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THE AKIMA'S FITTING METHOD FOR QUARTIC SPLINES

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Abstract. For the Hermite type quartic spline interpolating on the partition knots and at the midpoint of each subinterval, we consider the estimation of the derivatives on the knots, and the values of these derivatives are obtained by constructing an algorithm of Akima's type. For computing the derivatives on endpoints are also considered alternatives that request optimal properties near the endpoints. The error estimate in the interpolation with this quartic spline is generally obtained in terms of the modulus of continuity. In the case of interpolating smooth functions, the corresponding error estimate reveal the maximal order of approximation $\mathcal{O}(h^3)$. A numerical experiment is presented for making the comparison between the Akima's cubic spline and the Akima's variant quartic spline having deficiency 2 and natural endpoint conditions.

MSC. 65D07, 65D10.
Keywords. Quartic splines, Akima's fitting spline interpolation procedure, error estimates.

1. INTRODUCTION

Before the fundamental work of Schoenberg (see [15]) where the notion of B-spline is introduced in explicit way, according to de Boor and Pinkus [7], the first apparition of spline functions can be found in the pioneering works of Popoviciu (see [14]) and Chakalov (see [9]). Through polynomial spline functions, the widely used are cubic splines which can be expressed both in terms of the moments (second order derivatives of the spline on knots) and in terms of the local first order derivatives m_i, $i = \overline{0,n}$, as in the case of Hermite type cubic splines, that are presented in the following. On a partition $\Delta : a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ of $[a, b]$, the Hermite type cubic spline $s \in C^1[a, b]$ has the expression

(1)
$$\begin{aligned} s(x) =& \tfrac{(x_i-x)^2[2(x-x_{i-1})+h_i]}{h_i^3} y_{i-1} + \tfrac{(x-x_{i-1})^2[2(x_i-x)+h_i]}{h_i^3} y_i \\ &+ \tfrac{(x_i-x)^2(x-x_{i-1})}{h_i^2} m_{i-1} - \tfrac{(x-x_{i-1})^2(x_i-x)}{h_i^2} m_i, \qquad x \in [x_{i-1}, x_i], \quad i = \overline{1,n} \end{aligned}$$

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where $h_i = x_i - x_{i-1}$, $i = \overline{1,n}$, and $y_i = s(x_i)$, $i = \overline{0,n}$. For the computation of the local derivatives m_i, $i = \overline{0,n}$, were proposed several procedures. Imposing the smallest deficiency, that is $s \in C^2[a,b]$, and considering two endpoint conditions, various types of cubic splines are obtained such as complete cubic splines, not-a-knot splines, periodic cubic splines, $E(\alpha)$ cubic splines, natural cubic splines (see [3] and [13]). For instance, the natural cubic spline that minimizes the L_2-norm of s'' is generated by the endpoint conditions $s''(a) = s''(b) = 0$. Another idea is to determine the derivatives m_i, $i = \overline{0,n}$ under the smoothness property $s \in C^1[a,b]$ and to consider some geometric type procedures such as in [1] and [8], or by minimizing a functional related to the data polygon (see [4] and [11]).

The derivatives m_i, $i = \overline{0,n}$, are computed in [1] by using geometric reasoning based on the slopes $p_i = \frac{y_{i+1}-y_i}{x_{i+1}-x_i}$, $i = \overline{0,n-1}$, and are given as,

$$m_i = \frac{|p_{i+1}-p_i| \cdot p_{i-1} + |p_{i-1}-p_{i-2}| \cdot p_i}{|p_{i+1}-p_i| + |p_{i-1}-p_{i-2}|}, \quad i = \overline{2,n-2}. \tag{2}$$

In order to extend formula (2) for $i = \overline{0,n}$, the previously computed slopes are not enough and therefore, Akima proposes the construction of four new supplementary slopes $p_{-1}, p_{-2}, p_n, p_{n+1}$, as follows: $p_{-1} = 2p_0 - p_1$, $p_{-2} = 3p_0 - 2p_1$, $p_n = 2p_{n-1} - p_{n-2}$, $p_{n+1} = 3p_{n-1} - 2p_{n-2}$. As it is shown in [4] and [5], sometimes, this treatment near endpoints could generate significant oscillations. Therefore, in [5], the values of the derivatives on the first two and last two knots are computed by using optimal procedures.

In this work we focus our attention to quartic splines and propose an Akima's type procedure for computing the derivatives m_i, $i = \overline{0,n}$, of the deficient C^1-smooth quartic spline $S \in C^1[a,b]$ proposed in [12], which has the following expression on the intervals $[x_{i-1}, x_i]$, $i = \overline{1,n}$:

$$\begin{aligned} S_i(x) =& \frac{(x_i-x)^2\left[(x_i-x)^2+4(x_i-x)(x-x_{i-1})-5(x-x_{i-1})^2\right]}{h_i^4} y_{i-1} \\ &+ \frac{16(x-x_{i-1})^2(x_i-x)^2}{h_i^4} y_{i-1/2} \\ &+ \frac{(x-x_{i-1})^2\left[(x-x_{i-1})^2+4(x_i-x)(x-x_{i-1})-5(x_i-x)^2\right]}{h_i^4} y_i \\ &+ \frac{(x_i-x)(x-x_{i-1})(x_{i-1}+x_i-2x)\left[(x_i-x)m_{i-1}+(x-x_{i-1})m_i\right]}{h_i^3} \\ =& A_i(x)\, y_{i-1} + B_i(x)\, y_{i-1/2} + C_i(x)\, y_i + D_i(x)\, m_{i-1} + E_i(x)\, m_i \end{aligned} \tag{3}$$

where $m_i = S'(x_i)$, $y_i = S(x_i)$, $i = \overline{0,n}$, and $y_{i-1/2} = S\left(\frac{x_{i-1}+x_i}{2}\right)$, $i = \overline{1,n}$.

Error estimates in the interpolation with the C^2 quartic splines (4) were established in [12], [10] and [16]. In [6], the values of the derivatives m_i, $i = \overline{0,n}$, were determined in order to minimize the L_2-norm of S', S'', and S''', respectively. Here, the values of m_i, $i = \overline{0,n}$, will be obtained by using a new Akima's type method.

As will be viewed in the following, while for the Akima's method a special treatment is required on four knots (the first two and the last two), in our method the special treatment is involved only on endpoints. The reason is in the fact that on each interval $[x_{i-1}, x_i]$, $i = \overline{1,n}$, the derivatives m_{i-1} and m_i are computed on the points x_{i-1} and x_i by using the values on the midpoints $x_{i-1/2} = \frac{x_{i-1}+x_i}{2}$, $i = \overline{1,n}$, too. Therefore, the knowledge at midpoints is an advantage. In the treatment of endpoints, in order to avoid the introduction of supplementary slopes, we develop three special variants for computing the values m_0 and m_n. The interpolation error estimates of this Akima's variant quartic spline is given in terms of the modulus of continuity for less smooth class of functions. When smooth functions are interpolated we obtain the corresponding error estimates and prove that the order of approximation is $\mathcal{O}(h^3)$. Finally, a numerical experiment is presented in order to illustrate the behaviour of the proposed interpolation procedure, including a comparison with the classical Akima's cubic spline interpolation method.

2. THE CONSTRUCTION OF THE AKIMA'S TYPE PROCEDURE FOR QUARTIC SPLINES

Consider two neighbouring intervals $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ and the midpoints $x_{i-1/2}$, $x_{i+1/2}$ in each of these intervals. Suppose that the points to be interpolated are (x_0, y_0), (x_1, y_1), $\ldots$, (x_n, y_n) and define the slopes

$$d_i = \frac{y_{i-1/2}-y_{i-1}}{h_i/2}, \quad d_{i-1/2} = \frac{y_i-y_{i-1/2}}{h_i/2}, \quad d_{i+1/2} = \frac{y_{i+1/2}-y_i}{h_{i+1}/2}, \quad d_{i+1} = \frac{y_{i+1}-y_{i+1/2}}{h_{i+1}/2}$$

for $i = \overline{1, n-1}$. Let p_i and p_{i+1} be the quadratic Lagrange polynomials interpolating the points x_{i-1}, $x_{i-1/2}$, x_i and respectively, x_i, $x_{i+1/2}$, x_{i+1}, on the intervals $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$,

$$p_i(x) = \frac{2\left(y_{i-1}+y_i-2y_{i-1/2}\right)}{h_i^2}(x-x_{i-1})^2 + \frac{4y_{i-1/2}-y_i-3y_{i-1}}{h_i}(x-x_{i-1}) + y_{i-1},$$

$$p_{i+1}(x) = \frac{2\left(y_{i+1}+y_i-2y_{i+1/2}\right)}{h_{i+1}^2}(x-x_i)^2 + \frac{4y_{i+1/2}-y_{i+1}-3y_i}{h_{i+1}}(x-x_i) + y_i.$$

Computing the derivatives of p_i and p_{i+1} on the point x_i we get

$$p_i'(x_i) = \frac{3y_i+y_{i-1}-4y_{i-1/2}}{h_i},$$

$$p_{i+1}'(x_i) = \frac{4y_{i+1/2}-y_{i+1}-3y_i}{h_{i+1}}.$$

Let

$$\widetilde{y}_i' = \frac{-2h_{i+1}}{h_i(h_i+h_{i+1})}y_{i-1/2} + \frac{2(h_{i+1}-h_i)}{h_ih_{i+1}}y_i + \frac{2h_i}{h_{i+1}(h_i+h_{i+1})}y_{i+1/2} \tag{4}$$

be the three-point difference approximation formula of the derivative in the point x_i computed on the interval $[x_{i-1/2}, x_{i+1/2}]$. Now, we compute the left tangent and the right tangent in x_i by

$$T_-(x_i) = \frac{1}{2}\left(p_i'(x_i) + \widetilde{y}_i'\right) \quad \text{and} \quad T_+(x_i) = \frac{1}{2}\left(p_{i+1}'(x_i) + \widetilde{y}_i'\right).$$

If $|d_{i-1/2} - d_i| + |d_{i+1} - d_{i+1/2}| \neq 0$ we propose the value of the expected derivative m_i to be

$$(5) \qquad m_i = \frac{|d_{i+1} - d_{i+1/2}| \cdot T_- (x_i) + |d_{i-1/2} - d_i| \cdot T_+ (x_i)}{|d_{i-1/2} - d_i| + |d_{i+1} - d_{i+1/2}|}, \quad i = \overline{1, n-1}$$

and if $|d_{i-1/2} - d_i| + |d_{i+1} - d_{i+1/2}| = 0$, then this value will be

$$m_i = \tfrac{1}{2} \left(T_- (x_i) + T_+ (x_i) \right).$$

We see that the values m_0 and m_n at the endpoints remain free. If the values $y'(a)$ and $y'(b)$ are known, then we put $m_0 = y'(a)$ and $m_n = y'(b)$, but if these values $y'(a)$ and $y'(b)$ are not available we will compute the values m_0 and m_n by using three proposed variants that will be presented in what follows.

3. THE TREATMENT OF THE ENDPOINTS

Firstly, we can consider the endpoint type conditions $S''(a) = S''(b) = 0$, that usually appears at natural cubic splines. These conditions lead to the equations

$$(6) \qquad \begin{cases} 4m_0 - m_1 = \frac{1}{h_1} \left(-11y_0 + 16y_{1-1/2} - 5y_1 \right) \\ -m_{n-1} + 4m_n = \frac{1}{h_n} \left(5y_{n-1} - 16y_{n-1/2} + 11y_n \right) \end{cases}$$

obtaining in this way the Akima's quartic spline with natural endpoint conditions with $m_0 = \frac{m_1}{4} + \frac{-11y_0+16y_{1-1/2}-5y_1}{4h_1}$, $m_n = \frac{m_{n-1}}{4} + \frac{5y_{n-1}-16y_{n-1/2}+11y_n}{4h_n}$.

Another variant is to consider the local optimal condition involving the minimization of the integrals

$$\int_{x_0}^{x_1} \left(S''(x) \right)^2 dx \quad \text{and} \quad \int_{x_{n-1}}^{x_n} \left(S''(x) \right)^2 dx$$

near endpoints, resulting minimal local curvature on the first and on the last subinterval $[x_0, x_1]$ and $[x_{n-1}, x_n]$. In this purpose we consider the functionals

$$J_2(m_0) = \int_{x_0}^{x_1} \left(S''(x) \right)^2 dx \quad \text{and} \quad J_2(m_n) = \int_{x_{n-1}}^{x_n} \left(S''(x) \right)^2 dx,$$

$$J_2(m_0) = \int_{x_0}^{x_1} \Big[A_1''(x) y_0 + B_1''(x) y_{1-1/2} + C_1''(x) y_1 + D_1''(x) m_0 + E_1''(x) m_1 \Big]^2 dx,$$

$$J_2(m_n) =$$

$$= \int_{x_{n-1}}^{x_n} \Big[A_n''(x) y_{n-1} + B_n''(x) y_{n-1/2} + C_n''(x) y_n + D_n''(x) m_{n-1} + E_n''(x) m_n \Big]^2 dx,$$

the system of normal equations $J_2'(m_0)=0,\ J_2'(m_n)=0$ being

$$\begin{cases} \frac{36}{5h_1}m_0-\frac{6}{5h_1}m_1=-\frac{94}{5h_1^2}y_0+\frac{128}{5h_1^2}y_{1-1/2}-\frac{34}{5h_1^2}y_1 \\ -\frac{6}{5h_n}m_{n-1}+\frac{36}{5h_n}m_n=\frac{34}{5h_n^2}y_{n-1}-\frac{128}{5h_n^2}y_{n-1/2}+\frac{94}{5h_n^2}y_n \end{cases}$$

Then one obtains

$$m_0=\frac{m_1}{6}-\frac{47}{18h_1}y_0+\frac{32}{9h_1}y_{1-1/2}-\frac{17}{18h_1}y_1,$$
$$m_n=\frac{m_{n-1}}{6}+\frac{17}{18h_n}y_{n-1}-\frac{32}{9h_n}y_{n-1/2}+\frac{47}{18h_n}y_n.$$

The following variant is inspired by the idea of the work [5] but uses the technique from [11], minimizing the derivative oscillation on the intervals $[x_0,x_1]$ and $[x_{n-1},x_n]$. For this purpose we consider the functionals

$$J_1(m_0)=$$
$$=\int_{x_0}^{x_1}\left(S'(x)-\frac{y_1-y_0}{h_1}\right)^2dx$$
$$=\int_{x_0}^{x_1}\left[A_1'(x)y_0+B_1'(x)y_{1-1/2}+C_1'(x)y_1+D_1'(x)m_0+E_1'(x)m_1-\frac{y_1-y_0}{h_1}\right]^2dx,$$

$$J_1(m_n)=$$
$$=\int_{x_{n-1}}^{x_n}\left(S'(x)-\frac{y_n-y_{n-1}}{h_n}\right)^2dx$$
$$=\int_{x_{n-1}}^{x_n}\left[A_n'(x)y_{n-1}+B_n'(x)y_{n-\frac{1}{2}}+C_n'(x)y_n+D_n'(x)m_{n-1}+E_n'(x)m_n-\frac{y_n-y_{n-1}}{h_n}\right]^2dx.$$

By the normal equations $J_1'(m_0)=0,\ J_1'(m_n)=0$ we get the values

$$m_0=-\frac{5}{16}m_1-\frac{29}{16h_1}y_0+\frac{1}{h_1}y_{1-1/2}+\frac{13}{16h_1}y_1,$$
$$m_n=-\frac{5}{16}m_{n-1}-\frac{13}{16h_n}y_{n-1}-\frac{1}{h_n}y_{n-1/2}+\frac{29}{16h_n}y_n.$$

4. THE INTERPOLATION ERROR ESTIMATES

For the derivatives computed by the Akima's variant (5) we have the estimate

$$|m_i|\le\frac{\left|d_{i-1/2}-d_i\right|\max\{|T_-(x_i)|,|T_+(x_i)|\}+\left|d_{i+1}-d_{i+1/2}\right|\max\{|T_-(x_i)|,|T_+(x_i)|\}}{\left|d_{i-1/2}-d_i\right|+\left|d_{i+1}-d_{i+1/2}\right|}$$

and so, $|m_i|\le\max\{|T_-(x_i)|,|T_+(x_i)|\}$. Now, the estimates

$$|p_i'(x_i)|\le\frac{1}{h_i}\left(3|y_i-y_{i-1/2}|+|y_{i-1}-y_{i-1/2}|\right)\le\frac{4}{\underline{h}}\omega\left(y,\frac{h}{2}\right)$$
$$|p_{i+1}'(x_i)|\le\frac{1}{h_{i+1}}\left(3|y_{i+1/2}-y_i|+|y_{i+1/2}-y_{i+1}|\right)\le\frac{4}{\underline{h}}\omega\left(y,\frac{h}{2}\right)$$

are obtained in terms of the modulus of continuity, where $\underline{h} = \min\{h_i : i = \overline{1,n}\}$ and $h = \max\{h_i : i = \overline{1,n}\}$. By (4) we infer that

$$|\widetilde{y_i'}| \leq \frac{-2h_{i+1}}{h_i(h_i+h_{i+1})}|y_i - y_{i-1/2}| + \frac{2h_i}{h_{i+1}(h_i+h_{i+1})}|y_{i+1/2} - y_i| \leq \frac{2h}{\underline{h}^2}\omega\left(y, \frac{h}{2}\right)$$

and thus

$$|T_-(x_i)| \leq \frac{3h}{\underline{h}^2}\omega\left(y, \frac{h}{2}\right), \qquad |T_+(x_i)| \leq \frac{3h}{\underline{h}^2}\omega\left(y, \frac{h}{2}\right).$$

Consequently, it obtains the estimate

$$|m_i| \leq \frac{3h}{\underline{h}^2}\omega\left(y, \frac{h}{2}\right), \quad \forall i = \overline{1, n-1}.$$

In the case of the Akima quartic spline with natural type endpoint conditions we obtain the estimate

$$|m_0| \leq \frac{|m_1|}{4} + \frac{1}{4h_1}\left(11|y_{1-1/2} - y_0|\right) + 5|y_1 - y_{1-1/2}|) \leq \frac{19}{4\underline{h}}\omega\left(y, \frac{h}{2}\right) \tag{7}$$

and analogous, $|m_n| \leq \frac{19}{4\underline{h}}\omega\left(y, \frac{h}{2}\right)$. In the case of minimal local curvature on the intervals $[x_0, x_1]$ and $[x_{n-1}, x_n]$, we get the estimates

$$|m_n| \leq \frac{|m_{n-1}|}{6} + \frac{17}{18h_n}|y_{n-1} - y_{n-1/2}| + \frac{47}{18h_n}|y_n - y_{n-1/2}| \leq \frac{73}{18\underline{h}}\omega\left(y, \frac{h}{2}\right) \tag{8}$$

and $|m_0| \leq \frac{73}{18\underline{h}}\omega\left(y, \frac{h}{2}\right)$. For the Akima quartic spline with minimal derivative oscillation on the intervals $[x_0, x_1]$ and $[x_{n-1}, x_n]$ one obtains,

$$|m_0| \leq \frac{5|m_1|}{16} + \frac{29}{16h_1}|y_{1-1/2} - y_0| + \frac{13}{16h_1}|y_1 - y_{1-1/2}| \leq \frac{57}{16\underline{h}}\omega\left(y, \frac{h}{2}\right) \tag{9}$$

and $|m_n| \leq \frac{57}{16\underline{h}}\omega\left(y, \frac{h}{2}\right)$.

Now, observing that D_i and E_i have the same sign separately on the intervals $[x_{i-1}, x_{i-1/2}]$ and $[x_{i-1/2}, x_i]$, the estimate of $|S(x) - y(x)|$ will be performed on each half-subinterval $[x_{i-1}, x_{i-1/2}]$ and $[x_{i-1/2}, x_i]$, $i = \overline{1,n}$, similarly as in the proof of Corollary 7 from [6], obtaining

$$\begin{aligned}|S(x) - y(x)| \leq & \max_{x\in[x_{i-1},x_{i-1/2}]} |A_i(x)+B_i(x)| \max\{|y_{i-1} - y(x)|, |y_{i/2} - y(x)|\} \\ & + \max_{x\in[x_{i-1},x_{i-1/2}]} |C_i(x)| \cdot |y_i - y(x)| + \\ & + \max_{x\in[x_{i-1},x_{i-1/2}]} |D_i(x) + E_i(x)| \max\{|m_i| : i = \overline{0,n}\}\end{aligned}$$

for $x \in [x_{i-1}, x_{i-1/2}]$. Analogous, we have similar estimate for $x \in [x_{i-1/2}, x_i]$. Then we get

$$|S(x) - y(x)| \leq \frac{9317}{8192}\omega\left(y, \frac{h}{2}\right) + \frac{1125}{8192}\omega(y, h) + \frac{\sqrt{3}h_i}{18}\max\left\{|m_i| : i = \overline{0,n}\right\}$$

for $x \in [x_{i-1}, x_i]$, $i = \overline{1,n}$. Thus, concerning the interpolation error estimate, we obtain the following result.

THEOREM 1. *On the interval $[x_1, x_{n-1}]$ the error estimate for the Akima's interpolating quartic spline, in terms of the modulus of continuity, is*

$$|S(x) - y(x)| \le \left(\frac{9317}{8192} + \frac{\sqrt{3}h^2}{6\underline{h}^2}\right)\omega\left(y, \frac{h}{2}\right) + \frac{1125}{8192}\omega(y,h). \tag{10}$$

On the intervals $[x_0, x_1]$ and $[x_{n-1}, x_n]$ the error estimates are:

$$|S(x) - y(x)| \le \left(\frac{9317}{8192} + \frac{\sqrt{3}}{18}\left(\frac{3h^2}{\underline{h}^2} + \frac{19h}{4\underline{h}}\right)\right)\omega\left(y, \frac{h}{2}\right) + \frac{1125}{8192}\omega(y,h) \tag{11}$$

for taking the endpoint conditions $S''(a) = S''(b) = 0$,

$$|S(x) - y(x)| \le \left(\frac{9317}{8192} + \frac{\sqrt{3}}{18}\left(\frac{3h^2}{\underline{h}^2} + \frac{73h}{18\underline{h}}\right)\right)\omega\left(y, \frac{h}{2}\right) + \frac{1125}{8192}\omega(y,h) \tag{12}$$

when have minimal curvature on the endpoint intervals, and

$$|S(x) - y(x)| \le \left(\frac{9317}{8192} + \frac{\sqrt{3}}{18}\left(\frac{3h^2}{\underline{h}^2} + \frac{57h}{16\underline{h}}\right)\right)\omega\left(y, \frac{h}{2}\right) + \frac{1125}{8192}\omega(y,h) \tag{13}$$

in the case of minimal derivative oscillation near endpoints.

In contrast with the case of cubic splines, by comparing (7)–(8) and (12)–(13), we see that for the quartic splines (3) the condition of minimal curvature on the intervals $[x_0, x_1]$ and $[x_{n-1}, x_n]$, and the condition $s''(a) = s''(b) = 0$, lead to different spline interpolants.

5. ERROR ESTIMATES FOR SMOOTH FUNCTIONS

In this section we provide the error estimates for $\|S - f\|_\infty$ and $\|S' - f'\|_\infty$ when the Akima's variant quartic spline interpolates a smooth function $f \in C^4[a,b]$. First of all we prove a lemma related to Hermite quartic polynomial interpolation.

LEMMA 2. *If $f \in C^4[a,b]$ with Lipschitzian fourth order derivative and if $H_4(f)$ is the Hermite interpolation polynomial generated by the interpolation conditions given for $f(a)$, $f\left(\frac{a+b}{2}\right)$, $f(b)$, $f'(a)$, $f'(b)$, then the error estimate is*

$$|H_4(f)(x) - f(x)| \le \frac{\sqrt{5}(b-a)^5 L}{30000}, \quad \forall x \in [a,b] \tag{14}$$

where L is the Lipschitz constant of $f^{(4)}$.

Proof. Consider the fundamental polynomial

$$u(x) = (x-a)^2\left(x - \frac{a+b}{2}\right)(x-b)^2$$

and for arbitrary fixed $x \in [a,b]$ we define $\varphi_x : [a,b] \to \mathbb{R}$, by

$$\varphi_x(t) = \begin{vmatrix} u(t) & R(t) \\ u(x) & R(x) \end{vmatrix}$$

where $R = f - H_4(f)$ is the remainder. Because $f \in C^4[a,b]$ we infer that $R \in C^4[a,b]$ and $\varphi_x \in C^4[a,b]$. Since $\varphi_x(a) = \varphi_x(b) = \varphi_x\left(\frac{a+b}{2}\right) = \varphi_x(x) = 0$ and $\varphi_x'(a) = \varphi_x'(b) = 0$ after successive four times applications of the Rolle's mean value theorem we get $\varphi_x^{(4)}(v) = \varphi_x^{(4)}(w) = 0$ for some $v, w \in (a, b)$, $v \neq w$. Based on the fact that $f^{(4)} - R^{(4)}$ is constant we get

$$5!\,(v-w)\,R(x) - u(x)\left(f^{(4)}(v) - f^{(4)}(w)\right) = 0$$

and thus,

$$|R(x)| = \frac{|u(x)|\cdot\left|f^{(4)}(v)-f^{(4)}(w)\right|}{5!|v-w|} \leq \frac{L}{5!}\max_{x\in[a,b]}|u(x)|$$

obtaining (14). □

Of course, if $f \in C^5[a,b]$, then $L = \|f^{(5)}\|_\infty$ in (14). The error estimate for $|H_4(f)'(x) - f'(x)|$ can be obtained too. Since $\max_{x\in[a,b]}|u'(x)| = \frac{(b-a)^4}{16}$ and according to the proof of this lemma, we have

$$f'(x) - H_4(f)'(x) = R'(x) = \frac{u'(x)\left(f^{(4)}(v)-f^{(4)}(w)\right)}{5!(v-w)}.$$

Consequently, it obtains

$$\left|H_4(f)'(x) - f'(x)\right| \leq \frac{L(b-a)^4}{1920}, \quad \forall x \in [a,b]. \tag{15}$$

In the case $f \in C^4[a,b]$, if $f'(a)$ and $f'(b)$ are unknown we put

$$m_0 = \frac{-3y_0+4y_{1-1/2}-y_1}{h_1}, \quad m_n = \frac{y_{n-1}-4y_{n-1/2}+3y_n}{h_n}$$

where $y_i = f(x_i)$, $i = \overline{0,n}$, $y_{i-1/2} = f\left(\frac{x_{i-1}+x_i}{2}\right)$, $i = \overline{1,n}$, inspired by the technique from [2]. When $f'(a)$ and $f'(b)$ are known it is natural to consider $m_0 = f'(a)$, $m_n = f'(b)$. Concerning the interpolation error estimate of the Akima's variant quartic spline in the case of smooth functions we obtain the following main result.

Theorem 3. *If $f \in C^4[a,b]$ with Lipschitzian fourth order derivative and $S \in C^1[a,b]$ is the Akima's variant quartic spline interpolating f, then the error estimates are*

$$|S(x) - f(x)| \leq \begin{cases} \frac{h^3\sqrt{3}\|f'''\|_\infty}{288} + \frac{Lh^5\sqrt{5}}{30000}, & x \in [x_1, x_{n-1}] \\ \frac{h^3\sqrt{3}\|f'''\|_\infty}{54} + \frac{Lh^5\sqrt{5}}{30000}, & x \in [x_0, x_1] \cup [x_{n-1}, x_n] \end{cases} \tag{16}$$

and

$$|S'(x) - f'(x)| \leq \begin{cases} \frac{h^2\|f'''\|_\infty}{8} + \frac{Lh^4}{1920}, & x \in [x_1, x_{n-1}] \\ \frac{2h^2\|f'''\|_\infty}{3} + \frac{Lh^4}{1920}, & x \in [x_0, x_1] \cup [x_{n-1}, x_n] \end{cases} \tag{17}$$

where L is the Lipschitz constant of $f^{(4)}$ and $\|f'''\|_\infty = \max_{x\in[a,b]}|f'''(x)|$.

Proof. Consider $H(f)$ be the Hermite type piecewise quartic polynomial interpolating on each interval $[x_{i-1}, x_i]$ the values y_{i-1}, $y_{i-1/2}$, y_i, $f'(x_{i-1})$, $f'(x_i)$, $i = \overline{1,n}$, and by (14) we get

$$|H(f)(x) - f(x)| \leq \frac{L\sqrt{5}h_i^5}{30000} \leq \frac{L\sqrt{5}h^5}{30000}, \quad x \in [x_{i-1}, x_i].$$

Now, by (4) and by the Lagrange numerical differentiation formula will be $\eta_i \in (x_{i-1}, x_i)$, $\theta_i \in (x_i, x_{i+1})$, $c_i \in \left(x_{i-1/2}, x_{i+1/2}\right)$ such that

$$\begin{aligned} \left|f'(x_i) - \widetilde{y}_i'\right| &= \tfrac{h_i h_{i+1}}{4} \cdot \tfrac{|f'''(c_i)|}{3!} \\ \left|f'(x_i) - p_i'(x_i)\right| &= \tfrac{h_i^2}{2} \cdot \tfrac{|f'''(\eta_i)|}{3!}, \quad i = \overline{1, n-1}. \\ \left|f'(x_i) - p_{i+1}'(x_i)\right| &= \tfrac{h_{i+1}^2}{2} \cdot \tfrac{|f'''(\theta_i)|}{3!} \end{aligned}$$

Consequently,

$$\begin{aligned} \left|T_-(x_i) - f'(x_i)\right| &= \tfrac{1}{2}\left|p_i'(x_i) - f'(x_i) + \left(\widetilde{y}_i' - f'(x_i)\right)\right| \\ &\le \tfrac{1}{2}\left(\tfrac{h_i^2|f'''(\eta_i)|}{12} + \tfrac{h_i h_{i+1}|f'''(c_i)|}{24}\right) \le \tfrac{h^2}{16} \cdot \left\|f'''\right\|_\infty, \end{aligned}$$

$$\begin{aligned} \left|T_+(x_i) - f'(x_i)\right| &= \tfrac{1}{2}\left|p_{i+1}'(x_i) - f'(x_i) + \left(\widetilde{y}_i' - f'(x_i)\right)\right| \\ &\le \tfrac{1}{2}\left(\tfrac{h_{i+1}^2|f'''(\theta_i)|}{12} + \tfrac{h_i h_{i+1}|f'''(c_i)|}{24}\right) \le \tfrac{h^2}{16} \cdot \left\|f'''\right\|_\infty, \end{aligned}$$

for all $i = \overline{1, n-1}$, and by (5) we obtain

$$\left|m_i - f'(x_i)\right| \le \tfrac{h^2}{16} \cdot \left\|f'''\right\|_\infty, \quad i = \overline{1, n-1}. \tag{18}$$

On the other hand, at endpoints by the same Lagrange differentiation formula it obtains

$$\left|f'(x_0) - m_0\right| \le \tfrac{h^2}{3} \cdot \left\|f'''\right\|_\infty, \quad \left|f'(x_n) - m_n\right| \le \tfrac{h^2}{3} \cdot \left\|f'''\right\|_\infty. \tag{19}$$

Since S and $H(f)$ interpolates the same values y_{i-1}, $y_{i-1/2}$, y_i on $[x_{i-1}, x_i]$, $i = \overline{1, n}$, having the same structure, by (18) and (19) we get

$$\begin{aligned} \left|S(x) - H(f)(x)\right| &\le \max_{x \in [x_{i-1}, x_{i-1/2}] \cup [x_{i-1/2}, x_i]} \left|D_i(x) + E_i(x)\right| \cdot \left|m_i - f'(x_i)\right| \\ &\le \tfrac{h_i\sqrt{3}}{18} \cdot \tfrac{h^2}{16} \left\|f'''\right\|_\infty \le \tfrac{h^3\sqrt{3}}{288} \cdot \left\|f'''\right\|_\infty, \quad x \in [x_1, x_{n-1}] \end{aligned}$$

and

$$\left|S(x) - H(f)(x)\right| \le \tfrac{h_i\sqrt{3}}{18} \cdot \tfrac{h^2}{3} \left\|f'''\right\|_\infty \le \tfrac{h^3\sqrt{3}}{54} \cdot \left\|f'''\right\|_\infty$$

for $x \in [x_0, x_1] \cup [x_{n-1}, x_n]$, obtaining (16). Finally, by (15) we get

$$\left|H(f)'(x) - f'(x)\right| \le \tfrac{Lh^4}{1920} \quad \text{for all} \quad x \in [a, b]$$

and since

$$\max_{x \in [x_{i-1}, x_i]} \left|D_i'(x)\right| = \max_{x \in [x_{i-1}, x_i]} \left|E_i'(x)\right| = 1 \quad \text{for } i = \overline{1, n},$$

by (18) and (19) we obtain

$$\begin{aligned} \left|S'(x) - H'(f)(x)\right| &\le \tfrac{h^2}{8} \left\|f'''\right\|_\infty, \quad x \in [x_1, x_{n-1}] \\ \left|S'(x) - H'(f)(x)\right| &\le \tfrac{2h^2}{3} \left\|f'''\right\|_\infty, \quad x \in [x_0, x_1] \cup [x_{n-1}, x_n], \end{aligned}$$

and the estimate (17) follows. □

In (16) we see that the order of approximation by the Akima's variant quartic spline is $O\left(h^3\right)$ and, as in [12] and [16], we have obtained the corresponding estimate of $\|S' - f'\|_\infty$ for the derivative S', too.

6. NUMERICAL EXPERIMENT

In order to illustrate this method we present a numerical example considering the points (x_i, y_i), $i = \overline{0,5}$: $(0,16)$, $(2,20)$, $(3,28)$, $(5,21)$, $(6,24)$, $(7,28)$, while the values on midpoints are $y_{1-1/2} = 12$, $y_{2-1/2} = 23$, $y_{3-1/2} = 32$, $y_{4-1/2} = 18$, $y_{5-1/2} = 30$. The local derivatives m_i, $i = \overline{1,4}$ are computed by using the Akima's type procedure (5), while the values at endpoints m_0, m_5 are computed by using the alternatives presented in Section 3 such as, natural endpoint conditions $S''(a) = S''(b) = 0$, minimal curvature $J_2(m_0, m_5)$ near endpoints, and minimal derivative oscillation $J_1(m_0, m_5)$ near endpoints, respectively. These values of the local derivatives m_i, $i = \overline{0,5}$ are presented below and the obtained quartic splines are represented in Figs. 6.1 and 6.2.

$m_1 = 6.583$	$m_2 = 9.95$	$m_3 = -12.286$	$m_4 = 16.235$
end-cond.:	$s''(a) = s''(b) = 0$	$\min J_2(m_0, m_n)$	$\min J_1(m_0, m_n)$
m_0 :	-8.854	-7.9	-2.43
m_5 :	-8.94125	-8.183	-3.8234

Table 1. Numerical results.

In Fig. 6.1 we represent the Akima's quartic spline with natural type endpoint conditions $S''(a) = S''(b) = 0$ (denoted by (EN) and drawn as solid line curve) and the Akima's quartic spline with minimal derivative oscillation $J_1(m_0, m_5)$ near endpoints (denoted by (AD), the dots-line curve). Differences are observed in the first and in the last interval $[0,2]$ and $[6,7]$, respectively, where the curve (AD) has smaller oscillation. In Fig. 6.2 are represented by comparison the Akima's quartic spline with minimal curvature $J_2(m_0, m_5)$ on the intervals $[x_0, x_1] = [0,2]$ and $[x_4, x_5] = [6,7]$ (denoted by (CM) and plotted with dots), and the classical Akima's cubic spline (denoted by (AK) and drawn as solid line curve) interpolating the points $(x_0, y_0), \ldots, (x_5, y_5)$.

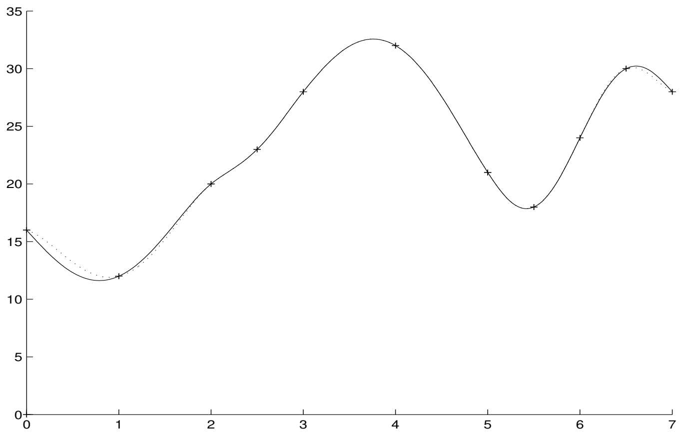

Fig. 6.1. Quartic splines, AD (...), EN (–)

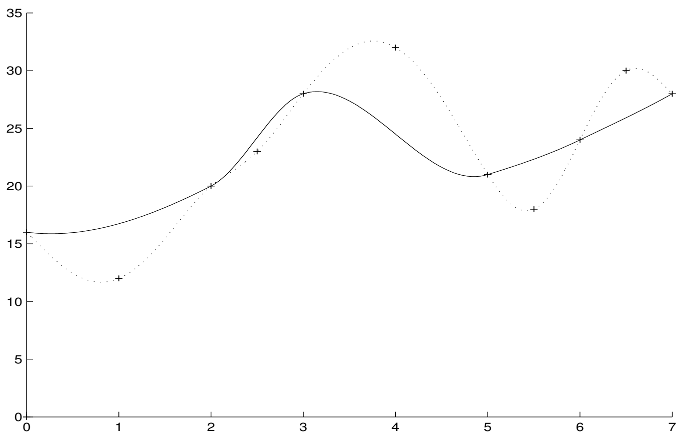

Fig. 6.2. Quartic and Akima splines, CM (...), AK (–)

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