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ON THE NUMERICAL SOLUTION OF VOLTERRA AND FREDHOLM INTEGRAL EQUATIONS USING THE FRACTIONAL SPLINE FUNCTION METHOD

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Abstract. In this article, the researchers develop a new type of Spline function with fractional order which constructs two distinct formulas for the proposed method by using fractional boundary conditions and fractional continuity conditions. These methods are used to solve linear Volterra and Fredholm-integral equations of the second kind. The convergence analysis is studied. Moreover, some numerical examples are provided and compared to illustrate the efficiency and applicability of the proposed methods.

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1. INTRODUCTION

Numerical analysis has been used more and more in the fields of applied mathematics since the beginning of the twentieth century, since spline functions are simple to analyze and work with on computer see [2], [3], [10] and [6], they are currently one of the most popular areas of approximation theory. These functions have an important role in mathematics and its technological applications, and also play a significant role in solving integral equations, integro-differential equations, and differential equations.

Currently, many researchers used fractional calculus to derive problems in mathematical science, computer science, physical science and engineering, see [7], [12], [9] and [13], the fractional operator has many definitions, including the Riemann's, Liouville's, Gruwald-Letnikov's, Riemann-Liouville's, and Caputo's fractional integrals and derivatives.

Recently, some works have been done on this model type, but with natural orders, see [1], [7],[14], [10], [8], [4], and [15], so that is why the proposed technique will be the beginning of a more detailed work on fractional models and perhaps it can progress the topic.

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This work is organized as follows: the researchers proposed two new types of fractional Spline method (FSM) to solve Volterra and Fredholm-integral equations by using two systems of equations as shown in Section 2 and Section 3, in Section 4 the convergence of these two fractional Spline models have been investigated, in Section 5 numerical examples are presented to illustrate the applications and effectiveness of the approach. In Section 6, a conclusion is given.

2. THE FRACTIONAL SPLINE METHOD

Consider the linear integral equations of the second kind with the unknown function $z(t)$:

$$z(t) - \lambda \int_0^{p(t)} k(t,s)z(s)ds = y(t), \quad t,s \in X = [0,p(t)], \quad 0 \le t \le b. \tag{1}$$

Where $k(t,s)$ and $y(t)$ are known continuous functions defined on X, $p(t) = t$ (VIE) or $p(t) = b$ (FIE), b and λ are constants. We approximate (1) by using fractional Spline model, divided the domain X into $(N-1)$ sub intervals, N is equally spaced mesh points $s_1, s_2, \dots, s_N$ where $s_i = a + ih$, (h is step size), $i = 0, 1, \dots, n-1$.

$$S_i(s) = a_i(s-s_i)^{5/2} + b_i(s-s_i)^{3/2} + c_i(s-s_i)^{1/2} + d_i. \tag{2}$$

To derive the approximate solution of (1) we used two boundary, continuity conditions on (2) in the following form:

$$FSM1 \begin{cases} S(s_i) = Z_i, \\ S(s_{i+1}) = Z_{i+1}, \\ S^{(\frac{1}{2})}(s_i) = F_i, \\ S^{(\frac{1}{2})}(s_{i+1}) = F_{i+1}, \end{cases} \qquad FSM2 \begin{cases} S(s_i) = Z_i, \\ S(s_{i+1}) = Z_{i+1}, \\ S^{(\frac{3}{2})}(s_i) = \bar{F}_i, \\ S^{(\frac{3}{2})}(s_{i+1}) = \bar{F}_{i+1}. \end{cases} \tag{3}$$

By using algebraic manipulation of (2) and (3) we obtain the following (it can be easily verified that the spline scheme approximation $S(s)$, is successfully uniquely determined using (2) recurrence formula for all h in the interval):

$$FSM1 \begin{cases} a_i = \frac{4}{h^{\frac{5}{2}}}\left[Z_i - Z_{i+1}\right] + \frac{8}{3\sqrt{\pi}h^2}\left[F_i + 2F_{i+1}\right], \\ b_i = \frac{5}{h^{\frac{3}{2}}}\left[Z_{i+1} - Z_i\right] - \frac{2}{3\sqrt{\pi}h}\left[7F_i + 8F_{i+1}\right], \\ c_i = \frac{2}{\sqrt{\pi}}F_i, \qquad d_i = Z_i. \end{cases} \tag{4}$$

$$FSM2 \begin{cases} a_i = \frac{8}{15\sqrt{\pi}h}\left[\bar{F}_{i+1} - \bar{F}_i\right], \qquad b_i = \dfrac{4}{3\sqrt{\pi}}\bar{F}_i, \\ c_i = \frac{1}{\sqrt{h}}\left[Z_{i+1} - Z_i\right] - \frac{4h}{15\sqrt{\pi}}\left[3\bar{F}_i + 2\bar{F}_{i+1}\right], \\ d_i = Z_i. \end{cases}$$

From $D^{\frac{3}{2}}S_i(s_i) = D^{\frac{3}{2}}S_{i-1}(s_i)$ and $D^{\frac{1}{2}}S_i(s_i) = D^{\frac{1}{2}}S_{i-1}(s_i)$ for FSM1 and FSM2, respectively, we get the following equations:

$$FSM1\left\{3F_{i-1}+19F_i+8F_{i+1}=\frac{15\sqrt{\pi}}{2\sqrt{h}}\Big[Z_{i+1}-Z_{i-1}\Big],\right. \tag{5}$$

$$FSM2\left\{3\bar{F}_{i-1}+19\bar{F}_i+8\bar{F}_{i+1}=\frac{15\sqrt{\pi}}{h^{\frac{3}{2}}}\Big[Z_{i-1}-2Z_i+Z_{i+1}\Big].\right. \tag{6}$$

According to the Taylor formula for traditional calculus for all s in that neighborhood a,the function $y = z(s)$ has derivatives up to the order $(n+1)$, and the following is the general form of fractional Taylor series.

$$z^{(n)}(s)=\sum_{k=-\infty}^{\infty}\frac{z^{(k)}(a)s^{k-n}}{\Gamma(k-n+1)}.$$

To get a unique solution for (5) and (6), we get the following equations by using fractional Taylor series.

$$FSM1\begin{cases}3F_{i-1}+8F_{i+1}=\frac{\sqrt{\pi}}{2\sqrt{h}}\Big[8Z_{i+2}-8Z_{i+1}+3Z_i-3Z_{i-1}\Big],\\ 19F_i+8F_{i+1}=\frac{\sqrt{\pi}}{2\sqrt{h}}\Big[8Z_{i+2}+11Z_{i+1}-19Z_i\Big],\end{cases} \tag{7}$$

$$FSM2\begin{cases}3\bar{F}_{i-1}+8\bar{F}_{i+1}=\frac{\sqrt{\pi}}{4h^{\frac{3}{2}}}\Big[24Z_{i+2}-24Z_{i+1}+9Z_i-9Z_{i-1}\Big],\\ 19\bar{F}_i+8\bar{F}_{i+1}=\frac{\sqrt{\pi}}{4h^{\frac{3}{2}}}\Big[24Z_{i+2}+33Z_{i+1}-57Z_i\Big].\end{cases}$$

By expanding (5) and (6) with fractional order Taylor series about s_i, we obtained the following Truncation error.

(8)

$$T_i=\begin{cases}[\frac{14}{\sqrt{\pi}}]h^{\frac{3}{2}}Z_i^{(2)}+[\frac{11}{2}]h^2Z_i^{(\frac{5}{2})}+[\frac{88}{15}]h^{\frac{5}{2}}Z_i^{(3)}+\dots, & (FSM1)\\ [\frac{42}{3\sqrt{\pi}}]hZ_i^{(\frac{3}{2})}+[\frac{42}{3\sqrt{\pi}}+15\sqrt{\pi}]h^{\frac{3}{2}}Z_i^{(2)}+[\frac{43}{2}]h^2Z_i^{(\frac{5}{2})}+\dots. & (FSM2)\end{cases}$$

The matrix representation of (5) and (6) are $A_1F = B_1Z$ and $A_2\bar{F} = B_2Z$ respectively, from (2) and (4) we get

$$F=A_1^{-1}B_1Z=M_1Z, \qquad and \qquad \bar{F}=A_2^{-1}B_2Z=M_2Z. \tag{9}$$

Where A_1, A_2, B_1 and B_2 are three diagonal matrices of (5) and (6) respectively, and $F = (F_0, F_1, \dots, F_N)^T$, $\bar{F} = (\bar{F}_0, \bar{F}_1, \dots, \bar{F}_N)^T$, and $Z = (Z_0, Z_1, \dots, Z_N)^T$.

3. THE METHOD OF SOLUTION

In this section, we used the proposed fractional spline model discussed in Section 2 to derive problem for (1).

From (2) and (4) we get

(10)

$$S_i(s)=\begin{cases}\left[\frac{4Z_i-4Z_{i+1}}{h^{\frac{5}{2}}}+\frac{8F_i+16F_{i+1}}{3\sqrt{\pi}h^2}\right]\sqrt{(s-s_i)^5}+\left[\frac{5Z_{i+1}-5Z_i}{h^{\frac{3}{2}}}\right.\\ \left.+\frac{14F_i+t16F_{i+1}}{3\sqrt{\pi}h}\right]\sqrt{(s-s_i)^3}+\left[\frac{2F_i}{\sqrt{\pi}}\right]\sqrt{(s-s_i)}+Z_i+\mathcal{O}(h^4), \quad (FSM1)\\ \left[\frac{8\bar{F}_{i+1}-8\bar{F}_i}{15\sqrt{\pi}h}\right]\sqrt{(s-s_i)^5}+\left[\frac{4\bar{F}_i}{3\sqrt{\pi}}\right]\sqrt{(s-s_i)^3}\\ +\left[\frac{Z_{i+1}-Z_i}{h^{\frac{1}{2}}}-\frac{12h\bar{F}_i+8h\bar{F}_{i+1}}{15\sqrt{\pi}}\right]\sqrt{(s-s_i)}+Z_i+\mathcal{O}(h^4). \quad (FSM2)\end{cases}$$

By substituting (10) in (1) we get

$$\begin{aligned}z(t_j)&=\sum_{i=0}^{n-1}\lambda\int_{s_i}^{s_{i+1}}k(t_j,s)z(s)ds+y(t_j)\\&\approx\sum_{i=0}^{n-1}\lambda\int_{s_i}^{s_{i+1}}k(t_j,s)S_i(s)ds+y(t_j)+\mathcal{O}(h^4).\end{aligned}$$

We divide the above equation of two FSM of the following (FSM1):

$$\begin{aligned}z(t_j)=&y(t_j)+\sum_{i=0}^{n-1}\left[\frac{4Z_i}{h^{\frac{5}{2}}}+\frac{8F_i}{3\sqrt{\pi}h^2}\right]\lambda\int_{s_i}^{s_{i+1}}k(t_j,s)\sqrt{(s-s_i)^5}ds\\&+\sum_{i=0}^{n-1}\left[\frac{-4Z_{i+1}}{h^{\frac{5}{2}}}+\frac{16F_{i+1}}{3\sqrt{\pi}h^2}\right]\lambda\int_{s_i}^{s_{i+1}}k(t_j,s)\sqrt{(s-s_i)^5}ds\\&+\sum_{i=0}^{n-1}\left[\frac{-5Z_i}{h^{\frac{3}{2}}}-\frac{14F_i}{3\sqrt{\pi}h}\right]\lambda\int_{s_i}^{s_{i+1}}k(t_j,s)\sqrt{(s-s_i)^3}ds\\&+\sum_{i=0}^{n-1}\left[\frac{5Z_{i+1}}{h^{\frac{3}{2}}}+\frac{16F_{i+1}}{3\sqrt{\pi}h}\right]\lambda\int_{s_i}^{s_{i+1}}k(t_j,s)\sqrt{(s-s_i)^3}ds\\&+\sum_{i=0}^{n-1}\left[\frac{2F_i}{\sqrt{\pi}}\right]\lambda\int_{s_i}^{s_{i+1}}k(t_j,s)\sqrt{(s-s_i)}ds+\sum_{i=0}^{n-1}\left[Z_i\right]\lambda\int_{s_i}^{s_{i+1}}k(t_j,s)ds+\mathcal{O}(h^4)\\=&y(t_j)+\sum_{i=0}^{n-1}\left[\frac{4Z_i}{h^{\frac{5}{2}}}+\frac{8F_i}{3\sqrt{\pi}h^2}\right]a_{j,i}+\left[\frac{-4Z_{i+1}}{h^{\frac{5}{2}}}+\frac{16F_{i+1}}{3\sqrt{\pi}h^2}\right]b_{j,i}\\&+\left[\frac{-5Z_i}{h^{\frac{3}{2}}}-\frac{14F_i}{3\sqrt{\pi}h}\right]c_{j,i}+\left[\frac{5Z_{i+1}}{h^{\frac{3}{2}}}+\frac{16F_{i+1}}{3\sqrt{\pi}h}\right]d_{j,i}+\left[\frac{2F_i}{\sqrt{\pi}}\right]e_{j,i}+\left[Z_i\right]f_{j,i}+\mathcal{O}(h^4).\end{aligned}$$

and the following (FSM2):

$$\begin{aligned}
z(t_j) =& y(t_j) + \sum_{i=0}^{n-1}\left[\tfrac{-8\bar{F}_i}{15\sqrt{\pi}h}\right]\lambda\int_{s_i}^{s_{i+1}} k(t_j,s)\sqrt{(s-s_i)^5}ds+\\
&+\sum_{i=0}^{n-1}\left[\tfrac{8\bar{F}_{i+1}}{15\sqrt{\pi}h}\right]\lambda\int_{s_i}^{s_{i+1}} k(t_j,s)\sqrt{(s-s_i)^5}ds+\\
&+\sum_{i=0}^{n-1}\left[\tfrac{4\bar{F}_i}{3\sqrt{\pi}}\right]\lambda\int_{s_i}^{s_{i+1}} k(t_j,s)\sqrt{(s-s_i)^3}ds+\\
&+\sum_{i=0}^{n-1}\left[\tfrac{-Z_i}{\sqrt{h}}-\tfrac{4h\bar{F}_i}{5\sqrt{\pi}}\right]\lambda\int_{s_i}^{s_{i+1}} k(t_j,s)\sqrt{(s-s_i)}ds\\
&+\sum_{i=0}^{n-1}\left[\tfrac{Z_{i+1}}{\sqrt{h}}-\tfrac{8h\bar{F}_i}{15\sqrt{\pi}}\right]\lambda\int_{s_i}^{s_{i+1}} k(t_j,s)\sqrt{(s-s_i)}ds+\\
&+\sum_{i=0}^{n-1}\left[Z_i\right]\lambda\int_{s_i}^{s_{i+1}} k(t_j,s)ds+\mathcal{O}(h^4)=\\
=& y(t_j)+\sum_{i=0}^{n-1}\left[\tfrac{-8\bar{F}_i}{15\sqrt{\pi}h}\right]\bar{a}_{j,i}+\sum_{i=0}^{n-1}\left[\tfrac{8\bar{F}_{i+1}}{15\sqrt{\pi}h}\right]\bar{b}_{j,i}+\sum_{i=0}^{n-1}\left[\tfrac{4\bar{F}_i}{3\sqrt{\pi}}\right]\bar{c}_{j,i}+\\
&+\sum_{i=0}^{n-1}\left[\tfrac{-Z_i}{\sqrt{h}}-\tfrac{4h\bar{F}_i}{5\sqrt{\pi}}\right]\bar{d}_{j,i}+\sum_{i=0}^{n-1}\left[\tfrac{Z_{i+1}}{\sqrt{h}}-\tfrac{8h\bar{F}_i}{15\sqrt{\pi}}\right]\bar{e}_{j,i}+\sum_{i=0}^{n-1}\left[Z_i\right]\bar{f}_{j,i}+\mathcal{O}(h^4),
\end{aligned}$$

where

$$a_{j,i}=b_{j,i+1}=\bar{a}_{j,i}=\bar{b}_{j,i+1}=\lambda\int_{s_i}^{s_{i+1}} k(t_j,s)\sqrt{(s-s_i)^5}ds,$$

$$c_{j,i}=d_{j,i+1}=\bar{c}_{j,i}=\lambda\int_{s_i}^{s_{i+1}} k(t_j,s)\sqrt{(s-s_i)^3}ds,$$

$$e_{j,i}=\bar{d}_{j,i}=\bar{e}_{j,i+1}=\lambda\int_{s_i}^{s_{i+1}} k(t_j,s)\sqrt{(s-s_i)}ds,$$

and $$f_{j,i}=\bar{f}_{j,i}=\lambda\int_{s_i}^{s_{i+1}} k(t_j,s)ds.$$

We introduce that $a_{j,n}=b_{j,0}=\bar{a}_{j,n}=\bar{b}_{j,0}=c_{j,n}=d_{j,0}=\bar{c}_{j,n}=e_{j,n}=\bar{d}_{j,n}=\bar{e}_{j,0}=f_{j,n}=\bar{f}_{j,n}=0$, so we can write the matrix representation as:

$$A=a_{j,i}, B=b_{j,i}, \bar{A}=\bar{a}_{j,i}, \bar{B}=\bar{b}_{j,i}, C=c_{j,i}, d=d_{j,i},$$

$$\bar{C}=\bar{c}_{j,i}, E=e_{j,i}, \bar{D}=\bar{d}_{j,i}, \bar{E}=\bar{e}_{j,i}, F=f_{j,i},$$

and $\bar{F}=\bar{F}_{j,i}$, also $Y=(y_0,y_1,\ldots,y_n)^T, Z=(z_0,z_1,\ldots,z_n)^T, F=(f_0,f_1,\ldots,f_n)^T$ and $\bar{F}=(\bar{f}_0,\bar{f}_1,\ldots,\bar{f}_n)^T$. So we have:

(11)

$$FSM1\left\{Z=Y+\left[\tfrac{4A-4B}{h^{\frac{5}{2}}}+\tfrac{5D-5C}{h^{\frac{3}{2}}}+F\right]Z+\left[\tfrac{8A+16B}{3\sqrt{\pi}h^2}+\tfrac{16D-14C}{3\sqrt{\pi}h}+\tfrac{2E}{\sqrt{\pi}}\right]F\right.,$$

$$FSM2\left\{Z = Y + \left[\frac{\bar{E}-\bar{D}}{\sqrt{h}} + \bar{F}\right]Z + \left[\frac{8\bar{B}-8\bar{A}}{15\sqrt{\pi}h} + \bar{F} + \frac{4\bar{C}}{3\sqrt{\pi}} - \frac{8h\bar{D}-8h\bar{E}}{15\sqrt{\pi}}\right]\bar{F}.\right.$$

By substituting (9) in (11) we get:

$$(12)\qquad FSM1\left\{[I-P-P_1M_1]Z = Y \implies Z = [I-P-P_1M_1]^{-1}Y\ ,\right.$$

$$FSM2\left\{[I-\bar{P}-\bar{P}_1M_2]Z = Y \implies Z = [I-\bar{P}-P_1M_2]^{-1}Y.\right.$$

We consider that the exact solution of (10) is:

$$(13)\quad \hat{S}_i(s) = \begin{cases} \left[\frac{4\hat{Z}_i-4\hat{Z}_{i+1}}{h^{\frac{5}{2}}} + \frac{8\hat{F}_i+16\hat{F}_{i+1}}{3\sqrt{\pi}h^2}\right]\sqrt{(s-s_i)^5} \\ +\left[\frac{5\hat{Z}_{i+1}-5\hat{Z}_i}{h^{\frac{3}{2}}} + \frac{14\hat{F}_i+16\hat{F}_{i+1}}{3\sqrt{\pi}h}\right]\sqrt{(s-s_i)^3} \\ +\left[\frac{2\hat{F}_i}{\sqrt{\pi}}\right]\sqrt{(s-s_i)} + \hat{Z}_i + \mathcal{O}(h^4).\ (FSM1) \\ \left[\frac{8\hat{\bar{F}}_{i+1}-8\hat{\bar{F}}_i}{15\sqrt{\pi}h}\right]\sqrt{(s-s_i)^5} + \left[\frac{4\hat{\bar{F}}_i}{3\sqrt{\pi}}\right]\sqrt{(s-s_i)^3} \\ +\left[\frac{\hat{Z}_{i+1}-\hat{Z}_i}{h^{\frac{1}{2}}} - \frac{12h\hat{\bar{F}}_i+8h\hat{\bar{F}}_{i+1}}{15\sqrt{\pi}}\right]\sqrt{(s-s_i)} + \hat{Z}_i + \mathcal{O}(h^4).(FSM2) \end{cases}$$

where $\hat{Z}_i, \hat{F}_i$ and $\hat{\bar{F}}_i$ are the exact solutions of $\hat{Z}_i, \hat{Z}_i^{(\frac{1}{2})}$ and $\hat{Z}_i^{(\frac{3}{2})}$ respectively. By subtracting (13) and (10) we get the following

$$\hat{e} = S_i(s)-\hat{S}_i(s) = \begin{cases} \left[\frac{4(Z_i-\hat{Z}_i)-4(Z_{i+1}-\hat{Z}_{i+1})}{h^{\frac{5}{2}}} + \frac{8(F_i-\hat{F}_i)+16(F_{i+1}-\hat{F}_{i+1})}{3\sqrt{\pi}h^2}\right]\sqrt{(s-s_i)^5} \\ +\left[\frac{5(Z_{i+1}-\hat{Z}_{i+1})-5(Z_i-\hat{Z}_i)}{h^{\frac{3}{2}}} + \frac{14(F_i-\hat{F}_i)+16(F_{i+1}-\hat{F}_{i+1})}{3\sqrt{\pi}h}\right]\sqrt{(s-s_i)^3} \\ +\left[\frac{2(F_i-\hat{F}_i)}{\sqrt{\pi}}\right]\sqrt{(s-s_i)} + (Z_i-\hat{Z}_i) + \mathcal{O}(h^4), \quad (FSM1) \\ \left[\frac{8(\bar{F}_{i+1}-\hat{\bar{F}}_{i+1})-8(\bar{F}_i-\hat{\bar{F}}_i)}{15\sqrt{\pi}h}\right]\sqrt{(s-s_i)^5} + \left[\frac{4(\bar{F}_i-\hat{\bar{F}}_i)}{3\sqrt{\pi}}\right]\sqrt{(s-s_i)^3} \\ +\left[\frac{(Z_{i+1}-\hat{Z}_{i+1})-(Z_i-\hat{Z}_i)}{h^{\frac{1}{2}}} - \frac{12h(\bar{F}_i-\hat{\bar{F}}_i)+8h(\bar{F}_{i+1}-\hat{\bar{F}}_{i+1})}{15\sqrt{\pi}}\right]\sqrt{(s-s_i)} \\ +\hat{Z}_i + \mathcal{O}(h^4). \quad (FSM2) \end{cases}$$

Hence

$$(14)\qquad |\hat{e}| \equiv \begin{cases} \beta_0 h^4, \\ \bar{\beta}_0 h^4, \end{cases}$$

where β_0 and $\bar{\beta}_0$ are constants.

4. CONVERGENCE ANALYSIS

The convergence of the proposed methods have been proved in this section.

LEMMA 1 (see [9]). *If L is a square Matrix with $\|L\|_\infty < 1$, then the matrix $(I-L)^{-1}$ is exist, and* $\|(I-L)^{-1}\|_\infty \le \frac{1}{1-\|L\|_\infty}$.

LEMMA 2. *The matrices $[I-P-P_1M_1]$ and $[I-\bar{P}-\bar{P}_1M_2]$ in (12) are invertible if:*

$$\|k\|_\infty(b-a)(10\mu_1+1) < 1, \quad (FSM1),$$

and

$$\|k\|_\infty(b-a)(8\mu_2+1) < 1, \quad (FSM2).$$

Proof. For $i = 0, 1, \dots, n$, we have

$$\begin{aligned}
\|A\|_\infty = \|B\|_\infty = \|\bar{A}\|_\infty = \|\bar{B}\|_\infty &\le \|k\|_\infty(b-a)\tfrac{2h^{\frac{5}{2}}}{7},\\
\|C\|_\infty = \|\bar{C}\|_\infty = \|D\|_\infty &\le \|k\|_\infty(b-a)\tfrac{2h^{\frac{3}{2}}}{5},\\
\|E\|_\infty = \|\bar{E}\|_\infty = \|\bar{D}\|_\infty &\le \|k\|_\infty(b-a)\tfrac{2h^{\frac{1}{2}}}{3},\\
\|F\|_\infty = \|\bar{F}\|_\infty &\le \|k\|_\infty(b-a),\\
\|P\|_\infty &\le \|k\|_\infty(b-a),\\
\|P_1\|_\infty &\le \tfrac{4\sqrt{h}}{3\sqrt{\pi}}\|k\|_\infty(b-a),\\
\|\bar{P}\|_\infty &\le \|k\|_\infty(b-a),\\
\|\bar{P}_1\|_\infty &\le \tfrac{8h^{\frac{3}{2}}}{15\sqrt{\pi}}\|k\|_\infty(b-a).
\end{aligned}$$

By using Lemma 1 the matrices $[I-P-P_1M_1]$ and $[I-\bar{P}-\bar{P}_1M_2]$ are invertible, if

$$[P+P_1M_1] < 1,$$
$$\|k\|_\infty(b-a)(10\mu_1+1) < 1, \quad (FSM1)$$

and

$$[\bar{P}+\bar{P}_1M_2] < 1,$$
$$\|k\|_\infty(b-a)(8\mu_2+1) < 1, \quad (FSM2)$$

where μ_1 is the matrix product of A_1^{-1} and B_1 from (9) and μ_2 is the matrix product of A_2^{-1} and B_2 from (9). □

THEOREM 3. *We assume that $y(t) \in C^4(I)$ and $K(t,s) \in C^4(I \times I)$, in a way that*

$$\|k\|_\infty(b-a)(10\mu_1+1) < 1,$$
$$\|k\|_\infty(b-a)(8\mu_2+1) < 1.$$

Considering s the unique approximation solutions and the error $e = Z - \bar{S}$ and $\bar{e} = Z - \hat{\bar{S}}$, they satisfy $\|e\|_\infty \le \mu_5(h^{\frac{3}{2}})$, and $\|\bar{e}\|_\infty \le \mu_6(h)$ respectively.

Proof. From (14) we can write

$$\begin{cases} \|S_i - \hat{S}_i\|_\infty \le \beta_0(h^4), \\ \|S_i - \hat{\hat{S}}_i\|_\infty \le \bar{\beta}_0(h^4). \end{cases} \tag{15}$$

Let $\hat{Z}$ be the exact solution, T and $\bar{T}$ are the local truncation errors vector of FSM1 and FSM2 respectively, so, from (12) we have

$$\begin{cases} [I - P - M_1P_1]\hat{Z} = Y + T, & (FSM1) \\ [I - \bar{P} - M_2\bar{P}_1]\hat{Z} = Y + \bar{T}. & (FSM2) \end{cases} \tag{16}$$

Subtracting (16) in (12) we get

$$\begin{cases} [I - P - M_1P_1]e = T \implies e = [I - P - M_1P_1]^{-1}T, \\ [I - \bar{P} - M_2\bar{P}_1]\hat{e} = \bar{T} \implies \hat{e} = [I - \bar{P} - M_2\bar{P}_1]^{-1}\bar{T}. \end{cases} \tag{17}$$

So, from the local truncation error (8) and (17)

$$\|T\|_\infty \le \frac{14}{\sqrt{\pi}}h^{\frac{3}{2}}\mu_3 \qquad \textit{and} \qquad \|\bar{T}\|_\infty \le \frac{42}{3\sqrt{\pi}}h\mu_4,$$

where $\mu_3 = \max_{t_i \le \alpha_1 \le t_{i+1}} z^{(2)}(\alpha_1)$, $\mu_4 = \max_{t_i \le \alpha_2 \le t_{i+1}} z^{(\frac{3}{2})}(\alpha_2)$, α_1 and α_2 are constants,

$$\begin{cases} \|e\|_\infty \le \|(I - P - M_1P_1)^{-1}\|_\infty \|T\|_\infty, \\ \|\hat{e}\|_\infty \le \|(I - \bar{P} - M_2\bar{P}_1)^{-1}\|_\infty \|\bar{T}\|_\infty. \end{cases} \tag{18}$$

Then, from Lemma 1, Lemma 2 and (18), we get

$$\begin{cases} \|e\|_\infty \le \frac{\|T\|_\infty}{1 - \|k\|_\infty (b-a)(10\mu_1 + 1)}, \\ \|\hat{e}\|_\infty \le \frac{\|\bar{T}\|_\infty}{1 - \|k\|_\infty (b-a)(8\mu_2 + 1)}, \end{cases} \tag{19}$$

$$\therefore \begin{cases} \|\hat{e}\|_\infty \le \mu_5 h^{\frac{3}{2}}, \\ \|\hat{\hat{e}}\|_\infty \le \mu_6 h, \end{cases} \tag{20}$$

where $\mu_5 = \frac{14h^{\frac{3}{2}}\mu_3}{\sqrt{\pi}\alpha_3}$, $\mu_6 = \frac{42h\mu_4}{3\sqrt{\pi}\alpha_4}$, $\alpha_3 = 1 - \|k\|_\infty(b-a)(10\mu_1 + 1) < 1$ and $\alpha_4 = 1 - \|k\|_\infty(b-a)(8\mu_2 + 1) < 1$. Now, from (15) and (19) we get:

$$\|Z - \hat{S}_i\|_\infty \le \|Z - S_i\|_\infty + \|S_i - \hat{S}_i\|_\infty \le \beta_1 h^{\frac{3}{2}}$$

$$\|Z - \hat{\hat{S}}_i\|_\infty \le \|Z - S_i\|_\infty + \|S_i - \hat{\hat{S}}_i\|_\infty \le \beta_2 h,$$

where $\beta_1 = \mu_5 + \beta_0$ and $\beta_2 = \mu_6 + \bar{\beta}_0$, so the convergence of the (one and a half) order and (first) order of FSM1 and FSM2 are explained respectively, because we can write $\|E\| \to 0$ as $h \to 0$. □

5. TEST PROBLEMS

In this section, we present some test problems to show the efficiency of the FSM, the Python program is used to obtain the results. The numerical results reported in tables includes the errors and comparisons of the exact and approximation solutions and explained in figures. The absolute error $\|E\|$ is used to measure the errors between different points, and mean absolute error(MAE) is used to measure the errors between different step sizes.

$$\|E_n\| = \|z_n(t) - z(t)\|_2, \quad t \in [0, p(t)], \quad p(t) = \begin{cases} b, & FIE, \\ t, & VIE. \end{cases}$$

$$MAE = \frac{\|E_n\|}{n}.$$

EXAMPLE 4. The Fredholm integral equation of the second kind

$$z(t) - \int_{-1}^{1} e^{t-s-3} z(s) ds = 1 - e^{t-2} + e^{t-4}.$$

$z(t) = 1$ is the exact solution, Table 1, Fig. 5.1 and Fig. 5.2 presents the errors, and it is clear that the proposed methods are more accurate than all methods mentioned in legend.

t	Best $\|E\|$ in [4]	Best $\|E\|$ in [5]	Best $\|E\|$ in [3]	Best $\|E\|$ in [6]	Best $\|E\|$ in FSM1 and FSM2
0	1.5×10^{-4}	3.9×10^{-5}	8.2×10^{-15}	2.2×10^{-16}	$< 1 \times 10^{-16}$
0.1	1.9×10^{-4}	4.8×10^{-5}	3.4×10^{-16}	2.3×10^{-16}	1.1×10^{-16}
0.2	2.3×10^{-4}	5.9×10^{-5}	1.2×10^{-16}	3.3×10^{-16}	$< 1 \times 10^{-16}$
0.3	2.9×10^{-4}	7.2×10^{-5}	4.2×10^{-16}	4.4×10^{-16}	1.1×10^{-16}
0.4	3.5×10^{-4}	8.8×10^{-5}	2.1×10^{-16}	1.1×10^{-16}	$< 1 \times 10^{-16}$
0.5	4.3×10^{-4}	1.0×10^{-4}	3.5×10^{-16}	4.4×10^{-16}	$< 1 \times 10^{-16}$
0.6	5.2×10^{-4}	1.3×10^{-4}	5.6×10^{-16}	2.2×10^{-16}	$< 1 \times 10^{-16}$
0.7	6.4×10^{-4}	1.6×10^{-4}	7.2×10^{-16}	2.2×10^{-16}	$< 1 \times 10^{-16}$
0.8	7.8×10^{-4}	1.9×10^{-4}	1.1×10^{-15}	2.2×10^{-16}	$< 1 \times 10^{-16}$
0.9	9.6×10^{-4}	2.4×10^{-4}	3.6×10^{-15}	1.1×10^{-16}	$< 1 \times 10^{-16}$

Table 1. Comparison of $\|E\|$ for Example 4 with h=0.1.

EXAMPLE 5. The Fredholm integral equation of the second kind

$$z(t) - \int_{0}^{1} e^{s-t-12} z(s) ds = \cos(t) - \tfrac{e^{-t-12}}{2}[e \sin(1) + e \cos(1) - 1].$$

$z(t) = \cos(t)$ is the exact solution, in Table 2 and Fig. 5.3 presented the numerical solutions and in Table 3 and Fig. 5.4 presented the errors, and it is clear that the FSMs are more accurate than all methods mentioned in legend.

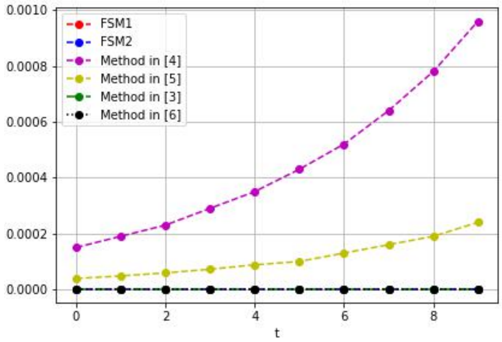

Fig. 5.1. Comparison $\|E\|$ between methods.

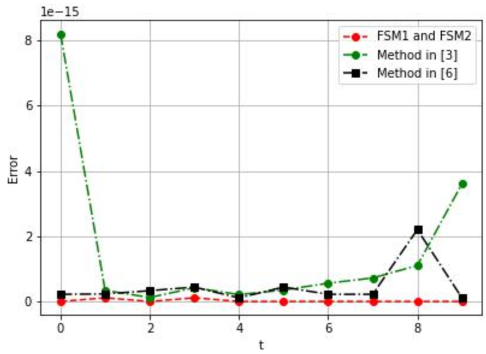

Fig. 5.2. Comparison $\|E\|$ between FSM1, FSM2, methods in [3] and [6].

t	Exact solution	FSM1	FSM2
0	1	1.0000051988836043	1.0000051476505354
0.1	0.9999619230641713	0.9999649297946219	0.9999650227955869
0.2	0.9998476951563913	0.9998497279264817	0.9998496004109405
0.3	0.9996573249755573	0.999658274299757	0.999658444784708
0.4	0.9993908270190958	0.9993917805147041	0.9993915522948859
0.5	0.9990482215818578	0.9990482934885683	0.9990485985854253
0.6	0.9986295347545738	0.9986302549465026	0.9986298468523224
0.7	0.9981347984218669	0.998134330240686	0.9981348759646125
0.8	0.9975640502598242	0.9975649764944626	0.997564246641432
0.9	0.996917333733128	0.996916276990419	0.9969172530472656

Table 2. Numerical solution of Example 5 of mesh points with h=0.5.

$\|E\|$ in [11] with M=20	$\|E\|$ in [3] with n=2	$\|E\|$ in FSM1 with n=2	$\|E\|$ in FSM2 with n=2
1.5×10^{-5}	1.2×10^{-5}	5.2×10^{-6}	5.4×10^{-6}
1.9×10^{-5}	2.3×10^{-5}	3.0×10^{-6}	3.1×10^{-6}
2.3×10^{-5}	3.3×10^{-6}	2.0×10^{-6}	1.9×10^{-6}
2.9×10^{-5}	2.0×10^{-6}	9.5×10^{-7}	1.1×10^{-6}
3.5×10^{-5}	1.0×10^{-6}	9.5×10^{-7}	7.3×10^{-7}
4.3×10^{-5}	6.2×10^{-7}	7.2×10^{-8}	3.8×10^{-7}
5.2×10^{-5}	7.9×10^{-7}	7.2×10^{-7}	3.1×10^{-7}
6.4×10^{-5}	3.4×10^{-6}	4.7×10^{-7}	7.8×10^{-8}
7.8×10^{-5}	3.0×10^{-6}	9.3×10^{-7}	1.9×10^{-7}
9.6×10^{-5}	4.4×10^{-6}	1.1×10^{-6}	8.1×10^{-8}

Table 3. Comparison of $\|E\|$ for Example 5.

EXAMPLE 6. The Volterra integral equation of the second kind

$$z(t) + \int_0^t (t-s)z(s)ds = t, t \in [0,1).$$

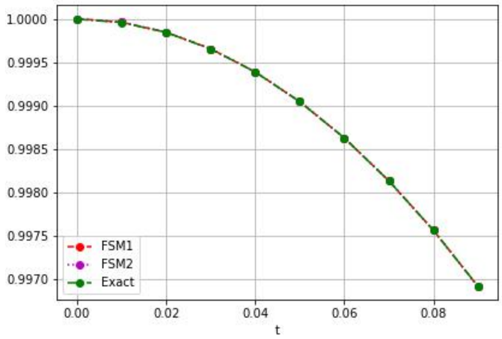

Fig. 5.3. Comparison between FSM1, FSM2 and Exact solutions.

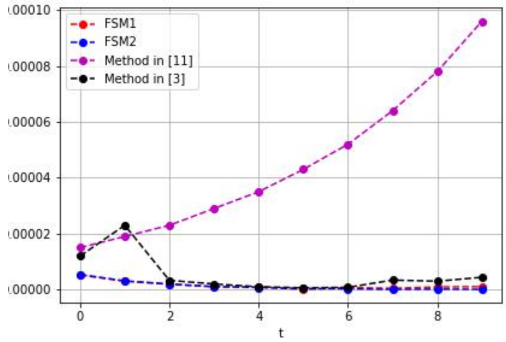

Fig. 5.4. Comparison $\|E\|$ between methods.

$z(t) = \sin(t)$ is the exact solution, in Table 4 and Fig. 5.5 presented the numerical solutions and in Table 5 and Fig. 5.6 presented the MAE.

t	Exact solution	FSM2	FSM1
0	0	0	0
0.1	0.0156243474	0.0156243474	0.0156243565
0.2	0.0312449139	0.0312446780	0.0312449251
0.3	0.0468578357	0.0468577235	0.0468452820
0.4	0.0624593178	0.0624592837	0.0624743507
0.5	0.0780455513	0.0780447447	0.0780325743
0.6	0.0936127312	0.0936110727	0.0936209802
0.7	0.1091570568	0.1091567654	0.1091126341
0.8	0.1246747333	0.1246741958	0.1246957544
0.9	0.1401619723	0.1401613196	0.1401529085

Table 4. Numerical solution of Example 6 of mesh points with n=64.

n	FSM2	FSM1	Method in [2]
64	4.3×10^{-7}	1.2×10^{-5}	7.3×10^{-6}
128	2.8×10^{-8}	7.7×10^{-7}	1.8×10^{-6}
256	1.7×10^{-9}	4.8×10^{-8}	4.6×10^{-7}
512	1.1×10^{-10}	3.0×10^{-9}	1.1×10^{-7}
1024	6.8×10^{-12}	1.9×10^{-10}	2.9×10^{-8}

Table 5. Comparison of MAE for Example 6.

Example 7. The Volterra integral equation of the second kind

$$z(t) + \int_0^t (s^2 t + t^2 x) z(s) ds = t + \tfrac{7}{12} t^5, t \in [0, 1).$$

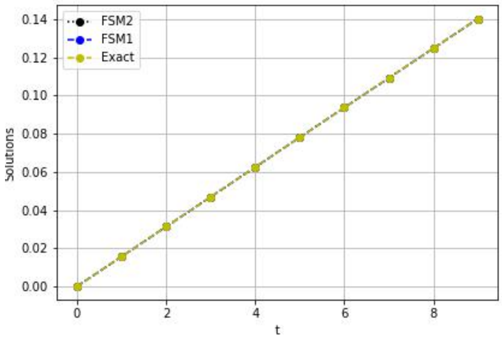

Fig. 5.5. Comparison between FSM1, FSM2 and Exact solutions.

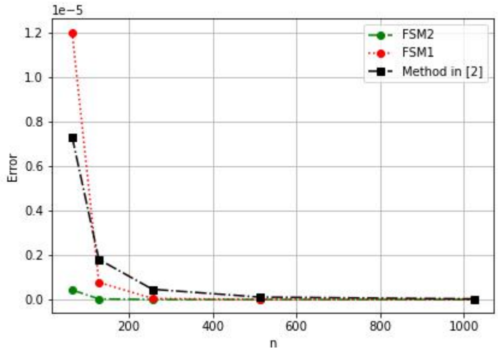

Fig. 5.6. Comparison MAE between methods.

$z(t) = t$ is the exact solution, in Table 6 and Fig. 5.7 presented the numerical solutions and in Table 7 and Fig. 5.8 presented the MAE.

t	Exact solution	FSM2	FSM1
0	0	0	0
0.1	0.015625	0.01562499	0.01562499
0.2	0.03125	0.03124997	0.03125011
0.3	0.046875	0.04687485	0.04686837
0.4	0.0625	0.06249943	0.06260679
0.5	0.078125	0.07812337	0.07801748
0.6	0.09375	0.09374616	0.09383638
0.7	0.109375	0.10936703	0.10933316
0.8	0.125	0.12498497	0.12504516
0.9	0.140625	0.14059868	0.14064388

Table 6. Numerical solution of Example 7 of mesh points with n=64.

n	FSM2	FSM1	Method in [2]
64	5.6×10^{-6}	4.1×10^{-5}	1.8×10^{-5}
128	1.3×10^{-6}	1.3×10^{-6}	4.5×10^{-6}
256	5.4×10^{-9}	4.0×10^{-8}	1.1×10^{-6}
512	1.7×10^{-10}	1.3×10^{-9}	2.8×10^{-7}
1024	5.3×10^{-12}	3.9×10^{-11}	7.0×10^{-8}

Table 7. Comparison of MAE for Example 7.

6. CONCLUSION

In this study, we suggested a new numerical scheme to solve the second kind of linear Volterra and Fredholm-integral equations by using a new idea for Spline function with fractional order, to investigate the convergence analysis

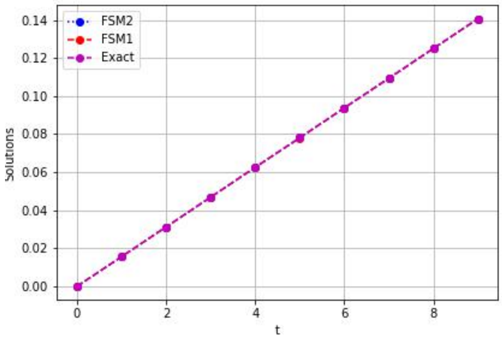

Fig. 5.7. Comparison between FSM1, FSM2 and Exact solutions.

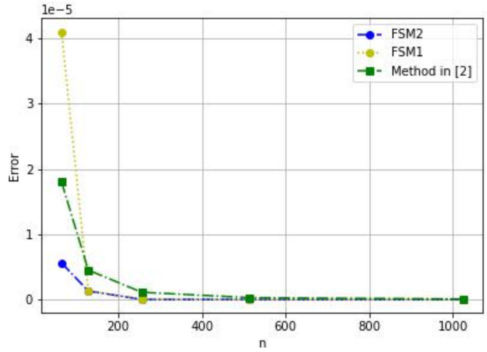

Fig. 5.8. Comparison MAE between methods.

some useful lemmas and theorem have been proved. To precisely the technique four examples have been illustrated and the results explained in tables and figures shows that proposed techniques are better than the previous methods, for this purpose the researchers are used Python program.

REFERENCES

[1] K. Parand, A.A. Aghaei, M. Jani, A. Ghodsi, *A new approach to the numerical solution of Fredholm integral equations using least squares-support vector regression*, Math. Comput. Simulation, **180** (2021), pp. 114–128. https://doi.org/10.1016/j.matcom.2020.08.010

[2] S. Hatamzadeh-Varmazyar, Z. Masouri, *Numerical solution of second kind Volterra and Fredholm integral equations based on a direct method via triangular functions*, Int. J. Ind. Math., **11** (2019) 2, pp. 79–87.

[3] K. Maleknejad, J. Rashidinia, H. Jalilian, *Quintic Spline functions and Fredholm integral equation*, Comput. Methods Differ. Equ., **9** (2021) 1, pp. 211–224. https://doi.org/10.22034/CMDE.2019.31983.1492

[4] F. Müller, W. Varnhorn, *On approximation and numerical solution of Fredholm integral equations of second kind using quasi-interpolation*, Appl. Math. Comput., **217** (2011) 13, pp. 6409–6416. https:doi.org/10.1016/j.amc.2011.01.022

[5] Panda, S.C.Martha, A. Chakrabarti, *A modified approach to numerical solution of Fredholm integral equations of the second kind*, Appl. Math. Comput., **271** (2015), pp. 102–112. https://doi.org10.1016/j.amc.2015.08.111

[6] J. Rashidinia, K. Maleknejad, H. Jalilian, *Convergence analysis of non-polynomial spline functions for the Fredholm integral equation*, Int. J. Comput. Math., **97** (2020) 6, pp. 1197–1211. https://doi.org/10.1080/00207160.2019.1609669

[7] D.A. Hammad, M.S. Semary, A.G. Khattab, *Ten non-polynomial cubic splines for some classes of Fredholm integral equations*, Ain Shams Eng. J., **13** (2022) 4, art. no. 101666. https://doi.org/10.1016/j.asej.2021.101666

[8] K. Maleknejad, J. Rashidinia, T. Eftekhari, *Numerical solution of three-dimensional Volterra–Fredholm integral equations of the first and second kinds based on Bernstein's approximation*, Appl. Math. Comput., **339** (2018), pp. 272–285. https://doi.org/10.1016/j.amc.2018.07.021

[9] R.J. Qadir, F. Hamasalh,, *Fractional Spline Model for Computing Fredholm Integral Equations*, In 2022 Int. Conf. Comput. Sci. Softw. Eng. (CSASE) (2022), pp. 343–347. IEEE.

[10] T. Tahernezhad, R. Jalilian, *Exponential spline for the numerical solutions of linear Fredholm integro-differential equations*, Adv. Difference Equ., (2020) 1, pp. 1–15. https://doi.org/10.1186/s13662-020-02591-3

[11] J. Rashidinia, M. Zarebnia, *Numerical solution of linear integral equations by using Sinc–collocation method*, Appl. Math. Comput., **168** (2005) 2, pp. 806–822. https://doi.org/10.1016/j.amc.2004.09.044

[12] P. Assari, F. Asadi-Mehrega, M. Dehghan, *On the numerical solution of Fredholm integral equations utilizing the local radial basis function method*, Int. J. Comput. Math., **96** (2019) 7, pp. 1416–1443. https://doi.org/10.1080/00207160.2018.1500693

[13] R. Jaza, F. Hamasalh, *Non-polynomial fractional spline method for solving Fredholm integral equations*, J. Innov. Appl. Math. Comput. Sci., **2** (2022) 3, pp. 1–14.

[14] K. Maleknejad, J. Rashidinia, H. Jalilian, *Non-polynomial spline functions and Quasi-linearization to approximate nonlinear Volterra integral equation*, Filomat, **32** (2018) 11, pp. 3947–3956. https://doi.org/10.2298/FIL1811947M

[15] K. Maleknejad, J. Rashidinia, T. Eftekhari, *Operational matrices based on hybrid functions for solving general nonlinear two-dimensional fractional integro-differential equations*, Comput. Appl. Math., **39** (2020) 2, pp 1–34. https://doi.org/10.1007/s40314-020-1126-8

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