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NEW ESTIMATES RELATED WITH THE BEST POLYNOMIAL APPROXIMATION

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Abstract. In some old results, we find estimates the best approximation $E_{n,p}(f)$ of a periodic function satisfying $f^{(r)} \in \mathbb{L}^p_{2\pi}$ in terms of the norm of $f^{(r)}$ (Favard inequality). In this work, we look for a similar result under the weaker assumption $f^{(r)} \in \mathbb{L}^q_{2\pi}$, with $1 < q < p < \infty$. We will present inequalities of the form $E_{n,p}(f) \leq C(n)\|D^{(r)} f\|_q$, where $D^{(r)}$ is a differential operator. We also study the same problem in spaces of non-periodic functions with a Jacobi weight.

MSC. 42A10, 41A10.

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1. INTRODUCTION

Let $C_{2\pi}$ denote the Banach space of all real 2π-periodic continuous functions f defined on the real line $\mathbb{R}$ with the sup norm

$$\|f\|_\infty = \max_{x \in [-\pi,\pi]} \mid f(x) \mid .$$

For $1 \leq p < \infty$, the Banach space $\mathbb{L}^p_{2\pi}$ consists of all 2π-periodic, p-th power Lebesgue integrable (class of) functions f on $\mathbb{R}$ with the norm

$$\|f\|_p = \Big(\tfrac{1}{2\pi}\int_{-\pi}^{\pi} \mid f(x) \mid^p dx\Big)^{1/p}.$$

In order to simplify we write $X^p_{2\pi} = \mathbb{L}^p_{2\pi}$ for $1 \leq p < \infty$ and $X^\infty_{2\pi} = C_{2\pi}$.

Let $\mathbb{T}_n$ denote the family of all trigonometric polynomials of degree non greater that n. For $n \in \mathbb{N}_0$ and $f \in X^p_{2\pi}$, the best approximation is defined by

$$E_{n,p}(f) = \inf_{T \in \mathbb{T}_n} \|f - T\|_p.$$

By $W^r_{p,2\pi}$ we mean the family of all functions $f \in X^p_{2\pi}$ such that $f, \dots, f^{(r-1)}$ are absolutely continuous and $f^{(r)} \in X^p_{2\pi}$.

The following result is known (for instance, see [4, p. 166] where other notations were used).

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THEOREM 1. *If* $1 \le p \le \infty$, $r, n \in \mathbb{N}$ *and* $g \in W^r_{p,2\pi}$, *then*

$$E_{n,p}(g) \le \frac{F_r}{(n+1)^r} E_{n,p}(g^{(r)}), \tag{1}$$

where F_r *is the Favard constant.*

Here we want to consider the following problem. Is there an estimate similar to (1), if $1 \le q < p \le \infty$, $r, n \in \mathbb{N}$, $g \in X^p_{2\pi}$ and $g \in W^r_{q,2\pi}$ (with the necessary adjustment)?

In [1], Ganzburg considered a similar problem, when the derivative of a continuous functions is not continuous.

THEOREM 2 (Ganzburg, [1]). *If* $f \in C_{2\pi}$ *is absolutely continuous and* $f' \in \mathbb{L}^1_{2\pi}$, *then*

$$E_{n,\infty}(f) \le \tfrac{1}{2} E_{n,1}(f').$$

In Section 2 we extend the result of Ganzburh to the case when $1 < q < p < \infty$, $f \in X^p_{2\pi}$ and $f \in W^r_{q,2\pi}$.

For $1 \le p < \infty$, we denote by $\mathbb{L}^p_{\alpha,\beta}$ the space of all measurable functions f satisfying

$$\|f\|_{p,\alpha,\beta} = \left\{ \int_{-1}^{1} \mid f(x) \mid^p \varrho_{\alpha,\beta}(x) dx \right\}^{1/p} < \infty,$$

where $\varrho_{\alpha,\beta}(x) = (1-x)^\alpha (1+x)^\beta$ and $\alpha, \beta > -1$.

For $n \in \mathbb{N}_0$ and $f \in L^p_{(\alpha,\beta)}$, define the best approximation of f of order n by

$$E_n(f)_{p,\alpha,\beta} = \inf \Big\{ \|f - P_n\|_{p,\alpha,\beta} \, ; \, P_n \in \mathbb{P}_n \Big\},$$

here $\mathbb{P}_n$ is the family of all algebraic polynomials of degree not bigger than n.

In Section 3, for $1 < q < p$ and functions $f \in L^p_{(\alpha,\beta)}$ such that $\mathcal{D}^r_{\alpha,\beta}(f) \in L^q_{(\alpha,\beta)}$ (see (9)), we estimate $E_n(f)_{p,\alpha,\beta}$ in terms of $E_n(\mathcal{D}^r_{\alpha,\beta}(f))_{q,\alpha,\beta}$. We will assume that $\alpha \ge \beta \ge -1/2$.

2. THE PERIODICAL CASE

Let

$$\mathfrak{B}_r(t) = \sum_{k=1}^{\infty} \frac{\cos(kt - r\pi/2)}{k^r} \tag{2}$$

be the Bernoulli kernel.

THEOREM 3. *Assume* $1 < q < p < \infty$ *and let* s *satisfy*

$$\tfrac{1}{q} - \tfrac{1}{p} = 1 - \tfrac{1}{s}. \tag{3}$$

If $r \in \mathbb{N}$, $f \in X^p_{2\pi}$ *and* $f \in W^r_{q,2\pi}$, *then*

$$E_{n,p}(f) \le 2\, E_{n,s}(\mathfrak{B}_r)\; E_{n,q}(f^{(r)}).$$

for each $n \in \mathbb{N}$.

Proof. Fix polynomials $T_n, B_n \in \mathbb{T}_n$ such that

$$E_{n,q}(f^{(r)}) = \|f^{(r)} - T_n\|_q \qquad \text{and} \qquad \|\mathfrak{B}_r - B_n\|_s = E_{n,s}(\mathfrak{B}_r).$$

Define

$$M_n(f,x) = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f^{(r)}(t)B_n(x-t)dt,$$

$$N_n(f,x) = \tfrac{1}{\pi}\int_{-\pi}^{\pi} T_n(t)\Big(\mathfrak{B}_r(x-t) - B_n(x-t)\Big)dt,$$

and

$$L_n(f,x) = \tfrac{1}{2\pi}\int_{0}^{2\pi} f(t)dt + M_n(f,x) + N_n(f,x).$$

Since $M_n(f,x)$ and $N_n(f,x)$ are convolution with trigonometric polynomials of degree not bigger than n, one has $M_n(f), N_n(f), L_n(f) \in \mathbb{T}_n$.

Recall that (see [4, p. 30])

$$f(x) = \tfrac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt + \tfrac{1}{\pi}\int_{-\pi}^{\pi} f^{(r)}(t)\mathfrak{B}_r(x-t)dt.$$

Therefore

$$L_n(f,x) - f(x) =$$

$$= \tfrac{1}{\pi}\int_{-\pi}^{\pi} f^{(r)}(t)B_n(x-t) + T_n(t)\Big(\mathfrak{B}_r(x-t) - B_n(x-t)\Big) - \Big(f^{(r)}(t)\mathfrak{B}_r(x-t)\Big)dt$$

$$= \tfrac{1}{\pi}\int_{-\pi}^{\pi}\Big(T_n(t) - f^{(r)}(t)\Big)\Big(\mathfrak{B}_r(x-t) - B_n(x-t)\Big)dt.$$

That is,

$$\tfrac{1}{2}(L_n(f,x) - f(x)) = \Big(T_n - f^{(r)}\Big) * \Big(\mathfrak{B}_r - B_n\Big)(x). \tag{4}$$

Now an application of the Young inequality yields

$$\tfrac{1}{2}\|L_n(f) - f\|_p \le \|T_n - f^{(r)}\|_q\ \|\mathfrak{B}_r - B_n\|_s = E_{n,s}(\mathfrak{B}_r)\ E_{n,q}(f^{(r)}). \qquad \square$$

In Theorem 4 we extend the result of Ganzburg presented in the introduction. We consider $r > 1$, because $\mathfrak{B}_1$ is not a continuous function. On the other hand the series (2) converges uniformly for $r \ge 2$.

THEOREM 4. *If $r \in \mathbb{N}$, $r > 1$, $1 < p \le \infty$, $f \in X^p_{2\pi}$ and $f \in W^r_1$, then*

$$E_{n,\infty}(f) \le\ 2E_{n,p}(\mathfrak{B}_r)\ E_{n,1}(f^{(r)}).$$

for each $n \in \mathbb{N}$.

Proof. It is known (see [4, p. 36]) that if $f \in X^p_{2\pi}$ and $h \in \mathbb{L}^1$, then $f * h \in X^p_{2\pi}$ and

$$\|f * h\|_p \le \|h\|_1\,\|f\|_p.$$

If $f \in X^p_{2\pi}$ and $f \in W^r_1$, we fix polynomials $T_n, B_n \in \mathbb{T}_n$ such that

$$E_{n,1}(f^{(r)}) = \|f^{(r)} - T_n\|_1 \qquad \text{and} \qquad \|\mathfrak{B}_r - B_n\|_p = E_{n,p}(\mathfrak{B}_r).$$

It follows from (4) that

$$E_{n,p}(f) \leq 2\|T_n - f^{(r)}\|_1 \; \|\mathfrak{B}_r - B_n\|_p = 2E_{n,p}(\mathfrak{B}_r)\; E_{n,1}(f^{(r)}). \qquad \square$$

REMARK 5. We have not found a study of the quantities $E_{n,s}(\mathfrak{B}_r)$ for $s > 1$. But, if $r, s > 1$ and $S_n(x)$ is the partial sum of the series (2), then

$$E_{n,s}(\mathfrak{B}_r) \leq \|\mathfrak{B}_r - S_n\|_s = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big|\sum_{k=n+1}^{\infty}\frac{\cos(kx)}{k^r}\Big|^s dx\right)^{1/s}$$
$$\leq \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big(\sum_{k=n+1}^{\infty}\frac{1}{k^r}\Big)^s dx\right)^{1/s} = \sum_{k=n+1}^{\infty}\frac{1}{k^r} \leq \frac{1}{(r-1)(n+1)^{r-1}}.$$

$\square$

3. THE CASE OF NON-PERIODIC FUNCTIONS

Let $\{J_n^{(\alpha,\beta)}\}$ be the orthogonal system of Jacobi polynomials on $[-1,1]$ with respect to the weight $\varrho_{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}$, normalized by the conditions $J_n^{(\alpha,\beta)}(1) = 1$.

If $\alpha \geq \beta \geq -1/2$ and $1 \leq p \leq \infty$, it is known that the exists a generalized translation $\tau_t : \mathbb{L}^p_{\alpha,\beta} \to \mathbb{L}^p_{\alpha,\beta}$ with the following properties

$$\|\tau_t(f)\|_{p,\alpha,\beta} \leq \|f\|_{p,\alpha,\beta}, \qquad t \in [-1,1],\; f \in \mathbb{L}^p_{\alpha,\beta},$$

and

$$c_k(\tau_t(f)) = c_k(f)\; J_k^{(\alpha,\beta)}(t), \qquad k \in \mathbb{N}_0, \tag{5}$$

where

$$c_k(g) = c_k^{(\alpha,\beta)}(g) = \int_{-1}^{1} g(x) J_k^{(\alpha,\beta)}(x)\varrho_{\alpha,\beta}(x)dx, \qquad k \in \mathbb{N}_0.$$

for all $g \in \mathbb{L}^1_{\alpha,\beta}$.

The generalized translation allows us to define a convolution in $\mathbb{L}^p_{\alpha,\beta}$ by setting

$$(f * g)(x) = \int_{-1}^{1} \tau_s(f,x) g(s) \varrho_{\alpha,\beta}(s)\, ds.$$

If $f \in L^p_{(\alpha,\beta)}$ and $g \in L^1_{(\alpha,\beta)}$, for almost every $x \in [-1,1]$ the integral

$$(f * g)(x) = \int_{-1}^{1} \tau_y(f,x)\, g(y)\, \varrho^{(\alpha,\beta)}(y) dy \tag{6}$$

exists, $f * g \in L^1_{(\alpha,\beta)}$, $f * g = g * f$ and

$$\|f * g\|_p \leq \|g\|_1 \|f\|_p. \tag{7}$$

Moreover

$$c_k(f * g) = c_k(f)\; c_k(g), \qquad k \in \mathbb{N}_0.$$

For the properties of the translation quoted above see [2] and [3].

We also need the following property (see [5]). Assume $1 \le r, q < \infty$, $r^{-1} + q^{-1} > 1$, and $p^{-1} = r^{-1} + q^{-1} - 1$. If $\varphi \in L^r_{(\alpha,\beta)}$ and $g \in L^q_{(\alpha,\beta)}$, then $\varphi * g \in L^p_{(\alpha,\beta)}$ and

$$\|\varphi * g\|_{p,\alpha,\beta} \le \|\varphi\|_{r,\alpha,\beta}\ \|g\|_{q,\alpha,\beta}. \tag{8}$$

As in Rafalson [5], let $\Omega^r_{\alpha,\beta}$ be the family of all functions $f \in C^{2r-1}(-1,1)$ such that, for $0 \le k \le r-1$, the function

$$\psi^k_r(f,x) := \Big((1-x)^{\alpha+r}(1+x)^{\beta+r} f^{(r)}(x)\Big)^{(k)}$$

is absolutely continuous in $(-1,1)$ and $\psi^k_r(f,\pm 1) = 0$.

For $f \in \Omega^r_{\alpha,\beta}$, define

$$\mathcal{D}^r_{\alpha,\beta}(f,x) = \tfrac{1}{(1-x)^\alpha}\tfrac{1}{(1+x)^\beta}\Big((1-x)^{\alpha+r}(1+x)^{\beta+r} f^{(r)}(x)\Big)^{(k)}. \tag{9}$$

For $r \in \mathbb{N}$ and $t \in (-1,1)$, consider the function

$$\Phi^r_{\alpha,\beta}(t) = C_r \int_{-1}^{t} \tfrac{(t-z)^{r-1}}{(1-z)^{1+\alpha}(1+z)^{r+\beta}} \int_{-1}^{z} (1-u)^\alpha (1+z)^{r+\beta-1} du dz,$$

where

$$C_r = C(r,\alpha,\beta) := \tfrac{(-1)^r}{2^{\alpha+\beta+r}} \tfrac{\Gamma(r+\alpha+\beta+1)}{\Gamma^2(r)\Gamma(\alpha+1)\Gamma(r+\beta)}.$$

It is known [5, Lemma 8] that, if $f \in \Omega^r_{\alpha,\beta}$, then

$$f(x) - S^{(\alpha,\beta)}_{r-1}(f,x) = \mathcal{D}^r_{\alpha,\beta}(f) * \Phi^r_{\alpha,\beta}(x) \tag{10}$$

almost everywhere with respect to the Lebesgue measure. Here $S^{(\alpha,\beta)}_r(f)$ is the partial sum of order r of the Jacobi expansion of f.

Let us recall a known result.

Theorem 6. (Rafalson, [5, Th. 1]) *Assume $\alpha \ge \beta \ge -1$, $r, n \in \mathbb{N}$, $n \ge r-1$, and one of the following conditions hold:*

$$\text{(i)} \qquad r > \alpha + 1, \quad \textit{and} \quad 1 \le p \le \infty,$$

$$\text{(ii)} \qquad r = \alpha + 1, \quad \textit{and} \quad 1 \le p < \infty,$$

or

$$\text{(iii)} \qquad r > \alpha + 1, \quad \textit{and} \quad 1 \le r < \tfrac{1+\alpha}{1+\alpha-r}.$$

If $f \in \Omega^r_{\alpha,\beta}$, then

$$E_n(f)_{p,\alpha,\beta} \le E_n(\Phi^r_{\alpha,\beta})_{p,\alpha,\beta}\ E_n(\mathcal{D}^r_{\alpha,\beta}(f))_{p,\alpha,\beta}.$$

We present an analogous of Theorem 3 in the frame of Jacobi spaces.

Theorem 7. *Assume $\alpha \ge \beta \ge -1/2$, $1 < q < p < \infty$ and chose s from the condition (3). If $r \in \mathbb{N}$, $f \in \Omega^r_{\alpha,\beta}$ and $\mathcal{D}^r_{\alpha,\beta}(f) \in L^q_{(\alpha,\beta)}$, then*

$$E_n(f)_{p,\alpha,\beta} \le E_n(\Phi^r_{\alpha,\beta})_{s,\alpha,\beta}\ E_n(\mathcal{D}^r_{\alpha,\beta}(f))_{q,\alpha,\beta}.$$

for each $n \in \mathbb{N}$, $n \ge r-1$.

Proof. Let $P_n \in \mathbb{P}_n$ $(B_n \in \mathbb{P}_n)$ be the polynomial of the best approximation of order n of $\mathcal{D}^r_{\alpha,\beta}(f)$ $(\Phi^r_{\alpha,\beta})$ in the norm of $L^q_{(\alpha,\beta)}$ $(L^s_{(\alpha,\beta)})$. That is,

$$E_n(\mathcal{D}^r_{\alpha,\beta}(f))_{q,\alpha,\beta} = \|\mathcal{D}^r_{\alpha,\beta}(f) - P_n\|_{q,\alpha,\beta}$$

and

$$E_n(\Phi^r_{\alpha,\beta})_{s,\alpha,\beta} = \|\Phi^r_{\alpha,\beta} - B_n\|_{s,\alpha,\beta}$$

Define

$$\begin{aligned} M_n(g,x) &= \int_{-1}^{1} (\mathcal{D}^r_{\alpha,\beta}(f)(t)\tau_t(B_n,x)\varrho^{(\alpha,\beta)}(t)dt, \\ N_n(g,x) &= \int_{-1}^{1} P_n(t)\tau_t\Big(\Phi^r_{\alpha,\beta} - B_n\Big)(x)\varrho^{(\alpha,\beta)}(t)dt, \end{aligned}$$

and

$$L_n(g,x) = S^{(\alpha,\beta)}_{r-1}(f,x) + M_n(g,x) + N_n(g,x).$$

Since $M_n(g)$ and $N_n(g)$ are convolutions with algebraic polynomials of degree non greater than n and $n \geq r-1$, it follows from (5) that $M_n(g), N_n(g)$, $L_n(g) \in \mathbb{P}_n$.

Taking into account (10), we obtain

$$\begin{aligned} &L_n(f,x) - f(x) = L_n(f,x) - S^{(\alpha,\beta)}_{r-1}(x) - (f - S^{(\alpha,\beta)}_{r-1}(f))(x) \\ &= L_n(f,x) - S^{(\alpha,\beta)}_{r-1}(x) - (\mathcal{D}^r_{\alpha,\beta}(f) * \Phi^r_{\alpha,\beta}(x)) \\ &= \int_{-1}^{1}\Big(\mathcal{D}^r_{\alpha,\beta}(f,t)\tau_t(B_n) + P_n(t)\tau_t(\Phi^r_{\alpha,\beta} - B_n) - \mathcal{D}^r_{\alpha,\beta}(f,t)\tau_t(\Phi^r_{\alpha,\beta})\Big)(x)\varrho^{(\alpha,\beta)}(t)dt \\ &= \int_{-1}^{1} \Big(P_n - \mathcal{D}^r_{\alpha,\beta}(f)\Big)(t)\tau_t\Big(\Phi^r_{\alpha,\beta} - B_n\Big)(x)dt \\ &= \Big((P_n - \mathcal{D}^r_{\alpha,\beta}(f)) * \tau_t(\Phi^r_{\alpha,\beta} - B_n)\Big)(x). \end{aligned}$$

Now, it follows from (8) that

$$\begin{aligned} \|L_n(f) - f\|_{p,\alpha,\beta} &\leq \|\Phi^r_{\alpha,\beta} - B_n\|_{s,\alpha,\beta}\ \|P_n - \mathcal{D}^r_{\alpha,\beta}(f))\|_{q,\alpha,\beta} \\ &= E_n(\Phi^r_{\alpha,\beta})_{s,\alpha,\beta}\ E_n(\mathcal{D}^r_{\alpha,\beta}(f))_{q,\alpha,\beta}. \end{aligned}$$ □

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