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CONVERGENCE AND ERROR ESTIMATES FOR PSEUDO-POLYHARMONIC DIV-CURL AND ELASTIC INTERPOLATION ON A BOUNDED DOMAIN

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Abstract. This paper establishes convergence rates and error estimates for the pseudo-polyharmonic div-curl and elastic interpolation. This type of interpolation is based on a combination of the divergence and the curl of a multivariate vector field and minimizing an appropriate functional energy related to the divergence and curl. Convergence rates and error estimates are established when the interpolated vector field is assumed to be in the classical fractional vectorial Sobolev space on an open bounded set with a Lipschitz-continuous boundary. The error estimates introduced in this work are sharp and its rate of convergence depends algebraically on the fill distance of the scattered data nodes. More precisely, the order of convergence depends, essentially, on the smoothness of the target vector field, on the dimension of the Euclidean space and on the null space of corresponding Sobolev semi-norm. A numerical example is given to illustrate the convergence shape.

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1. INTRODUCTION

This paper deals with convergence and error estimates for approximation of vector fields by div-curl and elastic pseudo-polyharmonic splines in n-dimensional vector space $n \geq 2$. Approximation of vector fields arises in many scientific applications such as meteorology, electromagnetic, optic flow [13, 14, 32, 33, 34]. Up to the work [8] this specific type of approximation has

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only been studied only in 2D or 3D dimensional spaces, both from the theoretical and applied point of view, only for fields in dimension $n = 2$ and $n = 3$, see [3, 7, 16]. Interpolation and approximation theory and computational algorithm for fields in $n \geq 4$, is an interesting problem for many applications. We recall, that there are many scientific fields where multivariate approximation in $n \geq 4$ is important. For instance, a four-dimensional (4D) respiratory correlated computed tomography (RCCT) has been widely used for studying organ motion [20], a moving computational domain method and its application to flow around a high-speed car passing through a hairpin curve [37] and robust 4D flow denoising using divergence-free wavelet transform [30]. We also mention many seismic processing techniques where interpolation algorithms that use multiple spatial dimensions have many advantages over one-dimensional methods. In particular, simultaneous interpolation in all five seismic data dimension (inline, crossline, offset, azimuth, and frequency) has great utility in predicting missing data with correct amplitude and phase variations [35, 36].

It is important to stress that multivariate interpolation and approximation for n-dimensional space with $n \geq 2$ have been extensively considered in the literature, for the case of multivariate and one dimensional valued functions. For instance, we cite [12] and the references therein and the list of papers studying scalar radial basis functions in the general dimension n is not exhaustive. In [7, 9], the div-curl approximation problem was studied by the authors both for interpolating and smoothing div-curl problem in only 3D dimensional space and was related to the thin plate splines under tension. Indeed, to extend the work given in [7, 9] to n-dimensional space with $n \geq 4$ still an open problem since the thin plate spline under a tension τ of order m are related to the differential operator $\Delta^{m+1} - \tau^2\Delta^m$ for which finding a fundamental solution in the distributional sense in n-dimensional space is a difficult task and it is an open problem. Here, Δ^m denotes the iterate Laplace operator of order m. It is important to underline that the present work is quite different from the one introduced in [7, 9]. In [8], the authors introduced and studied pseudo-polyharmonic Div-Curl and Elastic vector fields approximation their associated operator is the fractional pseudo-differential iterate Laplacian operator Δ^{m+s} for which a fundamental solution is known in any n-dimensional space [29]. It is not easy, to introduce the main problem we want to study here without introducing all the notations. To facilitate the readability of our present paper, we will dedicate a specific subsection where we will recall all the notations and definitions needed. We hope that once all the notations have been introduced, the paper will be easier to read. For more details on the pseudo-polyharmonic div-curl and elastic approximation, we refer to [8]. The smoothing and interpolating problems proposed in [8], are based on the minimization of a quadratic functional which linearly combines two energy terms related to the divergence and the curl of the vector field in n-dimensional space, respectively.

The aim of this paper is to study and establish some results on the error estimates and convergence for the the pseudo-polyharmonic div-curl and elastic interpolation, when the vector field to be interpolated belongs to the vectorial classical fractional Sobolev space on an open connected and bounded set in n-dimensional space. This theoretical setting is placed in Hilbert spaces, sometimes referred as Native spaces, which are contained in the standard vectorial Sobolev spaces. The specific mathematical results along with notations, functional spaces and energy used are stated in Section 2. However, we can briefly say that the main results of this paper, are the proof of the convergence in the Sobolev space $H^{m+s}(\Omega, \mathbb{R}^n)$, see Theorem 3.2 and Theorem 4.2, for div-curl minimization problem and for elastic minimization problem, respectively, together with the results on error estimates, see Theorem 3.4 and Theorem 4.3, for div-curl minimization problem and for elastic minimization problem, respectively.

The paper is organized as follows. In Section 2, we will give the notations and some preliminary results. Then, the minimal interpolating problem and the minimal extension problem to the classical Sobolev space $H^{m+s}(\Omega; \mathbb{R}^n)$ are studied. Their basic properties are given. In Section 3, the convergence Theorem 3.2 and error estimates Theorem 3.4 are proved for div-curl minimization problem. In Section 4, we give the interpolating minimization problem and results on convergence and error estimates for the elastic minimization problem. Theorem 4.2 and Theorem 4.3 are proved. In Section 5, some numerical experiments are given to illustrate some theoretical results. As usual, we finish our paper by giving a general conclusion.

2. PSEUDO-POLYHARMONIC DIV-CURL INTERPOLATION

In this section we analyzed the minimal interpolating problem and the minimal extension problem to the classical Sobolev space $H^{m+s}(\Omega; \mathbb{R}^n)$. We first introduce in a subsection the notations which we will use and some preliminary results.

2.1. Notations and preliminary results.

The notations used in this paper, are similar to those used in [8]. But, it should be more convenient to state here the definitions and properties on the classical Sobolev space that we will use. For more details, we invite the reader to consult, for example, the long-standing classic references on Sobolev spaces [1, 2, 26].

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the set of positive integers and $\Omega \subset \mathbb{R}^n$ be a nonempty open set in $\mathbb{R}^n$ with $n \in \mathbb{N}$ and $n \geq 2$. The standard Euclidean norm in $\mathbb{R}^n$ is denoted by $|x| = \sqrt{x_1^2 + \dots + x_n^2}$ for $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$, where the notation x^T stands for the transpose of x. The notation $x^T y$ stands for the classical scalar product in $\mathbb{R}^n$. We denote by $\mathscr{C}^k(\overline{\Omega}; \mathbb{R}^n)$ the space of continuous functions with derivatives up to the k-th order are continuous over the closure of Ω. For $p \in [1, \infty)$ and $r \in [0, \infty)$, the Sobolev spaces on Ω, denoted by $W^{r,p}(\Omega)$ are the spaces

$$W^{r,p}(\Omega) := \{u \in L^p(\Omega) : \forall \alpha \in \mathbb{N}^n,\ 0 \le |\alpha| \le r,\ \partial^\alpha u \in L^p(\Omega)\},$$

where $L^p(\Omega)$ is the classical Lebesgue space of order p and $\partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots x_n^{\alpha_n}}$ stands for the derivative of u of order $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ in the distributional sense. We recall that $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \times \cdots \times \alpha_n!$ stand for the module and the factorial of the multi-index α, respectively.

The Sobolev spaces $W^{r,p}(\Omega)$ are equipped with the following semi-norms and norms (see [1, 2, 26]) given as follows

- For any $r \in \mathbb{N}$, the usual semi-norms $|\,.\,|_{k,p,\Omega}$ with $k \in \{0, \ldots, r\}$, and the norm $\|\cdot\|_{r,p,\Omega}$ are defined by

$$\left\{\begin{array}{l} \text{For } 1 \le p < \infty : \\ |u|_{k,p,\Omega} := \left(\displaystyle\sum_{|\alpha|=k} \int_\Omega |\partial^\alpha u(x)|^p dx \right)^{1/p}, \quad \|u\|_{r,p,\Omega} := \left(\displaystyle\sum_{0 \le k \le r} |u|^p_{k,p,\Omega} \right)^{1/p}. \\ \text{For } p = \infty : \\ |u|_{k,\infty,\Omega} := \displaystyle\max_{|\alpha|=k} \left(\operatorname*{ess\,sup}_{x\in\Omega} |\partial^\alpha u(x)| \right), \qquad \|u\|_{r,\infty,\Omega} := \displaystyle\max_{0\le k\le r} |u|_{k,\infty,\Omega}. \end{array}\right. \tag{2.1}$$

- For any $r \in [0, \infty) \setminus \mathbb{N}$, the notations $\lfloor r \rfloor$ and $\lceil r \rceil$ stand for the integers (floor and ceiling of r) satisfying $\lfloor r \rfloor \le r < \lfloor r \rfloor + 1$ and $\lceil r \rceil - 1 < r < \lceil r \rceil$, respectively. The Sobolev space $W^{r,p}(\Omega)$ of non-integer order r, consists of the (equivalence classes of) functions $u \in W^{\lfloor r \rfloor, p}(\Omega)$ such that

$$\left\{\begin{array}{ll} |u|_{r,p,\Omega} := \left(\displaystyle\sum_{|\alpha|=\lfloor r \rfloor} \int_{\Omega\times\Omega} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^p}{|x-y|^{n+p(r-\lfloor r \rfloor)}} dxdy \right)^{1/p} < \infty, & \text{if } 1 \le p < \infty \\ |u|_{r,\infty,\Omega} := \displaystyle\max_{|\alpha|=\lfloor r \rfloor} \operatorname*{ess\,sup}_{\substack{x,y\in\Omega \\ x \ne y}} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^{r-\lfloor r \rfloor}} < \infty, & \text{if } p = \infty. \end{array}\right. \tag{2.2}$$

The norm $\|\cdot\|_{r,p,\Omega}$ defined in $W^{r,p}(\Omega)$ is given by

$$\|u\|_{r,p,\Omega} := \left\{\begin{array}{ll} \left(\|u\|^p_{\lfloor r \rfloor,p,\Omega} + |u|^p_{r,p,\Omega} \right)^{1/p}, & \text{if } 1 \le p < \infty, \\ \max\left(\|u\|_{\lfloor r \rfloor,\infty,\Omega},\ |u|_{r,\infty,\Omega} \right), & \text{if } p = \infty. \end{array}\right. \tag{2.3}$$

The semi-norms and the norms defined on the product space $W^{r,p}(\Omega; \mathbb{R}^n) = [W^{r,p}(\Omega)]^n$, are denoted by the similar notations as in the scalar case, and are given by

$$
\begin{aligned}
&\bullet \text{ For } 1 \le p < \infty: \\
&|u|_{r,p,\Omega} := \left(\sum_{i=1}^{n} |u_i|_{r,p,\Omega}^{p}\right)^{1/p}, \quad \|u\|_{r,p,\Omega} := \left(\sum_{i=1}^{n} \|u_i\|_{r,p,\Omega}^{p}\right)^{1/p}, \\
&\bullet \text{ For } p = \infty: \\
&|u|_{r,\infty,\Omega} := \max_{1\le i\le n} |u_i|_{r,\infty,\Omega}, \qquad \|u\|_{r,\infty,\Omega} := \max_{1\le i\le n} \|u_i\|_{r,\infty,\Omega},
\end{aligned}
\tag{2.4}
$$

for all $u = (u_1, \dots, u_n)^T \in W^{r,p}(\Omega;\mathbb{R}^n)$. When $p = 2$ the Sobolev space $W^{r,2}(\Omega;\mathbb{R}^n)$ is denoted as usual by $H^r(\Omega;\mathbb{R}^n)$. When the open set Ω is bounded and has a Lipschitz-continuous boundary (in the sense of Necăs [26]), for any $r > 0$, the space $W^{r,p}(\Omega;\mathbb{R}^n)$ satisfies the properties given in the following proposition.

Proposition 2.1. *The following properties hold.*

i) Sobolev embedding theorem:

$$\forall r > 0,\ \forall k \in \mathbb{Z}_+,\ k + \tfrac{n}{p} < r,\ \ W^{r,p}(\Omega;\mathbb{R}^n) \hookrightarrow \mathscr{C}^k(\overline{\Omega};\mathbb{R}^n) \tag{2.5}$$

ii) Existence theorem of an extension operator:
There exists a linear continuous operator E_Ω from $W^{r,p}(\Omega;\mathbb{R}^n)$ into $W^{r,p}(\mathbb{R}^n;\mathbb{R}^n)$ such that, for any $v \in W^{r,p}(\Omega;\mathbb{R}^n)$, $E_\Omega v|_\Omega = v$.

iii) Quotient norm:
Let $p \in [1,\infty)$, $r > 0$ and $k = \lceil r \rceil - 1$. Then, there exists a positive constant C such that

$$\min_{q\in\Pi_k(\Omega;\mathbb{R}^n)} \|v - q\|_{r,p,\Omega} \le C|v|_{r,p,\Omega}, \tag{2.6}$$

for all $v \in W^{r,p}(\Omega;\mathbb{R}^n)$, where $\Pi_k(\Omega;\mathbb{R}^n)$ is the space of vector-valued polynomials of n-variables with degree $\le k$.

Proof. The result is a vectorial version of the scalar case, namely Items 1. and 2. are generalizations of the corresponding scalar case results, (see [2, 26]).
Item 3. can be derived from Theorem 3.1.1, see [15], if $r \in \mathbb{N}^*$, and Theorem 5.1 [28] otherwise. □

In the following, the notation $L^1_{loc}(\mathbb{R}^n)$ stands for the classical Lebesgue space of locally integrable functions on $\mathbb{R}^n$. Let us also recall the Schwartz spaces that we will need, for more details see [29]: $\mathscr{D}'(\mathbb{R}^n;\mathbb{R}^n)$ stands for the space of the vector-valued distributions on $\mathbb{R}^n$ and $\mathscr{S}(\mathbb{R}^n)$ stands for the space of rapidly decreasing functions on $\mathbb{R}^n$. Its topological dual is the space $\mathscr{S}'(\mathbb{R}^n)$ is the Schwartz space of tempered distributions on $\mathbb{R}^n$. We also recall, that for a function $\varphi \in \mathscr{S}(\mathbb{R}^n)$, its Fourier transform, denoted by $\widehat{\varphi}$, is defined by $\widehat{\varphi}(\xi) = \displaystyle\int_{\mathbb{R}^n} \varphi(x) e^{-2i\pi x^T \xi} dx$ and for a tempered distribution $T \in \mathscr{S}'(\mathbb{R}^n)$, its Fourier transform $\widehat{T}$ is defined by duality $\langle \widehat{T}, \varphi\rangle = \langle T, \widehat{\varphi}\rangle$ for all $\varphi \in \mathscr{S}(\mathbb{R}^n)$. If it is necessary, we will also use the standard notation $\mathcal{F}[T]$.

2.2. Minimal interpolating problem and its extension to the classical Sobolev space. Let $s \in \mathbb{R}$, $n, m \in \mathbb{N}$ with $n \geq 2$, and consider the space

$$X^{m,s}(\mathbb{R}^n;\mathbb{R}^n) = \Big\{ u = (u_1,\ldots,u_n)^T \in \mathscr{D}'(\mathbb{R}^n;\mathbb{R}^n) \,:\, \forall \alpha \in \mathbb{Z}_+^n, |\alpha| = m,$$

$$\partial^\alpha u_i \in \widetilde{H}^s(\mathbb{R}^n), i = 1,\ldots,n \Big\}, \tag{2.7}$$

where the (scalar) space $\widetilde{H}^s(\mathbb{R}^n)$ (see [27]) is defined as

$$\widetilde{H}^s(\mathbb{R}^n) = \Big\{ v \in \mathscr{S}'(\mathbb{R}^n) \,:\, \widehat{v} \in L^1_{loc}(\mathbb{R}^n),\ \int_{\mathbb{R}^n} |\xi|^{2s} |\widehat{v}(\xi)|^2 d\xi < +\infty \Big\}.$$

We assume that the integer $m \geq 1$ and $s \in \mathbb{R}$ are such that

$$-m + \tfrac{n}{2} < s < \tfrac{n}{2}. \tag{2.8}$$

The space $X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$ is equipped with the following semi-scalar product and its associated semi-norm

$$(u|v)_{m,s} = \sum_{|\alpha|=m} \tfrac{m!}{\alpha!} \int_{\mathbb{R}^n} |\xi|^{2s} \big(\widehat{\partial^\alpha u}(\xi)\big)^T \big(\overline{\widehat{\partial^\alpha v}(\xi)}\big) d\xi, \quad |u|_{m,s} = \sqrt{(u|u)_{m,s}}. \tag{2.9}$$

The null space associated to the semi-scalar product is the space, denoted by $\Pi_{m-1}(\mathbb{R}^n;\mathbb{R}^n)$ of vector-valued polynomials of n-variables with degree $\leq m-1$. We have the following result.

Proposition 2.2. *i) The space $X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$ endowed with the semi-scalar product given by (2.9) is a semi-Hilbert space.*

ii) For any bounded open subset Ω of $\mathbb{R}^n$, the space $X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$ endowed with the following scalar product and its associated norm defined by

$$((u|v))^\Omega_{m,s} = \sum_{i=1}^n \int_\Omega u_i(x) v_i(x) dx + (u|u)_{m,s}, \quad \|u\|^\Omega_{m,s} = \sqrt{((u|u))^\Omega_{m,s}}, \tag{2.10}$$

for $u = (u_1,\ldots,u_n)^T$ and $v = (v_1,\ldots,v_n)^T \in X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$, is a Hilbert space and its topology is independent of Ω.

iii) The following continuous embedding

$$X^{m,s}(\mathbb{R}^n;\mathbb{R}^n) \hookrightarrow H^{m+s}_{loc}(\mathbb{R}^n;\mathbb{R}^n),$$

holds, which implies that $X^{m,s}(\mathbb{R}^n;\mathbb{R}^n) \hookrightarrow \mathscr{C}^k(\mathbb{R}^n;\mathbb{R}^n)$ for all integer k such that $k + n/2 < m + s$.

Proof. The proposition is an immediate consequence of a scalar version see [4, 5, 17, 18]. □

From now on, the scalar product and its associated norm given in (2.10), will be denoted by $((.|.))_{m,s}$ and $\|\cdot\|_{m,s}$, respectively, without making any particular reference to any particular open set Ω.

Let Ω be an open bounded connected nonempty subset of $\mathbb{R}^n$ having a Lipschitz-continuous boundary. Let R_Ω denote the operator of restriction from $\mathbb{R}^n$ to Ω.

We note here that as an immediate consequence of a scalar version given in [4, 5, 17, 18], we have that: The operator R_Ω is linear and continuous from $X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$ onto $H^{m+s}(\Omega;\mathbb{R}^n)$. Thus, there exists an extension operator, denoted by E_Ω which is linear and continuous from $H^{m+s}(\Omega;\mathbb{R}^n)$ into $X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$ such that $R_\Omega E_\Omega u = u$ for all u in $H^{m+s}(\Omega;\mathbb{R}^n)$.

In the remainder of this paper, an extension $E_\Omega u$ of u will be denoted by u, namely we use for simplicity $E_\Omega u = u$. We recall that, the div and curl operators are defined by

$$\operatorname{div} u = \nabla^T \cdot u = \sum_{i=1}^{n} \partial_i u_i, \ \ \operatorname{curl} u = \nabla \cdot u^T - (\nabla \cdot u^T)^T = (\partial_i u_j - \partial_j u_i)_{1\leq i,j\leq n},$$

where $\nabla = (\partial_1, \dots, \partial_n)^T$ stands for the gradient operator, $u = (u_1, \dots, u_n)^T$ is a vector-valued distribution and $\partial_i u_j = \frac{\partial u_j}{\partial x_i}$ is the i-th partial derivative of u_j. The general definition of the curl is classical in multidimensional harmonic analysis, see [19, 31]. Let $\rho > 0$ denote a positive real parameter. We consider the bilinear forms $D_{m,s}$, $R_{m,s}$ and $M^\rho_{m,s}$ given by

$$\begin{aligned}
D_{m,s}(u,v) &= \sum_{|\alpha|=m-1} \tfrac{(m-1)!}{\alpha!} \int_{\mathbb{R}^n} |\xi|^{2s} \partial^\alpha \widehat{(\operatorname{div} u)}(\xi) \overline{\partial^\alpha \widehat{(\operatorname{div} v)}(\xi)} d\xi, \\
R_{m,s}(u,v) &= \tfrac{1}{2} \sum_{|\alpha|=m-1} \tfrac{(m-1)!}{\alpha!} \int_{\mathbb{R}^n} |\xi|^{2s} \left\langle \partial^\alpha \widehat{(\operatorname{curl} u)}(\xi) \middle| \overline{\partial^\alpha \widehat{(\operatorname{curl} v)}(\xi)} \right\rangle_{n\times n} d\xi, \\
M^\rho_{m,s}(u,v) &= \rho D_{m,s}(u,v) + R_{m,s}(u,v).
\end{aligned}$$

Here the notation $\langle \,.|.\, \rangle_{n\times n}$ stands for the Frobenius scalar product $\langle z|z' \rangle_{n\times n} = \operatorname{trace}(z^T z')$ in the space $\mathbb{R}^{n\times n}$ of $n\times n$ matrices. Its associated norm is denoted by $\|\cdot\|_{n\times n}$. The square root of the quadratic forms associated to $D_{m,s}$, $R_{m,s}$ and $M^\rho_{m,s}$ are called the div-energy, the curl-energy and the div-curl energy, respectively. For short notation, we will write $D_{m,s}(u)$, $R_{m,s}(u)$ and $M^\rho_{m,s}(u)$ for $D_{m,s}(u,u)$, $R_{m,s}(u,u)$ and $M^\rho_{m,s}(u,u)$, respectively.

Note that as a direct consequence of Proposition 7 in [8], we have that, for all u, $v \in X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$, we have

$$M^1_{m,s}(u,v) = (u|v)_{m,s}, \tag{2.11}$$

and for all positive real numbers ρ, we have

$$\inf(\rho,1)|u|^2_{m,s} \leq M^\rho_{m,s}(u) \leq \sup(\rho,1)|u|^2_{m,s}. \tag{2.12}$$

Let $d(n) = \dim \Pi_{m-1}(\mathbb{R}^n)$. We recall that a set $\mathscr{B} = \left\{b_1, \dots, b_{d(n)}\right\} \subset \mathbb{R}^n$ is called a Π_{m-1}-unisolvent (see [15]) if and only if

$$\forall \{z_1, \dots, z_{d(n)}\} \subset \mathbb{R}, \ \exists! \vartheta \in \Pi_{m-1}(\mathbb{R}^n), \ \forall i = 1, \dots, d(n), \ \vartheta(b_i) = z_i.$$

Let $\mathscr{A}$ be a finite set of scattered data points in $\overline{\Omega} := \operatorname{closure}(\Omega)$. We assume that $\mathscr{A}$ contains a $\Pi_{m-1}(\mathbb{R}^n)$-unisolvent subset, which implies that any polynomial in $\Pi_{m-1}(\mathbb{R}^n)$ vanishing in $\mathscr{A}$ is identically zero.

For any vector-valued function $u : \bar{\Omega} \to \mathbb{R}^n$, we consider The subset of $X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$ defined as $\mathcal{I}_{\mathscr{A}}(u) = \{w \in X^{m,s}(\mathbb{R}^n;\mathbb{R}^n) \mid w(a) = u(a),\ \forall a \in \mathscr{A}\}$. The affine space $\mathcal{I}_{\mathscr{A}}(u)$ is the interpolating set of all the functions $w \in X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$ taking the same values on $\mathscr{A}$ as u. We also consider the following minimal interpolating problem:

PROBLEM 1. *Find $u^* \in \mathcal{I}_{\mathscr{A}}(u)$ minimizing the functional energy $M^{\rho}_{m,s}(w)$, namely*

$$u^* = \arg\Big[\min_{w\in\mathcal{I}_{\mathscr{A}}(u)} M^{\rho}_{m,s}(w)\Big]. \tag{2.13}$$

REMARK 2.3. We observe that as for the pointwise values of $w \in X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$, *i.e.*, $w(a)$, must to be well-defined, then w has to be a regular function which require the regularity condition (2.8). □

We have the following proposition.

PROPOSITION 2.4. *For all u: $\bar{\Omega} \to R^n$ one has that:*

i) The minimal interpolating problem Problem 1, introduced above, has a unique solution in $X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$. This unique solution will be denoted by $S^{\mathscr{A}}_{\rho}u$.

ii) The solution $S^{\mathscr{A}}_{\rho}u$ of Problem 1, is the unique element in $\mathcal{I}_{\mathscr{A}}(u)$ satisfying the following characterization

$$M^{\rho}_{m,s}(S^{\mathscr{A}}_{\rho}u, v) = 0, \tag{2.14}$$

for all $v \in X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$ such that $v(a) = 0$ for all $a \in \mathscr{A}$.

iii) The solution $S^{\mathscr{A}}_{\rho}u$ belongs to the space $\mathscr{C}^{\eta}(\mathbb{R}^n;\mathbb{R}^n)$ where η is the integer given by

$$\eta = \begin{cases} 2m+2s-n-1, & \text{for } 2m+2s-n \in \mathbb{N}^*, \\ \lfloor 2m+2s-n \rfloor, & \text{otherwise.} \end{cases} \tag{2.15}$$

iv) For all $q \in \Pi_{m-1}(\mathbb{R}^n;\mathbb{R}^n)$, we have the reproducing property

$$S^{\mathscr{A}}_{\rho}q = q. \tag{2.16}$$

Proof. See [8]. □

The unique solution $S^{\mathscr{A}}_{\rho}u$ of Problem 1, may be obtained explicitly and may be computed numerically from the values $\{u(a)\}_{a\in\mathscr{A}}$ by solving a linear system, see [8] for more details. However, it is not necessary here to have in mind all these details.

Now, we give a result on the minimal pseudo-polyharmonic div-curl extension problem in the classical vectorial Sobolev space $H^{m+s}(\Omega;\mathbb{R}^n)$ on the domain Ω.

We recall that R_Ω denotes the operator of restriction from $\mathbb{R}^n$ to Ω and we consider the subset $\mathcal{I}_\Omega(f) = \{w \in X^{m,s}(\mathbb{R}^n;\mathbb{R}^n) \mid R_\Omega w = f\}$ of $X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$. We define the following minimization div-curl extension problem:

Problem 2. *For any $f \in H^{m+s}(\Omega;\mathbb{R}^n)$, find $v^* \in \mathcal{I}_\Omega(f)$ minimizing the functional energy $M^\rho_{m,s}(w)$, namely*

$$v^* = \arg\Big[\min_{w\in\mathcal{I}_\Omega(f)} M^\rho_{m,s}(w)\Big]. \tag{2.17}$$

We have the following results.

Proposition 2.5. *For all $f \in H^{m+s}(\Omega;\mathbb{R}^n)$*

i) The minimal div-curl extension Problem 2 admits a unique solution in $X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$. This unique solution will be denoted by $S^\Omega_\rho f$.

ii) The solution $S^\Omega_\rho f$ of Problem 2 is the unique element in $\mathcal{I}_\Omega(f)$ satisfying the following characterization

$$M^\rho_{m,s}(S^\Omega_\rho f, v) = 0, \tag{2.18}$$

for all $v \in X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$ such that $R_\Omega v = 0$.

iii) For all $q \in \Pi_{m-1}(\Omega;\mathbb{R}^n)$, we have the reproducing property

$$S^\Omega_\rho q = q, \quad \text{in } \Omega. \tag{2.19}$$

Proof. According to the inequality (2.12), the symmetric positive bilinear form $M^\rho_{m,s}$ is continuous. Then, there exists a positive and symmetric continuous linear operator

$$S : X^{m,s}(\mathbb{R}^n;\mathbb{R}^n) \to X^{m,s}(\mathbb{R}^n;\mathbb{R}^n),$$

such that

$$M^\rho_{m,s}(u,v) = (Su \mid v)_{m,s},$$

for all $u, v \in X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$. The operator S admits a symmetric positive square-root. Namely, there exists a symmetric and positive continuous linear operator $T : X^{m,s}(\mathbb{R}^n;\mathbb{R}^n) \to X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$ such that $S = T^2$. In consequence, we have

$$M^\rho_{m,s}(u) = M^\rho_{m,s}(u,u) = (T^2 u \mid u)_{m,s} = (Tu \mid Tu)_{m,s} = |Tu|^2_{m,s},$$

for all $u \in X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$.

The operators R_Ω and T satisfy the following properties:

(i) R_Ω is continuous and surjective.

(ii) $\ker(T) = \Pi_{m-1}(\mathbb{R}^n;\mathbb{R}^n)$ and $T(X^{m,s}(\mathbb{R}^n;\mathbb{R}^n))$ is closed: This is a consequence of the fact that $\Pi_{m-1}(\mathbb{R}^n;\mathbb{R}^n)$ is the null space of the semi-scalar product given in (2.9) together with the inequality (2.12).

(iii) $\ker(T)+\ker(R_\Omega)$ is closed: It is a consequence of the fact that $\ker(R_\Omega)$ is closed and $\ker(T) = \Pi_{m-1}(\mathbb{R}^n;\mathbb{R}^n)$ is a finite dimensional space.

(iv) $\ker(T) \cap \ker(R_\Omega) = \{0\}$.

Thus all the results given in the proposition are a consequence of the general spline theory (see [6, 11, 21]). □

Remark 2.6. The items ii) and iii) in the previous proposition are not surprising as the results are consequences of a general spline theory, where the results are proved for three abstract Hilbert spaces with general two continuous operators satisfying items (a)-(b)-(c)-(d) proved in our cases for the operators R_Ω and T, see the abstract frame work given in [6, 11, 21]. $\square$

In the following proposition we prove the continuity of the operator S_ρ^Ω.

Proposition 2.7. *The linear operator $S_\rho^\Omega : H^{m+s}(\Omega;\mathbb{R}^n) \longrightarrow X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$ is continuous. Thus, there exists a positive constant C (depending on Ω, m and s) such that the inequality*

$$\|S_\rho^\Omega f\|_{m,s} \;\le\; C\,\|f\|_{m+s,2,\Omega}, \tag{2.20}$$

holds for all $f \in H^{m+s}(\Omega;\mathbb{R}^n)$. Moreover, if $s \le 0$, then there exists a constant $C_1 > 0$ such that

$$|S_\rho^\Omega f|_{m,s} \;\le\; C_1\,|f|_{m+s,2,\Omega}, \tag{2.21}$$

holds for all $f \in H^{m+s}(\Omega;\mathbb{R}^n)$.

Proof. Let $(g_\ell)_{\ell\in\mathbb{N}}$ be any sequence in $H^{m+s}(\Omega;\mathbb{R}^n)$ such that

$$\exists\, g \in H^{m+s}(\Omega;\mathbb{R}^n),\;\; g_\ell \longrightarrow g \text{ in } H^{m+s}(\Omega;\mathbb{R}^n), \tag{2.22}$$

$$\exists\, u \in X^{m,s}(\mathbb{R}^n;\mathbb{R}^n),\;\; S_\rho^\Omega g_\ell \longrightarrow u \text{ in } X^{m,s}(\mathbb{R}^n;\mathbb{R}^n). \tag{2.23}$$

Using (2.18), we have $M^\rho_{m,s}(S_\rho^\Omega g_\ell, v) = 0$ for all $\ell \in \mathbb{N}$ and all $v \in \mathcal{I}_\Omega(0)$. This implies together with (2.23), that $M^\rho_{m,s}(u, v) = 0$. From the continuity of the operator R_Ω and the convergence (2.23), we obtain the convergence $g_\ell = R_\Omega S_\rho^\Omega g_\ell \longrightarrow R_\Omega u$ in $H^{m+s}(\Omega;\mathbb{R}^n)$. Thus, from the convergence (2.22), we deduce that $R_\Omega u = g$, which means that u belongs to $\mathcal{I}_\Omega(g)$. Now, by the characterization (2.18), we conclude that $u = S_\rho^\Omega g$. Consequently the graph of the operator S_ρ^Ω is closed in $H^{m+s}(\Omega;\mathbb{R}^n) \times X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$ and the closed graph theorem implies that S_ρ^Ω is continuous. Then, Property (2.20) holds.

Let us suppose that $s \le 0$. Since $S_\rho^\Omega q = q$ (see (2.19)) and $|q|_{m,s} = 0$, for all q in $\Pi_{m-1}(\mathbb{R}^n;\mathbb{R}^n)$, we have

$$|S_\rho^\Omega f|_{m,s} = |S_\rho^\Omega f - q|_{m,s} = |S_\rho^\Omega (f-q)|_{m,s} \le \|S_\rho^\Omega (f-q)\|_{m,s}.$$

Using the continuity of the operator S_ρ^Ω, we get

$$|S_\rho^\Omega f|_{m,s} \le C\,\|f-q\|_{m+s,2,\Omega},$$

for all q in $\Pi_{m-1}(\mathbb{R}^n;\mathbb{R}^n)$ and for all $f \in H^{m+s}(\Omega;\mathbb{R}^n)$, where the constant C is given in (2.20).

Therefore,

$$|S_\rho^\Omega f|_{m,s} \le C \min_{q\in\Pi_k(\Omega;\mathbb{R}^n)} \|f-q\|_{m+s,2,\Omega},$$

for all $f \in H^{m+s}(\Omega;\mathbb{R}^n)$, where $k = \lceil m+s \rceil - 1 \le m-1$.

According to item 3 in Proposition 2.1, there exists a positive constant C_1 such that

$$|S_\rho^\Omega f|_{m,s} \leq C_1 \, |f|_{m+s,2,\Omega},$$

for all $f \in H^{m+s}(\Omega; \mathbb{R}^n)$. $\square$

Proposition 2.8. *We have*

$$M_{m,s}^\rho(S_\rho^\Omega f - S_\rho^{\mathscr{A}} S_\rho^\Omega f) = M_{m,s}^\rho(S_\rho^\Omega f) - M_{m,s}^\rho(S_\rho^{\mathscr{A}} S_\rho^\Omega f) \leq M_{m,s}^\rho(S_\rho^\Omega f), \tag{2.24}$$

where $S_\rho^{\mathscr{A}} S_\rho^\Omega f$ is the solution of the minimal interpolating problem (2.13) relative to the element $u = S_\rho^\Omega f$.

Proof. The solution $S_\rho^{\mathscr{A}} S_\rho^\Omega f$ of Problem (2.13) satisfies $M_{m,s}^\rho(S_\rho^{\mathscr{A}} S_\rho^\Omega f, u) = 0$ for all u in $X^{m,s}(\mathbb{R}^n; \mathbb{R}^n)$ vanishing in $\mathscr{A}$ (see (2.14)). Since the function $u = S_\rho^\Omega f - S_\rho^{\mathscr{A}}[S_\rho^\Omega f]$ vanishes on $\mathscr{A}$, we get

$$\begin{aligned} M_{m,s}^\rho(S_\rho^\Omega f) =& M_{m,s}^\rho(S_\rho^{\mathscr{A}} S_\rho^\Omega f + u, S_\rho^{\mathscr{A}} S_\rho^\Omega f + u) \\ =& M_{m,s}^\rho(S_\rho^{\mathscr{A}} S_\rho^\Omega f, S_\rho^{\mathscr{A}} S_\rho^\Omega f) + M_{m,s}^\rho(u, u) \\ =& M_{m,s}^\rho(S_\rho^{\mathscr{A}} S_\rho^\Omega f) + M_{m,s}^\rho(S_\rho^\Omega f - S_\rho^{\mathscr{A}} S_\rho^\Omega f). \end{aligned}$$ $\square$

3. CONVERGENCE AND ERROR ESTIMATES

In this section, we present results about convergence and error estimates in the Sobolev space $W^{m+s,p}(\Omega; \mathbb{R}^n)$. Henceforth, we assume that the following hypothesis

- (H_1): Ω is an open bounded connected subset of $\mathbb{R}^n$ having a Lipschitz-continuous boundary.
- (H_2): $-m + \frac{n}{2} < s < \frac{n}{2}$.

are satisfied. The fill-distance of a subset $\mathscr{A} \subset \overline{\Omega}$ is defined by

$$h := h(\mathscr{A}, \overline{\Omega}) = \sup_{x \in \overline{\Omega}} \inf_{a \in \mathscr{A}} |x - a|.$$

From (H_1), the domain Ω satisfies the cone property.

3.1. Convergence in $H^{m+s}(\Omega; \mathbb{R}^n)$.

Now we establish a norm equivalence that will be crucial for the convergence results and error estimates to follow.

Proposition 3.1. *Let $\mathscr{A}$ be a finite subset of $\overline{\Omega}$ containing a Π_{m-1}-unisolvent subset. Then, there exists $h_0 > 0$ (independent of $\mathscr{A}$) such that for any $h := h(\mathscr{A}, \Omega) < h_0$, there exists a $\Pi_{m-1}-$unisolvent subset $\mathscr{A}_0^h \subset \mathscr{A}$ such that the following inequality*

$$C_1 \|u\|_{m,s} \leq \left(\sum_{a \in \mathscr{A}_0^h} |u(a)|^2 + M_{m,s}^\rho(u) \right)^{1/2} \leq C_2 \|u\|_{m,s}, \tag{3.1}$$

holds for all $u \in X^{m,s}(\mathbb{R}^n; \mathbb{R}^n)$, where C_1 and C_2 are positive constants independent of h, $\mathscr{A}$ and u.

Proof. From [4, Proposition I-2.3 and II-5.1] or [5, Lemma 6.1], we get the vector version, *i.e.*, there exists $h_0 > 0$ (independent of $\mathscr{A}$) such that for any $h := h(\mathscr{A}, \Omega) < h_0$, there exists a Π_{m-1}-unisolvent subset $\mathscr{A}_0^h \subset \mathscr{A}$ such that the following inequality

$$D_1\|u\|_{m,s} \leq \left(\sum_{a \in \mathscr{A}_0^h} |u(a)|^2 + |u|^2_{m,s} \right)^{1/2} \leq D_2\|u\|_{m,s}, \tag{3.2}$$

holds for all $u \in X^{m,s}(\mathbb{R}^n; \mathbb{R}^n)$, where D_1 and D_2 are positive constants independent of h, $\mathscr{A}$ and u. Using the inequality (2.12), we get

$$\begin{aligned} \inf(1,\rho)\left(\sum_{a \in \mathscr{A}_0^h} |u(a)|^2 + |u|^2_{m,s} \right) &\leq \sum_{a \in \mathscr{A}_0^h} |u(a)|^2 + M^\rho_{m,s}(u) \\ &\leq \sup(1,\rho)\left(\sum_{a \in \mathscr{A}_0^h} |u(a)|^2 + |u|^2_{m,s} \right). \end{aligned} \tag{3.3}$$

Then the result (3.1) is a consequence of the inequalities (3.2) and (3.3) with $C_1 = D_1\sqrt{\inf(1,\rho)}$ and $C_2 = D_2\sqrt{\sup(1,\rho)}$. □

Now, we state the main results about convergence.

THEOREM 3.2. *Let $\mathscr{A}$ be a finite subset of $\overline{\Omega}$ containing a Π_{m-1}-unisolvent subset. For all $f \in H^{m+s}(\Omega; \mathbb{R}^n)$ we have the following strong convergence results*

i) $S^\Omega_\rho f = \lim_{h\to 0} S^{\mathscr{A}}_\rho S^\Omega_\rho f$ *in* $X^{m,s}(\mathbb{R}^n; \mathbb{R}^n)$.

ii) $f = \lim_{h\to 0} S^{\mathscr{A}}_\rho S^\Omega_\rho f$ *in* $H^{m+s}(\Omega; \mathbb{R}^n)$ *and consequently in* $\mathcal{C}^{m(s)-1}(\Omega; \mathbb{R}^n)$, *where* $m(s)$ *is the integer given by*

$$m(s) = \begin{cases} m+s-1, & \text{for } (m+s) \in \mathbb{N}^*, \\ \lfloor m+s \rfloor, & \text{otherwise.} \end{cases} \tag{3.4}$$

Proof. For all $f \in H^{m+s}(\Omega; \mathbb{R}^n)$, let $(f_h)_{h>0}$ be the sequence given by i) Using the inequalities (3.1) and the minimal norm property of the spline function f_h successively, we get

$$\begin{aligned} \|f_h\|_{m,s} &\leq \tfrac{1}{C_1}\left(\sum_{a \in \mathscr{A}_0^h} |f_h(a)|^2 + M^\rho_{m,s}(f_h) \right)^{1/2} \\ &\leq \tfrac{1}{C_1}\left(\sum_{a \in \mathscr{A}_0^h} |S^\Omega_\rho f(a)|^2 + M^\rho_{m,s}(S^\Omega_\rho f) \right)^{1/2} \\ &\leq \tfrac{C_2}{C_1}\|S^\Omega_\rho f\|_{m,s}. \end{aligned} \tag{3.5}$$

This implies that the sequence $(f_h)_{h>0}$ is bounded, as h tends to zero, in the Hilbert space $X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$ equipped with the norm $\|\cdot\|_{m,s}$ defined in (2.10).

Thus, there exists a subsequence $(f_{h(\ell)})_{\ell\in\mathbb{N}}$ which weakly converges to an element f^* in $X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$.

Let x be any point in Ω. By using item 3 of Proposition 2.2, the mapping $\delta_x : v \mapsto v(x)$ is strongly continuous from $X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$ into $\mathbb{C}$ and consequently it is also weakly continuous. Then

$$\lim_{\ell\to+\infty} f_{h(\ell)}(x) = f^*(x) = R_\Omega f^*(x). \tag{3.6}$$

The hypothesis $h = \sup_{x\in\overline{\Omega}} \inf_{a\in\mathscr{A}} |x-a| \to 0$, implies that there exists $x_h \in \mathscr{A}$ such that $|x - x_h| < h$. We have

$$\forall a \in \mathscr{A},\ f_h(a) = S_\rho^\Omega f(a) = f(a),$$

and by taking into account the fact that $R_\Omega S_\rho^\Omega f = f$, we have

$$f(x) - f_{h(\ell)}(x) = \Big(S_\rho^\Omega f(x) - S_\rho^\Omega f(x_{h(\ell)})\Big) + \Big(f_{h(\ell)}(x_{h(\ell)}) - f_{h(\ell)}(x)\Big).$$

The continuity of $S_\rho^\Omega f$ implies that $\lim\limits_{\ell\to+\infty} S_\rho^\Omega f(x_{h(\ell)}) = S_\rho^\Omega f(x)$. Using Sobolev embedding theorem for the space $H^{m+s}(\Omega;\mathbb{R}^n)$ (item 1 of Proposition 2.1), we obtain that $\lim\limits_{\ell\to+\infty}\Big(f_{h(\ell)}(x_{h(\ell)}) - f_{h(\ell)}(x)\Big) = 0$. Thus

$$\lim_{\ell\to+\infty} f_{h(\ell)}(x) = f(x). \tag{3.7}$$

Relations (3.6) and (3.7) provide the relation $R_\Omega f^* = f$.

The weak convergence of $\Big(f_{h(\ell)}\Big)_{\ell\in\mathbb{N}}$ to f^* and the strong continuity of the quadratic form $M_{m,s}^\rho$ imply that

$$M_{m,s}^\rho(f^*) \le \liminf_{\ell\to+\infty} M_{m,s}^\rho(f_{h(\ell)}) \le M_{m,s}^\rho(S_\rho^\Omega f). \tag{3.8}$$

According to the fact that $R_\Omega f^* = f$, Inequality (3.8) and the uniqueness of the solution $S_\rho^\Omega f$ of Problem (2.17), imply that $f^* = S_\rho^\Omega f$ in $X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$. Thus, from Inequality (3.8), we obtain the convergence

$$M_{m,s}^\rho(S_\rho^\Omega f) = \lim_{\ell\to+\infty} M_{m,s}^\rho(f_{h(\ell)}), \tag{3.9}$$

Using inequality (2.24), we obtain that

$$\lim_{\ell\to+\infty} M_{m,s}^\rho\left(S_\rho^\Omega f - f_{h(\ell)}\right) = 0. \tag{3.10}$$

By using (3.1) for $u = S_\rho^\Omega f - f_{h(\ell)}$, we get

$$\|S_\rho^\Omega f - f_{h(\ell)}\|_{m,s} \le \tfrac{1}{C_1}\Big(M_{m,s}^\rho\left(S_\rho^\Omega f - f_{h(\ell)}\right)\Big)^{1/2}. \tag{3.11}$$

Relation (3.10), together with (3.11) imply that the subsequence $\left(f_{h(\ell)}\right)_{\ell\in\mathbb{N}}$ is strongly convergent to $S^{\Omega}_{\rho}f$ in $X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$. In the same manner, we can show that every convergent subsequence of the sequence $(f_h)_{h>0}$ is necessarily convergent to $S^{\Omega}_{\rho}f$. In consequence, the sequence $(f_h)_{h>0}$ is strongly convergent to $S^{\Omega}_{\rho}f$ in $X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$.

ii) It is a consequence of the fact that the restriction operator R_Ω is continuous from $X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$ into $H^{m+s}(\Omega;\mathbb{R}^n)$ and $R_\Omega S^{\Omega}f = f$. The convergence in $\mathcal{C}^{m(s)-1}(\Omega;\mathbb{R}^n)$ is a consequence of item 3 of Proposition 2.2. □

3.2. Error estimates. The following theorem gives a result about the global error estimates in vector version. It is an immediate consequence of the scalar version given by Corollary 4.1 or by Theorem 4.1 in [5]. For more details see Theorem 4.1 in [5], which gives an extension of a bound for functions in general Sobolev space. This bound has known several precursors in the literature, see for instance [11, 17, 22, 23, 24, 25].

Theorem 3.3. *Let $p \in [1,\infty)$, let $r_0 = m+s-n(1/2-1/p)_+$. Then, there exist two positive constants h_0 (depending on Ω, n, and $m+s$) and C (depending on Ω, n, $m+s$ and p) satisfying the following property: for any finite set $\mathscr{A} \subset \overline{\Omega}$ such that $h = \sup_{x\in\overline{\Omega}} \inf_{a\in\mathscr{A}} |x-a| < h_0$, for any integer $k = 0,\dots,\lceil r_0\rceil - 1$, the following inequality*

$$|u|_{k,p,\Omega} \le C h^{m+s-k-n(1/2-1/p)_+} |u|_{m+s,2,\Omega}, \tag{3.12}$$

holds for any $u \in H^{m+s}(\Omega;\mathbb{R}^n)$ vanishing on $\mathscr{A}$. If $m+s \in \mathbb{N}^*$, this bound also holds with $k = r_0$ when either $2 < p < \infty$ and $r_0 \in \mathbb{N}$, or $p \le 2$. Here $(r)_+ = \max\{r,0\}$.

Now, let us prove the following main theorem on error estimates.

Theorem 3.4. *Let $p \in [1,\infty)$, let $r_0 = m+s-n(1/2-1/p)_+$. Then, there exist a positive constant h_0 (depending on Ω, n, and $m+s$) and two positive constants C_1 and C_2 (depending on Ω, n, $m+s$ and p) satisfying the following property: for any finite set $\mathscr{A} \subset \overline{\Omega}$ containing a Π_{m-1}-unisolvent subset, such that $h = \sup_{x\in\overline{\Omega}} \inf_{a\in\mathscr{A}} |x-a| < h_0$, for any integer $k = 0,\dots,\lceil r_0\rceil - 1$, the following inequalities:*

$$\begin{aligned} |f - S^{\mathscr{A}}_{\rho} S^{\Omega}_{\rho} f|_{k,p,\Omega} &\le C_1 h^{m+s-k-n(1/2-1/p)_+}\ |f - S^{\mathscr{A}}_{\rho} S^{\Omega}_{\rho} f|_{m+s,2,\Omega} \\ &\le C_2 \sqrt{\tfrac{\sup(1,\rho)}{\inf(1,\rho)}}\ h^{m+s-k-n(1/2-1/p)_+} \|f\|_{m+s,2,\Omega} \end{aligned} \tag{3.13}$$

hold for every function f belonging to $H^{m+s}(\Omega;\mathbb{R}^n)$. If $m+s \in \mathbb{N}^*$, this bound also holds with $k = r_0$ when either $2 < p < \infty$ and $r_0 \in \mathbb{N}$, or $p \le 2$. Moreover, if $s \le 0$, $\|f\|_{m+s,2,\Omega}$ in (3.14) may be replaced by $|f|_{m+s,2,\Omega}$.

Proof. To prove the theorem we apply Theorem 3.3. Let $\mathscr{A}$ be any Π_{m-1}-unisolvent subset of $\overline{\Omega}$ such that $h = \sup_{x\in\overline{\Omega}} \inf_{a\in\mathscr{A}} |x-a| < h_0$. Let f be any element belonging to $H^{m+s}(\Omega;\mathbb{R}^n)$. The function $u = S_\rho^\Omega f - S_\rho^{\mathscr{A}} S_\rho^\Omega f$ vanishes on $\mathscr{A}$, thus according to Inequality (3.12), we get

$$|S_\rho^\Omega f - S_\rho^{\mathscr{A}} S_\rho^\Omega f|_{k,p,\Omega} \leq C_1 h^{m+s-k-n(1/2-1/p)_+} |S_\rho^\Omega f - S_\rho^{\mathscr{A}} S_\rho^\Omega f|_{m+s,2,\Omega},$$

where $C_1 > 0$ is a positive constant (depending on Ω, n, $m+s$ and p). Since, $R_\Omega[S_\rho^\Omega f] = f$, we have

$$|S_\rho^\Omega f - S_\rho^{\mathscr{A}} S_\rho^\Omega f|_{k,p,\Omega} = |f - S_\rho^{\mathscr{A}} S_\rho^\Omega f|_{k,p,\Omega},$$

and

$$|S_\rho^\Omega f - S_\rho^{\mathscr{A}} S_\rho^\Omega f|_{m+s,2,\Omega} = |f - S_\rho^{\mathscr{A}} S_\rho^\Omega f|_{m+s,2,\Omega},$$

Thus, we obtain the first inequality

$$|f - S_\rho^{\mathscr{A}} S_\rho^\Omega f|_{k,p,\Omega} \leq C_1 h^{m+s-k-n(1/2-1/p)_+} |f - S_\rho^{\mathscr{A}} S_\rho^\Omega f|_{m+s,2,\Omega}.$$

But, we also have

$$|S_\rho^\Omega f - S_\rho^{\mathscr{A}} S_\rho^\Omega f|_{m+s,2,\Omega} = |R_\Omega(S_\rho^\Omega f - S_\rho^{\mathscr{A}} S_\rho^\Omega f)|_{m+s,2,\Omega}.$$

Then,

$$|f - S_\rho^{\mathscr{A}} S_\rho^\Omega f|_{k,p,\Omega} \leq C_1 h^{m+s-k-n(1/2-1/p)_+} |R_\Omega(S_\rho^\Omega f - S^{\mathscr{A}} S_\rho^\Omega f)|_{m+s,2,\Omega}.$$

The continuity of the operator $R_\Omega : H^{m+s}(\Omega;\mathbb{R}^n) \longrightarrow X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$ leads to the existence of a constant $C > 0$ (depending on Ω, n, $m+s$ and p) such that

$$|f - S_\rho^{\mathscr{A}} S_\rho^\Omega f|_{k,p,\Omega} \leq C h^{m+s-k-n(1/2-1/p)_+} \|S_\rho^\Omega f - S_\rho^{\mathscr{A}} S_\rho^\Omega f\|_{m,s}.$$

According to (3.2), it follows that there exists a constant C' (depending on Ω, n, $m+s$ and p) such that

$$|f - S_\rho^{\mathscr{A}} S_\rho^\Omega f|_{k,p,\Omega} \leq$$

$$\leq C' h^{m+s-k-n(1/2-1/p)_+} \left(\sum_{a \in \mathscr{A}_0^h} |(S_\rho^\Omega f - S_\rho^{\mathscr{A}} S_\rho^\Omega f)(a)|^2 + |S_\rho^\Omega f - S_\rho^{\mathscr{A}} S_\rho^\Omega f|_{m,s}^2 \right)^{\frac{1}{2}}$$

But $(S_\rho^\Omega f - S_\rho^{\mathscr{A}} S_\rho^\Omega f)(a) = 0$, for all $a \in \mathscr{A}$. Then

$$|f - S_\rho^{\mathscr{A}} S_\rho^\Omega f|_{k,p,\Omega} \leq C' h^{m+s-k-n(1/2-1/p)_+} \ |S_\rho^\Omega f - S_\rho^{\mathscr{A}} S_\rho^\Omega f|_{m,s}.$$

From (2.12), we get

$$|f - S_\rho^{\mathscr{A}} S_\rho^\Omega f|_{k,p,\Omega} \leq C' \frac{1}{\sqrt{\inf(1,\rho)}} \ h^{m+s-k-n(1/2-1/p)_+} \ \sqrt{M_{m,s}^\rho (S_\rho^\Omega f - S_\rho^{\mathscr{A}} S_\rho^\Omega f)}.$$

Inequality (2.24) states that $M_{m,s}^\rho(S_\rho^\Omega f - S_\rho^{\mathscr{A}} S_\rho^\Omega f) \leq M_{m,s}^\rho(S_\rho^\Omega f)$, then

$$|f - S_\rho^{\mathscr{A}} S_\rho^\Omega f|_{k,p,\Omega} \leq C' \frac{1}{\sqrt{\inf(1,\rho)}} \ h^{m+s-k-n(1/2-1/p)_+} \ \sqrt{M_{m,s}^\rho(S_\rho^\Omega f)} \ .$$

By using again (2.12), we obtain

$$|f - S_\rho^{\mathscr{A}} S_\rho^\Omega f|_{k,p,\Omega} \leq C' \sqrt{\frac{\sup(1,\rho)}{\inf(1,\rho)}} \ h^{m+s-k-n(1/2-1/p)_+} \ |S^\Omega f|_{m,s} \tag{3.14}$$

$$\leq C' \sqrt{\frac{\sup(1,\rho)}{\inf(1,\rho)}}\ h^{m+s-k-n(1/2-1/p)_+}\ \|S_\rho^\Omega f\|_{m,s}$$

The continuity of the operator $S_\rho^\Omega : W^{m+s,2}(\Omega;\mathbb{R}^n) \longrightarrow X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$ (see (2.20)), provides the existence of a constant C_2 (depending on Ω, n, $m+s$ and p) such that

$$|f - S_\rho^{\mathscr{A}} S_\rho^\Omega f|_{k,p,\Omega} \leq C_2 \sqrt{\frac{\sup(1,\rho)}{\inf(1,\rho)}}\ h^{m+s-k-n(1/2-1/p)_+}\ \|f\|_{m+s,2,\Omega}\ .$$

Now, let us suppose that $s \leq 0$. According to Proposition 2.7 together with inequality (3.14), there exists a constant $C_2' > 0$ such that

$$|f - S_\rho^{\mathscr{A}} S_\rho^\Omega f|_{k,p,\Omega} \leq C_2' \sqrt{\frac{\sup(1,\rho)}{\inf(1,\rho)}}\ h^{m+s-k-n(1/2-1/p)_+}\ |f|_{m+s,2,\Omega}.$$

This concludes the proof. □

We have the following corollary.

Corollary 3.5. *Let $\mathscr{A}$ be a finite subset of $\overline{\Omega}$ containing a Π_{m-1}-unisolvent subset. Let $p \in [1,\infty)$, let $r_0 = m+s-n(1/2-1/p)_+$. Let $f \in H^{m+s}(\Omega;\mathbb{R}^n)$. For any integer $k = 0,\dots,\lceil r_0\rceil - 1$, we have*

$$|f - S_\rho^{\mathscr{A}} S_\rho^\Omega f|_{m+s,2,\Omega} = o(1) \quad \text{as} \quad h \to 0, \tag{3.15}$$

and consequently

$$|f - S_\rho^{\mathscr{A}} S_\rho^\Omega f|_{k,q,\Omega} = o(h^{m+s-k-n(1/2-1/p)_+}) \quad \text{as} \quad h \to 0. \tag{3.16}$$

If $m+s \in \mathbb{N}^*$, this bound also holds with $k = r_0$ when either $2 < p < \infty$ and $r_0 \in \mathbb{N}$, or $p \leq 2$.

Proof. The results are obtained immediately from Theorem 3.2 together with Theorem 3.4. □

4. CONVERGENCE AND ERROR ESTIMATES FOR ELASTIC SPLINES

In this section, we study the convergence and the error estimates for the interpolating elastic splines (see [8]). In elasticity theory the strain tensor is given by

$$\mathcal{E}u = \tfrac{1}{2}(\nabla\cdot u^T + (\nabla\cdot u^T)^T) = \tfrac{1}{2}(\partial_i u_j + \partial_j u_i)_{1\leq i,j\leq n}.$$

We consider the bilinear forms $S_{m,s}$ and $E_{m,s}^{\mu,\lambda}$ defined on $X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$ as follows

$$\begin{aligned} S_{m,s}(u,v) = &\sum_{|\alpha|=m-1} \tfrac{(m-1)!}{\alpha!} \int_{\mathbb{R}^n} |\xi|^{2s} \langle \widehat{\partial^\alpha(\mathcal{E}u)}(\xi) | \overline{\widehat{\partial^\alpha(\mathcal{E}v)}(\xi)}\rangle_{n\times n} d\xi, \\ E_{m,s}^{\mu,\lambda}(u,v) = &\ 2\mu S_{m,s}(u,v) + \lambda D_{m,s}(u,v), \end{aligned} \tag{4.1}$$

for all $u = (u_1,\dots,u_n)^T$ and $v = (v_1,\dots,v_n)^T$ in $X^{m,s}(\mathbb{R}^n;\mathbb{R}^n)$. For short notation, the associated quadratic forms will be denoted by $S_{m,s}(u) = S_{m,s}(u,u)$ and $E_{m,s}^{\mu,\lambda}(u) = E_{m,s}^{\mu,\lambda}(u,u)$. The form $E_{m,s}^{\mu,\lambda}$ is the energy stored in the body and the real numbers μ and λ are called the Lamé constants of the isotropic body (see [10, 15]). We assume $\mu > 0$ and $\lambda + 2\mu > 0$.

As in the previous section, let $\mathscr{A}$ be a finite subset of $\overline{\Omega}$, we assume that $\mathscr{A}$ contains a Π_{m-1}-unisolvent subset.

For any vector-valued function $u : \bar{\Omega} \to \mathbb{R}^n$, we consider the following minimal elastic interpolating problem:

Problem 3. *Find $u^* \in \mathcal{I}_{\mathscr{A}}(u)$ minimizing the functional energy $E^{\lambda,\mu}_{m,s}(w)$, namely*

$$u^* = \arg\Big[\min_{w \in \mathcal{I}_{\mathscr{A}}(u)} E^{\lambda,\mu}_{m,s}(w)\Big]. \tag{4.2}$$

For any $f \in H^{m+s}(\Omega; \mathbb{R})$ we also consider the minimal elastic extension problem:

Problem 4. *Find $v^* \in I_\Omega(f)$ minimizing the functional energy $E^{\lambda,\mu}_{m,s}(w)$, namely*

$$v^* = \arg\Big[\min_{w \in \mathcal{I}_\Omega(u)} E^{\lambda,\mu}_{m,s}(w)\Big]. \tag{4.3}$$

We have the following proposition.

Proposition 4.1. *i) The minimal interpolating problem (4.2) introduced above, has a unique solution in $X^{m,s}(\mathbb{R}^n; \mathbb{R}^n)$. This unique solution will be denoted by $\mathscr{E}^{\mathscr{A}}_{\lambda,\mu} u$ it satisfies*

$$\mathscr{E}^{\mathscr{A}}_{\lambda,\mu} u = S^{\mathscr{A}}_{2+\lambda/\mu} u \tag{4.4}$$

where $S^{\mathscr{A}}_{2+\lambda/\mu} u$ is the solution of the minimal div-curl interpolating problem (2.13) corresponding to $\rho = 2 + \lambda/\mu$.

ii) The minimal elastic extension problem (4.3) introduced above, has a unique unique solution in $X^{m,s}(\mathbb{R}^n; \mathbb{R}^n)$. This unique solution will be denoted by $\mathscr{E}^{\Omega}_{\lambda,\mu} f$ it satisfies

$$\mathscr{E}^{\Omega}_{\lambda,\mu} f = S^{\Omega}_{2+\lambda/\mu} f \tag{4.5}$$

where $S^{\Omega}_{2+\lambda/\mu} u$ is the solution of the minimal div-curl extension problem (2.17) corresponding to $\rho = 2 + \lambda/\mu$.

Proof. See [8, Theorem 5], for item i). For Item ii), it is a direct consequence of Relation (45) in [8] and Proposition 2.5 in this paper. □

We have the following convergence result

Theorem 4.2. *Let $\mathscr{A}$ be a finite subset of $\overline{\Omega}$ containing a Π_{m-1}-unisolvent subset. For all $f \in H^{m+s}(\Omega; \mathbb{R}^n)$ we have*

i) $\mathscr{E}^{\Omega}_{\lambda,\mu} f = \lim_{h \to 0} \mathscr{E}^{\mathscr{A}}_{\lambda,\mu} \mathscr{E}^{\Omega}_{\lambda,\mu} f$ in $X^{m,s}(\mathbb{R}^n; \mathbb{R}^n)$.

ii) $f = \lim_{h\to 0} \mathscr{E}^{\mathscr{A}}_{\lambda,\mu}\mathscr{E}^{\Omega}_{\lambda,\mu} f$ *in* $H^{m+s}(\Omega;\mathbb{R}^n)$ *and consequently this strong convergence holds also in* $\mathcal{C}^{m(s)-1}(\Omega;\mathbb{R}^n)$*, where* $m(s)$ *is given by* (3.4).

Proof. The result is immediately obtained from Theorem 3.2 and Proposition 4.1. □

THEOREM 4.3. *Let* $p \in [1,\infty)$*, let* $r_0 = m + s - n(1/2 - 1/p)_+$*. Then, there exist a positive constant* h_0 *(depending on* Ω*,* n*, and* $m+s$*) and two positive constants* C_1 *and* C_2 *(depending on* Ω*,* n*,* $m+s$ *and* p*) satisfying the following property: for any finite set* $\mathscr{A} \subset \overline{\Omega}$ *containing a* Π_{m-1}*-unisolvent subset, such that* $h = \sup_{x\in\overline{\Omega}} \inf_{a\in\mathscr{A}} |x-a| < h_0$*, for any integer* $k = 0,\dots,\lceil r_0\rceil - 1$*, the following inequalities:*

$$
\begin{aligned}
|f - \mathscr{E}^{\mathscr{A}}_{\lambda,\mu}\mathscr{E}^{\Omega}_{\lambda,\mu} f|_{k,p,\Omega} &\le C_1 h^{m+s-k-n(1/2-1/p)_+}\ |f - \mathscr{E}^{\mathscr{A}}_{\lambda,\mu}\mathscr{E}^{\Omega}_{\lambda,\mu} f|_{m+s,2,\Omega} \\
&\le C_2 \sqrt{2+\tfrac{\lambda}{\mu}}\, h^{m+s-k-n(1/2-1/p)_+}\ \|f\|_{m+s,2,\Omega}
\end{aligned}
\tag{4.6}
$$

hold for every function f *belonging to* $H^{m+s,2}(\Omega;\mathbb{R}^n)$*. If* $m+s \in \mathbb{N}^*$*, this bound also holds with* $k = r_0$ *when either* $2 < p < \infty$ *and* $r_0 \in \mathbb{N}$*, or* $p \le 2$*. Moreover, if* $s \le 0$*,* $\|f\|_{m+s,2,\Omega}$ *in* (4.6) *may be replaced by* $|f|_{m+s,2,\Omega}$*.*

Proof. The results are obtained immediately from Theorem 3.4 together with Proposition 4.1, and the fact that

$$\sup(1, 2+\tfrac{\lambda}{\mu}) = 2+\tfrac{\lambda}{\mu} \quad \text{and} \quad \inf(1,2+\tfrac{\lambda}{\mu}) = 1. \qquad \square$$

Thanks to Theorem 4.2 and Theorem 4.3, we get the following corollary.

COROLLARY 4.4. *Let* $\mathscr{A}$ *be a finite subset of* $\overline{\Omega}$ *containing a* Π_{m-1}*-unisolvent subset. Let* $p \in [1,\infty)$ *and let* $r_0 = m+s-n(1/2-1/p)_+$*. For any* $f \in H^{m+s}(\Omega;\mathbb{R}^n)$ *and for any integer* $k = 0,\dots,\lceil r_0\rceil - 1$*, we have*

$$|f - \mathscr{E}^{\mathscr{A}}_{\lambda,\mu}\mathscr{E}^{\Omega}_{\lambda,\mu} f|_{m+s,2,\Omega} = o(1) \quad \text{as} \quad h \to 0, \tag{4.7}$$

and consequently

$$|f - \mathscr{E}^{\mathscr{A}}_{\lambda,\mu}\mathscr{E}^{\Omega}_{\lambda,\mu} f|_{k,q,\Omega} = o(h^{m+s-k-n(1/2-1/p)_+}) \quad \text{as} \quad h \to 0. \tag{4.8}$$

If $m+s \in \mathbb{N}^*$*, this bound also holds with* $k = r_0$ *when either* $2 < p < \infty$ *and* $r_0 \in \mathbb{N}$*, or* $p \le 2$*.*

5. NUMERICAL EXAMPLE

In this breve section, we will give a numerical example that illustrate the convergence results. More numerical tests may be found in [8]. We consider for instance the scalar function given by $f(x) = \sum_{i=1}^{5} e^{-(x_1-a_i)^2-(x_1-b_i)^2}$, for all two-variate $x = (x_1,x_2) \in \mathbb{R}^2$, where $(a_1,b_1) = (3,3)$, $(a_2,b_2) = (3,-3)$, $(a_3,b_3) = (-3,-3)$, $(a_4,b_4) = (-3,3)$ and $(a_5,b_5) = (0,0)$. Then, we consider the original vector field function constructed as

$$u(x) = \tfrac{50}{51}(\partial_2 f(x), -\partial_1 f(x)) + \tfrac{1}{50}(\partial_1 f(x), \partial_2 f(x)),$$

and restricted to the open square $\Omega =]-2,2[\times]-2,2[$. The vector field function u is in fact a convex combination between two fields one is div-free and the other one is curl-free. The original vector field function u is represented in Fig. 5.1.

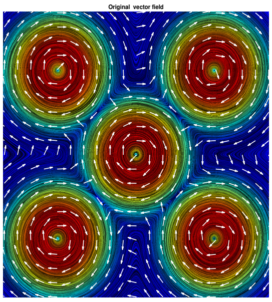

Fig. 5.1. The original vector field function

Now we interpolate the original field on a finite set $\mathcal{A}$ of N scattered data points on Ω by using the elastic pseudo-polyharmonic spline with $m = 2$ and $s = 0$, denoted here, for simplicity, by σ_h, we recall that h is the fill distance. The choose of optimal parameters μ and λ is another problem which deserves more investigations. For our example, we fix for instance the parameters to the values $\mu = 25$ and $\lambda = 10$. We will increase the number N and observe how the elastic pseudo-polyharmonic splines approximate the original vector field. To compare the convergence, we will compute the discrete relative error on a meshgrid points $\mathcal{G}$ on the domain Ω:

$$RE(u,\sigma_h) := \frac{\sqrt{\sum_{x\in\mathcal{G}} |u(x)-\sigma_h(x)|^2}}{\sqrt{\sum_{x\in\mathcal{G}} |u(x)|^2}}.$$

N	h	$RE(u,\sigma_h)$	cputime	N	h	$RE(u,\sigma_h)$	cputime
25	2.61e-02	8.69e-01	0.15 s	800	3.64e-03	5.68e-03	3.36 s
50	1.59e-02	5.73e-01	0.28 s	1600	3.64e-03	2.01e-03	7.71 s
100	1.59e-02	1.71e-01	0.40 s	3200	2.10e-03	1.16e-03	27.00 s
200	1.13e-02	7.12e-02	0.70 s	6400	2.10e-03	6.18e-04	163.57 s
400	3.64e-03	1.87e-02	1.71 s	12800	2.10e-03	3.93e-04	1076.75 s

Table 5.1. The summarized results corresponding to the pseudo-polyharmonic elastic spline with $\mu = 25$ and $\lambda = 10$.

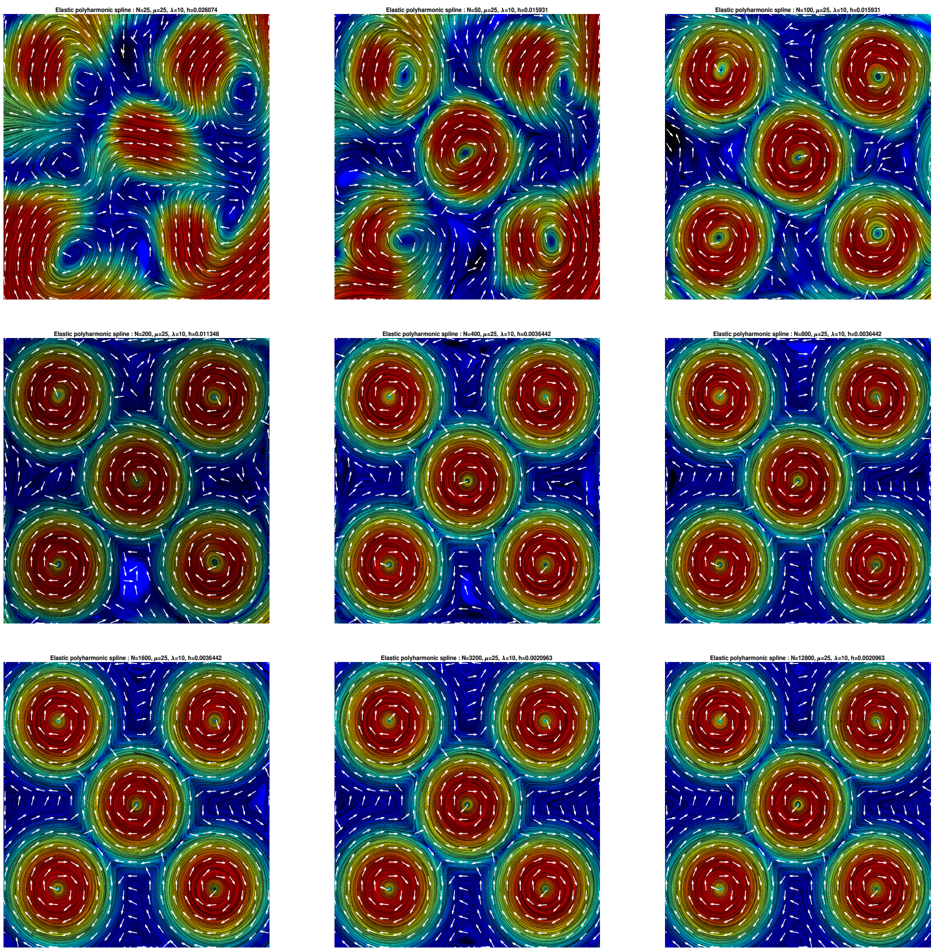

Fig. 5.2. The elastic pseudo-polyharmonic spline interpolating the original vector field u with $\mu = 25$ and $\lambda = 10$ for $N = 25$ (left-top), $N = 50$ (center-top) and $N = 100$ (right-top),
$N = 200$ (left-middle), $N = 400$ (center-middle) and $N = 800$ (right-middle)
$N = 1600$ (left-bottom), $N = 3200$ (center-bottom) and $N = 12800$ (right-bottom).

We observe in Fig. 5.2 that the elastic pseudo-polyharmonic is close to the original vector fields as the number of interpolating points becomes more and more large. The results are summarized in Table 5.1, where we observe that the relative error $R(u, \sigma_h)$ becomes more and more small as the fill distance $h \to 0$. We also give the `cputime` in seconds for computing the pseudo-polyharmonic spline. This results are in concordance with the theoretical convergence results.

6. CONCLUSION

In this paper, we have proved some results on convergence and error estimates for both minimization problems of div-curl and elastic field approximation. First, we have recalled the problem of div-curl and elastic approximation previously introduced by the authors and then all the results on convergence and error estimates are proved. A numerical experiment is given to illustrate briefly the theoretical results.

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