

ON UNIQUE SOLVABILITY  
OF THE PIECEWISE LINEAR EQUATION SYSTEM

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**Abstract.** In this article, we take the piecewise linear equation system  $x - W|x| = b$ , which is also known as the absolute value equation, where  $W \in \mathbb{R}^{n \times n}$ ,  $b \in \mathbb{R}^n$  are given and to determine the value of  $x \in \mathbb{R}^n$ . The absolute value equation (AVE) has many applications in various fields of mathematics like bi-matrix games, linear interval systems, linear complementarity problems (LCP) etc. By the equivalence relation of AVE with LCP, some necessary and sufficient conditions proved the existence and unique solvability of the AVE. Some examples are also provided to highlight the current singular value conditions for a unique solution that may revise in the future.

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1. INTRODUCTION

Here we study the AVE (absolute value equation) of the type

$$(1) \quad x - W|x| = b,$$

where  $W \in \mathbb{R}^{n \times n}$ ,  $b \in \mathbb{R}^n$  are known, and  $x \in \mathbb{R}^n$  is unknown;  $|\cdot|$  denotes the absolute value operator. For a matrix  $W \in \mathbb{R}^{n \times n}$  and a vector  $x \in \mathbb{R}^n$ ,  $|W|$  and  $|x|$  denote the component-wise absolute value of the matrix and the vector, respectively.

The generalized absolute value equation (GAVE) is the standard form of (1) defined as

$$(2) \quad Vx - W|x| = b,$$

where  $V, W \in \mathbb{R}^{n \times n}$ ,  $b \in \mathbb{R}^n$  are given while  $x \in \mathbb{R}^n$  is to be determined.

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In the literature, many authors discussed its unique solvability (see [1, 16, 18, 22, 23] and the references therein) and its particular case, when  $W = I$ , i.e.,

$$(3) \quad Vx - |x| = b,$$

is discussed in ([7, 8, 12, 20, 21] and the references therein).

To the best of our knowledge, another particular case of Eq.(2) as  $V = I$  have not been discussed for the sufficient condition of unique solvability aspects in the literature in a strong way to date. In 2014, Jiri Rohn [19] took the particular case of (1) as  $W \geq 0$  and discussed the unique solution condition with restrictions on  $W$  and  $b$ . In [6], authors provided some conditions for the bounding solution of AVE (1). The study of the AVE (1) is challenging and interesting, as it contains non-linear and non-differential terms  $W|x|$ . Currently, many methods to solve AVE available in the literature, like the linear programming method [11], globally and quadratically convergent method [3], matrix multi-splitting Picard-iterative method [5], generalized Newton method [10], iterative method [18] and for more, one may refer to [2, 13, 17]. In recent years, some authors investigated the existence and uniqueness of different types of AVE (see [8, 9, 15, 16, 20, 26] and references therein).

Throughout the paper, we will denote  $\hat{D} = \text{diag}(\hat{d}_i)$  with  $\hat{d}_i \in [0, 1]$  is a diagonal matrix. For a matrix  $A$ ,  $\sigma_1(A)$  or  $\sigma_{\max}(A)$  and  $\sigma_n(A)$  or  $\sigma_{\min}(A)$  will denote the maximum singular value and the minimum singular value of  $A$ , respectively.

The structure of the paper is the following: in Section 2 we recall some necessary definitions and results that will be used throughout the paper. In Section 3, with the help of LCP, some unique solution results are established for (1). Finally, the conclusions are discussed in Section 4.

## 2. PRELIMINARIES

Here, we recall some definitions and lemmas for additional results.

DEFINITION 1. *The LCP( $q, M$ ) is defined as [4]:*

$$(4) \quad 0 \leq z \perp Mz + q \geq 0$$

where  $q, z \in \mathbb{R}^n$ ,  $M \in \mathbb{R}^{n \times n}$ .

DEFINITION 2 ([14]). *A square matrix is called a P-matrix if all its principal minors are positive. Furthermore, every positive definite matrix is a P-matrix.*

LEMMA 3 ([4]). *A matrix  $M$  is a P-matrix if and only if the LCP( $q, M$ ) has a unique solution for any  $q$ .*

LEMMA 4 ([24]). *Let  $A \in C^{m \times m}$  satisfy the condition  $\|A\|_2 < 1$ . Then  $A \pm I$  is non-singular.*

LEMMA 5 ([3]). *Let  $\|A^{-1}\|_2 < 1$ . Then  $A + \bar{D}$  is non-singular for any diagonal matrix  $\bar{D} = \text{diag}(\bar{d}_i)$  with  $-1 \leq \bar{d}_i \leq 1$ .*

LEMMA 6 ([22]). *If the maximum singular value of the matrix  $W$  is strictly less than 1, then AVE (1) has a unique solution.*

LEMMA 7 ([4]). *A matrix  $M$  is called a  $Q$ -matrix if, for any  $q \in \mathbb{R}^n$ ,  $LCP(q, M)$  has a solution.*

LEMMA 8 ([23]). *The AVE (2) has a unique solution if and only if matrix  $V + W - 2W\hat{D}$  or  $V - W + 2W\hat{D}$  is non-singular for any diagonal matrix  $\hat{D}$ .*

Lemma 8 is equivalent to the following lemma when  $W$  is a non-singular matrix.

LEMMA 9. *The AVE (2) has a unique solution if and only if matrix*

$$W(W^{-1}V + I - 2\hat{D})$$

or

$$W(W^{-1}V - I + 2\hat{D})$$

is non-singular for any diagonal matrix  $\hat{D}$ .

LEMMA 10 ([25]). *For square matrices  $A$  and  $B$ , singular value inequality  $\sigma_i(A + B) \geq \sigma_i(A) - \sigma_1(B)$ ,  $i = 1, 2, \dots, n$  holds.*

### 3. MAIN RESULTS

In this section, we introduce significant results for the unique solution of AVE (1) and GAVE (2) and establish an equivalence relation between AVE and LCP. With the help of this relation, we get some results for the AVE (1).

THEOREM 11. *If matrix  $(I - W)$  is non-singular, then the following results are equivalent:*

1. *for any  $b$ , the AVE (1) has a unique solution;*
2. *matrix  $(I - W)^{-1}(I + W)$  is a  $P$ -matrix;*
3. *matrix  $(I - W)^{-1}(I + W)$  is a positive definite matrix;*
4. *all the principal minors of the matrix  $(I - W)^{-1}(I + W)$  are positive.*

*Proof.* Let  $x_i^+ = \max_i(0, x_i)$  and  $x_i^- = \max_i(0, -x_i)$ . So

$$(5) \quad x = x^+ - x^- \quad \text{and} \quad |x| = x^+ + x^-$$

Now by (5), AVE (1) is rewritten as

$$(6) \quad x^+ = (I - W)^{-1}(I + W)x^- + (I - W)^{-1}b.$$

(6) is a LCP form, where  $M = (I - W)^{-1}(I + W)$ .

Then, by Lemma 3 and Definition 2, our results hold.  $\square$

Based on Lemma 7, we get the existence result for (1) in Theorem 12.

THEOREM 12. *If  $(I - W)$  is non-singular matrix then for any  $b$ , AVE (1) has a solution, if  $(I - W)^{-1}(I + W)$  is a  $Q$ -matrix.*

*Proof.* If matrix  $(I - W)$  is non-singular, then (1) can be written as the given below form:

$$x^+ = (I - W)^{-1}(I + W)x^- + (I - W)^{-1}b,$$

where  $x^+$  and  $x^-$  are defined in (5). Which is a linear complementarity problem form. Then, with the help of Lemma 7, we get our result.  $\square$

Now, by substituting  $V = I$  in Lemma 8, we will get the necessary and sufficient conditions for AVE (1) see Theorem 13 and Theorem 14.

**THEOREM 13.** *For any diagonal matrix  $\hat{D}$ , the matrix*

$$(I - W + 2W\hat{D})$$

*is non-singular if and only if for any  $b$ , AVE (1) has a unique solution.*

**THEOREM 14.** *For any diagonal matrix  $\hat{D}$ , the matrix  $(I + W - 2W\hat{D})$  is non-singular if and only if there exists a unique solution of AVE (1) for any  $b$ .*

Based on the simple result “if the smallest singular value of a matrix is greater than 0, then the given matrix is non-singular”, we get the following two results, see Theorem 15 and Theorem 16.

**THEOREM 15.** *If  $\sigma_{\min}(I + W) > 2\sigma_{\max}(W\hat{D})$  or  $\sigma_{\min}(I - W) > 2\sigma_{\max}(W\hat{D})$  for any diagonal matrix  $\hat{D}$ , then there exists a unique solution of AVE (1) for any  $b$ .*

*Proof.* By Lemma 10, we have  $\sigma_{\min}(I + W - 2W\hat{D}) \geq \sigma_{\min}(I + W) - 2\sigma_{\max}(W\hat{D})$ .

If  $\sigma_{\min}(I + W) > 2\sigma_{\max}(W\hat{D})$ , then matrix  $I + W - 2W\hat{D}$  is nonsingular for any diagonal matrix  $\hat{D}$ . So by the previous result of the Theorem 14, AVE (1) has a unique solution. By Similar way with the help of Theorem 13 we can prove that if  $\sigma_{\min}(I - W) > 2\sigma_{\max}(W\hat{D})$  then AVE (1) has a unique solution.  $\square$

Moreover, the result of the Theorem 15 can be extended for the GAVE (2).

**THEOREM 16.** *If  $\sigma_{\min}(W(2\hat{D} - I)) > 1$  for any diagonal matrix  $\hat{D}$ , then AVE (1) has a unique solution for any  $b$ .*

*Proof.* Based on Lemma 10, we have

$$\sigma_{\min}(I - W + 2W\hat{D}) \geq \sigma_{\min}(2W\hat{D} - W) - \sigma_{\max}(I).$$

Since  $\sigma_{\max}(I) = 1$ , so if  $\sigma_{\min}(2W\hat{D} - W) > 1$  then matrix  $I + W - 2W\hat{D}$  is nonsingular for any diagonal matrix  $\hat{D}$ , so the AVE (1) has a unique solution for any  $b$ .  $\square$

Based on Lemma 4, we get the following result for GAVE (2).

**THEOREM 17.** For any diagonal matrix  $\hat{D}$ , if  $\|W^{-1}V - 2\hat{D}\|_2 < 1$  or  $\|W^{-1}V + 2\hat{D}\|_2 < 1$  then for any  $b$ , the AVE (2) has a unique solution, where  $W$  is a non-singular matrix.

*Proof.* By the help of Lemma 4, if  $\|W^{-1}V - 2\hat{D}\|_2 < 1$ , then  $W^{-1}V - 2\hat{D} + I$  is non-singular, since  $W$  is non-singular, so  $W[W^{-1}V - 2\hat{D} + I]$  is also non-singular, then by Lemma 9, AVE (2) has a unique solution for any  $b$ .

Or, when  $\|W^{-1}V + 2\hat{D}\|_2 < 1$ , then matrix  $W^{-1}V + 2\hat{D} - I$  is non-singular, so  $W[W^{-1}V + 2\hat{D} - I]$  is also non-singular, then by Lemma 9, AVE (2) has a unique solution for any  $b$ .  $\square$

**COROLLARY 18.** There exists a unique solution of AVE (3) if  $\|(V + 2\hat{D})\|_2 < 1$  or  $\|(V - 2\hat{D})\|_2 < 1$  for any diagonal matrix  $\hat{D}$ .

By substituting  $W = I$  in Theorem 17, then Corollary 18 will become the main result of the [21, Th. 3.4]).

Based on Theorem 13, Theorem 14 and Lemma 4, we get the following result.

**THEOREM 19.** For any diagonal matrix  $\hat{D}$ , matrix  $W$  satisfies  $\|W - 2W\hat{D}\|_2 < 1$  or  $\|-W + 2W\hat{D}\|_2 < 1$  then for any  $b$ , AVE (1) has a unique solution.

*Proof.* With the aid of Lemma 4, if  $\|W - 2W\hat{D}\|_2 < 1$ , then  $W - 2W\hat{D} + I$  is non-singular, then by Theorem 14, AVE (1) has unique solution.

Or, when  $\|-W + 2W\hat{D}\|_2 < 1$ , then matrix  $-W + 2W\hat{D} + I$  is non-singular, then by Theorem 13, AVE (1) has unique solution.  $\square$

Based on Lemma 5 and Lemma 8, we get the following result.

**THEOREM 20.** If matrix  $V$  and non-singular matrix  $W$  will satisfy the conditions  $\|(W^{-1}V - I)^{-1}\|_2 < 2$  or  $\|(W^{-1}V + I)^{-1}\|_2 < 2$  then for any  $b$ , AVE (2) has a unique solution.

*Proof.* Let

$$(7) \quad \begin{aligned} (i) \quad (V - W + 2W\hat{D}) &= 2W \left[ \frac{1}{2}W^{-1}(V - W) + \hat{D} \right] \\ (ii) \quad (V + W - 2W\hat{D}) &= 2W \left[ \frac{1}{2}W^{-1}(V + W) - \hat{D} \right]. \end{aligned}$$

With the aid of Lemma 5, when  $\|\frac{1}{2}(W^{-1}V - I)^{-1}\|_2 < 1$ , then  $\frac{1}{2}(W^{-1}V - I) + \hat{D}$  is non-singular. By (7) and Lemma 8 our proof is complete.

Or, when  $\|\frac{1}{2}(W^{-1}V + I)^{-1}\|_2 < 1$ , then matrix  $\frac{1}{2}(W^{-1}V + I) - \hat{D}$  is non-singular. Then, by (7) and Lemma 8 our proof is completed.  $\square$

**COROLLARY 21.** There exists a unique solution of AVE (3) if  $\|(V - I)^{-1}\|_2 < 2$  or  $\|(V + I)^{-1}\|_2 < 2$ .

By substituting  $W = I$  in Theorem 20, then Corollary 21 will become Theorem 3.5 as a main result in [21].

**COROLLARY 22.** *If  $\|(W^{-1} - I)^{-1}\|_2 < 2$  or  $\|(W^{-1} + I)^{-1}\|_2 < 2$ , then there exists a unique solution of AVE (1).*

We get our result by substituting  $V = I$  in [Theorem 20](#).

**REMARK 23.** Sometime the condition  $\|(V - I)^{-1}\|_2 < 2$  given in [Corollary 21](#) is not satisfying even AVE  $Vx - |x| = b$  has a unique solution.  $\square$

Let us consider the following example.

**EXAMPLE 24.** The AVE  $Vx - |x| = (2, -2)^T$  ( $T$  denotes the transpose of the vector  $(2, -2)$ ) has a unique solution  $x = (20, -0.909091)^T$ , for

$$V = \begin{bmatrix} 1.1 & 0.0 \\ 0.0 & 1.2 \end{bmatrix}.$$

Clearly,  $\|(V - I)^{-1}\|_2 = 10 \not< 2$ .  $\square$

**REMARK 25.** By the result  $\sigma_{\max}(W)\sigma_{\min}(W^{-1}) = 1$ , the condition  $\sigma_{\max}(W) < 1$  of [Lemma 6](#) is equivalent to the condition  $\sigma_{\min}(W^{-1}) > 1$ . Sometimes, matrix  $W$  does not satisfy the conditions  $\sigma_{\max}(W) < 1$  or  $\sigma_{\min}(W^{-1}) > 1$  but AVE (1) has a unique solution for some  $b$ .  $\square$

Let us consider the following example.

**EXAMPLE 26.** Consider the matrix

$$W = \begin{bmatrix} -0.9 & 0.2 \\ -0.4 & -0.8 \end{bmatrix}.$$

Here  $\sigma_{\min}(W^{-1}) = 0.9870 \not> 1$  and  $\sigma_{\max}(W) = 1.0132 \not< 1$ , even AVE  $x - W|x| = b$  has a unique solution  $x = (2.66667, -15.33334)^T$  for some  $b = (2, -2)^T$  ( $T$  denotes the transpose).
















Furthermore, the conditions  $\sigma_{\min}(W) < 1$  and  $\sigma_{\max}(W^{-1}) > 1$  maybe replace the existing conditions  $\sigma_{\max}(W) < 1$  and  $\sigma_{\min}(W^{-1}) > 1$  respectively, because of we have inequalities  $\sigma_{\min}(W) \leq \sigma_{\max}(W) < 1$  and  $\sigma_{\max}(W^{-1}) \geq \sigma_{\min}(W^{-1}) > 1$ . Also for [Example 26](#) we have  $\sigma_{\max}(W^{-1}) = 1.2665 > 1$  and  $\sigma_{\min}(W) = 0.7896 < 1$ . So existing singular value conditions may be revised in future. This needs further investigation. These conditions may be or may not be superior then the current singular value conditions.  $\square$





#### 4. CONCLUSION

This article discusses the existence of solution possibilities in the uniqueness of the GAVE  $Vx - W|x| = b$  and its particular cases. The conditions of [Lemma 6](#), [Corollary 21](#) and [Corollary 22](#) may be revised in the future. With the help of the equivalence relation of LCP and AVE, we established some more conditions for the unique solvability of (1). The numerical results for the absolute value equations are also an interesting topic in the future.

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