\xi \in \mathbb{K}$ *and each* $\gamma \in \mathcal{N}_1$, *then* $\left(X^{\text{per}}(\Psi), X^{\text{per}}(\widetilde{\Psi})\right)$ *is a periodic wavelet bi-frame for* $L^2(\mathbb{K})$.

Proof. We define $\widetilde{\Gamma}$ by

$$\widehat{\widetilde{\Gamma}}_0 = \varphi\widehat{\widetilde{\psi}}_0$$

Then clearly $\widetilde{\Gamma}$ is $\mathfrak{p}$-refinable with the symbol

$$\widetilde{H'}_\ell(\xi) = \tfrac{\varphi(\mathfrak{p}^{-1}\xi)\widetilde{H}_0(\xi)}{\varphi(\xi)}.$$

Further, we define $\widetilde{\Gamma}_\ell$ by

$$\widehat{\widetilde{\Gamma}}_\ell(\xi) = \widetilde{H'}_\ell(\xi)\widehat{\widetilde{\Gamma}}_0 \text{ with } \widetilde{H'}_\ell(\xi) = \frac{\widetilde{H'}_\ell(\xi)}{\Phi(\xi)} \text{ for } 1 \le \ell \le L.$$

Denote $\widetilde{\mho} = \{\widetilde{\Gamma}_\ell : 1 \le \ell \le L\}$, it is easy to check that the systems $\Psi \cup \{\psi_0\}$ and $\widetilde{\mho} \cup \{\widetilde{\Gamma}_0\}$ associated with $\widetilde{H'}_\ell$ satisfy the conditions of Theorem 4.7, and $\widetilde{\mho} = \widetilde{\Psi}$. Therefore, $(X^{\text{per}}(\Psi), X^{\text{per}}(\widetilde{\mho})$ i.e, $(X^{\text{per}}(\Psi), X^{\text{per}}(\widetilde{\Psi})$ is a periodic wavelet bi-frame by Theorem 4.7. The proof is complete. $\square$

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