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DIRECT METHODS FOR SINGULAR INTEGRAL EQUATIONS AND NON-HOMOGENEOUS PARABOLIC PDES

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Abstract. In this article, the author presented some applications of the Laplace, $\mathcal{L}_2$, and Post-Widder transforms for solving fractional singular integral equation, impulsive differential equation and systems of differential equations. Finally, analytic solution for a non-homogeneous partial differential equation with non-constant coefficients is given. The obtained results reveal that the integral transform method is an effective and convenient tool.

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1. INTRODUCTION

The use of integral transforms in applications is quite extensive. In applied probability, as inventory and risk theory, queueing theory, the Laplace transform is a valuable tool for finding underlying probability density functions. As the Laplace transforms, the $\mathcal{L}_2$-transform is used in a variety of applications, the most common usage of the $\mathcal{L}_2$-transform is in the solution of the singular integral equations and initial value problems. Our interest in this transform stems from the potential applications to boundary value problems. To the best of our knowledge, the properties of the $\mathcal{L}_2$-transform have not been studied in any detail.

For solving partial differential equations, two methods, have been more extensively used the Laplace type integral transformations on the one hand and separation of variables on the other hand. New methods have also been proposed, the first integral method, the (G'/G)-expansion method and many more. The main purpose of this work has been to employ the integral transform method for studying certain mathematical models. It is worth mentioning that the integral transform methods are mostly suitable for the solution of linear differential and partial differential equations. Finally, this article presents the exact solution of non-homogeneous partial differential equation with non-constant coefficients, which is solved by direct application of the $\mathcal{L}_2$-transform. The primary advantage of this approach is that it solves boundary

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value problems characterized by partial differential equation without classical differential equation theory. We confine ourselves here to a few non-trivial examples which illustrate the method and lead to some interesting new results needed in the paper.

DEFINITION 1.1. *The Laplace transform of function $f(t)$ is given by* [1, 4, 6]

$$\mathcal{L}\{f(t);s\} = \int_0^\infty e^{-st} f(t)dt = F(s). \tag{1.1}$$

If $\mathcal{L}\{f(t);s\} = F(s)$, then $\mathcal{L}^{-1}\{F(s)\}$ is as follows

$$f(t) = \tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F(s)ds, \tag{1.2}$$

where $F(s)$ is analytic in the region $\mathrm{Re}(s) > c$.

The above complex integral is known as Bromwich integral [7]. The existence of the Laplace transform will depend on the function $f(t)$ and the parameter s.

LEMMA 1.2. *The following identities hold true.*

(1) $\frac{1}{(\sqrt{s}+\lambda)} = \mathcal{L}\Big[\frac{1}{\sqrt{\pi t}} - \lambda e^{\lambda^2 t}\,\mathrm{Erfc}(\lambda\sqrt{t})\Big]$,

(2) $\frac{1}{\sqrt{s}(\sqrt{s}+\lambda)} = \mathcal{L}\Big[e^{\lambda^2 t}\,\mathrm{Erfc}(\lambda\sqrt{t})\Big]$,

Proof. See [1, 6]. □

Singular integral equations arise frequently in the mathematical modeling of continuum phenomena, and many a time cannot be treated by known analytical techniques. Though certain problems had received the attention of aerodynamicists long ago, by contrast, extensive development of theory and methods for the approximate numerical solution is of recent vintage. Many physical problems dealing with radiative transfer, neutron transport, dispersal of aerosol like particles, fluid flow and waveguides can be reduced to singular integral equations. Comprehensive accounts of techniques for numerical solution of integral equations can be found in the monograph by Prossdorf and Silberman [9].

THEOREM 1.3. *Let us consider fractional singular integro-differential equation*

$$D_{0,t}^{c,\alpha}\phi(t) = f(t) + \lambda\int_t^{+\infty}\phi(\xi)d\xi, \quad 0 < t < 1$$

$$\phi(0) = u_0, \quad \int_0^{+\infty}\phi(\xi)d\xi = k, \quad 0 < \alpha < 1.$$

Then the above fractional singular integro-differential equation has the following solution.

$$\phi(t) = u_0 \sum_{n=0}^{n=+\infty} \frac{(-\lambda)^n t^{(\alpha+1)n}}{\Gamma(1+(\alpha+1)n)} + \sum_{n=0}^{n=+\infty} (-\lambda)^n \int_0^t f(t-\eta) \frac{\eta^{(\alpha+1)n}}{\Gamma(1+(\alpha+1)n)} d\eta - $$
$$- \lambda k \sum_{n=0}^{n=+\infty} \frac{(-\lambda)^n t^{(\alpha+1)(1+n)-1}}{\Gamma(1+(\alpha+1)n)}.$$

NOTE. This kind of singular integral equation is not considered in the literature. □

SOLUTION. Taking the Laplace transform of the above fractional singular integral equation term wise, leads to

$$s^{\alpha}\Phi(s) - s^{\alpha-1}u_0 = F(s) + \lambda \frac{\Phi(s)-\Phi(0)}{s} = F(s) + \lambda \frac{\Phi(s)-k}{s}.$$

After solving the transformed equation, we obtain

$$\Phi(s) = \frac{sF(s)}{\lambda + s^{\alpha+1}} + \frac{u_0 s^{\alpha} - \lambda k}{\lambda + s^{\alpha+1}},$$

or,

$$\Phi(s) = \sum_{n=0}^{+\infty} (-\lambda)^n \left[\frac{F(s)}{s^{n(\alpha+1)+\alpha}} + \frac{u_0}{s^{(\alpha+1)n+1}} - \frac{\lambda k}{s^{(\alpha+1)(n+1)}} \right].$$

Taking the inverse Laplace transform term-wise, we get

$$\phi(t) = \sum_{n=0}^{+\infty} (-\lambda)^n \left[\int_0^t f(t-\xi) \frac{\xi^{n(\alpha+1)+\alpha-1}}{\Gamma(n(\alpha+1)+\alpha)} d\xi + \frac{u_0 t^{(\alpha+1)n}}{\Gamma(n(\alpha+1)+1)} - \frac{\lambda k t^{(\alpha+1)(n+1)-1}}{\Gamma((\alpha+1)(n+1))} \right],$$

$0 < t < 1$. It is easy to verify that $\phi(0) = u_0$. □

In applied mathematics, engineering and mathematical physics, Bessel functions are associated most commonly with the partial differential equations of the wave or diffusion equations in cylindrical or spherical coordinates. No other special functions have received such a detailed treatment as have the Bessel functions [2].

LEMMA 1.4. *By using an appropriate integral representation for the modified Bessel functions of the second kind of order ν, $K_\nu(s)$, we have the following*

$$\mathcal{L}^{-1}\left\{ \frac{K_\nu(s)}{s^\nu} \right\} = \frac{\sqrt{\pi}}{\Gamma\left(\nu+\frac{1}{2}\right)2^\nu} (t^2-1)^{\nu-\frac{1}{2}}. \tag{1.3}$$

Proof. In view of the Definition 1.1, taking the inverse Laplace transform of the given $\frac{K_\nu(s)}{s^\nu}$, we obtain

$$h(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{st} \left(\frac{K_\nu(s)}{s^\nu} \right) ds, \tag{1.4}$$

by using the following integral representation for $K_\nu(s)$

$$\frac{K_\nu(s)}{s^\nu} = \frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})2^\nu}\int_0^\infty e^{-s\cosh t}\sinh^{2\nu} t dt. \tag{1.5}$$

By inserting relation (1.5) in (1.4), we get

$$h(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st}\Big(\frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})2^\nu}\int_0^\infty e^{-s\cosh r}\sinh^{2\nu} r dr\Big)ds, \tag{1.6}$$

and changing the order of integration in relation (1.6) leads to

$$h(t) = \frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})2^\nu}\int_0^\infty \sinh^{2\nu} r\Big(\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{s(t-\cosh r)}ds\Big)dr. \tag{1.7}$$

The inner integral is $\delta(t-\cosh r)$, therefore

$$h(t) = \frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})2^\nu}\int_0^\infty \delta(t-\cosh r)\sinh^{2\nu} r dr. \tag{1.8}$$

Making the change of variable $t-\cosh r = u$, and considerable algebra and elimination process, we obtain

$$h(t) = \frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})2^\nu}\int_{-\infty}^{t-1} \delta(u)\frac{((t-u)^2-1)^\nu}{\sqrt{(t-u)^2-1}}du = \frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})2^\nu}(t^2-1)^{\nu-\frac{1}{2}}. \tag{1.9}$$

For the special case $\nu = 0$, we get the following relation

$$\mathcal{L}^{-1}\{K_0(s)\} = (t^2-1)^{-\frac{1}{2}}. \qquad \square$$

Example 1.5. Consider the following generalized Abel singular integral equation of the second kind.

$$\phi(t) = \frac{e^{-\frac{\lambda^2}{t}}}{\sqrt[k]{t^\nu+\beta}} + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\phi(\xi)}{(t-\xi)^{1-\alpha}}d\xi, \qquad \phi(0) = 0, \qquad 0 < \alpha, \nu < 1. \qquad \square$$

This type of integral equation arises in the theory of wave propagation over a flat surface. Such integral equations occur rather frequently in mathematical physics and possess very interesting properties.

Note. The above mentioned singular integral equation can be written in terms of the Riemann-Liouville fractional integral as below

$$\phi(t) = \frac{e^{-\frac{\lambda^2}{t}}}{\sqrt[k]{t^\nu+\beta}} + I_{0,t}^{R-L,\alpha}\phi(t), \qquad 0 < \alpha, \nu < 1. \qquad \square$$

Solution. By taking the Laplace transform of the given integral equation, after simplifying we arrive at

$$\Phi(s) = \mathcal{L}\Big(\frac{e^{-\frac{\lambda^2}{t}}}{\sqrt[k]{t^\nu+\beta}}\Big) + \frac{\Phi(s)}{s^\alpha}. \tag{1.10}$$

Solving transformed equation leads to

$$\Phi(s) = \mathcal{L}\Big(\frac{e^{-\frac{\lambda^2}{t}}}{\sqrt[k]{t^\nu+\beta}}\Big)(1-\frac{1}{s^\alpha})^{-1} = \mathcal{L}\Big(\frac{e^{-\frac{\lambda^2}{t}}}{\sqrt[k]{t^\nu+\beta}}\Big) + \mathcal{L}\Big(\frac{e^{-\frac{\lambda^2}{t}}}{\sqrt[k]{t^\nu+\beta}}\Big)(\frac{1}{s^\alpha-1}). \tag{1.11}$$

Upon taking the inverse Laplace transform, we get

$$\phi(t) = \frac{e^{-\frac{\lambda^2}{t}}}{\sqrt[k]{t^\nu+\beta}} + \int_0^t \frac{e^{-\frac{\lambda^2}{t-\eta}}}{\sqrt[k]{(t-\eta)^\nu+\beta}} \mathcal{L}^{-1}(\tfrac{1}{s^\alpha-1}) d\eta. \tag{1.12}$$

In the special case $\alpha = 0.5$, we have the following relation

$$\phi(t) = \frac{e^{-\frac{\lambda^2}{t}}}{\sqrt[k]{t^\nu+\beta}} + \int_0^t \frac{e^{-\frac{\lambda^2}{t-\eta}}}{\sqrt[k]{(t-\eta)^\nu+\beta}} \Big(\tfrac{1}{\sqrt{\pi\eta}} + e^\eta \operatorname{Erfc}(-\sqrt{\eta})\Big) d\eta. \qquad \square$$

Note. We may check that $\phi(0) = 0$. $\square$

In recent years, fractional calculus appeared as an important tool to deal with anomalous diffusion processes. A more physical approach of anomalous diffusion processes has several applications in many fields such as diffusion in porous media or long range correlation of DNA sequence [8]. The closed form solution of the time fractional impulsive heat equation has been presented. At this stage we use the joint transform method to obtain a solution of a time fractional impulsive heat equation. The joint transform method provides an effective procedure for exact solution of a wide class of systems representing real physical problems.

Problem 1.6. *Let us consider the following non-homogeneous time fractional impulsive heat equation,*

$$D_{0,t}^{c,0.5} u = \lambda u_{xx} + \eta\delta(t-a)\delta(x-b) \qquad t > 0, -\infty < x < +\infty, \tag{1.13}$$

with initial and boundary conditions

$$u(x,0) = \sqrt{2\pi}\delta(x), \qquad \lim_{|x|\to+\infty} |u(x,t)| < M_0,$$

where a, b, η are constants and λ, M_0 are positive constants.

Note. Fractional derivative is in the Caputo sense. $\square$

Solution. The joint Laplace-Fourier transform of function $u(x,t)$ is defined as follows

$$\mathcal{L}\Big[\mathcal{F}[u(x,t); x \to w]; t \to s\Big] = \int_0^{+\infty} e^{-st} \Big[\tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{iwx} u(x,t) dx\Big] dt = U(w,s).$$

Let us take the joint Laplace-Fourier transform of the above equation (1.13) term wise and using boundary conditions, we get

$$\sqrt{s}U(w,s) - \tfrac{1}{\sqrt{s}} = -\lambda w^2 U(w,s) + \eta e^{-bs} e^{iwa}.$$

Solving the transformed equation leads to

$$U(w,s) = \tfrac{1}{\sqrt{s}(\sqrt{s}+\lambda w^2)} + \eta e^{iaw}[\tfrac{e^{-bs}}{\sqrt{s}+\lambda w^2}].$$

At this point taking the inverse joint Laplace-Fourier transforms and by virtue of the Lemma 1.2 we obtain

$$u(x,t) = \tfrac{1}{\sqrt{2\pi}} \Big[\int_{-\infty}^{+\infty} e^{-ixw} [e^{t\lambda^2 w^4} \operatorname{Erfc}(\lambda\sqrt{t} w^2)] dw$$

$$+ \eta \int_{-\infty}^{+\infty} e^{-i(x-a)w}\Big[\frac{1}{\sqrt{\pi(t-b)}} - \lambda w^2 e^{\lambda^2(t-b)w^4}\,\mathrm{Erfc}(\lambda\sqrt{t-b}w^2)\Big]dw\Big]. \square$$

2. THE $\mathcal{L}_2$-TRANSFORM

In the literature, we have significant generalizations of the integral transforms and new uses of the transformation method in engineering, applied mathematics and physics applications. The $\mathcal{L}_2$-transformation was first introduced by Yurekli [9] and denoted as follows

$$\mathcal{L}_2\{f(t); s\} = \int_0^{\infty} te^{-s^2t^2} f(t)dt. \tag{2.1}$$

In the absence of methods for the inversion of the $\mathcal{L}_2$-transform, recently the authors [3], established a simple formula to invert the $\mathcal{L}_2$-transform of a desired function. We present certain new inversion techniques for the $\mathcal{L}_2$-transform and an application of generalized product theorem for solving singular integral equations and boundary value problems are given.

Lemma 2.1. *The following identity holds true*

$$\mathcal{L}_2\{t^{\eta}\delta(at^m - \lambda); s\} = \tfrac{1}{am}(\tfrac{\lambda}{a})^{\frac{\eta-m+2}{m}} e^{-s^2(\frac{\lambda}{a})^{\frac{2}{m}}}, \qquad \lambda, \eta > 0, k, m > 1.$$

Solution. By definition of the $\mathcal{L}_2$-transform, we have

$$\mathcal{L}_2\{t^{\eta}\delta(at^m - \lambda); s\} = \int_0^{+\infty} t^{\eta+1} e^{-s^2t^2}\delta(at^m - \lambda)dt.$$

Making a change of variable $at^m - \lambda = \xi$, then we have

$$\mathcal{L}_2\Big\{t^{\eta}\delta(at^m - \lambda); s\Big\} = \int_{-\lambda}^{+\infty} (\tfrac{\xi+\lambda}{a})^{\frac{\eta-m+2}{m}} e^{-s^2(\frac{\xi+\lambda}{a})^{\frac{2}{m}}}\delta(\xi)\tfrac{d\xi}{am},$$

using elementary property of Dirac-delta function, yields

$$\mathcal{L}_2\Big\{t^{\eta}\delta(at^m - \lambda); s\Big\} = \tfrac{1}{am}(\tfrac{\lambda}{a})^{\frac{\eta-m+2}{m}} e^{-s^2(\frac{\lambda}{a})^{\frac{2}{m}}}.$$

Consider the special case $\eta = 0, m = 1, a = 1$, we get

$$\mathcal{L}_2\Big[\delta(t - \lambda); s\Big] = \lambda e^{-\lambda^2 s^2}. \qquad \square$$

3. ELEMENTARY PROPERTIES OF THE $\mathcal{L}_2$-TRANSFORM

Here, we will derive a relation between the $\mathcal{L}_2$-transform of the derivative of the function and the $\mathcal{L}_2$-transform of the function itself. We recall a useful lemma about the $\mathcal{L}_2$ -transform of the δ-derivatives.

Lemma 3.1. *If $f, f', \ldots, f^{(n-1)}$ are all continuous and of exponential order $\exp(c^2t^2)$ as $t \to \infty$ for some real constant c and piecewise continuous derivative $f^{(n)}$ on the interval $t \geq 0$*

(1) *For* $n = 1, 2, \ldots$ *then*

$$\mathcal{L}_2\Big\{\delta_t^n f(t); s\Big\} = 2^n s^{2n} \mathcal{L}_2\{f(t); s\} - 2^{n-1} s^{2(n-1)} f(0^+) - 2^{n-2} s^{2(n-2)} (\delta_t f)(0^+) - \ldots - (\delta_t^{n-1} f)(0^+). \tag{3.1}$$

(2) *For* $n = 1, 2, \ldots$

$$\mathcal{L}_2\Big\{t^{2n} f(t); s\Big\} = \tfrac{(-1)^n}{2^n} \delta_s^n \mathcal{L}_2\Big\{f(t); s\Big\}, \tag{3.2}$$

where the differential operators δ_t,δ_t^2, *are defined as below*

$$\delta_t = \tfrac{1}{t}\tfrac{d}{dt}, \qquad \delta_t^2 = \delta_t \delta_t = \tfrac{1}{t^2}\tfrac{d^2}{dt^2} - \tfrac{1}{t^3}\tfrac{d}{dt}.$$

Proof. See [10]. □

4. INVERSION FORMULA FOR THE $\mathcal{L}_2$-TRANSFORM AND EFROS' THEOREM

LEMMA 4.1. *Let us assume that* $F(\sqrt{s})$ *is analytic function* ($s = 0$ *is not a branch point) except at finite number of poles each of which lies to the left hand side of the vertical line* $\mathrm{Re}(s) = c$ *and if* $F(\sqrt{s}) \to 0$ *as* $s \to \infty$ *through the left plane* $\mathrm{Re}(s) \leq c$, *and*

$$\mathcal{L}_2\{f(t); s\} = \int_0^\infty t \exp(-s^2 t^2) f(t) dt = F(s), \tag{4.1}$$

then

(4.2)

$$\mathcal{L}_2^{-1}\{F(s)\} = \tfrac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2F(\sqrt{s}) e^{st^2} ds = \sum_{k=1}^{m} \Big[Res\{2F(\sqrt{s}) e^{st^2}\}, s = s_k\Big].$$

Proof. See [3]. □

EXAMPLE 4.2. Solving the following Cauchy's problem attached to a second order impulsive differential equation with non-constant coefficients

$$\tfrac{1}{t^2} y'' - t\tfrac{1}{t^3} y' + 4\lambda^2 y(t) = t^\beta \delta(t - \xi), \qquad y(0^+) = (\delta_t y)(0^+) = 0. \qquad \square$$

SOLUTION. By taking the $\mathcal{L}_2$-transform of the above equation term wise, we get

$$\mathcal{L}_2\Big(\delta_t^2 y(t)\Big) + 4\lambda^2 \mathcal{L}_2(y(t)) = \mathcal{L}_2(t^\beta \delta(t - \xi)). \tag{4.3}$$

Let us assume that $\mathcal{L}_2(y(t)) = Y(s)$, then after evaluation of the $\mathcal{L}_2$-transform each term, we arrive at

$$4s^4 Y(s) + 4\lambda^2 Y(s) = \xi^{\beta+1} e^{-\xi^2 s^2}, \tag{4.4}$$

or,

$$Y(s) = \tfrac{\xi^{\beta+1} e^{-\xi^2 s^2}}{4s^4 + 4\lambda^2}, \tag{4.5}$$

using inversion formula for the $\mathcal{L}_2$-transform, we have

$$y(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{\xi^{\beta+1}e^{-\xi^2 s}}{2(s^2+\lambda^2)}\right)e^{st^2}ds, \tag{4.6}$$

finally, using the second part of the Lemma 3.1 leads to the following solution:

$$y(t) = \xi^{\beta+1}\frac{1}{2\lambda}\sin(\lambda(t^2-\xi^2)). \qquad \square$$

Lemma 4.3 (Efros' Theorem for $\mathcal{L}_2$-Transforms). *Let $\mathcal{L}_2(f(t)) = F(s)$ and assuming $\Phi(s)$, $q(s)$ are analytic and such that, $\mathcal{L}_2(\Phi(t,\tau)) = \Phi(s)\tau e^{-\tau^2 q^2(s)}$, then the following relation holds true,*

$$\mathcal{L}_2\left\{\int_0^\infty f(\tau)\Phi(t,\tau)d\,\tau\right\} = F(q(s))\Phi(s). \tag{4.7}$$

Proof. By definition of the $\mathcal{L}_2$-transform

$$\mathcal{L}_2\Big\{\int_0^\infty f(\tau)\phi(t,\tau)d\tau\Big\} = \int_0^\infty te^{-s^2t^2}\Big(\int_0^\infty f(\tau)\phi(t,\tau)d\tau\Big)dt, \tag{4.8}$$

and changing the order of integration we arrive at

$$\int_0^\infty f(\tau)\Big(\int_0^\infty te^{-s^2t^2}\phi(t,\tau)dt\Big)d\tau = \Phi(s)\int_0^\infty f(\tau)\tau e^{-\tau^2q^2(s)}d\tau = \Phi(s)F(q(s)). \tag{4.9}$$

$\square$

In the sequel we will show that the $\mathcal{L}_2$-transform is suitable for solving singular integral equation with trigonometric kernel.

Example 4.4. By means of the above Lemma 4.3, we may solve the singular integral equation with trigonometric kernel

$$\int_0^\infty f(\xi)\sin(t\xi)d\tau = H(t-\lambda), \tag{4.10}$$

is

$$f(t) = \frac{2}{\pi}\frac{\cos(\lambda t)}{t}. \tag{4.11}$$

Solution. Applying the $\mathcal{L}_2$-transform followed by the generalized product theorem and using the fact that

$$\mathcal{L}_2\{\sin(\tau t)\} = \frac{\pi}{4s^3}\tau e^{-\frac{\tau^2}{4s^2}}, \tag{4.12}$$

and

$$\mathcal{L}_2\Big[H(t-\lambda, t\to s\Big] = \frac{1}{2s^2}e^{-\lambda^2 s^2}, \tag{4.13}$$

or,

$$F\left(\frac{1}{2s}\right)\frac{\sqrt{\pi}}{4s^3} = \frac{1}{2s^2}e^{-\lambda^2s^2}, \tag{4.14}$$

finally,

$$F(s) = \frac{1}{\sqrt{\pi}s}e^{-\frac{\lambda^2}{4s^2}}, \tag{4.15}$$

using inversion formula for the $\mathcal{L}_2$-transform leads to

$$f(t) = \mathcal{L}_2^{-1}\left[\frac{1}{\sqrt{\pi}s}e^{-\frac{\lambda^2}{4s^2}}; s \to t\right] = \frac{2}{\pi}\frac{\cos(\lambda t)}{t}. \qquad \square$$

5. SOLUTION TO THE SYSTEM OF SECOND ORDER DIFFERENTIAL EQUATIONS WITH NON CONSTANT COEFFICIENTS VIA THE $\mathcal{L}_2$-TRANSFORM

THEOREM 5.1. *We may consider the system of non-homogeneous second order differential equations in general form as follows*

$$\frac{1}{t^2}\overrightarrow{X}''(t) - \frac{1}{t^3}\overrightarrow{X}'(t) = A\overrightarrow{X}(t) + B \tag{5.1}$$

where A and B are coefficient and constants matrices of type $(n\times n)$ and $(n\times 1)$ respectively and $\overrightarrow{X}''(t)$, $\overrightarrow{X}'(t)$ $\overrightarrow{X}(t)$ are column vectors. Thus, the above system of equations has the following solution.

$$\overrightarrow{X}(t) = \mathcal{L}_2^{-1}\left\{(4s^4I - A)^{-1}(2s^2\overrightarrow{X}(0) + \overrightarrow{\delta_t X}(0) + \frac{1}{2s^2}B)\right\}.$$

Proof. For the solution of the above system, first, we take $\mathcal{L}_2$-transform of the above system, we get,

$$\mathcal{L}_2\left[\frac{1}{t^2}\overrightarrow{X}'' - \frac{1}{t^3}\overrightarrow{X}'(t)\right] = A\mathcal{L}_2\left[\overrightarrow{X}(t)\right] + B\mathcal{L}_2[1]$$

or,

$$4s^4\mathcal{L}_2\left[\overrightarrow{X}(t)\right] - 2s^2\overrightarrow{X}(0) - \overrightarrow{\delta_t X}(0) = A\mathcal{L}_2\left[\overrightarrow{X}(t)\right] + \frac{1}{2s^2}B$$

after simplifying the above relation, we obtain

$$(4s^4I - A)\mathcal{L}_2\left[\overrightarrow{X}(t)\right] = 2s^2\overrightarrow{X}(0) + \overrightarrow{\delta_t X}(0) + \frac{1}{2s^2}B$$

then,

$$\mathcal{L}_2\left[\overrightarrow{X}(t)\right] = (4s^4I - A)^{-1}(2s^2\overrightarrow{X}(0) + \overrightarrow{\delta_t X}(0) + \frac{1}{2s^2}B)$$

finally,

$$\overrightarrow{X}(t) = \mathcal{L}_2^{-1}\left\{(4s^4I - A)^{-1}(2s^2\overrightarrow{X}(0) + \overrightarrow{\delta_t X}(0) + \frac{1}{2s^2}B)\right\}. \qquad \square$$

In the sequel, we give certain illustrative examples and lemmas related to the $\mathcal{L}_2$, Post-Widder transforms, and inversion formula for the Post-Widder transform.

6. ILLUSTRATIVE LEMMAS AND EXAMPLES

LEMMA 6.1. *By using an integral representation for the modified Bessel functions of the second kind of order ν, $K_\nu(s)$, the following identity holds true*

$$\mathcal{L}_2^{-1}\left\{\frac{K_\nu(s^2)}{s^{2\nu}}\right\} = \frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})2^\nu}(t^4 - 1)^{\nu-\frac{1}{2}}. \tag{6.1}$$

Solution. By applying Lemma 4.1 and taking the inverse $\mathcal{L}_2$-transform of the given $\frac{K_\nu(s^2)}{s^{2\nu}}$, we arrive at

$$g(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st^2}\left(\frac{K_\nu(s)}{s^\nu}\right)ds, \tag{6.2}$$

at this point, we use an integral representation for $K_\nu(s)$

$$\frac{K_\nu(s)}{s^\nu} = \frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})2^\nu}\int_0^\infty e^{-s\cosh t}\sinh^{2\nu} t dt. \tag{6.3}$$

By inserting relation (1.3) in (1.2) , we get

$$g(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{st^2}\left(\frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})2^\nu}\int_0^\infty e^{-s\cosh r}\sinh^{2\nu} r dr\right)ds, \tag{6.4}$$

in relation (1.4), we may change the order of integration to obtain

$$g(t) = \frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})2^\nu}\int_0^\infty \sinh^{2\nu} r\left(\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{s(t^2-\cosh r)}ds\right)dr, \tag{6.5}$$

the inner integral is $\delta(t^2-\cosh r)$, therefore

$$g(t) = \frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})2^\nu}\int_0^\infty \delta(t^2-\cosh r)\sinh^{2\nu} r dr, \tag{6.6}$$

let us introduce a change of variable $t^2-\cosh r = u$, and considerable algebra and elimination process, we obtain

$$g(t) = \frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})2^\nu}\int_{-\infty}^{t^2-1}\delta(u)\frac{((t^2-u)^2-1)^\nu}{\sqrt{(t^2-u)^2-1}}du = \frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})2^\nu}(t^4-1)^{\nu-\frac{1}{2}}. \tag{6.7}$$

In view of the definition of the $\mathcal{L}_2$-transform we have

$$\frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})2^\nu}\int_0^{+\infty} e^{-s^2t^2}t(t^4-1)^{\nu-\frac{1}{2}}dt = \frac{K_\nu(s^2)}{s^{2\nu}}.$$

Let us consider the special case $\nu=0$, we get the following relation

$$\mathcal{L}_2^{-1}\{K_0(s^2)\} = (t^4-1)^{-\frac{1}{2}}. \qquad \square$$

Lemma 6.2. *The following singular integral equation of Post-Widder type*

$$\int_0^{+\infty}\frac{u\phi(u)}{u^2+s^2}du = \frac{c}{\sqrt[k]{s^2-a^2}}, \tag{6.8}$$

has solution as below

$$\phi(u) = \frac{4c\sin(\frac{\pi}{k})}{\pi\sqrt[k]{u^2+a^2}}. \tag{6.9}$$

Proof. Using the inverse Post-Widder transform (second iteration of the $\mathcal{L}_2$-transform) we have

$$\phi(u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{su^2}\left(\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{e^{ps^2}}{\sqrt[k]{p-a^2}}dp\right)_{s\to\sqrt{s}} ds, \tag{6.10}$$

introducing the new variable $w = p - a^2$ leads to

$$\phi(u) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2e^{su^2}\left(\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{2e^{ws^2+a^2s^2}}{\sqrt[k]{w}}dw\right)_{s\to\sqrt{s}} ds = .. \tag{6.11}$$

$$.. = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 4ce^{s(a^2+u^2)}\left(\frac{s^{\frac{1}{k}-1}}{\Gamma(\frac{1}{k})}\right) ds,$$

hence

$$\phi(u) = \frac{4c}{\Gamma(\frac{1}{k})}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{(a^2+u^2)s}\frac{1}{s^{1-\frac{1}{k}}}ds, \tag{6.12}$$

therefore, the final solution is as below

$$\phi(u) = \frac{4c}{\Gamma(\frac{1}{k})}\frac{1}{\Gamma(1-\frac{1}{k})}\cdot\frac{1}{\sqrt[k]{u^2+a^2}} = \frac{4c\sin(\frac{\pi}{k})}{\pi\sqrt[k]{u^2+a^2}}.$$ □

LEMMA 6.3. *Let us show the following Post-Widder type singular integral equation*

$$\int_0^{+\infty}\frac{u\phi(u)}{u^2+s^2}du = \frac{\sqrt[n]{s}}{s^{2k}-\lambda^2}, \tag{6.13}$$

has a solution as below

$$\phi(u) = \frac{2}{\pi}\left[\frac{u^{\frac{2}{n}}\sin(\frac{\pi}{n})}{\lambda-u^{4k}}\right]. \tag{6.14}$$

Proof. Let us consider the following inversion formula for the Post-Widder transform [1]

$$\mathcal{P}^{-1}\{F(s)\} = \frac{1}{\pi i}\Big\{F(u^2e^{-i\pi}) - F(u^2e^{i\pi})\Big\}. \tag{6.15}$$

In view of the above inversion formula for the Post-Widder transform, we have that

$$\begin{aligned}\mathcal{P}^{-1}\Big\{\frac{2\sqrt[n]{s}}{s^{2k}-\lambda}\Big\} &= \frac{1}{\pi i}\Big\{\frac{\sqrt[n]{u^2e^{-i\pi}}}{(u^2e^{-i\pi})^{2k}-\lambda} - \frac{\sqrt[n]{u^2e^{i\pi}}}{(u^2e^{i\pi})^{2k}-\lambda}\Big\}\\ &= \frac{2}{\pi}\left[\frac{u^{\frac{2}{n}}\sin(\frac{\pi}{n})}{\lambda-u^{4k}}\right].\end{aligned}$$

Since the obtained solution satisfies integral equation, we get the following interesting integral identity,

$$\frac{4\sin(\frac{\pi}{n})}{\pi}\int_0^{+\infty}\frac{u^{1+\frac{2}{n}}}{(u^2+s^2)(\lambda-u^{4k})}du = \frac{\sqrt[n]{s}}{s^{2k}-\lambda^2}. \tag{6.16}$$

Let us take $s = 1$, then after simplifying we get the following integral

$$\int_0^{+\infty}\frac{u^{1+\frac{2}{n}}}{(u^2+1)(\lambda-u^{4k})}du = \frac{\pi}{4\sin(\frac{\pi}{n})}.$$ □

EXAMPLE 6.4. Let us consider the following singular integral equation with trigonometric kernel

$$\int_0^{+\infty} x\phi(x)\cos\xi x dx = e^{\xi}\,\mathrm{Erfc}(\sqrt{\xi}). \tag{6.17}$$

Solution. Taking the Laplace transform of both sides of the integral equation with respect to variable ξ, we arrive at

$$\mathcal{L}\Big\{\int_0^{+\infty} x\phi(x)\cos\xi x dx\Big\} = \tfrac{1}{(s-1)\sqrt{s}}, \tag{6.18}$$

or, equivalently

$$\int_0^{+\infty} \tfrac{x}{x^2+s^2}\phi(x)dx = \tfrac{1}{s(s-1)\sqrt{s}}, \tag{6.19}$$

the left hand side of the above relation can be written as Widder potential transform of $\phi(x)$. We have

$$\mathcal{P}\{\phi(x);s\} = \tfrac{1}{s(s-1)\sqrt{s}}, \tag{6.20}$$

or,

$$\phi(x) = \tfrac{1}{\pi i}\Big(\tfrac{1}{x^2e^{-i\pi}(x^2e^{-i\pi}-1)\sqrt{x^2e^{-i\pi}}} - \tfrac{1}{x^2e^{i\pi}(x^2e^{i\pi}-1)\sqrt{x^2e^{i\pi}}}\Big), \tag{6.21}$$

after simplifying, we get

$$\phi(x) = \tfrac{2}{\pi x^3(x^2+1)}, \tag{6.22}$$

we deduce that

$$\int_0^{+\infty} \tfrac{2\cos\xi x}{\pi x^2(x^2+1)}dx = e^{\xi}\operatorname{Erfc}(\sqrt{\xi}). \tag{6.23}$$

Finally, from the above integral and inversion formula for the Fourier-cosine transforms we arrive at

$$\int_0^{+\infty} e^{\xi}\cos x\xi \operatorname{Erfc}(\sqrt{\xi})d\xi = \tfrac{1}{x^2(x^2+1)}. \tag{6.24}$$

Through an application of Leibnitz's rule, by differentiating the above integral with respect to x under the integral sign and after simplifying we arrive at

$$\int_0^{+\infty} \xi e^{\xi}\sin x\xi \operatorname{Erfc}(\sqrt{\xi})d\xi = \tfrac{4x^2+2}{x^4(x^2+1)^2}. \tag{6.25}$$

At this point upon using inverse Fourier-sine transform we have

$$\tfrac{2}{\pi}\int_0^{+\infty} \sin(\xi x)\tfrac{4x^2+2}{x^4(x^2+1)^2}dx = \xi e^{\xi}\operatorname{Erfc}(\sqrt{\xi}) \tag{6.26}$$

In special case if we choose $x=1$ in the above integral (6.26), we have the following

$$\int_0^{+\infty} \xi e^{\xi}\sin\xi \operatorname{Erfc}(\sqrt{\xi})d\xi = \tfrac{3}{2}. \qquad \square$$

Note. In the next section we will briefly illustrate the method of residues as it applies to both the $\mathcal{L}_2$-transform and its inversion formula. $\square$

7. MAIN RESULTS. THE $\mathcal{L}_2$-TRANSFORM FOR NON-HOMOGENEOUS PARABOLIC PDES.

The second order PDEs, with non-constant coefficients have a number of applications in electrical and mechanical engineering, medical sciences and economics. The heat equation plays an important role in a number of fields of science. In this section, we will study the application of such PDEs. Note that separation of variables will only work if both the partial differential equation and the boundary conditions are linear and homogeneous.

PROBLEM 7.1. *Let us consider the following non-homogeneous parabolic PDE*

$$\frac{1}{2t}u_t = u_{rr} + \frac{1}{r}u_r - \lambda^2 u - k \qquad 0 < r < 1, t > 0, \tag{7.1}$$

with initial and boundary conditions

$$u(1,t) = u(r,0) = 0, |u(r,t)| < M_0.$$

where $k > 0, M_0 > 0$.

SOLUTION. By taking the $\mathcal{L}_2$-transform of the above equation, we have

$$(s^2+\lambda^2)U(r,s) - 0.5u(r,0) - U_{rr}(r,s) - \frac{1}{r}U_r(r,s) = -\frac{k}{2s^2}, \tag{7.2}$$

or,

$$U_{rr} + \frac{1}{r}U_r - (s^2+\lambda^2)U = \frac{k}{2s^2} \quad U(1,s) = 0, \quad |U(r,s)| < M'. \tag{7.3}$$

The general solution of the transformed equation is given in terms of the modified Bessel functions of order zero as follows

$$U(r,s) = c_1 I_0(r\sqrt{s^2+\lambda^2}) + c_2 K_0(r\sqrt{s^2+\lambda^2}) - \frac{k}{2s^2(s^2+\lambda^2)}, \tag{7.4}$$

since $K_0(sr)$ is unbounded as $r \to 0$, we have to choose $c_2 = 0$, thus

$$U(r,s) = c_1 I_0(r\sqrt{s^2+\lambda^2}) - \frac{k}{2s^2(s^2+\lambda^2)}, \tag{7.5}$$

from $U(1,s) = 0$, we find $c_1 = \frac{k}{2s^2(s^2+\lambda^2)I_0(\sqrt{s^2+\lambda^2})}$, therefore

$$U(r,s) = -\frac{k}{2s^2(s^2+\lambda^2)} + \frac{kI_0(r\sqrt{s^2+\lambda^2})}{2s^2(s^2+\lambda^2)I_0(\sqrt{s^2+\lambda^2})}. \tag{7.6}$$

Using complex inversion formula for the $\mathcal{L}_2$-transform, we have

$$u(r,t) = -\frac{k}{\lambda^2}(1 - e^{-\lambda^2 t^2}) + \frac{k}{2\pi i}\int_{c-i\infty}^{c+\infty} \frac{e^{st^2} I_0(\sqrt{s+\lambda^2}r)}{s(s+\lambda^2)I_0(\sqrt{s+\lambda^2})}ds, \tag{7.7}$$

the integrand in the above integral has simple poles at at $s+\lambda^2 = 0, s = 0$ and $s+\lambda^2 = -\alpha_n^2, n = 1,2,3,\ldots.$ hence, the residue of integrand at $s = -\lambda^2$ is

$$\lim_{s\to-\lambda^2}(s+\lambda^2)\frac{e^{st^2} I_0(\sqrt{s+\lambda^2}r)}{s(s+\lambda^2)I_0(\sqrt{s+\lambda^2})} = \frac{e^{-\lambda^2 t^2}}{-\lambda^2}, \tag{7.8}$$

and the residue of integrand at $s = 0$ is

$$\lim_{s\to 0}(s)\frac{I_0(\sqrt{s+\lambda^2}r)}{s(s+\lambda^2)I_0(\sqrt{s+\lambda^2})} = \frac{I_0(\lambda r)}{\lambda^2 I_0(\lambda)}, \tag{7.9}$$

the residue of integrand at $s = -\lambda^2 - \alpha_n^2$ is

$$\lim_{s\to -\lambda^2-\alpha_n^2}(s+\alpha_n^2+\lambda^2)\frac{e^{st^2}I_0(\sqrt{s+\lambda^2}r)}{s(s+\lambda^2)I_0(\sqrt{s+\lambda^2})}, \tag{7.10}$$

the integrand in the above integral has simple poles at $s+\lambda^2 = -\alpha_n^2, n = 1,2,3,\dots$ and also at $s+\lambda^2 = 0$ where η_n are simple zeros of the modified Bessel function of order zero I_0 , as $\sqrt{s+\lambda^2} = \eta_1, \eta_2, \dots, \eta_n, \dots$, hence, we have

$$\lim_{s\to -\alpha_n^2-\lambda^2}(s+\lambda^2+\alpha_n^2)\frac{e^{st^2}I_0(\sqrt{s+\lambda^2}r)}{(s+\lambda^2)I_0(\sqrt{s+\lambda^2})} =$$
$$= \left(\lim_{s\to -\alpha_n^2-\lambda^2}\frac{s+\lambda^2+\alpha_n^2}{I_0(\sqrt{s+\lambda^2})}\right)\left(\lim_{s\to -\alpha_n^2-\lambda^2}\frac{e^{st^2}I_0(\sqrt{s+\lambda^2}r)}{s+\lambda^2}\right)$$
$$= \left(\lim_{s\to -\alpha_n^2-\lambda^2}\frac{1}{I_0'(\sqrt{s+\lambda^2})\frac{1}{2\sqrt{s+\lambda^2}}}\right)\left(\frac{e^{\alpha_n^2 t^2}I_0(i\alpha_n r)}{i\alpha_n^2}\right) = \frac{-2e^{\alpha_n^2 t^2}I_0(i\alpha_n r)}{i\alpha_n I_1(i\alpha_n)}$$

and

$$u(r,t) = -\frac{2k}{\lambda^2} + \frac{2kI_0(\lambda r)}{\lambda^2 I_0(\lambda)} - 2\sum_{n=1}^{n=\infty}\frac{e^{\alpha_n^2 t^2}I_0(i\alpha_n r)}{i\alpha_n I_1(i\alpha_n)}. \tag{7.11}$$

In view of the properties of the Bessel function, we have

$$I_0(t) = J_0(it) = J_0(-it), I_1(t) = iJ_1(it), J_1(-t) = -J_1(t),$$

therefore, we obtain the exact solution as below

$$u(r,t) = \frac{2kI_0(\lambda r)}{\lambda^2 I_0(\lambda)} - \frac{2k}{\lambda^2} + 2\sum_{n=1}^{+\infty}\frac{e^{\alpha_n^2 t^2}J_0(\alpha_n r)}{\alpha_n J_1(\alpha_n)}. \qquad \square$$

Note. We can check that $u(1,t) = 0$ and $|u(r,t)| < M_0$. $\square$

8. CONCLUSION

The main goal of the present paper is to extend the application of the $\mathcal{L}_2$-transform to derive an analytic solution of boundary value problems. We have presented certain methods of solution for singular integral equations and boundary value problem using the Laplace and $\mathcal{L}_2$-transform. Certain non-trivial examples are also provided. The formulations presented in this article are simple and can be extended to other problems in the field of integral transforms.

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