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ON THE SEMI-LOCAL CONVERGENCE OF A SIXTH ORDER METHOD IN BANACH SPACE

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Abstract. High convergence order methods are important in computational mathematics, since they generate sequences converging to a solution of a non-linear equation. The derivation of the order requires Taylor series expansions and the existence of derivatives not appearing on the method. Therefore, these results cannot assure the convergence of the method in those cases when such high order derivatives do not exist. But, the method may converge. In this article, a process is introduced by which the semi-local convergence analysis of a sixth order method is obtained using only information from the operators on the method. Numerical examples are included to complement the theory.

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1. INTRODUCTION

Finding a locally unique solution of the system of non-linear equations of the form

$$G(x) = 0 \tag{1}$$

is a major problem with extensive applications in the field of mathematical and engineering sciences. Presently, there are numerous efficient methods to solve (1) [1, 8–10, 13, 14, 16–18]. But, in most of the cases non-linear equations and systems arising from mathematical modeling of physical systems does not have exact solutions. Because of this problem, scientists and researchers have focused on proposing iterative methods for solving non-linear systems. Newton's method is a popular iterative process for dealing with non-linear equations. Many novel, higher-order iterative strategies for dealing with nonlinear equations have been discovered and are currently being used in recent years [1, 3, 4, 7–10, 13–18]. However, the theorems on convergence of these schemes in most of these publications are derived by applying high

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order derivatives. Furthermore, no results are discussed regarding the error distances, radii of convergence or the region in which the solution is unique.

The study of local and semi-local analysis of an iterative formula allows to estimate the convergence balls, bounds on error and uniqueness region for a solution. The results of local and semi-local convergence of efficient iterative procedures have been deduced in [2, 3, 5–7, 11]. In these works, important results containing convergence radii, measurements on error estimates and expanded utility of these iterative approaches have been given. Outcomes of these type of analysis are valuable because they illustrate the complexity of selecting initial points.

In this article, we develop semi-local convergence theorem for a method with sixth order convergence proposed in [1]. The method can be stated as

$$\begin{aligned} y_n &= x_n - G'(x_n)^{-1}G(x_n) \\ z_n &= y_n - G'(y_n)^{-1}G(y_n) \\ x_{n+1} &= z_n - G'(y_n)^{-1}G(z_n), \qquad x_0 \in \Omega, \forall n = 0, 1, 2, \ldots \end{aligned} \tag{2}$$

where $G : \Omega \subset B_1 \to B_2$ is a Fréchet differentiable and continuous non-linear operator, B_1 and B_2 are Banach spaces and $\Omega \neq \phi$ is a convex and open set. Let x^* represent a root of the equation (1) which is locally unique.

The local convergence of method (2) was established assuming that the seventh derivative (at least) of the operator G exists. As a consequence, productivity of this method is limited. To see this, we define G on $\Omega = [-0.5, 1.5]$ by

$$G(t) = \begin{cases} t^3 \ln(t^2) + t^5 - t^4, & \text{if } t \neq 0 \\ 0, & \text{if } t = 0 \end{cases} \tag{3}$$

Now, it is easy to find that due to the unboundedness of G''' the results on convergence of (2) does not hold for this example. Also, previous research articles do not produce any formula for approximating the error $\|x_n - x^*\|$, the convergence region or the uniqueness and accurate location of the root x^*. This is the major motivation for developing the ball convergence theorems by considering assumptions only on G'. Our research provides important formulas for the estimation of $\|x_n - x^*\|$ and convergence radii. This study also discusses about an exact location and the uniqueness of x^*.

The other contents of this material can be summarized as follows: Section 2 discusses the development of majorizing sequence for the method (2). Section 3 discusses the semi-local convergence properties of the prsented method (2). Numerical testing of convergence outcomes are placed in Section 4. Concluding remarks are also stated.

2. MAJORIZING SEQUENCE

A scalar sequence is introduced in this section that is shown to be majorizing for the method (2) in Section 3. Let $M = [0, +\infty)$.
Suppose:

There exists a function $\phi_0 : M \to \mathbb{R}$ which is continuous and non-decreasing such that the equation

$$\phi_0(t) - 1 = 0$$

has a smallest positive solution. Denote such a solution by α. Set $M_0 = [0, \alpha]$. Let $\phi : M_0 \to \mathbb{R}$ be a continuous and non-decreasing function. Let $d \geq 0$ denote a given parameter.

Define the scalar sequences $\{t_n\}$, $\{s_n\}$ and $\{v_n\}$ for each $n = 0, 1, 2, \dots$ by

$$\begin{aligned}
t_0 &= 0, \quad s_0 = d,\\
a_n &= \int_0^1 \phi(\lambda(s_n - t_n))d\lambda(s_n - t_n),\\
v_n &= s_n + \frac{a_n}{1 - \phi_0(s_n)},\\
b_n &= (1 + \int_0^1 \phi_0(s_n + \lambda(v_n - s_n))d\lambda)(v_n - s_n) + a_n,\\
t_{n+1} &= v_n + \frac{b_n}{1 - \phi_0(s_n)},\\
c_{n+1} &= \int_0^1 \phi(\lambda(t_{n+1} - t_n))d\lambda(t_{n+1} - t_n) + (1 + \phi_0(t_n))(t_{n+1} - s_n)\\
\text{and } s_{n+1} &= t_{n+1} + \frac{c_{n+1}}{1 - \phi_0(t_{n+1})}.
\end{aligned} \tag{4}$$

Next, we develop a general convergence result for the sequence $\{t_n\}$ given by the formula (4).

LEMMA 1. *Suppose that for each $n = 0, 1, 2, \dots$ there exists a parameter $\delta > 0$ such that*

$$\phi_0(s_n) < 1, \quad \phi_0(t_{n+1}) < 1 \quad \text{and } t_{n+1} \leq \delta. \tag{5}$$

Then, the following assertions hold

$$0 \leq t_n \leq s_n \leq v_n \leq t_{n+1} \quad \text{and } \lim_{n\to\infty} t_n = t^* \leq \delta. \tag{6}$$

Proof. The definition of the sequences $\{t_n\}$, $\{s_n\}$, $\{v_n\}$ and the condition (5) imply that the sequence $\{t_n\}$ is bounded above by λ and non-decreasing. Hence, it converges to its unique least upper bound t^* satisfying $t^* \in [0, \delta]$. □

REMARK 2. (i) If the function ϕ_0 is strictly increasing, then we can set $\delta = \phi_0^{-1}(1)$.

(ii) Another possible choice for δ is any number $\gamma_0 \in [0, \delta]$ or $\gamma_0 \in [0, \alpha)$. □

3. CONVERGENCE ANALYSIS

The semi-local convergence analysis of the method (2) is developed in this section. For this, we need an auxiliary Ostrowski-type result for the method (2), which is stated below.

Lemma 3. *Suppose that the iterates $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ appearing in the method (2) exist for each $n = 0, 1, 2, \ldots$. Then, the following assertions hold:*

$$G(y_n) = p_n = \int_0^1 (G'(x_n + \lambda(y_n - x_n))d\lambda - G'(x_n))(y_n - x_n), \tag{7}$$

$$z_n - y_n = -G(y_n)^{-1} p_n, \tag{8}$$

$$q_n = \int_0^1 G'(y_n + \lambda(z_n - y_n))d\lambda(z_n - y_n), \tag{9}$$

$$G(z_n) = q_n + p_n, \tag{10}$$

$$x_{n+1} - z_n = -G'(y_n)^{-1}(q_n + p_n), \tag{11}$$

$$G(x_{n+1}) = \delta_{n+1} = \int_0^1 (G'(x_n + \lambda(x_{n+1} - x_n)) - G'(x_n))d\lambda(x_{n+1} - x_n) \tag{12}$$

$$+ G'(x_n)(x_{n+1} - y_n) \quad \textit{and} \tag{13}$$

$$y_{n+1} - x_{n+1} = -G'(x_{n+1})\delta_{n+1}. \tag{14}$$

Proof. We can write by the second sub-step of the method (2) in turn that

$$\begin{aligned} G(y_n) &= G(y_n) - G(x_n) - G'(x_n)(y_n - x_n) \\ &= \int_0^1 (G'(x_n + \lambda(y_n - x_n))d\lambda - G'(x_n))(y_n - x_n) = p_n \end{aligned}$$

showing (7) and consequently (8).

Moreover, by using (7), (9) and the following identity,

$$\begin{aligned} G(z_n) &= G(z_n) - G(y_n) + G(y_n) \\ &= \int_0^1 G'(y_n + \lambda(z_n - y_n))d\lambda(z_n - y_n) + G(y_n) \end{aligned}$$

we obtain (10).

Hence, the identity (11) holds by the third sub-step of the method (2) and (10).

Furthermore, by the first sub-step of the method (2), we get (13) as follows:

$$\begin{aligned} &G(x_{n+1}) = \\ &= G(x_{n+1}) - G(x_n) - G'(x_n)(y_n - x_n) - G'(x_n)(x_{n+1} - x_n) + G'(x_n)(x_{n+1} - x_n) \\ &= (G(x_{n+1}) - G(x_n) - G'(x_n)(x_{n+1} - x_n)) + G'(x_n)(x_{n+1} - y_n) \\ &= \int_0^1 (G'(x_n + \lambda(x_{n+1} - x_n))d\lambda - G'(x_n))(x_{n+1} - x_n) + G'(x_n)(x_{n+1} - y_n) \\ &= \delta_{n+1}. \end{aligned}$$

Finally, the identity (14) follows from the first sub-step of the method (2) and (13). □

The following conditions are required to prove the semi-local convergence analysis of the method (2). Further, these conditions that guarantee the existence of the iterates $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$. Suppose:

(H_1) There exists an element $x_0 \in \Omega$ and a parameter $d \geq 0$ such that $G'(x_0)^{-1} \in \mathscr{L}(B_2, B_1)$ and $\|G'(x_0)^{-1}G(x_0)\| \leq d$.

(H_2) $\|G'(x_0)^{-1}(G'(u) - G'(x_0))\| \leq \phi_0(\|u - x_0\|)$ for each $u \in \Omega$.
Set $\Omega_0 = U(x_0, \alpha) \cap \Omega$.

(H_3) $\|G'(x_0)^{-1}(G'(u_2) - G'(u_1))\| \leq \phi(\|u_2 - u_1\|)$ for each $u_1, u_2 \in \Omega_0$.

(H_4) The conditions (5) hold and

(H_5) $U[x_0, t^*] \subset \Omega$, where the parameter t^* is given in the Lemma 1.

Next, we state and prove the semi-local convergence result for method (2).

Theorem 4. *Suppose that the conditions (H_1)–(H_5) hold. Then, the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ generated by the method (2) are well-defined, remain in $U[x_0, t^*]$ and converge to a solution $x^* \in U[x_0, t^*]$ of the equation $G(x) = 0$. Moreover, the following assertions hold*

$$\|y_n - x_n\| \leq s_n - t_n, \tag{15}$$

$$\|z_n - y_n\| \leq v_n - s_n, \tag{16}$$

$$\|x_{n+1} - z_n\| \leq t_{n+1} - v_n \quad \textit{and} \tag{17}$$

$$\|x^* - x_n\| \leq t^* - t_n. \tag{18}$$

Proof. Mathematical induction on n is applied to first show that the iterates $\{x_k\}$, $\{y_k\}$, $\{z_k\}$ exist and the estimates (15)–(17) hold.

It follows by the condition (H_1) and the first sub-step of the method (2) for $n = 0$ that

$$\|y_0 - x_0\| = \|G'(x_0)^{-1}G(x_0)\| \leq d = s_0 - t_0 = s_0 < t^*,$$

so, the estimate (15) holds for $n = 0$. Let $w \in U(x_0, t^*)$. Then, by applying the condition (H_2), we get in turn that

$$\|G'(x_0)^{-1}(G'(w) - G'(x_0))\| \leq \phi_0(\|w - x_0\|) \leq \phi_0(t^*) < 1. \tag{19}$$

The estimate (19) together with the Banach lemma on linear operators that have inverses [12] imply that $G'(w) \in \mathscr{L}(B_2, B_1)$ and

$$\|G'(w)^{-1}G'(x_0)\| \leq \tfrac{1}{1-\phi_0(\|w-x_0\|)}. \tag{20}$$

In particular, for $w = y_0, G'(y_0)^{-1} \in \mathscr{L}(B_2, B_1)$. Consequently, the iterates z_0 and x_1 exist by the second and the third sub-step of the method (2). Then, by Lemma 1, (H_3), (20) for $w = y_0$ and (4), we obtain in turn

$$\begin{aligned}
\|z_k - y_k\| &\leq \|G'(y_k)^{-1}G'(x_0)\|\|G'(x_0)^{-1}G(y_k)\| \\
&\leq \frac{\int_0^1 \phi(\lambda\|y_k - x_k\|)d\lambda\|y_k - x_k\|}{1 - \phi_0(\|y_k - x_0\|)} \\
&\leq \frac{\int_0^1 \phi(\lambda(s_k - t_k))d\lambda(s_k - t_k)}{1 - \phi_0(s_k)} = v_k - s_k,
\end{aligned}$$

$$\begin{aligned}
\|z_k - x_0\| &\le \|z_k - y_k\| + \|y_k - x_0\| \\
&\le v_k - s_k + s_k - t_0 = v_k \le t^*, \\
\|x_{k+1} - z_k\| &\le \|G'(y_k)^{-1} G'(x_0)\| \|G'(x_0)^{-1} G(z_k)\| \\
&\quad \times \Big(\frac{(1 + \int_0^1 \phi_0(\|y_k - x_0\| + \lambda \|z_k - y_k\|) d\lambda) \|z_k - y_k\|}{1 - \phi_0(s_k)} \\
&\quad + \frac{\int_0^1 \phi(\lambda \|y_k - x_k\|) d\lambda \|y_k - x_k\|}{1 - \phi_0(s_k)} \Big) \\
&\le \tfrac{b_k}{1 - \phi_0(s_k)}, \\
\|x_{k+1} - x_0\| &\le \|x_{k+1} - y_k\| + \|y_k - x_0\| \\
&\le t_{k+1} - s_k + s_k - t_0 = t_{k+1} \le t^*, \\
\|y_{k+1} - x_{k+1}\| &\le \|G'(y_k + 1)^{-1} G'(x_0)\| \|G'(x_0)^{-1} G(x_{k+1})\| \\
&\le \tfrac{c_{k+1}}{1 - \phi_0(t_{k+1})} = s_{k+1} - t_{k+1}, \\
\|y_{k+1} - x_0\| &\le \|y_{k+1} - x_{k+1}\| + \|x_{k+1} - x_0\| \\
&\le s_{k+1} - t_{k+1} + t_{k+1} - t_0 = s_{k+1} \le t^*,
\end{aligned}$$

where we also used

$$\begin{aligned}
&\Big\| \int_0^1 G'(x_0)^{-1} (G'(y_k + \lambda(z_k - y_k)) d\lambda) \Big\| = \\
&= \Big\| G'(x_0)^{-1} (G'(y_k + \lambda(z_k - y_k)) d\lambda - G'(x_0) + G'(x_0)) \Big\| \\
&\le 1 + \int_0^1 \phi_0(\|y_k - x_0\| + \lambda \|z_k - y_k\|) d\lambda \\
&\le 1 + \int_0^1 \phi_0(s_k + \lambda(v_k - s_k)) d\lambda
\end{aligned}$$

and

$$\begin{aligned}
\|G'(x_0)^{-1} G(x_{k+1})\| &\le \Big\| \| \int_0^1 G'(x_0)^{-1} (G'(x_k + \lambda(x_{k+1} - x_k)) d\lambda \\
&\quad - G'(x_k))(x_{k+1} - x_k) \Big\| \| \\
&\quad + \|G'(x_0)^{-1} (G'(x_k) - G(x_0) + G'(x_0))(x_{k+1} - y_k)\| \\
&\le \int_0^1 \phi(\lambda \|x_{k+1} - x_k\|) d\lambda \|x_{k+1} - x_k\| \\
&\qquad + (1 + \phi_0(\|x_k - x_0\|)) \|x_{k+1} - y_k\| \\
&\le \delta_{k+1}.
\end{aligned}$$

Therefore, the assertions (15)–(17) hold and the iterates $\{x_k\}, \{y_k\}, \{z_k\} \in U[x_0, t^*]$. Moreover, it follows that the sequence $\{x_k\}$ is fundamental in a Banach space B_1 and as such it converges to some $x^* \in U[x_0, t^*]$.

By letting $k \to \infty$ in the estimate $\|G'(x_0)^{-1}G(x_{k+1})\| \leq \delta_{k+1}$ and using the continuity of G, we deduce that $G(x^*) = 0$. Let $m \geq 0$ be an integer. Then, we can write in turn that

$$\begin{aligned}\|x_{n+m} - x_n\| &\leq \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_n\| \\ &\leq t_{n+m} - t_{n+m-1} + t_{n+m-1} - t_n \\ &\leq \ldots \leq t_{n+m} - t_n.\end{aligned}$$

By letting $m \to \infty$ in the preceding estimate we show the assertion (18). □

Next, the uniqueness of the solution x^* in a certain region is determined.

PROPOSITION 5. *Suppose:*

(i) *There exists a solution $\Lambda \in U(x_0, \rho)$ of the equation $G(x) = 0$ for some $\rho > 0$.*

(ii) *The condition (H_2) holds on $U(x_0, \rho)$.*

(iii) *There exists $R \geq \rho$ such that*

$$\int_0^1 \phi_0((1-\lambda)\rho + \lambda R)d\lambda < 1.$$

Set $\Omega_1 = U[x_0, R] \cap \Omega$.

Then, the equation $G(x) = 0$ is uniquely solvable by Λ in the region Ω_1.

Proof. Let $A = \int_0^1 G'(\Lambda + \lambda(\Lambda_1 - \Lambda))d\lambda$ for some $\Lambda_1 \in \Omega_1$ with $G(\Lambda_1) = 0$. Then, in view of *(ii)* and *(iii)*, we have in turn that

$$\begin{aligned}\|G'(x_0)^{-1}(A - G'(x_0))\| &\leq \int_0^1 \phi_0((1-\lambda)\|\Lambda - x_0\| + \lambda\|\Lambda_1 - x_0\|)d\lambda \\ &\leq \int_0^1 \phi_0((1-\lambda)\rho + \lambda\rho_1)d\lambda \\ &< 1.\end{aligned}$$

It follows that the linear operator A is invertible. Therefore, we can write

$$\begin{aligned}\Lambda_1 - \Lambda &= A^{-1}(G(\Lambda_1) - G(\Lambda)) \\ &= A^{-1}(0 - 0) = A^{-1}(0) \\ &= 0,\end{aligned}$$

showing that $\Lambda_1 = \Lambda$. □

REMARK 6. (1) The conditions ($H_1 - H_5$) were not used in the Proposition 5. But if they were used, then we can set $\rho = t^*$.

(2) The limit point t^* in the condition (H_5) can be replaced by δ. □

4. NUMERICAL EXAMPLES

EXAMPLE 7. We reconsider the motivational example from the introduction part of this work. Select $x_0 = 0.9955$. Conditions (H_1)-(H_3) are verified for

$$\|G'(x_0)^{-1}G(x_0)\| = 0.00456182 = d,$$

$\phi_0(t) = 12.8089t$, $\alpha = 0.0780704$, $\Omega_0 = U(x_0, \alpha) \cap \Omega$, $\phi(t) = 1.12091t$. Let $\delta = 0.06$. The conditions (5) are tested and the results are given in Table 1. Hence, we can observe that conditions (H_4) and (H_5) hold and therefore we

n	0	1	2	3
$\phi_0(s_n)$	0.0584319	0.0596158	0.0596159	0.0596159
$\phi_0(t_{n+1})$	0.0589276	0.0596158	0.0596159	0.0596159
t_{n+1}	0.00460052	0.00465425	0.00465426	0.00465426

$t^* = 0.00465426.$

Table 1. Estimates for Example 7.

can conclude that the sequence $\{x_n\}$ generated by the method (2) converges to a solution x^* of the equation $G(x) = 0$ in $U[x_0, t^*]$. □

EXAMPLE 8. The applicability of our work in the real world can be demonstrated by considering the quartic equation for fractional conversion which represents the fraction of the nitrogen-hydrogen feed that gets converted to ammonia. At $500°C$ and $250atm$, this equation can be formulated as

$$G(x) = x^4 - 7.79075x^3 + 14.7445x^2 + 2.511x - 1.674.$$

Let $\Omega = (0.3, 0.4)$ and $x_0 = 0.3$. Then, the conditions (H_1) - (H_3) are valid if

$$\|G'(x_0)^{-1}G(x_0)\| = 0.0217956 = d,$$

$\phi_0(t) = \phi(t) = 1.56036\text{t}$, $\alpha = 0.640877$ and $\Omega_0 = U[x_0, \alpha] \cap \Omega$. Choose $\delta = 0.3$. Conditions (5) are tested and the outcomes are given in Table 2.

n	0	1	2	3
$\phi_0(s_n)$	0.034009	0.0384204	0.0384459	0.0384459
$\phi_0(t_{n+1})$	0.0358473	0.038431	0.0384459	0.0384459
t_{n+1}	0.0229737	0.0246296	0.0246391	0.0246391

$t^* = 0.0246391.$

Table 2. Estimates for Example 8.

Thus, we can observe that conditions (H_4) and (H_5) hold and therefore we can conclude that the sequence $\{x_n\}$ generated by the method (2) converges to a solution x^* of the equation $G(x) = 0$ in $U[x_0, t^*]$. □

EXAMPLE 9. Define the cubic polynomial

$$G(x) = x^3 - a.$$

on the open ball $\Omega = U(x_0, 1-a)$ for some $a \in [0,1)$. If one chooses $x_0 = 1$, then the conditions (A_1)-(A_3) are verified for $d = \frac{1-a}{3}$, $\phi_0(t) = (3-a)t$, $\alpha = \delta = \frac{1}{3-a}$, $\Omega_0 = U(x_0, \frac{1}{3-a})$ and $\phi(t) = 2(1+\frac{1}{3-a})t$. Let $a = 0.95$. Table 3 depicts the outcomes on testing of conditions (5).

n	0	1	2	3
$\phi_0(s_n)$	0.0341667	0.0358153	0.0358166	0.0358166
$\phi_0(t_{n+1})$	0.0348607	0.0358158	0.0358166	0.0358166
t_{n+1}	0.0170052	0.0174711	0.0174715	0.0174715

$t^* = 0.0174715$.

Table 3. Estimates for Example 9.

Thus, we find that conditions (H_4) and (H_5) also hold and therefore we can conclude that the sequence $\{x_n\}$ generated by the method (2) converges to a solution x^* of the equation $G(x) = 0$ in $U[x_0, t^*]$. □

Example 10. Consider the non-linear system

$$G(w_1, w_2) = \left(w_1 + e^{w_2} - \cos w_2, 3w_1 - w_2 - \sin w_2\right)^T$$

defined on $\Omega = U(0,1)$ for $w = (w_1, w_2)^T$. We obtain the operator G' as follows

$$G'(w) = \begin{bmatrix} 1 & e^{w_2} + \sin w_2 \\ 3 & -1 - \cos w_2 \end{bmatrix}.$$

We find $x^* = (0,0)^T$ is a solution. Choose $x_0 = (0.1, 0.1)^T$. Then, by applying method (2), we find that the above system converges to x^* and the error estimates are given Table 4. □

$\|x_0 - x^*\|$	$\|x_1 - x_*\|$	$\|x_2 - x_*\|$	$\|x_3 - x_*\|$	$\|x_4 - x_*\|$
0.100	$6.28455 * 10^{-3}$	$3.99276 * 10^{-10}$	$2.62582 * 10^{-53}$	$2.12431 * 10^{-312}$

Table 4. Error Estimates for Example 10.

Example 11. Consider the 4×4 system of non-linear equations defined on the open ball $\Omega = U(0,1)$ given by

$$v_i - \cos\left(2v_i - \sum_{i=1}^{4} v_i\right) = 0, \quad i = 1,2,3,4.$$

$x^* = (0.5149, \ldots, 0.5149)^T$ is a solution. We choose the initial approximation as $x_0 = (0.5, \ldots, 0.5)^T$. Then, the method (2) is applied to this system and is found to be convergent to x^* and the error estimates obtained are given in Table 5. □

$\|x_0 - x^*\|$	$\|x_1 - x_*\|$	$\|x_2 - x_*\|$	$\|x_3 - x_*\|$	$\|x_4 - x_*\|$
0.01490	$1.21463 * 10^{-3}$	$3.87516 * 10^{-10}$	$4.08658 * 10^{-49}$	$5.62059 * 10^{-283}$

Table 5. Error Estimates for Example 11.

EXAMPLE 12. Let us solve the non-linear system of logarithmic type where for $w = (q_1, q_2, \ldots, q_m)^T$ and $Q(w) = \ln(q_i + 1) - \frac{q_i}{20}$, $i = 1, 2, \ldots, m$,

$$Q(w) = 0$$

Select $m = 100$ and $x_0 = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})^T$ (50 times). Then, by applying the method (2), the solution $x^* = (1, 1, \ldots, 1)^T$ (50 times) is obtained after three iterations. □

5. CONCLUSION

The semi-local convergence analysis for the method (2) is established by applying generalized Lipschitz condition only on the first derivative. Estimates on convergence balls, measurable error distances and the existence-uniqueness regions for the solution are deduced. At the end, the suggested theoretical outcomes are verified on numerical examples including one application problem. The major advantage is that the technique does not really depend on the method (2). Hence, it can be extended on other single and multi-step methods in the same manner.

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