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# NON-HOMOGENEOUS IMPULSIVE TIME FRACTIONAL HEAT CONDUCTION EQUATION

ARMAN AGHILI*

**Abstract.** This article provides a concise exposition of the integral transforms and its application to fractional partial differential equations. The author implemented an analytical technique, the transform method for solving the boundary value problems of impulsive time fractional heat conduction equation. Integral transforms method is a powerful tool for the evaluation of certain integrals involving special functions and solution of partial fractional differential equations. The proposed method is extremely concise, attractive as a mathematical tool. The obtained result reveals that the transform method is very convenient and effective. Certain new integrals involving the Airy functions are given.

**MSC.** 26A33, 44A10, 44A20, 35A22.

**Keywords.** Laplace transforms, Fourier transform, modified Bessel function, Airy function, Gross-Levi.

## 1. INTRODUCTION AND PRELIMINARIES

The time fractional heat equation, which is a mathematical model of a wide range of important physical phenomena, is a partial differential equation obtained from the classical heat equation by replacing the first time derivative of a fractional derivative of order $0 < \alpha < 1$. The author used the integral transform method for solving partial fractional differential equations which arise in applications.In the literature, different methods have been introduced to solve fractional differential equations, the popular Laplace transform method [1, 2, 6], the Fourier transform method, the iteration method and operational method. However, most of these methods are suitable for special types of fractional differential equations, mainly the linear with constant coefficients.

### 1.1. Definitions and Notations.

Definition 1. *The left Caputo fractional derivative of order* $\alpha$ $(0 < \alpha < 1)$ *of* $\phi(t)$ *is defined as follows* [7],

$$D_a^{c,\alpha}\phi(t) = \tfrac{1}{\Gamma(1-\alpha)} \int_a^t \tfrac{1}{(t-\xi)^\alpha}\phi'(\xi)d\xi. \tag{1}$$

*Department of Applied Mathematics, University of Guilan, Rasht, Iran, P.O. Box 1841, e-mail: `arman.aghili@gmail.com`, https://orcid.org/0000-0002-3758-2599. The author is partially supported by the University of Guilan.

Definition 2. *Let $f(t)$ be a continuous and single-valued function of real variable $t$ defined for all $t$, $0 < t < +\infty$, and is of exponential order. Then the Laplace transform of the function $f(t)$ is defined as a function $F(s)$ denoted by the integral* [3]

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt := F(s). \tag{2}$$

*If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}^{-1}\{F(s)\}$ is as follows*

$$f(t) = \tfrac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s) ds, \tag{3}$$

*where $F(s)$ is analytic in the region $\mathrm{Re}(s) > c$. The expression in* (3) *is the inverse Laplace transform for the function $F(s)$, and is often called the Bromwich integral.*

In many practical situations, determination of the inverse Laplace transform is very complex. Once we know the inverse Laplace transform of some elementary functions, we can find the inverse transform of many functions with the help of the properties associated with the inverse Laplace transform.

Lemma 3. *Let $L\{f(t)\} = F(s)$ then, the following identities hold true:*

1. $\mathcal{L}\left\{f(\tfrac{1}{t})\right\} = \int_0^\infty \sqrt{\tfrac{\xi}{s}} J_1\left(2\sqrt{s\xi}\right) F(\xi)\, d\xi;$
2. $\mathcal{L}\left\{f(t^3)\right\} = \int_0^\infty \sqrt{\tfrac{s}{\xi}} K_{\frac{1}{3}}\left(\left(\tfrac{s}{3\sqrt[3]{\xi}}\right)^{\frac{2}{3}}\right) F(\xi)\, d\xi;$
3. $\mathcal{L}^{-1}\left(e^{-\omega s^\beta}\right) = \tfrac{1}{\pi} \int_0^\infty e^{-r^\beta(\omega\cos\beta\pi)} \sin(\omega r^\beta \sin\beta\pi) \left(\int_0^\infty e^{-s\tau - r\tau} d\tau\right) dr;$
4. $\mathcal{L}^{-1}\left(F(s^\alpha)\right) = \tfrac{1}{\pi} \int_0^\infty f(u) \int_0^\infty e^{-tr-ur^\alpha\cos\alpha\pi} \sin(ur^\alpha \sin\alpha\pi) dr du;$
5. $\mathcal{L}^{-1}\left(F\left(\sqrt{(s+a)^2 - b^2}\right)\right) = e^{-at} f(t) + be^{-at} \int_0^t f\left(\sqrt{t^2-\xi^2}\right) I_1(b\xi) d\xi;$
6. $\mathcal{L}^{-1}\left(\tfrac{F(\sqrt{s^2-a^2})}{\sqrt{s^2-a^2}}\right) = \int_0^t I_0\left(a\sqrt{t^2-\xi^2}\right) d\xi.$

*Proof.* See [3]. □

Corollary 4 (A new class of inverse Laplace transform). *In the above Lemma 3 in part 5 and 6, let us take $f(t) = \delta(t-k)$, then we get the following Laplace transform relations:*

1. $\mathcal{L}^{-1}\left(e^{-k\sqrt{(s+a)^2-b^2}}\right) = e^{-at}\delta(t-k) + be^{-at} \int_0^t \delta(\sqrt{t^2-\xi^2} - k^2) I_1(b\xi) d\xi;$
2. $\mathcal{L}^{-1}\left(\tfrac{e^{-k\sqrt{(s+a)^2-b^2}}}{\sqrt{(s+a)^2-b^2}}\right) = e^{-at} I_0(b\sqrt{t^2-k^2}).$

Corollary 5. *The following integral relation holds true:*

$$\mathcal{L}^{-1}\left[\frac{e^{-ks^\alpha}}{s^\nu+\lambda}; s\to t\right] = f(t) =$$

$$= \frac{1}{\pi}\int_0^{+\infty} e^{-tr-kr^\alpha\cos(\pi\alpha)}\left[\frac{r^\nu\sin(\pi\nu+kr^\alpha\sin(\pi\alpha))+\lambda\sin(kr^\alpha\sin(\pi\alpha))}{r^{2\nu}+2\lambda r^\nu\cos(\pi\nu)+\lambda^2}\right]dr,$$

*Proof.* In view of the Gross-Levi theorem [3] we have the following

$$f(t) = \frac{1}{\pi}\int_0^{+\infty} e^{-tr}\,\mathrm{Im}\left[\frac{e^{-kr^\alpha e^{-i\pi\alpha}}}{(re^{-i\pi})^\nu+\lambda}\right]dr,$$

or

$$f(t) = \frac{1}{\pi}\int_0^{+\infty} e^{-tr-kr^\alpha\cos(\pi\alpha)}\,\mathrm{Im}\left[\frac{e^{ikr^\alpha\sin(\pi\alpha)}}{r^\nu(\cos(\pi\nu)-i\sin(\pi\nu))+\lambda}\right]dr,$$

after simplifying we have

$$f(t) =$$

$$= \frac{1}{\pi}\int_0^{+\infty} e^{-tr-kr^\alpha\cos(\pi\alpha)}\,\mathrm{Im}\,\frac{[\cos(kr^\alpha\sin(\pi\alpha))+i\sin(kr^\alpha\sin(\pi\alpha))][r^\nu\cos(\pi\nu)+\lambda+ir^\nu\sin(\pi\nu)]}{r^{2\nu}+2\lambda r^\nu\cos(\pi\nu)+\lambda^2}dr,$$

or

$$f(t) = \frac{1}{\pi}\int_0^{+\infty} e^{-tr-kr^\alpha\cos(\pi\alpha)}\left[\frac{r^\nu\sin(\pi\nu+kr^\alpha\sin(\pi\alpha))+\lambda\sin(kr^\alpha\sin(\pi\alpha))}{r^{2\nu}+2\lambda r^\nu\cos(\pi\nu)+\lambda^2}\right]dr.$$

Let us consider the following special cases

1. $\lambda = 0$ we have

$$\mathcal{L}^{-1}\left[\frac{e^{-ks^\alpha}}{s^\nu}; s\to t\right] = f(t) = \frac{1}{\pi}\int_0^{+\infty} e^{-tr}\left[\frac{r^\nu\sin(\pi\nu+kr^\alpha\sin(\pi\alpha))}{r^{2\nu}}\right]dr,$$

2. $k = 0,\ \nu = 0.5$ we have

$$\mathcal{L}^{-1}\left[\frac{1}{s^\nu+\lambda}; s\to t\right] = f(t) = \frac{1}{\pi}\int_0^{+\infty} e^{-tr}\left[\frac{\sqrt{r}}{r+\lambda^2}\right]dr = \frac{1}{\sqrt{\pi t}} - \lambda e^{\lambda^2 t}\,\mathrm{Erfc}(\lambda\sqrt{t}).$$

□

Lemma 6. *The following integral relation holds true:*

$$\left(\frac{\lambda}{s}\right)^{\frac{\nu}{2}} K_\nu(2\sqrt{\lambda s}) = \int_0^\infty \sqrt{\frac{\xi}{s}}J_1(2\sqrt{s\xi})\frac{\Gamma(\nu+2)}{(\xi+\lambda)^{\nu+2}}d\xi.$$

*Proof.* Let us take $f(t) = \frac{1}{2}t^{\nu+1}e^{-\lambda t}$ , then we get $F(s) = \frac{\Gamma(\nu+2)}{(s+\lambda)^{\nu+2}}$, on the other hand we have $f\left(\frac{1}{t}\right) = g(t) = \frac{1}{2t^{\nu+1}}e^{-\frac{\lambda}{t}}$

$$\mathcal{L}\left\{f\left(\frac{1}{t}\right)\right\} = \mathcal{L}[g(t)] = \int_0^{+\infty} e^{-st}\frac{1}{2t^{\nu+1}}e^{-\frac{\lambda}{t}}dt = \int_0^{+\infty} e^{-st-\frac{\lambda}{t}}\frac{dt}{2t^{\nu+1}},$$

at this stage using an integral representation for the modified Bessel function of the second kind of order $\nu$ as below

$$K_\nu(2\sqrt{pq}) = K_{-\nu}(2\sqrt{pq}) = \left(\frac{p}{q}\right)^{\frac{\nu}{2}}\int_0^{+\infty} e^{-pt-\frac{q}{t}}\frac{dt}{2t^{\nu+1}}.$$

Let us consider the case $p = q = \frac{\xi}{2}$, we get

$$K_\nu(\xi) = K_{-\nu}(\xi) = \int_0^{+\infty} e^{-\frac{\xi}{2}\left(t+\frac{1}{t}\right)} \frac{dt}{2t^{\nu+1}}$$

or,

$$\mathcal{L}\left\{f\left(\frac{1}{t}\right)\right\} = \mathcal{L}\left[g(t)\right] = \left(\frac{\lambda}{s}\right)^{\frac{\nu}{2}} K_\nu(2\sqrt{\lambda s}).$$

By setting all of the information in the first part of the Lemma 3, we get the following

$$\mathcal{L}\left\{f\left(\frac{1}{t}\right)\right\} = \mathcal{L}[g(t)] = \left(\frac{\lambda}{s}\right)^{\frac{\nu}{2}} K_\nu(2\sqrt{\lambda s}) = \int_0^\infty \sqrt{\frac{\xi}{s}} J_1(2\sqrt{s\xi}) \frac{\Gamma(\nu+2)}{(\xi+\lambda)^{\nu+2}} d\xi.$$

In special case $\nu = 0$, we have the following

$$K_0(2\sqrt{\lambda s}) = \int_0^\infty \sqrt{\frac{\xi}{s}} \frac{J_1(2\sqrt{s\xi})}{(\xi+\lambda)^2} d\xi.$$

□

LEMMA 7. *The following double integral relation holds true:*

$$\frac{1}{12\pi} \int_0^\infty \eta J_1\left(2\sqrt{\eta}\right) \left(\int_0^\infty K_{\frac{1}{3}}\left(\left(\frac{\eta}{3\sqrt[3]{\xi}}\right)^{\frac{2}{3}}\right) \left(\frac{\ln\xi}{\xi\sqrt{\xi}}\right) d\xi\right) d\eta = \gamma.$$

*Proof.* Let us take $f(t) = \gamma + \ln t$, then we get $F(s) = -\frac{\ln s}{s}$ , on the other hand we have

$$\mathcal{L}\{f(t^3)\} = \mathcal{L}[g(t)] = \mathcal{L}\{\gamma + 3\ln t\} = -\frac{(2\gamma + 3\ln s)}{s}.$$

By setting all of the information in part 2 of Lemma 3, we obtain

$$\mathcal{L}\{f(t^3)\} = \mathcal{L}\{g(t)\} = \frac{1}{3\pi} \int_0^\infty \sqrt{\frac{s}{\xi}} K_{\frac{1}{3}}\left(\left(\frac{s}{3\sqrt[3]{\xi}}\right)^{\frac{2}{3}}\right) \left(\frac{\ln\xi}{\xi}\right) d\xi = -\frac{(2\gamma + 3\ln s)}{s}.$$

Let us take $g(t) = \gamma + 3\ln t$, then we have

$$G(s) = -\frac{2\gamma + 3\ln s}{s} = \frac{1}{3\pi} \int_0^\infty \sqrt{\frac{s}{\xi}} K_{\frac{1}{3}}\left(\left(\frac{s}{3\sqrt[3]{\xi}}\right)^{\frac{2}{3}}\right) \left(\frac{\ln\xi}{\xi}\right) d\xi,$$

on the other hand

$$\mathcal{L}g\left(\frac{1}{t}\right) = \mathcal{L}(\gamma - 3\ln t) = \frac{4\gamma + 3\ln s}{s},$$

using the first part of Lemma 3 we obtain

$$\mathcal{L}\{g\left(\frac{1}{t}\right)\} = \int_0^\infty \sqrt{\frac{\eta}{s}} J_1(2\sqrt{s\eta}) \left(\frac{1}{3\pi} \int_0^\infty \sqrt{\frac{\eta}{\xi}} K_{\frac{1}{3}}\left(\left(\frac{\eta}{3\sqrt[3]{\xi}}\right)^{\frac{2}{3}}\right) \left(\frac{\ln\xi}{\xi}\right) d\xi\right) d\eta$$
$$= \frac{4\gamma + 3\ln s}{s}.$$

Now, by choosing $s = 1$ and after simplifying, we arrive at

$$\frac{1}{12\pi} \int_0^\infty \eta J_1\left(2\sqrt{\eta}\right) \left(\int_0^\infty K_{\frac{1}{3}}\left(\left(\frac{\eta}{3\sqrt[3]{\xi}}\right)^{\frac{2}{3}}\right) \left(\frac{\ln\xi}{\xi\sqrt{\xi}}\right) d\xi\right) d\eta = \gamma.$$

In the above relation $J_1(\cdot)$ stands for the Bessel function of the first kind and of first order.

$\square$

DEFINITION 8. *Let us consider the following homogeneous second order differential equation*

$$y'' - xy = 0,$$

*known as the Airy equation or the Stokes equation with the two linearly independent solutions* $\mathrm{Ai}(x)$ *and* $\mathrm{Bi}(x)$. *The Airy function also underlies the form of the intensity near a directional caustic, such as the rainbow. Historically, this was the problem that led Airy to develop this function* [1, 8].

LEMMA 9. *We have the following integral representations for Airy function:*

1. $\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^{\infty} \cos(\frac{\xi^3}{3} + x\xi)d\xi;$
2. $\mathrm{Ai}^2(x) = \frac{1}{\pi\sqrt[3]{4}}\int_0^{\infty} \mathrm{Ai}(\sqrt[3]{4}x + \xi)\frac{d\xi}{\xi};$
3. $\mathrm{Ai}(x)\,\mathrm{Ai}(-x) = \frac{1}{\pi\sqrt[3]{2}}\int_{-\infty}^{+\infty} \mathrm{Ai}(\sqrt[3]{4}\xi^2)\exp(2ix\xi)d\xi.$

*Proof.* See [8]. $\square$

LEMMA 10. *The following integral identity for Airy function holds true:*

$$\mathcal{F}\left[\frac{1}{\sqrt[3]{3\eta}}\,\mathrm{Ai}\left(\frac{x}{\sqrt[3]{3\eta}}\right)\right] = e^{i\eta\omega^3}.$$

*Proof.* See [1, 8]. $\square$

LEMMA 11. *The following identity for Fourier transform of the product of Airy functions holds true:*

$$\mathcal{F}[\mathrm{Ai}(x)\,\mathrm{Ai}(-x)] = \frac{1}{\sqrt{2\pi}\,\sqrt[3]{2}}\,\mathrm{Ai}\left(\frac{\omega^2}{\sqrt[3]{16}}\right).$$

*Proof.* By definition of the Fourier transform and in view of part 3 of Lemma 9, we have the following

$$\mathcal{F}[\mathrm{Ai}(x)\,\mathrm{Ai}(-x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{i\omega x}\left(\frac{1}{\pi\sqrt[3]{2}}\int_{-\infty}^{\infty}\mathrm{Ai}\left(\sqrt[3]{4}\xi^2\right)\exp(2ix\xi)d\xi\right)dx.$$

At this stage we change the order of integration which is permissible to obtain

$$\mathcal{F}[\mathrm{Ai}(x)\,\mathrm{Ai}(-x)] = \frac{1}{\pi\sqrt[3]{2}}\int_{-\infty}^{+\infty}\mathrm{Ai}\left(\sqrt[3]{4}\xi^2\right)\frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{+\infty} e^{i\omega x}e^{(2ix\xi)}dx\right)d\xi,$$

or

$$\mathcal{F}[\mathrm{Ai}(x)\,\mathrm{Ai}(-x)] = \frac{1}{\pi\sqrt[3]{2}}\int_{-\infty}^{+\infty}\mathrm{Ai}\left(\sqrt[3]{4}\xi^2\right)\frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{+\infty} e^{i(\omega+2\xi)x}dx\right)d\xi,$$

but the value of the inner integral is $\sqrt{2\pi}\delta(\omega + 2\xi)$, therefore we get

$$\mathcal{F}[\mathrm{Ai}(x)\,\mathrm{Ai}(-x)] = \frac{1}{\pi\sqrt[3]{2}}\int_{-\infty}^{+\infty}\mathrm{Ai}\left(\sqrt[3]{4}\xi^2\right)\sqrt{2\pi}\delta(\omega+2\xi)\,d\xi,$$

at this point, let us make a change of variable $\omega + 2\xi = \eta$, we have

$$\mathcal{F}[\mathrm{Ai}(x)\,\mathrm{Ai}(-x)] = \frac{1}{\pi\sqrt[3]{2}}\int_{-\infty}^{+\infty} \mathrm{Ai}\left(\frac{\sqrt[3]{4}(\eta-\omega)^2}{4}\right)\sqrt{2\pi}\delta\left(\eta\right)\frac{d\eta}{2},$$

after simplifying, we arrive at

$$\mathcal{F}[\mathrm{Ai}(x)\,\mathrm{Ai}(-x)] = \frac{1}{\sqrt{2\pi}\sqrt[3]{2}}\,\mathrm{Ai}\left(\frac{\omega^2}{2\sqrt[3]{2}}\right).$$

□

Lemma 12. *The following integral identity for the product of Airy functions holds true:*

1. $\displaystyle\int_{-\infty}^{+\infty} \mathrm{Ai}(x)\,\mathrm{Ai}(-x)dx = \frac{1}{\sqrt[3]{18}\Gamma\left(\frac{2}{3}\right)}$;
2. $\displaystyle\int_{0}^{+\infty} \mathrm{Ai}(x)\,\mathrm{Ai}(-x)\cos\left(\omega x\right)dx = \frac{1}{2\sqrt[3]{2}}\,\mathrm{Ai}\left(\frac{\omega^3}{2\sqrt[3]{2}}\right)$.

*Proof.* 1. By definition of the Fourier transform and in view of Lemma 9 we have the following

$$\mathcal{F}[\mathrm{Ai}(x)\,\mathrm{Ai}(-x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{i\omega x}\,\mathrm{Ai}(x)\,\mathrm{Ai}(-x)dx = \frac{1}{\sqrt{2\pi}\sqrt[3]{2}}\,\mathrm{Ai}\left(\frac{\omega^2}{2\sqrt[3]{2}}\right). \tag{4}$$

At this point, let us choose $\omega = 0$ in the above relation, after simplifying we get

$$\int_{-\infty}^{+\infty} \mathrm{Ai}(x)\,\mathrm{Ai}(-x)dx = \frac{1}{\sqrt[3]{2}}\,\mathrm{Ai}(0) = \frac{1}{\sqrt[3]{2}}\frac{1}{\Gamma(\frac{2}{3})\sqrt[3]{9}} = \frac{1}{\Gamma(\frac{2}{3})\sqrt[3]{18}}.$$

2. By taking the real part of (4) and after simplifying we arrive at,

$$\int_{0}^{+\infty} \mathrm{Ai}(x)\,\mathrm{Ai}(-x)\cos(\omega x)dx = \frac{1}{2\sqrt[3]{2}}\,\mathrm{Ai}\left(\frac{\omega^3}{2\sqrt[3]{2}}\right). \tag{5}$$

Finally, by taking the inverse Fourier-cosine transform, we obtain a new integral representation for the product of Airy functions.

$$\mathrm{Ai}(x)\,\mathrm{Ai}(-x) = \frac{1}{\pi\sqrt[3]{2}}\int_{0}^{+\infty} \cos(x\omega)\,\mathrm{Ai}\left(\frac{\omega^3}{2\sqrt[3]{2}}\right)d\omega. \tag{6}$$

□

Corollary 13. *We have the following integral identities:*

1. $\displaystyle\frac{1}{\pi}\int_{0}^{+\infty} \sqrt{\frac{x}{3}}K_{\frac{1}{3}}\left(\frac{2}{3}x\sqrt{x}\right)\mathrm{Ai}(-x)\cos(\omega x)dx = \frac{1}{2\sqrt[3]{2}}\,\mathrm{Ai}\left(\frac{\omega^3}{2\sqrt[3]{2}}\right)$;
2. $\displaystyle\int_{0}^{+\infty} \frac{1}{2\sqrt[4]{\pi^2 x}}W_{0,\frac{1}{3}}\left(\frac{4}{3}x\sqrt{x}\right)\mathrm{Ai}(-x)\cos(\omega x)dx = \frac{1}{2\sqrt[3]{2}}\,\mathrm{Ai}\left(\frac{\omega^3}{2\sqrt[3]{2}}\right)$.

Note. If we substitute $\mathrm{Ai}(x) = \sqrt{\frac{x}{3}}K_{\frac{1}{3}}\left(\frac{2}{3}x\sqrt{x}\right)$ and $\mathrm{Ai}(x) = \frac{1}{2\sqrt[4]{\pi^2 x}}W_{0,\frac{1}{3}}(\frac{4}{3}x\sqrt{x})$ in (5) respectively, we obtain the above integral identities [3, 8].

In the above integral identity, $W_{k,\mu}(\cdot)$ stands for the Wittaker function [3, 8].

*Proof.* If we substitute (6) in (5), we obtain the above integral identities. □

REMARK 14. This method of replacing the Airy function can be employed to obtain a number of interesting integrals, some of which will be considered in the future publications. The Airy theory are given in the chapter on the optics of a raindrop in detail by van de Hulst (1957). The function $\operatorname{Ai}(x)$ first appears as an integral in two articles by G. B. Airy. Reference to many of these applications as well as to the theory of elasticity and to the heat equation are given in Valle and Soares [8].

LEMMA 15. *Let us assume that* $\operatorname{Ai}(x)$ *is the Airy function, solution to the second order differential equation* $y'' - xy = 0$*. Then we have the following integral relation*

$$\int_0^{+\infty} \operatorname{Ai}(x) \operatorname{Ai}'''(x) dx = \frac{\sqrt[3]{3}}{8\pi^2} \Gamma^2\left(\frac{2}{3}\right).$$

*Proof.* It is well-known that the Airy functions $\operatorname{Ai}(x)$ and $Bi(x)$ are two linearly independent solution of the second order differential equation known as the Airy differential equation $y'' - xy = 0$, therefore, we have $\operatorname{Ai}''(x) - x\operatorname{Ai}(x) = 0$. By taking derivative of the Airy differential equation, we get $y''' - xy' - y = 0$, or $\operatorname{Ai}'''(x) = x\operatorname{Ai}'(x) + \operatorname{Ai}(x)$. At this stage the left hand side of the above integral can be written as follows

$$\begin{aligned}
\int_0^{+\infty} \operatorname{Ai}(x) \operatorname{Ai}'''(x) dx &= \int_0^{+\infty} \operatorname{Ai}(x)[x \operatorname{Ai}'(x) + \operatorname{Ai}(x)] dx \\
&= \int_0^{+\infty} \operatorname{Ai}^2(x) dx + \int_0^{+\infty} [x \operatorname{Ai}(x) \operatorname{Ai}'(x)] dx \\
&= \int_0^{+\infty} \operatorname{Ai}^2(x) dx + \frac{1}{2} \int_0^{+\infty} 2 \operatorname{Ai}'(x) \operatorname{Ai}''(x) dx.
\end{aligned}$$

At this point we evaluate the first integral by parts, we have

$$= [x \operatorname{Ai}^2(x)]_0^{+\infty} - \int_0^{+\infty} 2[x \operatorname{Ai}(x)] \operatorname{Ai}'(x) dx + \frac{1}{2} \int_0^{+\infty} d[\operatorname{Ai}'^2(x)].$$

Finally, after simplifying we obtain

$$\int_0^{+\infty} \operatorname{Ai}(x) \operatorname{Ai}'''(x) dx = \frac{1}{2} \operatorname{Ai}'^2(0) = \frac{\sqrt[3]{3}}{8\pi^2} \Gamma^2\left(\frac{2}{3}\right).$$

□

NOTE. We used the fact that

1. $\lim_{x \to +\infty} \operatorname{Ai}(x) = \lim_{x \to +\infty} \operatorname{Ai}'(x) = 0;$
2. $\operatorname{Ai}(0) = \frac{1}{\sqrt[3]{9}\Gamma(\frac{2}{3})}, \operatorname{Ai}'(0) = -\frac{1}{\sqrt[3]{3}\Gamma(\frac{1}{3})}.$

PROBLEM 16. *Let us solve the following non-homogeneous impulsive hyperbolic time fractional heat conduction equation.*

$$D_{0,t}^{C,2\alpha} u + 2\lambda D_{0,t}^{C,\alpha} u = \frac{\partial^2 u}{\partial x^2} + \eta u(x,t) + \delta(t), \qquad x,\ t > 0, \quad 0.5 < \alpha < 1;$$

1. $u(x,0) = u_t(x,0) = 0;$
2. $\lim_{x\to+\infty} u(x,t) = 0, \quad 0 < x < +\infty;$
3. $u(0,t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}.$

Solution. Taking the Laplace transform of the partial differential equation and using boundary condition (1.), we get the following non-homogeneous second order differential equation

$$U_{xx} - (s^{2\alpha} + 2\lambda s^{\alpha} - \eta)U = 1.$$

After solving the above equation, we arrive at

$$U(x,s) = c_1 e^{-x\sqrt{s^{2\alpha}+2\lambda s^{\alpha}-\eta}} + c_2 e^{x\sqrt{s^{2\alpha}+2\lambda s^{\alpha}-\eta}} - \frac{1}{s^{2\alpha}+2\lambda s^{\alpha}-\eta}.$$

In view of the boundary condition (2.), we have $c_2 = 0$, therefore

$$U(x,s) = c_1 e^{-x\sqrt{s^{2\alpha}+2\lambda s^{\alpha}-\eta}} - \frac{1}{s^{2\alpha}+2\lambda s^{\alpha}-\eta}.$$

At this point, in order to evaluate $c_1$, taking the Laplace transform of the boundary condition (3.) yields

$$c_1 = \frac{1}{s^{\alpha}} + \frac{1}{s^{2\alpha}+2\lambda s^{\alpha}-\eta},$$

from which we deduce that

$$U(x,s) = G(s^{\alpha}) = \left(\frac{1}{s^{\alpha}} + \frac{1}{s^{2\alpha}+2\lambda s^{\alpha}-\eta}\right) e^{-x\sqrt{s^{2\alpha}+2\lambda s^{\alpha}-\eta}} - \frac{1}{s^{2\alpha}+2\lambda s^{\alpha}-\eta}.$$

From the above relation, we have

$$G(s) = \left(\frac{1}{s} + \frac{1}{s^{2}+2\lambda s-\eta}\right) e^{-x\sqrt{s^{2}+2\lambda s-\eta}} - \frac{1}{s^{2}+2\lambda s-\eta} = \mathcal{L}[g(t)]$$

If we find $\mathcal{L}^{-1}[G(s)] = g(t)$, then from Lemma 3 and Corollary 4 we obtain $u(x,t)$ and $g(t)$ as follows

$$u(x,t) = \mathcal{L}^{-1}G(s^{\alpha}) = \frac{1}{\pi}\int_0^{\infty} g(u)\int_0^{\infty} e^{-tr-ur^{\alpha}\cos\alpha\pi}\sin(ur^{\alpha}\sin\alpha\pi)\,dr\,du,$$

where

$$\begin{aligned} g(t) = {} & \int_0^t \left[e^{-a\tau}\delta(\tau - x) + be^{-a\tau}\int_0^{\tau}\delta\left(\sqrt{\tau^2-\xi^2} - x^2\right)I_1(b\xi)d\xi\right]d\tau \\ & + \int_0^t e^{-x(t-\tau)} I_0\left((t-\tau)\sqrt{\lambda^2+\eta}\right)e^{-a\tau}I_0\left(b\sqrt{\tau^2-x^2}\right)d\tau \\ & + e^{-\lambda t}\sinh\left(t\sqrt{\lambda^2+\eta}\right). \end{aligned}$$

Note. Analytic solutions are more important than numerical solutions, because these are valid in the whole domain of definition whereas the numerical solutions are only valid at chosen points in the domain of definition.

## 2. SOLUTION FOR THE NON-HOMOGENEOUS IMPULSIVE TIME FRACTIONAL HEAT CONDUCTION EQUATION VIA THE LAPLACE TRANSFORM

Fractional calculus deals with the fractional integrals and derivatives of arbitrary order. It provides better models for systems having long range memory and non-local effects and it has important applications in several fields of engineering and sciences. Fractional differential equations are widely used for modeling anomalous diffusion phenomena. In this section, the author implemented the Laplace transforms to construct the exact solution for the time fractional heat conduction equation. The diffusion phenomena such as conduction of heat in solids and diffusion of vorticity in the case of viscous fluid flow past a body are governed by parabolic type PDE. In the past three decades, considerable research work has been invested for the study of the anomalous diffusion using the time fractional equation.

In the sequel we consider a generalization to problem which is not considered in the literature.

PROBLEM 17. *Let us consider the following non-homogeneous impulsive time fractional heat conduction equation defined in cylindrical coordinates.*

$$D_t^{c,\alpha}u - \tfrac{1}{r}\left[\tfrac{\partial}{\partial r}\left(r\tfrac{\partial u}{\partial r}\right)\right] + \lambda u = k\delta(t-\tau), \qquad t>0,\ 0<r<1,$$

*with the boundary conditions as follows*

$$\begin{array}{ll} 1. & u(r,0)=1, \qquad u(1,t)=\phi(t); \\ 2. & \lim_{r\to 0}|u(r,t)| < +\infty. \end{array}$$

SOLUTION. Let us define the Laplace transform as below

$$\mathcal{L}[u(r,t);t\to s] = \int_0^{+\infty} e^{-st}u(r,t)dt = U(r,s).$$

Direct application of the Laplace transforms to partial differential equation and using boundary conditions leads to the following relation

$$s^\alpha U(r,s) - \tfrac{\partial^2 U}{\partial r^2} - \tfrac{1}{r}\tfrac{\partial U}{\partial r} + \lambda U(r,s) = k\exp(-\tau s) + s^{\alpha-1},$$

or

$$\tfrac{\partial^2 U}{\partial r^2} + \tfrac{1}{r}\tfrac{\partial U}{\partial r} - (s^\alpha+\lambda)U = k\exp(-\tau s) + s^{\alpha-1},$$

solving the above non-homogeneous modified Bessel differential equation yields

$$U(r,s) = c_1 I_0(r\sqrt{s^\alpha+\lambda}) + c_2 K_0(r\sqrt{s^\alpha+\lambda}) - \tfrac{k\exp(-\tau s)+s^{\alpha-1}}{(s^\alpha+\lambda)}.$$

In view of the second boundary condition, we should have $c_2(s)=0$, therefore, we arrive at

$$U(r,s) = c_1(s) I_0(r\sqrt{s^\alpha+\lambda}) - \tfrac{k\exp(-\tau s)+s^{\alpha-1}}{(s^\alpha+\lambda)}.$$

At this stage we need to find $c_1(s)$, by using the first boundary condition, we have

$$U(1,s) = \Phi(s) = c_1(s) I_0(\sqrt{s^\alpha + \lambda}) - \frac{k\exp(-\tau s)+s^{\alpha-1}}{(s^\alpha+\lambda)}.$$

Finally, we get the solution to the transformed equation as follows

$$U(r,s) = \frac{\Phi(s) I_0(r\sqrt{s^\alpha+\lambda})}{I_0(\sqrt{s^\alpha+\lambda})} + \frac{k\exp(-\tau s)+s^{\alpha-1}}{(s^\alpha+\lambda) I_0(\sqrt{s^\alpha+\lambda})} I_0(r\sqrt{s^\alpha + \lambda}),$$

and thus by taking the inverse Laplace transform, we obtain

$$u(r,t) = \mathcal{L}^{-1}\left[\Phi(s)\frac{I_0(r\sqrt{s^\alpha+\lambda})}{I_0(\sqrt{s^\alpha+\lambda})}\right] + \mathcal{L}^{-1}\left[\frac{k\exp(-\tau s)+s^{\alpha-1}}{(s^\alpha+\lambda) I_0(\sqrt{s^\alpha+\lambda})} I_0(r\sqrt{s^\alpha + \lambda})\right].$$

1. The first term can be evaluated by convolution, thus we have

$$\mathcal{L}^{-1}\left[\Phi(s)\frac{I_0(r\sqrt{s^\alpha+\lambda})}{I_0(\sqrt{s^\alpha+\lambda})}\right] = \int_0^t \phi(t-\xi)\left[\mathcal{L}^{-1}\left[\frac{I_0(r\sqrt{s^\alpha+\lambda})}{I_0(\sqrt{s^\alpha+\lambda})}\right]; s \to \xi\right] d\xi,$$

but

$$\mathcal{L}^{-1}\left[\frac{I_0(r\sqrt{s^\alpha+\lambda})}{I_0(\sqrt{s^\alpha+\lambda})}\right] = \left[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{I_0(r\sqrt{s^\alpha+\lambda})}{I_0(\sqrt{s^\alpha+\lambda})}\right] e^{\xi s} ds.$$

In order to evaluate complex integral, let us set

$$\frac{I_0(r\sqrt{s^\alpha+\lambda})}{I_0(\sqrt{s^\alpha+\lambda})} = G(s^\alpha).$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{I_0(r\sqrt{s^\alpha+\lambda})}{I_0(\sqrt{s^\alpha+\lambda})} e^{\xi s} ds = g^*(\xi).$$

From which we deduce that

$$\frac{I_0(r\sqrt{s+\lambda})}{I_0(\sqrt{s+\lambda})} = G(s),$$

and

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{I_0(r\sqrt{s+\lambda})}{I_0(\sqrt{s+\lambda})} e^{\xi s} ds = g(\xi).$$

A linear change of variable $s + \lambda = \psi$ in the above complex integral leads to

$$\frac{e^{-\lambda\xi}}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} \frac{I_0(r\sqrt{\psi})}{I_0(\sqrt{\psi})} e^{\xi\psi} d\psi = g(\xi).$$

By method of residues we may evaluate the above complex integral. The function under integral sign has branch points at $\psi = -\phi_n^2,\ n = 1, 2, 3, \ldots$, where $I_0(i\phi_n) = J_0(\phi_n) = 0$. Therefore,

$$g(\xi) = \sum_{n=1}^{+\infty} \lim_{\psi \to -\phi_n^2} (\psi + \phi_n^2) \frac{I_0(r\sqrt{\psi})}{I_0(\sqrt{\psi})} e^{\xi\psi}.$$

After simplifying we obtain

$$g(\xi) = -e^{-\lambda\xi} \sum_{n=1}^{+\infty} \frac{2 J_0(r\phi_n) e^{-\phi_n^2 \xi}}{\phi_n J_1(\phi_n)}.$$

In view of part 3 of Lemma 3, we have

$$\mathcal{L}^{-1}(G(s^\alpha)) = g^*(t) = \frac{1}{\pi}\int_0^\infty g(\xi)\left[\int_0^\infty e^{-tr-\xi r^\alpha \cos\alpha\pi}\sin(\xi r^\alpha \sin\alpha\pi)dr\right]d\xi.$$

2. In order to evaluate the second term, we have the following

$$\mathcal{L}^{-1}\left[\frac{k\exp(-\tau s)+s^{\alpha-1}}{(s^\alpha+\lambda)I_0(\sqrt{s^\alpha+\lambda})}I_0(r\sqrt{s^\alpha+\lambda})\right] = \mathcal{L}^{-1}\left[\frac{k\exp(-\tau s)+s^{\alpha-1}}{(s^\alpha+\lambda)}\right]*\mathcal{L}^{-1}\left[\frac{I_0(r\sqrt{s^\alpha+\lambda})}{I_0(\sqrt{s^\alpha+\lambda})}\right].$$

But we have the following relations

$$\gamma(t) = \mathcal{L}^{-1}\left[\frac{k\exp(-\tau s)+s^{\alpha-1}}{(s^\alpha+\lambda)}\right] = \frac{1}{\pi}\int_0^{+\infty} e^{-t\xi}\,\mathrm{Im}\,\frac{e^{-\tau(\xi e^{-i\pi})}+(\xi e^{-i\pi})^{\alpha-1}}{(\xi e^{-i\pi})^\alpha+\lambda}d\xi,$$

after simplifying we have

$$\gamma(t) = \frac{1}{\pi}\int_0^{+\infty} e^{-t\xi}\left[\frac{\xi\sin\alpha\pi e^{\tau\xi}+\xi^{\alpha-1}[\xi^\alpha\cos\pi(2\alpha-1)-\lambda\sin\pi(\alpha-1)]}{\xi^{2\alpha}+2\lambda\xi^\alpha\cos\pi\alpha+\lambda^2}\right]d\xi,$$

and finally

$$u(r,t) = \int_0^t[\gamma(t-\eta)+\phi(t-\eta)]g^*(\eta)d\eta.$$

## 3. CONCLUSION

The paper is devoted to studying and application of the Laplace transform for solving non-homogeneous impulsive time fractional heat conduction equation. Methods in which techniques are used in applications are illustrated, and problems are included. The main purpose of this work is to develop a method for evaluation of certain integrals and time fractional PDEs. These results should be applicable to obtaining solutions of a wide class of problems in applied mathematics and mathematical physics.

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