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ON GENERAL FIXED POINT METHOD BASED ON MATRIX SPLITTING FOR SOLVING LINEAR COMPLEMENTARITY PROBLEM

BHARAT KUMAR[1], DEEPMALA[2] and A.K. DAS[3]

Abstract. In this article, we introduce a modified fixed point method to process the large and sparse linear complementarity problem (LCP) and formulate an equivalent fixed point equation for the LCP and show the equivalence. Also, we provide convergence conditions when the system matrix is a P-matrix and two sufficient convergence conditions when the system matrix is an H_+-matrix. To show the efficiency of our proposed method, we illustrate two numerical examples for different parameters.

MSC. 65F10, 65F50.

Keywords. Linear complementarity problem, H_+-matrix, P-matrix, Matrix splitting, Convergence.

1. INTRODUCTION

Given $A_1 \in \mathbb{R}^{n\times n}$ and a vector $q \in \mathbb{R}^n$, the linear complementarity problem denoted as LCP(q, A_1) is to find the solution $z \in \mathbb{R}^n$ to the following system

$$z \geq 0,\ A_1 z + q \geq 0,\ z^T(A_1 z + q) = 0. \tag{1}$$

The LCP has many applications, including operations research, control theory, mathematical economics, optimization theory, stochastic optimal control, the American option pricing problem, economics, elasticity theory, the free boundary problem, and the Nash equilibrium point of the bimatrix game, which has been extensively studied in the literature on mathematical programming. For details see [14], [10], [21]. For recent works on this problem see [8], [9].

The methods available for solving the LCP may be classified into two groups namely pivoting method [2], [5], [23] and iterative method [22], [18]. The

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[1]Mathematics Discipline, PDPM-Indian Institute of Information Technology, Design and Manufacturing, Jabalpur, M.P., India, e-mail: bharatnishad.kanpu@gmail.com.

[2]Mathematics Discipline, PDPM-Indian Institute of Information Technology, Design and Manufacturing, Jabalpur, M.P., India, e-mail: dmrai23@gmail.com.

[3]Indian Statistical Institute, 203 B.T. Road, Kolkata-700108, India, e-mail: akdas@isical.ac.in.

basic idea of the pivotal method is to obtain a basic feasible complementary vector through a series of pivot steps, whereas the iterative method generates a series of iterations that converge to a solution [4], [7]. Lemke and Howson [20] introduced the complementary pivot method. Following this method, Lemke introduced a technique known as Lemke's algorithm, which is well known for finding the solution to the LCP.

In order to develop effective iteration methods, we commonly use matrix splittings to find a numerical solution of the large and sparse LCP(q, A_1), such as the projected methods [16], [19], [25], the modulus algorithm [11] and the modulus based matrix splitting iterative methods [26], [27].

A general fixed point method (GFP) is proposed by Fang [17] assuming the case where $\Omega = \omega D_1^{-1}$ with $\omega > 0$ and D_1 is the diagonal matrix of A_1. The GFP approach takes less iterations than the modulus based successive over relaxation method (MSOR) [11]. We present a modified form of GFP [17] that incorporates projected type iteration techniques by including two positive diagonal parameter matrices Ω_1, Ω_2 and ϕ is a strictly lower triangular matrix in this article. We also show that the fixed point equation and the LCP is equivalent and discuss the convergence conditions and along with several convergence domains for our method.

The paper is organised as follows: we review some notation, definitions and lemmas in Section 2 in order to establish our key findings. The iterative fixed point approach for solving LCP(q, A_1) with convergence analysis is proposed in Section 3. We present two numerical examples in Section 4 to demonstrate the efficiency of the proposed methods. Section 5 ends the paper with some conclusions.

2. PRELIMINARIES

Some notation, preliminary definitions, and required lemmas are reviewed.

Here $A_1 = (\bar{a}_{ij}) \in \mathbb{R}^{n\times n}$ and $B_1 = (\bar{b}_{ij}) \in \mathbb{R}^{n\times n}$ are square matrices. For $A_1 = (\bar{a}_{ij}) \in \mathbb{R}^{n\times n}$ and $B_1 = (\bar{b}_{ij}) \in \mathbb{R}^{n\times n}$, $A_1 \geq (>)\ B_1$ means $\bar{a}_{ij} \geq (>)\ \bar{b}_{ij}$ for all $i, j \in \{1, 2, \ldots, n\}$.

Definition 1 ([17]). *Let $A_1 = (\bar{a}_{ij}) \in \mathbb{R}^{n\times n}$ be a square matrix, then $|A_1| = (\bar{b}_{ij})$ is defined by $\bar{b}_{ij} = |\bar{a}_{ij}|\ \forall\, i, j$ and $|A_1|$ represent that $\bar{a}_{ij} \geq 0\ \forall\, i, j$.*

Definition 2 ([17]). *Let $A_1, B_1 \in \mathbb{R}^{n\times n}$ be two square matrices. Then $|A_1 + B_1| \leq |A_1| + |B_1|$ and $|A_1B_1| \leq |A_1| \cdot |B_1|$. Moreover, when $a_1, b_1 \in \mathbb{R}^n$ then $|a_1 + b_1| \leq |a_1| + |b_1|$ and $||a_1| - |b_1|| \leq |a_1 - b_1|$.*

Definition 3 ([4]). *Let $A_1 \in \mathbb{R}^{n\times n}$ be a square matrix. A_1 is said to be a P-matrix if all its principle minors are positive such that $det(A_{1_{\alpha_1\alpha_1}}) > 0$ for all $\alpha_1 \subseteq \{1, 2, \ldots, n\}$.*

Definition 4 ([17]). *Suppose $A_1 \in \mathbb{R}^{n\times n}$ is a square matrix, then its comparison matrix is defined as $\langle \bar{a}_{ij} \rangle = |\bar{a}_{ij}|$ if $i = j$ and $\langle \bar{a}_{ij} \rangle = -|\bar{a}_{ij}|$ if $i \neq j$.*

Definition 5 ([6]). *Suppose $A_1 \in \mathbb{R}^{n\times n}$ is a square matrix. It is said to be a Z-matrix if all of its non diagonal elements are less than or equal to zero; an M-matrix if $A_1^{-1} \geq 0$ as well as Z-matrix. The matrix A_1 is said to be an H-matrix if $\langle A_1\rangle$ is an M-matrix. A_1 is said to be an H_+-matrix if A_1 is an H_+-matrix with $\bar{a}_{ii} > 0 \;\forall\; i \in \{1,2,\ldots,n\}$.*

Definition 6 ([6]). *The splitting $A_1 = M_1 - N_1$ is called an M-splitting if M_1 is a nonsingular M-matrix and $N_1 \geq 0$; an H-splitting if $\langle M_1\rangle - |N_1|$ is an M-matrix; an H-compatible splitting if $\langle A_1\rangle = \langle M_1\rangle - |N_1|$.*

Lemma 7 ([1]). *Let $a_1, b_1 \in \mathbb{R}^n$. Then $a_1 \geq 0$, $b_1 \geq 0$, $a_1^T b_1 = 0$ if and only if $a_1 + b_1 = |a_1 - b_1|$.*

Lemma 8 ([6]). *Suppose $A_1, B_1 \in \mathbb{R}^{n\times n}$. If A_1 and B_1 are M and Z-matrices respectively with $A_1 \leq B_1$, then B_1 is an M-matrix. If A_1 is an H-matrix, then $|A_1^{-1}| \leq \langle A_1\rangle^{-1}$. If $A_1 \leq B_1$, then $\rho(A_1) \leq \rho(B_1)$.*

Lemma 9 ([17]). *Let $A_1 \in \mathbb{R}^{n\times n}$ be an M-matrix and $A_1 = M_1 - N_1$ be an M-splitting. Let ρ be the spectral radius, then $\rho(M_1^{-1}N_1) < 1$.*

Lemma 10 ([13]). *If splitting is an H-compatible of an H-matrix, then it is an H-splitting but converse is not true.*

Lemma 11 ([6]). *Suppose $A_1 \geq 0$. If there exist $v > 0 \in \mathbb{R}^n$ and a scalar $\alpha_1 > 0$ such that $A_1 v \leq \alpha_1 v$, then $\rho(A_1) \leq \alpha_1$. Moreover, if $A_1 v<\alpha_1 v$, then $\rho(A_1)<\alpha_1$.*

3. MAIN RESULTS

For a given vector $s \in \mathbb{R}^n$, we consider the vectors $s_+ = \max\{0, s\}$, $s_- = \max\{0, -s\}$ and $A_1 = (D_1 + \phi) - (L_1 + U_1 + \phi)$, where ϕ is a strictly lower triangular matrix, U_1 is a strictly upper triangular matrix of A_1. U_1^T denotes the transpose of U_1, L_1 is strictly lower triangular matrix of A_1 and α is a positive real number. In the following theorem we convert the LCP(q, A_1) into a fixed point equation.

Theorem 12. *Let $A \in \mathbb{R}^{n\times n}$ with the splitting $A_1 = (D_1+\phi)-(L_1+U_1+\phi)$. If $z = \Omega_1 s_+$ and $\omega = \Omega_2 s_-$, then the equivalent formulation of the LCP(q, A_1) in the form of fixed point equation is*

$$s = (I_1 - \Omega_2^{-1}(D_1 + \phi - U_1)\Omega_1)s_+ + \Omega_2^{-1}(L_1 + \phi)\Omega_1 s_+ - \Omega_2^{-1}q. \tag{2}$$

Proof. Let $z = \Omega_1 s_+$ and $w = \Omega_2 s_-$, and $s = s_+ - s_-$. From LCP(q, A_1)

$$\begin{aligned}
\Omega_2 s_- &= A_1\Omega_1 s_+ + q,\\
s &= s_+ - \Omega_2^{-1}(A_1\Omega_1 s_+ + q),\\
s &= (I_1 - \Omega_2^{-1}A_1\Omega_1)s_+ - \Omega_2^{-1}q,\\
s &= (I_1 - \Omega_2^{-1}(D_1 + \phi - U_1)\Omega_1)s_+ + \Omega_2^{-1}(L_1 + \phi)\Omega_1 s_+ - \Omega_2^{-1}q.
\end{aligned}$$

□

Based on (2) we propose the following iteration method which is referred to as modified general fixed point method (MGFP) for solving the LCP(q, A_1),

(3) $s^{(k+1)} = (I_1 - \Omega_2^{-1}(D_1 + \phi - U_1)\Omega_1)s_+^{(k)} + \Omega_2^{-1}(L_1 + \phi)\Omega_1 s_+^{(k+1)} - \Omega_2^{-1}q.$

Let Res be the Euclidean norm of the error vector, which is defined [17] as follows,

$$\text{Res}(z^{(k)}) = \|\min(z^{(k)}, A_1 z^{(k)} + q)\|_2.$$

Consider the nonnegative initial vector $z^{(0)} \in \mathbb{R}^n$. The iteration process continues until the iteration sequence $\{z^{(k)}\}_{k=0}^{+\infty} \subset \mathbb{R}^n$ converges. The iteration process stop if $\text{Res}(z^{(k)}) < 10^{-5}$. For computing $z^{(k+1)} \in \mathbb{R}^n$ we use the following algorithm.

Algorithm 1 (Modified General Fixed Point Method)

- Given any initial vector $s^{(0)} \in \mathbb{R}^n$, $\epsilon > 0$ and set $k = 0$.
- **for** $k = 0, 1, 2, \ldots$ do
- $s_+^{(k)} = \max\{0, s^{(k)}\}$
- compute $\text{Res} = norm(\min(s_+^{(k)}, A_1 s_+^{(k)} + q))$
- **if** $\text{Res} < \epsilon$ then
- $s = s^{(k)}$
- **break**
- **else**
- Using the following scheme, create the sequence $s^{(k)}$:
  $$s_1^{(k+1)} = ((I_1 - \Omega_2^{-1}(D_1 + \phi - U_1)\Omega_1)s_+^{(k)} - \Omega_2^{-1}q)_1,$$
- **for** $i = 2, 3, \ldots, n$ do
  $$\begin{aligned} s_i^{(k+1)} = {} & ((I_1 - \Omega_2^{-1}(D_1 + \phi - U_1)\Omega_1)s_+^{(k)} - \Omega_2^{-1}q)_i \\ & + (\Omega_2^{-1}(L_1 + \phi)\Omega_1 s_+^{(k+1)})_i \end{aligned}$$
  and set $z^{(k+1)} = \Omega_1 s_+^{(k+1)}$.
- **end for**
- **end if**
- **end for**.

Moreover, the MGFP provides a general structure for solving LCP(q, A_1). Using some particular values of the parameter matrices Ω_1, Ω_2 and we obtain an iterative method. In particular,

- When $\Omega_1 = I$, $\Omega_2 = \Omega^{-1}$ and $\phi = 0$, from (3) we have,

(4) $$s^{(k+1)} = (I_1 - \Omega(D_1 - U_1))s_+^{(k)} + \Omega L_1 s_+^{(k+1)} - \Omega q,$$

  this is a GFP [17].

THEOREM 13. *Suppose $A_1 = (D_1 + \phi) - (L_1 + U_1 + \phi) \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$. Then s^* is a solution of (2) if and only if $z^* = \Omega_1 s_+^*$ is a solution of LCP(q, A_1).*

Proof. Let s^* be a solution of (2). Then

$$
\begin{aligned}
s^* &= (I_1 - \Omega_2^{-1}(D_1 + \phi - U_1)\Omega_1)s_+^* + \Omega_2^{-1}(L_1 + \phi)\Omega_1 s_+^* - \Omega_2^{-1}q,\\
s^* &= (I_1 - \Omega_2^{-1}A_1\Omega_1)s_+^* - \Omega_2^{-1}q,\\
\Omega_2 s_-^* &= A_1\Omega_1 s_+^* + q.
\end{aligned}
$$

Since $\Omega_2 s_-^* \geq 0$,

$$A_1\Omega_1 s_+^* + q \geq 0.$$

Moreover,

$$(\Omega_1 s_+^*)^T(A_1\Omega_1 s_+^* + q) = (\Omega_1 s_+^*)^T(\Omega_2 s_-^*) = 0,$$

and $\Omega_1 s_+^* \geq 0$. Therefore $z^* = \Omega_1 s_+^*$ is a solution of LCP(q, A_1).

Let $z^* = \Omega_1 s_+^*$, $w^* = \Omega_2 s_-^*$ and $s^* = s_+^* - s_-^*$. From LCP(q, A_1)

$$
\begin{aligned}
\Omega_2 s_-^* &= A_1\Omega_1 s_+^* + q,\\
s^* &= s_+^* - \Omega_2^{-1}(A_1\Omega_1 s_+^* + q),\\
s^* &= (I_1 - \Omega_2^{-1}A_1\Omega_1)s_+^* - \Omega_2^{-1}q,\\
s^* &= (I_1 - \Omega_2^{-1}(D_1 + \phi - U_1)\Omega_1)s_+^* + \Omega_2^{-1}(L_1 + \phi)\Omega_1 s_+^* - \Omega_2^{-1}q.
\end{aligned}
$$

Thus, s^* is a solution of (2). □

In the following theorem, we show that the solution of (2) is unique when the system matrix A_1 of LCP(q, A_1) is a P-matrix.

THEOREM 14. *Let A_1 be a P-matrix and $A_1 = (D_1 + \phi) - (L_1 + U_1 + \phi) \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$. Then for any positive diagonal matrices Ω_1 and Ω_2,* (2) *has a unique solution.*

Proof. Since A_1 is a P-matrix, for any $q \in \mathbb{R}^n$ LCP(q, A_1) has a unique solution. Let y^* and u^* be the solutions of (2). Then

$$
\begin{aligned}
s^* &= (I_1 - \Omega_2^{-1}(D_1 + \phi - U_1)\Omega_1)s_+^* + \Omega_2^{-1}(L_1 + \phi)\Omega_1 s_+^* - \Omega_2^{-1}q,\\
u^* &= (I_1 - \Omega_2^{-1}(D_1 + \phi - U_1)\Omega_1)u_+^* + \Omega_2^{-1}(L_1 + \phi)\Omega_1 u_+^* - \Omega_2^{-1}q.
\end{aligned}
$$

Since $\Omega_1 y_+^* = \Omega_1 u_+^* \implies y_+^* = u_+^*$, therefore

$$y^* = u^*.$$

□

In the following, we prove the convergence conditions when A_1 is a P-matrix.

THEOREM 15. *Let A_1 be a P-matrix with $A_1 = (D_1 + \phi) - (L_1 + U_1 + \phi) \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$. Assume*

$$\rho\Big(\big(I - |\Omega_2^{-1}(L_1 + \phi)\Omega_1|\big)^{-1}\big|I_1 - \Omega_2^{-1}(D_1 + \phi - U_1)\Omega_1\big|\Big) < 1$$

and s^* is the solution of (2)*. Then the sequence $\{z^{(k)}\}_{k=1}^{+\infty}$ generated by Algorithm 1 converges to z^* for any initial vector $s^{(0)} \in \mathbb{R}^n$.*

Proof. Suppose A_1 is a P-matrix, then s^* is a unique solution of (2). Thus

$$s^* = (I_1 - \Omega_2^{-1}(D_1 + \phi - U_1)\Omega_1)s_+^* + \Omega_2^{-1}(L_1 + \phi)\Omega_1 s_+^* - \Omega_2^{-1}q.$$

From (3), this implies

$$\begin{aligned} s^{(k+1)} - s^* = &(I_1 - \Omega_2^{-1}(D_1 + \phi - U_1)\Omega_1)(s_+^{(k)} - s_+^*) \\ &+ \Omega_2^{-1}(L_1 + \phi)\Omega_1(s_+^{(k+1)} - s_+^*). \end{aligned}$$

It follows that

$$\begin{aligned} &|s^{(k+1)} - s^*| = \\ &= |(I_1 - \Omega_2^{-1}(D_1 + \phi - U_1)\Omega_1)(s_+^{(k)} - s_+^*) + \Omega_2^{-1}(L_1 + \phi)\Omega_1(s_+^{(k+1)} - s_+^*)| \\ &\leq |(I_1 - \Omega_2^{-1}(D_1 + \phi - U_1)\Omega_1)(s_+^{(k)} - s_+^*)| + |\Omega_2^{-1}(L_1 + \phi)\Omega_1(s_+^{(k+1)} - s_+^*)| \\ &\leq |(I_1 - \Omega_2^{-1}(D_1 + \phi - U_1)\Omega_1)||s^{(k)} - s^*| + |\Omega_2^{-1}(L_1 + \phi)\Omega_1||s^{(k+1)} - s^*| \end{aligned}$$

$$|s^{(k+1)} - s^*| - |\Omega_2^{-1}(L_1 + \phi)\Omega_1||s^{(k+1)} - s^*| \leq |(I_1 - \Omega_2^{-1}(D_1 + \phi - U_1)\Omega_1)||s^{(k)} - s^*|$$

$$(I - |\Omega_2^{-1}(L_1 + \phi)\Omega_1|)|s^{(k+1)} - s^*| \leq |(I_1 - \Omega_2^{-1}(D_1 + \phi - U_1)\Omega_1)||s^{(k)} - s^*|$$

and

$$|s^{(k+1)} - s^*| \leq (I - |\Omega_2^{-1}(L_1 + \phi)\Omega_1|)^{-1}|(I_1 - \Omega_2^{-1}(D_1 + \phi - U_1)\Omega_1)||s^{(k)} - s^*|.$$

Therefore, if $\rho((I - |\Omega_2^{-1}(L_1 + \phi)\Omega_1|)^{-1}|(I_1 - \Omega_2^{-1}(D_1 + \phi - U_1)\Omega_1)|) < 1$, for any initial vector $s^{(0)} \in \mathbb{R}^n$ the sequence $\{z^{(k)}\}_{k=1}^{+\infty}$ converges to the z^*. □

Now, when A_1 is an H_+-matrix, we analyze the convergence domains for parameter matrices Ω_1 and Ω_2 for MGFP.

THEOREM 16. *Let A_1 be a H_+-matrix with $A_1 = (D_1 + \phi) - (L_1 + U_1 + \phi) \in \mathbb{R}^{n \times n}$ and either one of the following is true:*

(1) $\Omega_2^{-1}\Omega_1 > (D_1 + \phi)^{-1}$ *and* $(2\Omega_2^{-1}\Omega_1 - (D_1 + \phi) - |B + \phi|)$, *where* $B = L_1 + U_1$.

(2) $0 < \Omega_2^{-1}\Omega_1 \leq (D_1 + \phi)^{-1}$.

Then the sequence $\{z^{(k)}\}_{k=1}^{+\infty}$ generated by Algorithm 1 converges to z^* for any initial vector $s^{(0)} \in \mathbb{R}^n$.

Proof. Since A_1 is an H_+-matrix. there LCP(q, A_1) has unique solution [11]. Now we will look at the splitting,

$$\begin{aligned} &(I_1 - |\Omega_2^{-1}(L_1 + \phi)\Omega_1|) - |I_1 - \Omega_2^{-1}(D_1 + \phi - U_1)\Omega_1| = \\ &= (I_1 - |I_1 - \Omega_2^{-1}(D_1 + \phi)\Omega_1|) - \Omega_2^{-1}|B + \phi|\Omega_1. \end{aligned}$$

(1) If $\Omega_2^{-1}\Omega_1 > (D_1 + \phi)^{-1}$ then,

$$\begin{aligned} &(I_1 - |I_1 - \Omega_2^{-1}(D_1 + \phi)\Omega_1|) - \Omega_2^{-1}|B + \phi|\Omega_1 = \\ &= 2I_1 - \Omega_2^{-1}(D_1 + \phi)\Omega_1 - \Omega_2^{-1}|B + \phi|\Omega_1 \\ &= \Omega_2^{-1}(2\Omega_2^{-1}\Omega_1 - (D_1 + \phi) - |B + \phi|)\Omega_1. \end{aligned}$$

Since $(2\Omega_2^{-1}\Omega_1 - (D_1+\phi) - |B+\phi|)$ is an M-matrix. Then the splitting $(I_1 - |\Omega_2^{-1}(L_1+\phi)\Omega_1|) - |I_1 - \Omega_2^{-1}(D_1+\phi-U_1)\Omega_1|$ represent an M-splitting of the M-matrix $\Omega_2^{-1}(2\Omega_2^{-1}\Omega_1 - (D_1+\phi) - |B+\phi|)\Omega_1$, hence $\rho((I - |\Omega_2^{-1}(L_1+\phi)\Omega_1|)^{-1}|(I_1 - \Omega_2^{-1}(D_1+\phi-U_1)\Omega_1)|) < 1$.

(2) If $\Omega_2^{-1}\Omega_1 \le (D_1+\phi)^{-1}$ then,

$$\begin{aligned}(I_1 - |I_1 - \Omega_2^{-1}(D_1+\phi)\Omega_1|) - \Omega_2^{-1}|B+\phi|\Omega_1 &= \Omega_2^{-1}(D_1+\phi-|B+\phi|)\Omega_1\\ &= \Omega_2^{-1}\langle A_1\rangle\Omega_1.\end{aligned}$$

Therefore, $(I_1 - |\Omega_2^{-1}(L_1+\phi)\Omega_1|) - |I_1 - \Omega_2^{-1}(D_1+\phi-U_1)\Omega_1|$ represents an M-splitting of M-matrix $\Omega_2^{-1}\langle A_1\rangle\Omega_1$ [13]. Therefore, from *Lemma* 8 $\rho((I - |\Omega_2^{-1}(L_1+\phi)\Omega_1|)^{-1}|(I_1 - \Omega_2^{-1}(D_1+\phi-U_1)\Omega_1)|) < 1$. □

4. NUMERICAL EXAMPLES

In this section, two numerical examples are provided to show the effectiveness of our new method. We use some notation as, IT=number of iteration steps, CPU = CPU time in seconds. The system matrix A_1 is generated by

$$A_1(p_1,p_2,p_3) = Q + p_1I_1 + p_2G + p_3H,$$

where p_1, p_2 and p_3 are given constants, I_1 is the identity matrix of order n and $G = \text{tridiag}\,(0,0,1) = \begin{bmatrix}0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ 0 & 0 & 0 & 1 & \vdots\\ \vdots & \dots & 0 & \ddots & 1\\ 0 & \dots & 0 & 0 & 0\end{bmatrix} \in \mathbb{R}^{n\times n}$ and

$$H\text{=diag}([1\ ,2,\ 1,\ 2,\ \dots])= \begin{bmatrix}1 & 0 & 0 & \dots & 0\\ 0 & 2 & 0 & \dots & 0\\ 0 & 0 & 1 & 0 & 0\\ \vdots & \dots & 0 & 2 & \vdots\\ 0 & \dots & \dots & 0 & \ddots\end{bmatrix} \in \mathbb{R}^{n\times n}.$$

Let $s^{(0)} = (0,0,0,0,\dots 0,0,\dots)^T \in \mathbb{R}^n$ be an initial vector. We consider the LCP(q, A_1) which has always a unique solution, where $q = (1\ \ -1\ \ 1\ \ -1\ \ \dots 1\ \ -1\dots)^T \in \mathbb{R}^n$. We set $\Omega_1 = I_1$ and $\Omega_2 = \omega^{-1}D_1$ in the MGFP. The suggested method is compared with GFP [17], which is effective in solving LCP(q, A_1).

Matlab version 2021a was used for all calculations. Table 1 and Table 2 show the numerical results for GFP [17] and MGFP respectively.

EXAMPLE 17. The system matrix $A_1, B_1 \in \mathbb{R}^{n\times n}$ are generated by $A_1(p_1,p_2,p_3) = Q + p_1 I_1 + p_2 G + p_3 H$, where p_1, p_2 and p_3 are given constants and

$$Q = \text{tridiag}(-I_2, L_1, -I_2) = \begin{bmatrix} L_1 & -I_2 & 0 & \dots & 0 \\ -I_2 & L_1 & -I_2 & \dots & 0 \\ \vdots & -I_2 & L_1 & -I_2 & \vdots \\ 0 & \dots & -I_2 & \ddots & -I_2 \\ 0 & \dots & 0 & -I_2 & L_1 \end{bmatrix} \in \mathbb{R}^{n\times n},$$

$$L_1 = \text{tridiag}(-1, 4, -1) = \begin{bmatrix} 4 & -1 & \dots & \dots & 0 \\ -1 & 4 & -1 & \dots & 0 \\ \vdots & -1 & 4 & -1 & \vdots \\ 0 & \dots & -1 & \ddots & -1 \\ 0 & \dots & \dots & -1 & 4 \end{bmatrix} \in \mathbb{R}^{m\times m},$$

where I_2 is the identity matrix of order m, where $n = m^2$ with m being a positive integer. □

EXAMPLE 18. The system matrix $A_1 \in \mathbb{R}^{n\times n}$ is generated by

$$A_1(p_1,p_2,p_3) = Q + p_1 I_1 + p_2 G + p_3 H,$$

where p_1, p_2 and p_3 are given constants, I_1 is the identity matrix of order n and

$$\begin{aligned} Q &= \text{tridiag}(-1.5I_2, L_1, -0.5I_2) \\ &= \begin{bmatrix} L_1 & -0.5I_2 & 0 & \dots & 0 \\ -1.5I_2 & L_1 & -0.5I_2 & \dots & 0 \\ \vdots & -1.5I_2 & L_1 & -0.5I_2 & \vdots \\ 0 & \dots & -1.5I_2 & \ddots & -0.5I_2 \\ 0 & \dots & 0 & -1.5I_2 & L_1 \end{bmatrix} \in \mathbb{R}^{n\times n}, \end{aligned}$$

$$L_1 = \text{tridiag}(-1.5, 4, -0.5) = \begin{bmatrix} 4 & -0.5 & \dots & \dots & 0 \\ -1.5 & 4 & -0.5 & \dots & 0 \\ \vdots & -1.5 & 4 & -0.5 & \vdots \\ 0 & \dots & -1.5 & \ddots & -0.5 \\ 0 & \dots & \dots & -1.5 & 4 \end{bmatrix} \in \mathbb{R}^{m\times m},$$

where I_2 is the identity matrix of order m. □

From Table 1 and Table 2, we see that our proposed MGFP have requires less iteration steps than the GFP [17] respectively.

Table 1. Results for GFP [17] and MGFP with $\phi = \alpha(L_1 + U_1^T)$.

$\mathbf{A_1(1,1,-1)}$	**n**	100	400	900	1600	2500
GFP	**IT**	18	21	22	23	23
$\omega = 1$	**CPU**	0.0058	0.1267	0.8356	12.8507	76.8588
	Res	7.8e-06	7.5e-06	7.3e-06	5.3e-06	7.0e-06
MGFP	**IT**	15	18	19	20	20
$\omega = 1$	**CPU**	0.0050	0.1028	0.6105	11.2363	67.0586
$\alpha = 0.1$	**Res**	6.4e-06	7.8e-06	6.8e-06	4.4e-06	5.9e-06
$\mathbf{A_1(0,1,0)}$	**n**	100	400	900	1600	2500
GFP	**IT**	13	14	15	15	15
$\omega = 1$	**CPU**	0.0045	0.0582	0.3728	5.4135	31.8341
	Res	5.3e-06	7.6e-06	9.3e-06	2.3e-06	2.6e-06
MGFP	**IT**	12	13	13	13	13
$\omega = 1$	**CPU**	0.0041	0.0817	0.4127	7.2064	42.3931
$\alpha = 0.1$	**Res**	3.1e-06	3.2e-06	5.3e-06	7.4e-06	9.5e-06
$\mathbf{A_1(1,1,1)}$	**n**	100	400	900	1600	2500
GFP	**IT**	9	9	10	10	10
$\omega = 1$	**CPU**	0.0046	0.0580	0.447	5.2868	31.3160
	Res	2.1e-06	6.7e-06	1.8e-06	2.5e-06	3.2e-06
MGFP	**IT**	9	9	9	10	10
$\omega = 1$	**CPU**	0.0050	0.0530	0.3177	5.3091	31.5628
$\alpha = 0.02$	**Res**	1.6e-06	5.0e-06	8.3e-06	1.8e-06	2.3e-06
$\mathbf{A_1(1,0,1)}$	**n**	100	400	900	1600	2500
GFP	**IT**	9	9	9	10	10
$\omega = 1.1$	**CPU**	0.0045	0.0582	0.3728	5.4135	31.8341
	Res	5.3e-06	7.6e-06	9.3e-06	2.3e-06	2.6e-06
MGFP	**IT**	9	9	9	9	10
$\omega = 1.1$	**CPU**	0.0040	0.0507	0.3671	5.4161	31.9112
$\alpha = 0.05$	**Res**	4.7e-06	6.6e-06	8.1e-06	9.3e-06	2.1e-06

5. CONCLUSION

In this article, we introduced a modified general fixed point method based on new matrix splitting for solving the LCP(q, A_1) with parameter matrices Ω_1 and Ω_2. Also, we showed how the iterative form is linked to the new matrix splitting and the parameter matrices Ω_1 and Ω_2 . This iterative form preserves the big and sparse structure of A_1 during the iteration process. Moreover, we showed the convergence condition for P-matrix and presented sufficient convergence domains for Ω_1 and Ω_2 when system matrix A_1 is H_+-matrix. At the end, two examples are discussed to show the efficiency of our proposed method.

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Table 2. Results for GFP [17] and MGFP with $\phi = \alpha(L_1 + U_1^T)$.

$\mathbf{A_1(1,1,-1)}$	**n**	100	400	900	1600	2500
GFP	**IT**	13	14	15	15	15
$\omega = 1$	**CPU**	0.0061	0.0807	0.5871	8.3912	48.8538
	Res	5.0e-06	6.2e-06	3.5e-06	5.0e-06	6.5e-06
MGFP	**IT**	11	12	13	15	15
$\omega = 1$	**CPU**	0.0051	0.0607	0.5392	8.44191	49.4862
$\alpha = 0.1$	**Res**	3.0e-06	4.8e-06	7.9e-06	3.1e-06	8.7e-06
$\mathbf{A_1(0,1,0)}$	**n**	100	400	900	1600	2500
GFP	**IT**	10	10	10	11	11
$\omega = 1$	**CPU**	0.0041	0.0548	0.3621	5.999	35.2935
	Res	1.9e-06	5.8e-06	9.4e-06	2.6e-06	3.3e-06
MGFP	**IT**	8	9	9	9	9
$\omega = 1$	**CPU**	0.0045	0.0.0507	0.3412	4.5795	28.4975
$\alpha = 0.1$	**Res**	7.8e-06	2.05e-06	2.7e-06	3.4e-06	4.0e-06
$\mathbf{A_1(1,1,1)}$	**n**	100	400	900	1600	2500
GFP	**IT**	7	7	8	8	8
$\omega = 1$	**CPU**	0.0041	0.0407	0.3266	4.1892	24.6431
	Res	2.7e-06	6.8e-06	9.7e-06	1.3e-06	1.7e-06
MGFP	**IT**	6	7	7	7	7
$\omega = 1$	**CPU**	0.0043	0.0474	0.2796	3.4536	22.6106
$\alpha = 0.1$	**Res**	9.7e-06	1.3e-06	1.6e-06	2.0e-06	2.2e-06
$\mathbf{A_1(1,0,1)}$	**n**	100	400	900	1600	2500
GFP	**IT**	8	8	8	8	8
$\omega = 1.1$	**CPU**	0.0041	0.0407	0.2699	4.4741	24.6650
	Res	1.9e-06	2.7e-06	3.2e-06	3.8e-06	4.2e-06
MGFP	**IT**	8	8	8	8	8
$\omega = 1.1$	**CPU**	0.0047	0.05303	0.2554	4.0929	24.3851
$\alpha = 0.1$	**Res**	1.7e-06	2.4e-06	2.9e-06	3.4e-06	3.8e-06

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