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BOOK REVIEWS

Paolo Gatto, *Mathematical Foundations of Finite Elements and Iterative Solvers*, SIAM, Philadelphia, 2022, x + 176 pp., ISBN 978-1-61197-708-0 (paperback), ISBN 978-1-61197-709-7 (ebook).

Originating from lectures given by the author at RWTH Aachen University, this book aims to give a self-contained and short presentation of the *mathematical foundations underpinning finite elements* – one of the most important numerical methods for solving partial differential equations (PDEs). With a clear and captivating style that makes use of brief historical notes and insightful quotes, the author succeeds in writing a compelling mathematical story. Proofs are given for all the results, the main threads and the logical steps of the arguments are thoroughly explained, and carefully chosen theoretical exercises are inserted inside each chapter.

The book starts with a gentle *Incipit (Chapter 1)*, introducing through conservations laws the fundamental PDEs that will represent the main interest of the book (Laplace, Stokes, convection-diffusion), and containing also a brief discussion about open questions regarding the Navier-Stokes equation.

The action begins with *Elliptic problems (Chapter 2)*: the Poisson equation is introduced together with its possible boundary conditions, harmonic functions are discussed (maximum principle, regularity inside and up to the boundary, examples with nonsmooth solutions at the boundary, classical solutions and the need for weak solutions). After a light passage through distributions, elliptic regularity is discussed together with examples of solutions which are in H^2 for (sharp) L^2 sources and smooth domains, and of solutions which are not in H^2 for infinitely smooth sources due to the low regularity of the domain (re-entrant/concave corners). The classical abstract framework for discussing well-posedness of elliptic problems is then presented: coercivity and the Lax-Milgram lemma (for which two proofs are given) when the solution and test (Hilbert) spaces coincide, and the general Babuška-Nečas theorem (inf-sup condition) in the setting of distinct Banach spaces. The ideas are exemplified on an advection-convection-difussion equation, where integration by parts and trace operators are carefully discussed. Well-posedness for weakly coercive problems (satisfying a Gårding's inequality) is finally discussed using the Fredholm alternative.

The transition from functional analysis to numerical analysis takes places in *Discretization by Galerkin Methods (Chapter 3)*, considering first the setting of the Lax-Milgram lemma, where coercivity together with Galerkin orthogonality provide the convergence of the discrete solution in terms of the best approximation error (Céa's lemma). The discrete inf-sup condition is then introduced for proving well-posedness and convergence for Galerkin methods more generally (Babuška theorem). This is followed by a discussion about asymptotic discrete stability and an example of instability in the preasymptotic regime is given through the one dimensional Helmholtz equation. The bulk of the chapter describes the construction and analysis of basic finite element spaces, the red line being represented by the main steps in the error estimate: interpolation, pullbacks, scaling arguments to the reference element and the Bramble-Hilbert lemma. A detailed discussion on the $H(\mathrm{curl})$ Sobolev space combined with the Laplace, Nédélec, Raviart-Thomas interpolation operators conclude the chapter by obtaining the seminorm error estimate.

Numerical linear algebra comes into play for the first time in *Solvers (Chapter 4)*, with the SVD, matrix decompositions and eigenvalues. The second order finite difference matrix for the Laplacian is considered in great details for two dimensional problems, by comparing several methods for efficiently solving the linear system (ordering, elimination, eigenvalue solver). Iterative methods based on Krylov subspaces (Conjugate gradients, MINRES, GMRES) together with preconditioning techniques end this chapter. *Numerical Solution of Saddle Point Problems (Chapter 5)* continues the numerical linear algebra incursion, with saddle point matrices that arise in quadratic optimization or Stokes-type problems. Eigenvalues are discussed first, then the augmented Lagrangian approach, followed by different solvers (segregated or coupled) using the Schur complement, penalty methods or Uzawa's iteration. Preconditioning techniques conclude the chapter.

A return to finite elements is done in *Steady Advection-Diffusion Equations (Chapter 6)*, with convection-dominated (so-called confusion) equations for which the coercivity constant decreases and the discrete stability deteriorates. Boundary and interior layers fail to be correctly captured by naive numerical methods that are unstable. A one dimensional example with an analytical solution shows spurious oscillations for high Péclet numbers of the discrete solution (also analytical). The idea of upwind schemes that improve stability by increasing the numerical diffusion, at the price of decreasing the approximation order, is presented very clearly. Standard stabilized methods such as bubble functions, Galerkin Least Squares or SUPG are also briefly discussed.

The last chapter *The Stokes Problem (Chapter 6)* considers mixed-problems in their weak form and gives a historical account of various proofs for well-posedness, following a presentation given by Brezzi in 2004. The historical overview also considers the challenges of constructing stable pairs of finite elements for the Stokes problem. The chapter ends with the MINI element and a review of different stable and unstable discretizations using basic or more sophisticated elements.

Assuming few prerequisites in functional analysis, Sobolev spaces (for which the author has already announced an appendix) and linear algebra, the book is helpful for graduate students, and lecturers all the more. One can nowadays solve standard PDEs in few lines of user-friendly code using open source software, akin to a black blox. With programming efforts being greatly reduced, it can be argued that mathematical ideas regarding *well-posedness*, *numerical stability* or *efficient solvers* deserve to get emphasized – the book shows they really do, especially when presented in such a conversational but rigorous way.

Mihai Nechita

Philippe G. Ciarlet, *Locally Convex Spaces and Harmonic Analysis. An Introduction*, SIAM, Philadelphia, 2021, viii + 195 pp., ISBN 9781611976649 (paperback), ISBN 9781611976656 (ebook).

The book complements author's previous work Linear and Nonlinear Functional Analysis with Applications (SIAM, 2013), with two basic topics of linear functional analysis: locally convex spaces and harmonic analysis. It is also intended to serve as an introduction to more advanced texts in the field and as basic material for graduate courses and can be useful for a self-study in applied functional analysis.

The book has two chapters. Chapter 1: Locally Convex Spaces, begins with a detailed general analysis of locally convex spaces and Fréchet spaces. It continues with the study of some fundamental examples of such spaces, namely fundamental function spaces of distributions theory, the Schwartz space of rapidly decreasing functions, the space of distributions, the Schwartz space of tempered distributions, the space of distributions with compact support. The chapter finishes by the basic properties of two fundamental locally convex topologies: the weak and weak star topologies, and by two important results of linear functional analysis, the Banach-Bourbaki-Alaoglu and Milman-Pettis theorems.

Chapter 2: Harmonic Analysis, deals with the Fourier transform in the spaces L^1, $\mathcal{S}$, L^2, on the spaces L^p, $1 < p < 2$, thanks to the Riesz-Thorin interpolation theorem, and on the Schwartz space of tempered distributions. The last five sections are devoted to the Hilbert transform in L^2, to Calderón-Zygmund singular operators of convolution type in L^2, Marcinkiewicz interpolation theorem, Hardy-Littlewood maximal function, and Calderón-Zygmund singular operators of convolution type in L^p, $1 < p < \infty$.

The Bibliography includes more than 110 titles, and a List of Notations and an Index facilitate the use of this book.

Like all other works of the author, this book is also very carefully written, with complete proofs and numerous remarks and comments. A number of 93 problems that come to complete the theory, some accompanied by hints, make this book a useful source for a master's course or for individual study. Due to its theme, the book is also extremely useful to researchers in the field of partial differential equations and applied mathematics, offering an exact mathematical framework for the analysis of complex models.

Radu Precup