A NOTE ON THE FIXED POINT METHOD
AND THE LINEAR COMPLEMENTARITY PROBLEM

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Abstract. In this paper, we propose an extended form of a fixed point method
for processing the large and sparse linear complementarity problem (LCP). We
obtain an equivalent form of LCP by using two positive diagonal matrices and
prove the equivalence. For the proposed method, we provide some convergence
conditions when the system matrix is a $P$-matrix or an $H_+$-matrix or a symmetric
positive definite matrix.

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Keywords. Linear complementarity problem, $P$-matrix, $H_+$-matrix, symmetric
positive definite matrix, matrix splitting, convergence.

1. INTRODUCTION

In the literature on mathematical programming, the linear complementarity
problem has received considerable attention. It also appears in a number of
applications in operations research applications, control theory, mathematical
economics, optimization theory, stochastic optimal control, the American op-
tion pricing problem, economics, elasticity theory, the free boundary problem,
and the Nash equilibrium point of the bimatrix game. For details, see [18],
[19]. For recent works on this problem, see [14], [15] and references therein.

Consider $A_1 \in \mathbb{R}^{n \times n}$ and a vector $q \in \mathbb{R}^n$. The linear complementarity
problem denoted as LCP($q, A_1$) is to find the solution $z \in \mathbb{R}^n$ to the following
system

\begin{align}
  z & \geq 0, \quad A_1 z + q \geq 0, \quad z^T (A_1 z + q) = 0.
\end{align}

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There are various methods of solving the LCP using an iterative process, namely the projected methods [4, 9, 12, 16, 20, 21], the modulus algorithm [2, 3, 5, 10] and the modulus based matrix splitting iterative methods [8, 11, 22].

A general fixed point method (GFP) is proposed by Fang [6] assuming the case where \( \phi = \omega A_1^{-1} \) with \( \omega > 0 \) and \( A_{1D} \) is a diagonal matrix of \( A_1 \). The GFP approach takes less iterations than the modulus-based successive over-relaxation (MSOR) iteration method [2]. In this paper we present an extended form of the fixed point method [6] that incorporates projected type iteration techniques by including two positive diagonal parameter matrices \( \phi_1 \) and \( \phi_2 \). We also show that the fixed point equation and the linear complementarity problem are equivalent and discuss the convergence conditions as well as provide some convergence domains for the proposed method.

The rest of this paper is organized as follows: In Section 2, we present some notation, definitions and lemmas in order to establish our key findings. Section 3 discusses an extended form of a fixed point method for solving LCP(\( q, A_1 \)) with convergence analysis. In the last section, we give the conclusion.

2. PRELIMINARIES

Some notations, introductory definitions and required lemmas are introduced. For details, see [2], [6].

Let \( A_1 = (a_{ij}) \in \mathbb{R}^{n \times n} \) and \( B_1 = (b_{ij}) \in \mathbb{R}^{n \times n} \). We use \( A_1 \geq (>) B_1 \) to denote \( a_{ij} \geq (>) b_{ij} \) for all \( i, j \). The comparison matrix \( \langle A_1 \rangle = (\langle a_{ij} \rangle) \) of \( A_1 \) is defined by \( \langle a_{ij} \rangle = |a_{ij}| \) with \( i = j \) and \( \langle a_{ij} \rangle = -|a_{ij}| \) with \( i \neq j \) for \( i, j = 1, 2, \ldots, n \). The matrix \( A_1 \) is called a Z-matrix if all of its non-diagonal elements are less than or equal to zero; an Z-matrix is called an M-matrix if \( A_1^{-1} \geq 0 \); an H-matrix if \( \langle A_1 \rangle \) is an M-matrix. The splitting \( A_1 = M_1 - N_1 \) is called an M-splitting if \( \det(M_1) \neq 0 \) and \( N_1 \geq 0 \), an H-splitting if \( \langle M_1 \rangle - |N_1| \) is an M-matrix [7]. An H-matrix is called an Hₜ-matrix [1] if \( a_{ii} > 0 \) for \( i = 1, 2, \ldots, n \). Let \( A_1 \in \mathbb{R}^{n \times n} \), then \( A_1 \) is said to be a P-matrix if all its principle minors are positive that is \( \det(A_{1\alpha\alpha}) > 0 \) for all \( \alpha \subseteq \{1, 2, \ldots, n\} \). Suppose \( A_1 = (a_{ij}) \in \mathbb{R}^{n \times n} \) be a square matrix, then \( |A_1| = (c_{ij}) \) is defined by \( c_{ij} = |a_{ij}| \forall i, j \) and \( A_1^T \) denotes the transpose of \( A_1 \).

**Lemma 1** ([13]). The LCP\( (q, A_1) \) has a unique solution for any \( q \in \mathbb{R}^n \) if \( A_1 \in \mathbb{R}^{n \times n} \) is a P-matrix.

**Lemma 2** ([6]). Let \( A_1 \in \mathbb{R}^{n \times n} \) and \( A_1 = M_1 - N_1 \) be an M-splitting with \( M_1 \) an M-matrix and \( N_1 \geq 0 \). Then \( \rho(M_1^{-1}N_1) < 1 \).

3. MAIN RESULTS

For a given vector \( \xi \in \mathbb{R}^n \), we indicate the vectors \( \xi_+ = \max\{0, \xi\} \), \( \xi_- = \max\{0, -\xi\} \). Since \( \xi_+ \geq 0 \), \( \xi_- \geq 0 \), \( \xi = \xi_+ - \xi_- \), \( \xi_+^T \xi_- = 0 \). Let \( z = \phi_1 \xi_+ \) and
\( w = \phi_2 \xi_- \), where \( \phi_1 \) and \( \phi_2 \) are positive diagonal matrices of order \( n \). Now we convert the LCP into a fixed point formulation that is
\[
\xi = (I_1 - \phi_2^{-1} A_1 \phi_1) \xi_+ - \phi_2^{-1} q,
\]
where \( I_1 \) is an identity matrix of order \( n \).

In the following result, we provide an equivalent formulation of the \( \text{LCP}(q, A_1) \) for solving \( \text{LCP}(q, A_1) \).

**Theorem 1.** Suppose \( A_1 \in \mathbb{R}^{n \times n} \) and \( q \in \mathbb{R}^n \). Then \( \xi^* \) is a solution of (2) if and only if \( z^* = \phi_1 \xi_+^* \) is a solution of \( \text{LCP}(q, A_1) \).

**Proof.** Let \( \xi^* \) be a solution of (2). Then
\[
\xi^* = (I_1 - \phi_2^{-1} A_1 \phi_1) \xi_+^* - \phi_2^{-1} q.
\]
Since \( \phi_2 \xi_+^* \geq 0 \),
\[
A_1 \phi_1 \xi_+^* + q \geq 0.
\]
Moreover,
\[
(\phi_1 \xi_+^*)^T (A_1 \phi_1 \xi_+^* + q) = (\phi_1 \xi_+^*)^T (\phi_2 \xi_-^*),
\]
and \( \phi_1 \xi_+^* \geq 0 \). Therefore \( z^* = \phi_1 \xi_+^* \) is a solution of \( \text{LCP}(q, A_1) \).

Let \( z^* = \phi_1 \xi_+^* \) and \( w^* = \phi_2 \xi_-^* \), and \( \xi^* = \xi_+^* - \xi_-^* \). Now from \( \text{LCP}(q, A_1) \)
\[
\phi_2 \xi_-^* = A_1 \phi_1 \xi_+^* + q,
\]
\[
\xi_-^* = \phi_2^{-1} (A_1 \phi_1 \xi_+^* + q),
\]
\[
\xi^* = (I_1 - \phi_2^{-1} A_1 \phi_1) \xi_+^* - \phi_2^{-1} q.
\]
Thus, \( \xi^* \) is a solution of (2). \( \square \)

Now we show that the solution of (2) is unique when the matrix \( A_1 \) is a \( P \)-matrix.

**Theorem 2.** For any positive diagonal matrices \( \phi_1 \) and \( \phi_2 \), (2) has a unique solution if \( A_1 \in \mathbb{R}^{n \times n} \) is a \( P \)-matrix.

**Proof.** Since \( A_1 \) is a \( P \)-matrix, for any \( q \in \mathbb{R}^n \) \( \text{LCP}(q, A_1) \) has a unique solution. Let \( y^* \) and \( u^* \) be the solutions of (2). Then
\[
y^* = y_+^* - \phi_2^{-1} (A_1 \phi_1 y_+^* + q),
\]
and
\[
u^* = u_+^* - \phi_2^{-1} (A_1 \phi_1 u_+^* + q).
\]
As \( y_+^* = u_+^* \),
\[
y^* + \phi_2^{-1} (A_1 \phi_1 y_+^* + q) = u^* + \phi_2^{-1} (A_1 \phi_1 u_+^* + q).
\]
Hence
\[
y^* = u^*.
\]
\( \square \)
Based on (2), we obtain an extended form of the fixed point method which is referred to as Method 1.

**METHOD 1.** Let \( A_1 \in \mathbb{R}^{n \times n} \) and \( q \in \mathbb{R}^n \). Suppose \( \xi^{(0)} \in \mathbb{R}^n \) an initial vector and the sequence \( \{z^{(k)}\}^{+\infty}_{k=1} \subset \mathbb{R}^n \). Let Residue be an Euclidean norm of the error vector and define the Residue as

\[
\text{Res}(z^{(k)}) = \|\min(z^{(k)}, A_1z^{(k)} + q)\|_2,
\]

where \( z^{(k)} \) is the \( k^{th} \) approximate solution of the LCP \((q, A_1)\). The iteration process stop if \( (z^{(k)}) < 10^{-5} \) or the number of iteration reached 900. For computing \( \xi^{(k+1)} \in \mathbb{R}^n \) is as follows:

1. Given an initial vector \( \xi^{(0)} \in \mathbb{R}^n \), error \( \epsilon > 0 \) and set \( k = 0 \).
2. Using the following scheme, create the sequence \( \xi^{(k)} \):

\[
(3) \quad \xi^{(k+1)} = (I_1 - \phi_2^{-1}A_1\phi_1)\xi^{(k)} - \phi_2^{-1}q
\]

and set \( z^{(k+1)} = \phi_1\xi^{(k+1)} \).

3. If \( (z^{(k)}) < \epsilon \) then stop; otherwise, set \( k = k + 1 \) and return to step 2.

**Remark 3.** Fang [6] introduced a fixed point method, which is a special case of (3) with \( \phi_2 = \phi^{-1} \) and \( \phi_1 = I_1 \), where \( \phi \) is a positive diagonal matrix. \( \square \)

In the following result, we discuss the convergence condition when the system matrix \( A_1 \) is a P-matrix.

**Theorem 4.** Let \( A_1 \in \mathbb{R}^{n \times n} \) be a P-matrix. Let \( \rho(|I_1 - \phi_2^{-1}A_1\phi_1|) < 1 \) and \( \xi^* \) be the solution of (2). Then the sequence \( \{z^{(k)}\}^{+\infty}_{k=1} \) generated by Method 1 converges to \( z^* \) for any initial vector \( \xi^{(0)} \in \mathbb{R}^n \).

**Proof.** Suppose \( A_1 \) is a P-matrix. Then \( \xi^* \) is a unique solution of (2). Thus

\[
\xi^* = (I_1 - \phi_2^{-1}A_1\phi_1)\xi^*_+ - \phi_2^{-1}q.
\]

From (3), it results

\[
\xi^{(k+1)} - \xi^* = (I_1 - \phi_2^{-1}A_1\phi_1)(\xi^{(k)} - \xi^*_+).
\]

It follows that

\[
|\xi^{(k+1)} - \xi^*| = |(I_1 - \phi_2^{-1}A_1\phi_1)| \cdot |\xi^{(k)} - \xi^*_+| \leq |(I_1 - \phi_2^{-1}A_1\phi_1)| \cdot |\xi^{(k)} - \xi^*|.
\]

Since \( \rho(|I_1 - \phi_2^{-1}A_1\phi_1|) < 1 \) and for any initial vector \( \xi^{(0)} \in \mathbb{R}^n \) the sequence \( \{z^{(k)}\}^{+\infty}_{k=1} \) converges to the \( z^* \). \( \square \)

In the following result, we provide the convergence conditions for Method 1 when the system matrix is an \( H_+ \)-matrix.
THEOREM 5. Assume $A_1 \in \mathbb{R}^{n \times n}$ is an $H_+$-matrix, $A_{1D} = \text{diag}(A_1)$ and $B = A_{1D} - A_1 \in \mathbb{R}^{n \times n}$. Let $\phi_1 = \alpha_1 D_1$, $\phi_2 = \omega^{-1} A_{1D}$ and

$$
\rho(A_{1D}^{-1}|B|D_1) \leq \rho(A_{1D}^{-1}|B|)\rho(D_1),
$$

where $\alpha_1$, $\omega$ are positive constants and $D_1$ is a positive diagonal matrix. Let $\omega_1 = \beta$ and $\xi^*$ be the solution of (2). Then the sequence $\{z^{(k)}\}_{k=1}^{\infty}$ generated by Method 1 converges to $z^*$ for any initial vector $\xi^{(0)} \in \mathbb{R}^n$ if

$$
0 < \beta < \frac{2}{(1+\rho(A_{1D}^{-1}|B|))\rho(D_1)}.
$$

Proof. We have $A_1$ is an $H_+$-matrix, $A_{1D} = \text{diag}(A_1)$, $B = A_{1D} - A_1$ and $\rho(A_{1D}^{-1}|B|) < 1$. For $\phi_1 = \alpha_1 D_1$ and $\phi_2 = \omega^{-1} A_{1D}$, we obtain

$$
|I_1 - \phi_2^{-1} A_1 \phi_1| = |I_1 - (\omega^{-1} A_{1D})^{-1} A_1 \alpha_1 D_1| = |I_1 - (\omega^{-1} A_{1D})^{-1} (A_{1D} - B) \alpha_1 D_1| = |I_1 - \omega_1 D_1 + \omega A_{1D}^{-1} B \alpha_1 D_1| \leq |I_1 - \omega_1 D_1| + |\omega A_{1D}^{-1} B \alpha_1 D_1|
$$

$$
\leq |I_1 - \beta D_1| + \beta A_{1D}^{-1}|B|D_1.
$$

It follows that

$$
|I_1 - \beta D_1| + \beta A_{1D}^{-1}|B|D_1 = \begin{cases} (I_1 - \beta D_1) + \beta A_{1D}^{-1}|B|D_1, & \text{if } 0 < \beta D_1 \leq I_1, \\ (\beta D_1 - I_1) + \beta A_{1D}^{-1}|B|D_1, & \text{if } \beta D_1 > I_1. \end{cases}
$$

Now we write

$$
(4) \quad \rho(|I_1 - \phi_2^{-1} A_1 \phi_1|) \leq \begin{cases} 1 - (1 - \rho(A_{1D}^{-1}|B|))\beta \rho(D_1), & \text{if } 0 < \beta \rho(D_1) \leq 1, \\ (1 + \rho(A_{1D}^{-1}|B|))\beta \rho(D_1) - 1, & \text{if } \beta \rho(D_1) > 1. \end{cases}
$$

From (4) we can see that $\rho(|I_1 - \phi_2^{-1} A_1 \phi_1|) < 1$ for $\beta \rho(D_1) \in (0, 1]$ and for $\beta \rho(D_1) > 1$, $\rho(|I_1 - \phi_2^{-1} A_1 \phi_1|) < 1$ if and only if

$$
(1 + \rho(A_{1D}^{-1}|B|))\beta \rho(D_1) - 1 < 1
$$

such that $\beta < \frac{2}{(1+\rho(A_{1D}^{-1}|B|))\rho(D_1)}$. Therefore, if

$$
0 < \beta < \frac{2}{(1+\rho(A_{1D}^{-1}|B|))\rho(D_1)},
$$

for any initial vector $\xi^{(0)} \in \mathbb{R}^n$, the sequence $\{z^{(k)}\}_{k=1}^{\infty}$ converges to $z^*$. \( \square \)

In the following result, we provide the convergence conditions for Method 1 when the system matrix is a symmetric positive definite (SPD) matrix.

THEOREM 6. Let $A_1 \in \mathbb{R}^{n \times n}$ be the SPD matrix. Let $\phi_2 = \omega^{-1} I_1$ and $\phi_1 = \alpha_1 D_1$, where $D_1$ is a scalar matrix and denote the minimum and the maximum eigenvalues of $A_1 D_1$ by $\nu_{\text{min}}$ and $\nu_{\text{max}}$ respectively. Let $\xi^*$ be the solution of (2). Then the sequence $\{z^{(k)}\}_{k=1}^{\infty}$ generated by Method 1 converges to $z^*$ for any initial vector $\xi^{(0)} \in \mathbb{R}^n$ if $0 < \beta < \frac{2}{\nu_{\text{max}}}$. 


Proof. From Theorem 4, we have
\[ \xi^{(k+1)} - \xi^* = (I_1 - \phi_2^{-1} A_1 \phi_1)(\xi^{(k)}_+ - \xi^*_+). \]
This implies that
\[ \|\xi^{(k+1)} - \xi^*\|_2 = \|(I_1 - \phi_2^{-1} A_1 \phi_1)(\xi^{(k)}_+ - \xi^*_+}\|_2. \]
Since \( \|(\xi^{(k)}_+ - \xi^*_+)\|_2 \leq \|(\xi^{(k)} - \xi^*)\|_2 \),
\[ \|\xi^{(k+1)} - \xi^*\|_2 \leq \|(I_1 - \phi_2^{-1} A_1 \phi_1)\|_2 \|\xi^{(k)} - \xi^*\|_2 \]
\[ \leq \|(I_1 - \phi_2^{-1} A_1 \phi_1)\|_2 \|\xi^{(k)} - \xi^*\|_2. \]
If \( \|I_1 - \phi_2^{-1} A_1 \phi_1\|_2 < 1 \), then Method 1 is convergent. Therefore
\[ \|I_1 - \phi_2^{-1} A_1 \phi_1\|_2 = \|I_1 - \omega I_1 A_1 \alpha_1 D_1\|_2 \]
\[ = \|I_1 - \beta A_1 D_1\|_2. \]
We have
\[ \|I - \beta A_1 D_1\|_2 = \max\{|1 - \beta \nu_{\min}|, |1 - \beta \nu_{\max}|\}. \]
It follows that
\[ \|I_1 - \beta A_1 D_1\|_2 = \begin{cases} \{1 - \beta \nu_{\min}\}, & \text{if } |1 - \beta \nu_{\min}| \geq |1 - \beta \nu_{\max}|, \\ \{1 - \beta \nu_{\max}\}, & \text{if } |1 - \beta \nu_{\max}| \geq |1 - \beta \nu_{\min}|. \end{cases} \]
Thus \( \|I_1 - \beta A_1 D_1\|_2 < 1 \) if and only if
\[ (a) \begin{cases} |1 - \beta \nu_{\min}| < 1, \\ |1 - \beta \nu_{\min}| \geq |1 - \beta \nu_{\max}|, \end{cases} \]
and
\[ (b) \begin{cases} |1 - \beta \nu_{\max}| < 1, \\ |1 - \beta \nu_{\max}| \geq |1 - \beta \nu_{\min}|. \end{cases} \]
From (a) and (b) we obtain the convergence condition of Method 1 that is 0 < \( \beta \leq \frac{2}{\nu_{\min} + \nu_{\max}} \) and \( \frac{2}{\nu_{\min} + \nu_{\max}} \leq \beta < \frac{2}{\nu_{\max}} \), from these two inequalities we obtain
\[ 0 < \beta < \frac{2}{\nu_{\max}}. \]
4. CONCLUSION

We introduced an extended form of a fixed point method for solving the linear complementarity problem LCP(ϕ1, A1) with parameter matrices ϕ₁ and ϕ₂. Also, we have shown how the iterative form relates to the parameter matrices ϕ₁ and ϕ₂. We have presented some convergence conditions and some sufficient convergence domains for the proposed method.

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