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QUENCHING FOR DISCRETIZATIONS OF A SEMILINEAR PARABOLIC EQUATION WITH NONLINEAR BOUNDARY OUTFLUX

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Abstract. In this paper, we study numerical approximations of a semilinear parabolic problem in one-dimension, of which the nonlinearity appears both in source term and in Neumann boundary condition. By a semidiscretization using finite difference method, we obtain a system of ordinary differential equations which is an approximation of the original problem. We obtain conditions under which the positive solution of our system quenches in a finite time and estimate its semidiscrete quenching time. Convergence of the numerical quenching time to the theoretical one is established. Next, we show that the quenching rate of the numerical scheme is different from the continuous one. Finally, we give numerical results to illustrate our analysis.

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1. INTRODUCTION

Consider the following semilinear parabolic equation with nonlinear boundary outflux

$$(1)\quad \begin{cases} u_t(x,t) = u_{xx}(x,t) + f(x)(1-u(x,t))^{-p},\ 0<x<1,\ 0<t<\infty, \\ u_x(0,t) = u^{-q}(0,t), \quad 0<t<\infty, \\ u_x(1,t) = 0, \quad 0<t<\infty, \\ u(x,0) = u_0(x), \quad 0\le x\le 1. \end{cases}$$

Here p, q are positive constants, f is a non-negative function and initial datum $u_0 : [0,1] \to (0,1)$ is smooth enough and satisfies boundary conditions. We can regard this problem as a heat conduction model that incorporates the effects of nonlinear reaction (source) and nonlinear boundary outflux (emission).

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Our problem derives from the general the problem (1.1) of [18], where the authors proved the existence and uniqueness of the classical solution.

We say that the classical solution $u(x,t)$ of (1) quenches in a finite time if there exists a finite time T_q such that:

$$\lim_{t\to T_q^-} \min_{0\leq x\leq 1} u(x,t) = 0, \quad \textit{or} \quad \lim_{t\to T_q^-} \max_{0\leq x\leq 1} u(x,t) = 1.$$

From now, we denote the quenching time with T_q.

The quenching problem has been the subject of investigations of many authors since several decades (see [7, 13, 19, 20, 22, 23, 6, 17] and the references cited therein). It was in 1975 that Kawarada [13] introduced the first concept of quenching, he found sufficient conditions under which the solutions of the following problem quenches in finite time,

$$\begin{aligned} &u_t = u_{xx} + \tfrac{1}{1-u}, \quad t > 0, \quad x \in (0,l), \\ &u(t,0) = u(t,l) = 0, \quad t > 0, \\ &u(0,x) = 0, \quad x \in (0,l), \end{aligned}$$

where l is positive.

In our problem we have two singular heat sources, namely, a source term $f(x)(1-u)^{-p}$ and the boundary outflux term u^{-q}. Some authors studied such quenching problems with the nonlinearity both in source and in boundary conditions [7, 20, 22, 23, 17]. In [20], Selcuk and Ozalp studied the quenching behavior of the solution of a semilinear heat equation with a singular boundary outflux. They showed that quenching occurs on the boundary under certain conditions and that the time derivative blows up at a quenching point.

Concerning problem (1), Zhi [22] showed that if the initial datum satisfies

$$u_0''(x) + f(x)(1-u_0(x))^{-p} \leq 0 \text{ and not equals } 0 \text{ identically}, \; x \in (0,1),$$

then the classical solution u of (1) quenches in a finite time T_q with the following estimate $u(0,t) \sim (T_q - t)^{\frac{1}{2(q+1)}}$. This condition excludes the formation of a singularity for the source term, thus it is only sufficient to consider the case of quenching formation on the boundary for the solution of problem (1). He also asserted that the quenching can only occur on the point $x = 0$ if the given initial datum is monotone. A related problem was studied earlier in [23] where the author prescribed $f(x) = 1$, similar results have been also obtained.

Here we are interested in the numerical study of the phenomenon of quenching (For other works on numerical approximations of quenching solutions we refer to [15, 5, 14, 17]). We give assumptions under which the solution of a semidiscrete form of (1) quenches in a finite time and estimate its semidiscrete quenching time. We also prove that, under suitable assumptions on the initial datum, the semidiscrete quenching time converges to the theoretical one when the mesh size goes to zero. Our work was motived by the papers in [9, 1, 16, 2, 10, 4], where the authors have used semidiscrete forms for some parabolic equations to study the phenomenon of blow-up (we say that a solution blows up in a finite time if it reaches the value infinity in a finite time).

The paper is written in the following manner. In the next Section, we present a semidiscrete scheme of (1). In Section 3, we give properties concerning our semidiscrete scheme. In Section 4, under appropriate conditions, we prove that the solution of the semidiscrete form quenches in a finite time, estimate its semidiscrete quenching time and give results on the numerical quenching rate. In Section 5, we study the convergence of semidiscrete quenching time. Finally, in the last section, we give numerical experiments.

2. THE SEMIDISCRETE PROBLEM

Let I be a positive integer, we set $h = \frac{1}{I}$, and we define the grid $x_i = ih$, $i = 0, \dots, I$. We approximate the solution u of the problem (1) by the solution $U_h = (U_0, U_1, \dots, U_I)^T$ and the initial datum u_0 by $\varphi_h = (\varphi_0, \varphi_1, \dots, \varphi_I)^T$ of the semidiscrete equations

$$\frac{dU_i(t)}{dt} = \delta^2 U_i(t) + f_i(1 - U_i(t))^{-p}, \quad i = 1, \dots, I-1,\ t \in [0, T_q^h), \tag{2}$$

$$\frac{dU_0(t)}{dt} = \delta^2 U_0(t) - \frac{2}{h} U_0^{-q}(t) + f_0(1 - U_0(t))^{-p}, \quad [0, T_q^h), \tag{3}$$

$$\frac{dU_I(t)}{dt} = \delta^2 U_I(t) + f_I(1 - U_I(t))^{-p}, \quad [0, T_q^h), \tag{4}$$

$$U_i(0) = \varphi_i > 0, \quad i = 0, \dots, I, \tag{5}$$

where

$$f_i \simeq f(x_i), \quad i = 0, \dots, I,$$
$$\delta^2 U_i(t) = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2}, \quad i = 1, \dots, I-1,$$
$$\delta^2 U_0(t) = \frac{2U_1(t) - 2U_0(t)}{h^2}, \quad \delta^2 U_I(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2}.$$

Here $[0, T_q^h)$ is the maximal time interval on which $\|U_h(t)\|_{\inf} > 0$, where $\|U_h(t)\|_{\inf} = \min_{0 \le i \le I} |U_i(t)|$. The time T_q^h can be finite or infinite. When T_q^h is finite, we say that U_h quenches in finite time and T_q^h is called the quenching time of the solution U_h. Whereas when T_q^h is infinite we say that U_h quenches globally.

In the following, we give important results which will be used later. Here, V_i and W_i denote the respective $(i+1)$-th components of V_h and W_h.

Definition 1. *A function $V_h \in \mathcal{C}^1([0, T], \mathbb{R}^{I+1})$ is an upper solution of (2)–(5) if*

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) \ge f_i(1 - V_i(t))^{-p}, \quad i = 1, \dots, I,\ t \in (0, T],$$
$$\frac{dV_0(t)}{dt} - \delta^2 V_0(t) + \frac{2}{h} V_0^{-q}(t) \ge f_0(1 - V_0(t))^{-p},\ t \in (0, T],$$
$$V_i(0) \ge \varphi_i, \quad i = 0, \dots, I.$$

On the other hand, we say that $V_h \in \mathcal{C}^1([0,T],\mathbb{R}^{I+1})$ *is a lower solution of* (2)–(5) *if these inequalities are reversed.*

The following results are semidiscrete forms of a Maximum Principle and will be an important tool in the study of the semidiscrete problem (2)–(5). Their proofs are standard and will be omitted.

Lemma 2. *Let* $a_h \in C^0([0,T],\mathbb{R}^{I+1})$ *and* $V_h \in C^1([0,T],\mathbb{R}^{I+1})$ *such that*

$$\frac{d}{dt}V_i - \delta^2 V_i + a_i V_i \geq 0, \quad i = 0,\dots,I,\ t \in [0,T], \tag{6}$$

$$V_i(0) \geq 0, \quad i = 0,\dots,I, \tag{7}$$

then we have $V_i(t) \geq 0, \quad i = 0,\dots,I,\ t \in [0,T].$

Another form of the Maximum Principle for semidiscrete equations are the following comparison lemma.

Lemma 3. *Let* $g \in C(\mathbb{R},\mathbb{R})$ *and* $W_h, V_h \in C^1([0,T],\mathbb{R}^{I+1})$ *such that*

$$\frac{d}{dt}V_i - \delta^2 V_i - g(V_i) \leq \frac{d}{dt}W_i - \delta^2 W_i - g(W_i),\ i = 1,\dots,I,\ t \in (0,T], \tag{8}$$

$$\frac{d}{dt}V_0 + \frac{2}{h}V_0^{-q} - \delta^2 V_0 - g(V_0) \leq \frac{d}{dt}V_0 + \frac{2}{h}W_0^{-q} - \delta^2 W_0 - g(W_0), t \in (0,T], \tag{9}$$

$$V_i(0) \leq W_i(0),\ i = 0,\dots,I, \tag{10}$$

then $V_i(t) \leq W_i(t), \quad i = 0,\dots,I,\ t \in [0,T].$

3. QUENCHING IN THE SEMIDISCRETE PROBLEM

In this section, under appropriate assumptions, we show that solution U_h of the semidiscrete problem (2)–(5) quenches in a finite time T_q^h and we estimate its semidiscrete quenching time.

The following result gives a property of the operator δ^2.

Lemma 4. *Let* $U_h \in \mathbb{R}^{I+1}$ *be such that* $U_h > 0$. *Then, we have*

$$\delta^2(U_i^{-q}) \geq -qU_i^{-q-1}\delta^2 U_i,\ i = 0,\dots,I.$$

Proof. Let us introduce function $f(s) = s^{-q}$. Using Taylor's expansion we get

$$\delta^2 f(U_0) = f'(U_0)\delta^2 U_0 + \frac{(U_1-U_0)^2}{h^2} f''(\zeta_0),$$

$$\delta^2 f(U_I) = f'(U_I)\delta^2 U_I + \frac{(U_{I-1}-U_I)^2}{h^2} f''(\zeta_I),$$

$$\delta^2 f(U_i) = f'(U_i)\delta^2 U_i + \frac{(U_{i+1}-U_i)^2}{2h^2} f''(\eta_i) + \frac{(U_{i-1}-U_i)^2}{2h^2} f''(\zeta_i),\ i = 1,\dots,I-1.$$

where η_i is an intermediate value between U_i and U_{i+1} and ζ_i the one between U_i and U_{i-1}.

The result follows taking into account the fact that U_h is nonnegative. □

In the rest of this paper, we assume that these conditions are satisfied

$$0 \leq f_i \leq f_{i+1}, \quad i = 0,\dots,I-1, \tag{11}$$

$$0 < \varphi_i < \varphi_{i+1}, \quad i = 0,\dots,I-1. \tag{12}$$

Lemma 5. *Let U_h be the solution of (2)–(5), then, we have*

(1) $U_i(t) \geq \varphi_i$ *for* $i = 0, \dots, I,\ t \in [0, T]$;
(2) $U_i(t) < U_{i+1}(t)$ *for* $i = 0, \dots, I-1,\ t \in [0, T]$.

Proof.

(1) Using Lemma 3, we obtain $U_i(t) \geq \varphi_i > 0, \quad i = 0, \dots, I,\ t \in [0, T]$.
(2) For $i = 0, \dots, I-1$, $t \in [0, T]$, introduce Z_i such that $Z_i(t) = U_{i+1}(t) - U_i(t)$. Let t_0 be the first $t > 0$ such that $Z_i(t) > 0$ for $t \in [0, t_0)$, but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \dots, I-1\}$. Without loss of generality, we suppose that i_0 is the smallest integer checking the equality above. We observe that

$$\tfrac{d}{dt} Z_{i_0}(t_0) = \lim_{k \to 0} \tfrac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0, \quad i_0 = 0, \dots, I-1,$$

$$\delta^2 Z_{i_0}(t_0) = \tfrac{Z_{i_0-1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0+1}(t_0)}{h^2} > 0, \quad i_0 = 1, \dots, I-2,$$

$$\delta^2 Z_0(t_0) = \tfrac{Z_1(t_0) - 3Z_0(t_0)}{h^2} > 0,$$

$$\delta^2 Z_{I-1}(t_0) = \tfrac{Z_{I-2}(t_0) - 3Z_{I-1}(t_0)}{h^2} > 0.$$

By a computation, we get

$$\tfrac{d}{dt} Z_0(t_0) - \delta^2 Z_0(t_0) - f_1(1 - U_1(t_0))^{-p} + f_0(1 - U_0(t_0))^{-p} - \tfrac{2}{h} U_0^{-q}(t_0) < 0,$$

$$\tfrac{d}{dt} Z_{i_0}(t_0) - \delta^2 Z_{i_0}(t_0) - f_{i_0+1}(1 - U_{i_0+1}(t_0))^{-p} + f_{i_0}(1 - U_{i_0}(t_0))^{-p} < 0,\ i_0 = 1, \dots, I-1.$$

But these inequalities contradict (2)–(4) and the proof is complete. □

Theorem 6. *Let U_h be the solution of (2)–(5). Assume that the initial data at (5) verifies*

$$\delta^2 \varphi_i + f_i(1 - \varphi_i)^{-p} \leq -\epsilon \varphi_i^{-q}, \quad i = 1, \dots, I, \tag{13}$$

$$\delta^2 \varphi_0 + f_0(1 - \varphi_0)^{-p} - \frac{2}{h} \varphi_0^{-q} \leq -\epsilon \varphi_0^{-q}, \tag{14}$$

for a certain constant $\epsilon \in (0, 1]$.

Then, the solution U_h quenches in a finite time T_q^h and we have the following estimate

$$T_q^h \leq \frac{\|\varphi_h\|_{\inf}^{q+1}}{\epsilon(q+1)}.$$

Proof. Since $(0, T_q^h)$ is the maximal time interval which on which $\|U_h\|_{\inf} > 0$. We want to show that T_q^h is finite and satisfies the above inequality. Introduce the vector $J_h(t)$ defined as follows

$$J_i(t) = \tfrac{dU_i(t)}{dt} + \epsilon U_i^{-q}(t), \quad i = 0, \dots, I, \tag{15}$$

by a straightforward computation, we get

$$\frac{d}{dt}J_i - \delta^2 J_i = \frac{d}{dt}(\frac{dU_i}{dt} - \delta^2 U_i) - \epsilon(qU_i^{-q-1}\frac{dU_i}{dt} + \delta^2 U_i^{-q}), \quad i = 0, \dots, I.$$

Using Lemma 4 and equalities (2)–(4) and (15), the above equalities give

$$\frac{d}{dt}J_i - \delta^2 J_i \le pf_i(1-U_i)^{-p-1}J_i, \quad i = 1, \dots, I, \tag{16}$$

$$\frac{d}{dt}J_0 - \delta^2 J_0 \le (\frac{2q}{h}U_0^{-q-1} + pf_0(1-U_0)^{-p-1})J_0. \tag{17}$$

We observe from (13)–(14) that

$$J_i(0) = \delta^2\varphi_i + f_i(1-\varphi_i)^{-p} + \epsilon\varphi_i^{-q} \le 0, \quad i = 1, \dots, I,$$
$$J_0(0) = \delta^2\varphi_0 + f_0(1-\varphi_0)^{-p} + (\epsilon - \frac{2}{h})\varphi_0^{-q} \le 0.$$

We deduce from Lemma 2 that $J_h(t) \le 0$, for $t \in (0, T_q^h)$, which implies that

$$\frac{dU_i}{dt} + \epsilon U_i^{-q} \le 0 \quad \text{for} \quad t \in (0, T_q^h),\ i = 0, \dots, I. \tag{18}$$

By the above inequality we obtain the following form $U_i^q dU_i \le -\epsilon dt$ for $t \in (0, T_q^h)$ and $i = 0, \dots, I$. Integrating this inequality over $[0, T_q^h)$, we obtain

$$T_q^h - t \le \frac{U_i^{q+1}(t)}{\epsilon(q+1)}. \tag{19}$$

From Lemma 5, we have $U_0(0) = \|\varphi_h\|_{\inf}$ and taking $t = 0$ in (19) we get the desired result. □

The following result concerns the lower bound for the quenching rate.

Remark 7. *Using the inequality (19) we obtain* $T_q^h - t_0 \le \frac{U_0^{q+1}(t_0)}{\epsilon(q+1)}$ *for* $t_0 \in (0, T_q^h)$, *which implies that* $U_0(t) \ge (\epsilon(q+1))^{\frac{1}{q+1}}(T_q^h - t)^{\frac{1}{q+1}}$ *for* $t \in (0, T_q^h)$.

Theorem 8. *Assume that (13)–(14) remains true. Then, near the quenching time* T_q^h, *the solution* U_h *to problem (2)–(5) has following quenching rate estimate*

$U_0(t) \sim (T_q^h - t)^{\frac{1}{(q+1)}}$, *in the sense that there exist two positive constants* C_1, C_2 *such that*

$$C_1(T_q^h - t)^{\frac{1}{q+1}} \le U_0(t) \le C_2(T_q^h - t)^{\frac{1}{q+1}}, \quad \textit{for} \quad t \in (0, T_q^h).$$

Proof. Remark 7 ensures the term of left hand side.

Let i_0 be such that $U_{i_0}(t) = \min_{0\le i\le I} U_i(t)$, $t \in (0, T_q^h)$. From Lemma 5 we obtain

$$\delta^2 U_{i_0}(t) = \frac{U_{i_0+1}(t) - 2U_{i_0}(t) + U_{i_0-1}(t)}{h^2} \ge 0, \quad i_0 = 1, \dots, I-1,$$
$$\delta^2 U_0(t) = \frac{2U_1(t) - 2U_0(t)}{h^2} \ge 0,$$
$$\delta^2 U_I(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2} \ge 0.$$

Which leads to $\frac{dU_0}{dt} \geq -\frac{2}{h}U_0^{-q}$, integrating over (t, T_q^h) we have

$$U_0 \leq (\frac{2(q+1)}{h})^{\frac{1}{q+1}}(T_q^h - t)^{\frac{1}{q+1}}.$$

Hence, we have $U_0 \leq C_2(T_q^h - t)^{\frac{1}{q+1}}$ and the proof is completed. □

REMARK 9. *Let us point out that the quenching rate for the numerical scheme,*
$(T_q^h - t)^{\frac{1}{q+1}}$*, is different from the continuous one,* $(T_q - t)^{\frac{1}{2(q+1)}}$ [22].

4. CONVERGENCE OF THE SEMIDISCRETE QUENCHING TIME

In this section, with suitable assumptions, we establish the convergence of the quenching time of the approximate semidiscrete solution to the quenching time of the theoretical solution.

The next theorem establishes that, for each fixed time interval $[0, T]$, $(T < T_q)$ where u is defined, the solution of the semidiscrete problem (2)-(5) approximates u, as $h \to 0$.

THEOREM 10. *Assume that the problem* (1) *has a solution* $u \in C^{4,1}([0,1] \times [0,T])$ *such that* $\inf_{t\in[0,T]} \|u(.,t)\|_\infty = \alpha > 0$ *and the initial condition* φ_h *at* (5) *verifies*

$$\|\varphi_h - u_h(0)\|_\infty = \circ(1), \quad as \quad h \to 0, \tag{20}$$

where $u_h(t) = (u(x_0,t), \ldots, u(x_I,t))^T$, $t \in [0,T]$. *Then, for* h *small enough, the semidiscrete problem* (2)–(5) *has a unique solution* $U_h \in C^1([0,T], \mathbb{R}^{I+1})$ *such that*

$$\max_{t\in[0,T]}(\|U_h(t) - u_h(t)\|_\infty) = O(\|\varphi_h - u_h(0)\|_\infty + h^2) \quad as \quad h \to 0.$$

Proof. The semidiscrete problem (2)–(5) has for each h, a unique solution $U_h \in C^1([0, T_q^h), \mathbb{R}^{I+1})$. Let $t(h)$ be the greatest value of $t > 0$ such that

$$\|U_h(t) - u_h(t)\|_\infty < \tfrac{\alpha}{2} \quad for \quad t \in (0, t(h)). \tag{21}$$

The relation (20) implies $t(h) > 0$ for h small enough.

Let $t^\star(h) = \min\{t(h), T\}$, using the triangle inequality we obtain
$\|U_h(t)\|_{\inf} \geq \|u(.,t)\|_{\inf} - \|U_h(t) - u_h(t)\|_\infty$ for $t \in (0, t^\star(h))$, which implies that

$$\|U_h(t)\|_{\inf} \geq \tfrac{\alpha}{2} \quad for \quad t \in (0, t^\star(h)). \tag{22}$$

Let $e_h(t) = U_h(t) - u_h(t)$ be the error discretization and the vector $z_h(t)$ defined by $z_i(t) = e^{(K+1)(t-x^2+2x)}(\|\varphi_h - u_h(0)\|_\infty + Lh^2), \quad 0 \leq i \leq I, : \ t \in (0, t^\star(h))$. Using the Lemma 2 we can prove that $z_i(t) > |e_i(t)|$ for $t \in (0, t^\star(h))$, $i = 0, \ldots, I$, which implies that

$$\|U_h(t) - u_h(t)\|_\infty \leq e^{(K+1)(t-x^2+2x)}(\|\varphi_h - u_h(0)\|_\infty + Lh^2), \quad t \in (0, t^\star(h)).$$

Suppose that $T > t(h)$, from (21) we have $\frac{\alpha}{2} = \|U_h(t(h)) - u_h(t(h))\|_\infty \leq e^{(K+1)(T-x^2+2x)}(\|\varphi_h - u_h(0)\|_\infty + Lh^2)$. Since the term on the right hand side of the above inequality goes to zero as h goes to zero, we deduce that $\frac{\alpha}{2} \leq 0$, which is impossible. □

THEOREM 11. *Let T_q be the quenching time of the solution u of (1) such that*
$u \in C^{4,1}([0,1] \times [0,T_q))$. We assume that the initial data at (5) satisfies
$\|\varphi_h - u_h(0)\|_\infty = o(1)$, as $h \to 0$. Then under the hypothesis of Theorem 6, the solution U_h of the problem (2)–(5) quenches in finite time T_q^h and we have

$$\lim_{h\to 0} T_q^h = T_q.$$

Proof. Let $\eta > 0$, there exists a positive constant α such that

$$\frac{1}{\epsilon}\frac{s^{q+1}}{q+1} \leq \frac{\eta}{2} \quad for \quad s \in [0,\alpha]. \tag{23}$$

Since u quenches in a finite time T_q, there exists $T_0 \in (T_q - \frac{\eta}{2}, T_q)$ such that $0 < \|u(.,t)\|_{\inf} \leq \frac{\alpha}{2}$ for $t \in [T_0, T_q)$. From Theorem 10, the problem (2)-(5) has a solution $U_h(t)$ such that $\|U_h(t) - u_h(t)\|_\infty \leq \frac{\alpha}{2}$ for $t \in [0, T_1]$, where $T_1 = \frac{T_0+T_q}{2}$, which gives $\|U_h(T_1) - u_h(T_1)\|_\infty \leq \frac{\alpha}{2}$. Using the triangle inequality, we get $\|U_h(T_1)\|_{\inf} \leq \|U_h(T_1) - u_h(T_1)\|_\infty + \|u_h(T_1)\|_{\inf} \leq \alpha$. From Theorem 6, U_h quenches in a finite time T_q^h. We deduce from Remark 7 and (23) that

$$|T_q^h - T_q| \leq |T_q^h - T_1| + |T_1 - T_q| \leq \frac{1}{\epsilon}\frac{\|U_h(T_1)\|_{\inf}^{q+1}}{q+1} + \frac{\eta}{2} \leq \eta.$$

□

5. NUMERICAL EXPERIMENTS

In the section, we present numerical approximations to the quenching time of problem (1) in the case where $u_0(x) = -x^\varepsilon + \varepsilon x + \varepsilon^{-1/q}$ with $0 < p \leq 1$, $0 < q \leq \frac{1}{2}$, $\varepsilon = \frac{101}{100}$ and $f(x) = \frac{1}{1000}$. We also discuss the quenching sets. To do this, we transform the semidiscrete scheme (2)–(5) into the following semidiscrete equations. Set us $V_i(t) = \frac{1}{U_i(t)}$, we obtain

(24)
$$\frac{d}{dt}V_i(t) = \frac{1}{h^2}\left(2V_i(t) - \frac{V_i^2(t)}{V_{i-1}(t)} - \frac{V_i^2(t)}{V_{i+1}(t)}\right) - f_i V_i^{p+2}(t)(V_i(t)-1)^{-p}, i = 0,\dots,I-1,$$

$$\frac{d}{dt}V_0(t) = \frac{2}{h^2}\left(V_0(t) - \frac{V_0^2(t)}{V_1(t)}\right) + \frac{2}{h}V_0^{2+q}(t) - f_0 V_0^{2+p}(t)(V_0(t)-1)^{-p}, \tag{25}$$

$$\frac{d}{dt}V_I(t) = \frac{2}{h^2}\left(V_I(t) - \frac{V_I^2(t)}{V_{I-1}(t)}\right) - f_I V_I^{p+2}(t)(V_I(t)-1)^{-p}, \tag{26}$$

$$V_i(0) = (\varphi_i)^{-1}, \quad i = 0,\dots,I. \tag{27}$$

Using the method presented by Hirota and Ozawa [12], we transform the semidiscrete scheme (24)–(27) into a tractable form by the arc length transformation technique (see [21], [S. Moriguti, C. Okuno, R. Suekane, M. Iri, K.

Takeuchi, Ikiteiru Suugaku - Suuri Kougaku no Hatten (in Japanese), Baifukan, Tokyo, 1979.]) like this:

$$\left\{\begin{array}{l} \dfrac{d}{d\eta}\begin{pmatrix} t \\ V_0 \\ \vdots \\ V_I \end{pmatrix} = \dfrac{1}{\sqrt{1+\sum_{i=0}^{I} f_i^2}}\begin{pmatrix} 1 \\ f_0 \\ \vdots \\ f_I \end{pmatrix}, \quad 0<\eta<\infty, \\ t(0)=0, \quad V_i(0)=(\varphi_i)^{-1}, \quad i=0,\dots,I, \end{array}\right. \tag{28}$$

where

$$f_0 = \tfrac{2}{h^2}\Big(V_0(t) - \tfrac{V_0^2(t)}{V_1(t)}\Big) + \tfrac{2}{h} V_0^{2+q}(t) - f_0 V_0^{2+p}(t)(V_0(t)-1)^{-p},$$
$$f_i = \tfrac{1}{h^2}\Big(2V_i(t) - \tfrac{V_i^2(t)}{V_{i-1}(t)} - \tfrac{V_i^2(t)}{V_{i+1}(t)}\Big) - f_i V_i^{p+2}(t)(V_i(t)-1)^{-p}, \quad i=1,\dots,I-1,$$
$$f_I = \tfrac{2}{h^2}\Big(V_I(t) - \tfrac{V_I^2(t)}{V_{I-1}(t)}\Big) - f_I V_I^{p+2}(t)(V_I(t)-1)^{-p},$$
$$V_i(0) = (\varphi_i)^{-1}, \quad i=0,\dots,I.$$

"η" is the arc length and we have $d\eta^2 = dt^2 + \sum_{i=0}^{I} dV_i^2$. Note that in the transformation below the variables t and V_i are functions of η such that $\lim_{\eta\to\infty} t(\eta) = T^h$ *and* $\lim_{\eta\to\infty} \|V_h(\eta)\|_\infty = \infty$. Now we introduce $\{\eta_j\}$ which is the sequence of the arc lengths and we apply an ODE solver to (28) for each value of η_j in order to generate a linearly convergent sequence to the blow-up time. The resulting sequence is accelerated by the Aitken Δ^2 method [3]. We use the DOP54 [11] as the adaptive code for the integration of the ODEs. It has been written by Hairer *et al.* [11] based on explicit Runge-Kutta method of order (4)5 due to Dormand and Prince [8]. Let us define the sequence η_j by $\eta_j = 2^{12}.2^j$ $(j=0,\dots,10)$, and the parameters in the DOP54 are $InitialStep = 0$ and $AbsTol = RelTol = 1.e-15$. The parameters $AbsTol$ and $RelTol$ specify the tolerances of the absolute and relative errors, respectively, and $InitialStep$ is used to choose the manner in which the errors are controlled. In the following tables, in rows, we present the numerical quenching times T^h of problem (2)-(5), the Step and the orders of the approximations corresponding to meshes of 16, 32, 64, 128, 256 and 512. The order (s) of the method is computed from

$$\frac{\log((T^{4h}-T^{2h})/(T^{2h}-T^h))}{\log(2)}.$$

Remark 12. *From the tables, we can see the convergence of T^h to the quenching time of the solution of* (1)*, since the rate of convergence is near* 2*, which is just the accuracy of the difference approximation in space. The tables*

I	T^h	*Steps*	s
16	0.320236408	6304	-
32	0.318644976	12122	-
64	0.318150680	23261	1.69
128	0.318001392	44785	1.73
256	0.317957379	87402	1.76
512	0.317944663	178187	1.79

Table 1. Semidiscrete solution for $p = 1/4$, $q = 1/4$.

I	T^h	*Steps*	s
16	0.200847014	4490	-
32	0.198905750	8498	-
64	0.198304689	16154	1.69
128	0.198124110	30905	1.73
256	0.198071185	59982	1.77
512	0.198055981	120646	1.80

Table 2. Semidiscrete solution for $p = 1/4$, $q = 1/2$.

I	T^h	*Steps*	s
16	0.320475664	6306	-
32	0.318883760	12127	-
64	0.318389339	23270	1.69
128	0.318240017	44802	1.73
256	0.318195997	87437	1.76
512	0.318183278	178267	1.79

Table 3. Semidiscrete solution for $p = 1/2$, $q = 1/4$.

I	T^h	*Steps*	s
16	0.201026888	4492	-
32	0.199084652	8502	-
64	0.198483334	16161	1.69
128	0.198302688	30920	1.74
256	0.198249747	60011	1.77
512	0.198234539	120710	1.80

Table 4. Semidiscrete solution for $p = 1/2$, $q = 1/2$.

of our numerical results show that there is a relationship between the quenching time and the flow on the boundary on the one hand and the absorption in the interior of the domain on the other hand. Indeed, when the absorption in the interior of the domain is constant $(p = 1/4)$ *and that the flow on the boundary increases from* $1/4$ *to* $1/2$*, the quenching time decreases from* 0.318 *to* 0.198 *whereas when the flow on the boundary is constant* $(q = 1/4)$ *and that the absorption in the interior of the domain increases from* $1/4$ *to* $1/2$*, the quenching time remains substantially the same at* 0.318*. The absorption in the interior of the domain has in fact no essential effect upon the quenching behavior of problem* (1)*, whereas the flow on the boundary leads to the quenching, which is in agreement with the theoretical results* [23, 22]*.*

For other illustrations, in what follows, we present several graphs to illustrate our analysis. In Figures 1–4, we have used the case where $I = 64$ and $p = 1/4$. We can appreciate in Fig. 1 and Fig. 2 that the discrete solution quenches in a finite time at the first node, which is well known in a theoretical point of view [23, 22]. In Fig. 3 and Fig. 4 we see that the approximation of $u(x, T)$ increases and gives the value zero at the first node. The time T represents the quenching time of the solution u.

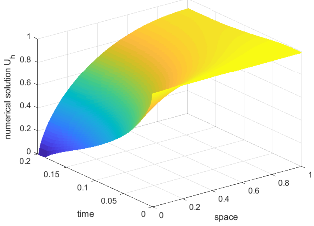

Fig. 1. Evolution of the semidiscrete solution, $q = 1/4$.

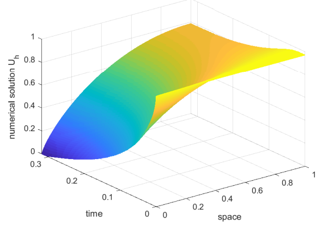

Fig. 2. Evolution of the semidiscrete solution, $q = 1/2$.

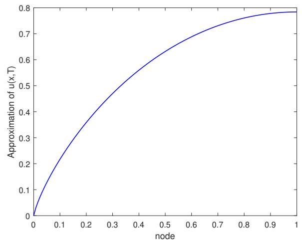

Fig. 3. Profile of the approximation of $u(x,T)$, $q = 1/4$.

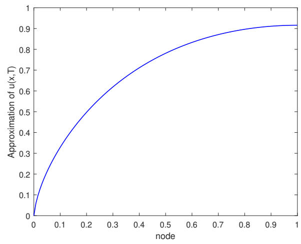

Fig. 4. Profile of the approximation of $u(x,T)$, $q = 1/2$.

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