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THE RATE OF CONVERGENCE OF BOUNDED LINEAR PROCESSES ON SPACES OF CONTINUOUS FUNCTIONS[1]

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Abstract. Quantitative Korovkin-type theorems for approximation by bounded linear operators defined on $C(X, d)$ are given, where (X, d) is a compact metric space. Special emphasis is on positive linear operators.

As is known from previous work of Newman and Shapiro, Jimenez Pozo, Nishishiraho and the author, among others, there are two possible ways to obtain error estimates for bounded linear operator approximation: the so-called direct approach, and the smoothing technique.

We give various generalizations and refinements of earlier results which were obtained by using both techniques. Furthermore, it will be shown that, in a certain sense, none of the two methods is superior to the other one.

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INTRODUCTION

The present paper discusses error estimates for the approximation of continuous functions defined on compact metric spaces. It is a contribution to quantitative approximation theory, and deals with pointwise error estimates for the approximation by bounded linear operators of continuous functions defined on compact metric spaces (X, d). Most estimates will be given in terms of moduli of continuity or their least concave majorants.

While we conducted our preliminary research on this topic, it turned out that, although such problems are now considered as being classical, and in spite of the research of T. Nishishiraho and M.A. Jiménez Pozo, there was still a large discrepancy between the general theory and estimates for continuous functions defined on, say, a compact interval $[a, b]$. This observation was one of the main motivations for writing the author's "Habilitationsschrift" from which the material presented here is taken. However, the thesis mentioned

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was never made available in the regular literature for no good reason, a fact which we would like to make up for (at least in part) by writing this note.

The main goals of this article are twofold. We present a new technique for deriving error estimates on approximation of functions defined on an arbitrary compact metric space (Section 3) and we give a more careful analysis of the so-called direct technique for special metric spaces (Section 4). According to the goals stated above, the paper is divided into the following parts.

Chapter 1 is of an auxiliary nature. In it we shall discuss several properties of (least concave majorants of) moduli of continuity and K-functionals which will be used throughout the remainder of this work. Among others, this part contains the refinement of a lemma due to M.A. Jiménez Pozo and the generalization of a lemma of N.P. Korneichuk to the case of continuous functions defined on a space with a finite coefficient of convex deformation.

Chapter 2 has its historical roots in a paper by G. Freud [13] and was mostly motivated by the work of M.A. Jiménez Pozo and T. Nishishiraho. In their work they gave error estimates for functions defined on compact metric spaces satisfying certain additional conditions. The aim of Section 3 is to show that these constraints can be completely removed, if least concave majorants of moduli of continuity are used rather than the corresponding moduli themselves. It is also shown that our approach may be even better than theirs, provided (X, d) satisfies the additional assumptions they imposed. Our new approach to give error estimates for the case of continuous functions defined on (X, d) has some unexpected consequences even in the case where $X = [a, b]$ is a compact interval of the real axis. Various further aspects are also discussed, such as a change of the test functions and the fact that the estimates of the main results of Section 3 are best possible in a certain sense.

Since the results of Section 3 are achieved via the intermediate use of a certain K-functional, we chose to call this the indirect approach. In Section 4 we discuss several consequences of the results of Section 3, and also present a careful analysis and a refinement of the direct technique. As is the case in Section 3, a general result for approximation by bounded linear operators is given. Subsequently the instance where L is also positive is discussed in depth. It will be shown in particular that the refined estimates of Section 4 are now best possible in a certain sense. As a consequence, we achieve a significant improvement over Nishishiraho's technique inasmuch as best possible constants can now be obtained by evaluating the upper bounds of our inequalities.

The critical function $L(d(\cdot, x); x)$ and several upper bounds of it are discussed in detail. However, more explicit upper bounds for $L(d(\cdot, x); x)$ and similar expressions occurring in particular in Sections 3 and 4 were discussed in some detail by Nishishiraho. Since the use of neither of these bounds improves an order of an estimate or a constant, the reader is referred to Nishishiraho's work for further information. It can be said in general that the construction of these upper bounds very closely follows the pattern suggested by classical Korovkin theory in order to introduce a finite number of test functions.

At certain central points of Sections 3 and 4 we also give a partial survey of related results which had been obtained earlier. Many of the examples discuss a univariate setting. This is the most convenient way to check the quality of both the order of approximation and of the constants.

The present article solely covers the general theory. For many applications the reader is referred to the author's "Habilitationsschrift". Among them are applications of the general results to certain operators of Vaida, Badea, and to the well-known Shepard operators. Part of the material given here was presented at the Third International Conference on Functional Analysis and Approximation Theory (Acquafredda di Maratea/Italy, September 1996).

CHAPTER I: SOME PROPERTIES OF MODULI OF SMOOTHNESS AND K-FUNCTIONALS

Most of the error estimates in this paper will be given in terms of moduli of smoothness (continuity) of various kinds and orders. Frequently so-called K-functionals will be used as intermediate tools for deriving these estimates. The aim of this section is to compile some information on both measures of smoothness, and to establish some new relationships between them and related quantities which will be useful in the sequel.

Section 1 lists several properties of moduli of continuity (smoothness) for real-valued and continuous functions of a single variable defined on a compact metric space (X, d). Section 2 contains a brief discussion of certain K-functionals in $C(X)$, where X is again a compact metric space.

As is clear from this summary, we do not attempt to present a comprehensive survey on moduli of smoothness and K-functionals in this chapter. It is solely intended to list those properties which will be needed in Chapter 2.

1. MODULI OF CONTINUITY OF FUNCTIONS DEFINED ON COMPACT METRIC SPACES

In this section we shall collect some information on the metric modulus of continuity of a function $f \in C(X) = C_{\mathbb{R}}((X, d))$. Here $C_{\mathbb{R}}((X, d))$ denotes the space of all real-valued and continuous functions defined on the compact metric space (X, d) having a diameter $d(X) > 0$.

Throughout this paper we shall need the following

Definition 1.1. *Let (X, d) be a compact metric space. The mapping*

$$\omega_d(f, \cdot) : \mathbb{R}_+ \ni \delta \mapsto \omega_d(f, \delta) := \sup\{|f(x) - f(y)| : x, y \in X,\ d(x, y) \le \delta\} \in \mathbb{R}$$

is called the (metric) modulus of continuity *of f. Note that for $\delta \ge d(X)$ one has $\omega_d(f, \delta) = \omega_d(f, d(X))$.*

If the metric space has the property that for $x, y \in X$ and $d(x, y) = a + b$ with $a, b > 0$, there exists always an element $z \in X$ such that $d(x, z) = a$ and $d(z, y) = b$, then the space is called *metrically convex.* This is a notion which was introduced by K. Menger [37]. For such spaces the modulus of continuity satisfies the properties given in the following

LEMMA 1.2. *Let (X,d) be metrically convex. Then for any $f \in C(X)$ its modulus of continuity $\omega_d(f,\cdot)$ has the following properties:*

(1) $\omega_d(f,0)=0$.
(2) $\omega_d(f,\cdot)$ *is positive and non-decreasing on* $\mathbb{R}_+$.
(3) $\omega_d(f,r\cdot\delta) \le (1+]r[)\cdot\omega_d(f,\delta), \quad r,\delta\in\mathbb{R}_+$;
here $]r[$ *denotes the largest integer less than* r.
(4) $\omega_d(f,\cdot)$ *is a subadditive function, i.e.,*

$$\omega_d(f,\delta_1+\delta_2) \le \omega_d(f,\delta_1)+\omega_d(f,\delta_2) \quad \textit{for all } \delta_1,\delta_2 \ge 0.$$

In particular, $\omega_d(f,\cdot)$ *is continuous on* $\mathbb{R}_+$.
(5) *If* $\delta \ge 0$ *is fixed, then* $\omega_d(\cdot,\delta)$ *is a seminorm on* $C(X)$.

There is no need to give a proof of Lemma 1.2 at this point because Lemma 1.6 covers a more general situation.

Examples of compact metric spaces being metrically convex are for instance given by compact and convex subsets of $\mathbb{R}^m$, $m \ge 1$, equipped with the Euclidean distance function, or the unit circle S^1 with metric d given by $d(e^{it},e^{ix}) = \min\{2\pi-|t-x|,|t-x|\},\ 0\le t,x<2\pi$.

There is a generalization of the notion of a metrically convex space. To our knowledge, it was introduced by M.A. Jiménez Pozo [23] and uses the concept of the so-called coefficient of convex deformation. A brief description follows.

DEFINITION 1.3. *Let (X,d) be a metric space, and let $[a,b]$ be a compact interval of the real axis. If $\varphi:[a,b]\to X$ is a parametrization of the simple Jordan arc $\Gamma[\varphi(a),\varphi(b)]=\Gamma$, and if $P=\{a=t_0<t_1<\ldots<t_n=b\}$ is a subdivision of $[a,b]$, then we define*

$$\ell(\varphi,P) := \sum_{i=1}^{n} d\left(\varphi(t_{i-1}),\varphi(t_i)\right).$$

Γ *is said to be* rectifiable with length $\ell(\Gamma)$, *if*

$$\ell(\Gamma) := \sup\{\ell(\varphi,P) : P \textit{ a subdivision of } [a,b]\} < \infty.$$

DEFINITION 1.4. *The metric space (X,d) is said to have a* coefficient of convex deformation $\varrho=\varrho(X)$, *if the following hold.*

(1) *For all $x,y\in X$ the set $R(x,y)$ of rectifiable arcs $\Gamma[x,y]$ connecting x and y is non-empty.*
(2) $\varrho := \sup\Big\{\inf\left\{\ell(\Gamma[x,y])/d(x,y) : \Gamma[x,y]\in R(x,y)\right\} : x,y\in X, x\ne y\Big\} < \infty$.

The following remarks are due to M.A. Jiménez Pozo.

REMARK 1.5 (M.A. Jiménez Pozo [23]). (1) $\varrho=\varrho(X)\ge 1$ *for every metric space (X,d).*

(2) *A compact metric space is metrically convex if and only if* $\varrho(X) = 1$.
(3) *Any semi-circle of* $\mathbb{R}^2$ *(equipped with the Euclidean distance) has a coefficient of convex deformation* $\varrho = \pi/2$.

Jiménez Pozo's motivation for introducing the coefficient of convex deformation ϱ was to make sure that an inequality of type

$$\omega_d(f, \lambda\alpha) \leq [1 + \varrho\lambda] \cdot \omega_d(f, \alpha), \quad \lambda, \alpha \in \mathbb{R}_+,$$

holds for the modulus of continuity as defined above. Here for $a \in \mathbb{R}$, $[a]$ denotes the largest integer $\leq a$. One of the observations of the following lemma is that a somewhat stronger inequality holds. It constitutes a generalization of Lemma 1.2.

Lemma 1.6. *Let* (X, d) *be a compact metric space with coefficient of convex deformation* $\varrho = \varrho(X) \geq 1$. *Then for any* $f \in C(X)$ *the following are true:*

(1) $\omega_d(f, 0) = 0$.
(2) $\omega_d(f, \cdot)$ *is a positive and non-decreasing function on* $\mathbb{R}_+$.
(3) $\omega_d(f, r \cdot \delta) \leq (1+]\varrho \cdot r[) \cdot \omega_d(f, \delta), \quad r, \delta \in \mathbb{R}_+$.
(4) $\omega_d(f, \delta_1+\delta_2) \leq \omega_d(f, \varrho\delta_1)+\omega_d(f, \varrho\delta_2) \leq (1+]\varrho^2[)\cdot(\omega_d(f, \delta_1) + \omega_d(f, \delta_2))$.
(5) *If* $\delta \geq 0$ *is fixed, then* $\omega_d(\cdot, \delta)$ *is a seminorm on* $C(X)$.

Proof. (1) and (2) are immediate consequences of the definition of ω_d.
(3) is obtained as follows. For $r = 0$ the inequality obviously holds. So let $r > 0$; because $\varrho \geq 1$ this implies $\varrho \cdot r > 0$.

If $\varrho \cdot r \notin \mathbb{N}$, then $1+]\varrho \cdot r[= [1 + \varrho \cdot r]$, where $[a] := \max\{z \in \mathbb{Z} \ : \ z \leq a\}$. For this case it was shown by M.A. Jiménez Pozo [23, Theorème 2] that the above inequality holds.

Now let $\varrho \cdot r \in \mathbb{N}$. Since (X, d) is compact, it is also finitely compact, and thus, by a theorem of Hilbert (see W. Rinow [51, Statement 7, p. 141]), assumption (1) of Definition 1.4 implies that for each two points $x, y \in X$, $x \neq y$, there is a shortest rectifiable arc Γ_0 (in German: "Kürzeste") connecting x and y such that

$$\inf\left\{\ell(\Gamma[x, y])/d(x, y) \ : \ \Gamma[x, y] \in R(x, y)\right\} = \ell(\Gamma_0[x, y])/d(x, y).$$

Because of (2) of Definition 1.4, we have in particular that

$$\ell(\Gamma_0[x, y])/d(x, y) \leq \varrho.$$

Using a slight modification of Jiménez Pozo's approach for the case of $\varrho\cdot r \notin \mathbb{N}$, we proceed further as follows.

Let x and y be such that $0 < d(x, y) \leq r\delta$. Choosing $\varrho \cdot r + 1$ points $t_i = i/(\varrho \cdot r)$, $0 \leq i \leq \varrho \cdot r$, and the parametrization ψ such that (see M.A. Jiménez Pozo [23, Lemme 1])

$$(\psi(t_{i-1}, \psi(t_i)) \leq \ell\Big(\Gamma_0[\psi(t_{i-1}), \psi(t_i)]\Big) = (t_i - t_{i-1}) \cdot \ell(\Gamma_0[x, y]),$$

where $\Gamma_0[x,y]$ is chosen as above and in particular such that

$$\ell(\Gamma_0[x,y]) \leq \varrho \cdot d(x,y) \leq \varrho \cdot r\delta,$$

one obtains (with $z_i := \psi(t_i)$ and $n := \varrho r$) the inequalities

$$\begin{aligned} |f(x) - f(y)| &\leq \sum_{i=1}^{n} |f(z_i) - f(z_{i-1})| \leq \sum_{i=1}^{n} \omega_d(f, d(z_i, z_{i-1})) \\ &\leq \sum_{i=1}^{n} \omega_d(f, (\varrho r)^{-1} \cdot \varrho \cdot r\delta) = \varrho r \cdot \omega_d(f,\delta). \end{aligned}$$

This implies

$$\omega_d(f, r\delta) \leq \varrho \cdot r \cdot \omega_d(f,\delta) = (1+]\varrho r[) \cdot \omega_d(f,\delta),$$

the latter equality being true by our assumption $\varrho r \in \mathbb{N}$.

(4) If $\delta_1 = 0$ or $\delta_2 = 0$, because $\varrho \geq 1$ the statement is trivial.
Let $\delta_1, \delta_2 > 0$ be given and fixed, and let $x, y \in X$ be such that $d(x,y) \leq \delta_1 + \delta_2$. Choosing $t_0 = 0$, $t_1 = \delta_1/(\delta_1+\delta_2)$, and $t_2 = 1$, the parametrization ψ and the arc Γ_0 from (3) satisfy for $i = 1, 2$

$$d\Big(\psi(t_{i-1}), \psi(t_i)\Big) \leq \ell\Big(\Gamma_0[\psi(t_{i-1}), \psi(t_i)]\Big) = (t_i - t_{i-1}) \cdot \ell(\Gamma_0[x,y]).$$

Hence

$$\begin{aligned} |f(x) - f(y)| &= |f(\psi(t_0)) - f(\psi(t_2))| \\ &\leq |f(\psi(t_0)) - f(\psi(t_1))| + |f(\psi(t_1)) - f(\psi(t_2))| \\ &\leq \omega_d\Big(f, d(\psi(t_0), \psi(t_1))\Big) + \omega_d\Big(f, d(\psi(t_1), \psi(t_2))\Big) \\ &\leq \omega_d\Big(f, (t_1 - t_0) \cdot \ell(\Gamma_0[x,y])\Big) + \omega_d\Big(f, (t_2 - t_1) \cdot \ell(\Gamma_0[x,y])\Big) \\ &\leq \omega_d\Big(f, \delta_1 \cdot (\delta_1+\delta_2)^{-1} \cdot \varrho \cdot (\delta_1+\delta_2)\Big) \\ &\quad + \omega_d\Big(f, \delta_2 \cdot (\delta_1+\delta_2)^{-1} \cdot \varrho \cdot (\delta_1+\delta_2)\Big) \\ &= \omega_d(f, \varrho \cdot \delta_1) + \omega_d(f, \varrho \cdot \delta_2), \end{aligned}$$

showing that

$$|f(x) - f(y)| \leq \omega_d(f, \varrho \cdot \delta_1) + \omega_d(f, \varrho \cdot \delta_2).$$

This immediately implies

$$\omega_d(f, \delta_1 + \delta_2) \leq \omega_d(f, \varrho \cdot \delta_1) + \omega_d(f, \varrho \cdot \delta_2),$$

and the second inequality is a consequence of (3).

(5) is an immediate corollary of the definition. □

Corollary 1.7. *Under the assumptions of Lemma 1.6 we also have for* $r, \delta \in \mathbb{R}_+$*:*

$$\omega_d(f, r\cdot\delta) \le [1+\varrho\cdot r]\cdot\omega_d(f,\delta) \le (1+\varrho\cdot r)\cdot\omega_d(f,\delta). \tag{1}$$

Here $[1+\varrho\cdot r]$ *is the largest integer* $\le 1+\varrho\cdot r$*.*

Hence metric spaces X having a coefficient of convex deformation ϱ constitute an example of metric spaces such that for any $f \in C(X)$ one has

$$\omega_d(f, r\cdot\delta) \le (1+\eta r)\cdot\omega_d(f,\delta), \quad r,\delta\in\mathbb{R}_+,$$

where $\eta > 0$ is a fixed constant. For further properties of moduli of continuity given for $f \in C(X)$ and satisfying an inequality of this type see Section 2.

2. K-FUNCTIONALS

In this section we shall compile several properties of certain classical and modified K-functionals, including their relationship to certain moduli of continuity and their least concave upper bounds.

2.1. Definition and elementary properties.

A further means for measuring the smoothness of functions is Peetre's K-functional. This measures the distance of an element f of a vector space E from a subspace U, at the same time reflecting certain properties of the approximating elements $g \in U$. The definition is as follows.

Definition 2.1. *Let* E *be a vector space and* U *a subspace of* E*. If* p *and* p^* *are seminorms on* E *and* U*, respectively, then the mapping*

$$\begin{aligned}(\cdot,*;E,U)_{p,p^*} : \mathbb{R}_+\times E \ni (t,f) \mapsto K(t,f;E,U)_{p,p^*} :=\\ := \inf\{p(f-g)+tp^*(g) : g\in u\}\in\mathbb{R}\end{aligned}$$

is the K*-functional with respect to* (E,p) *and* (U,p^*)*. Sometimes* $K(t,f;E,U)_{p,p^*}$ *will be abbreviated by* $K(t,f)$*.*

The following lemma collects some of the properties of K.

Lemma 2.2. (cf. P.L. Butzer and H. Berens [6, Proposition 3.2.3])
If E *is a vector space,* U *a subspace of* E*, and if* p *and* p^* *are seminorms on* E *and* U*, respectively, then* K *has the following properties.*

(1) *U is dense in E with respect to the topology generated by p if and only if for each $f \in E$ one has*

$$\lim_{t\to 0^+} K(t,f) = 0,$$

i.e., $K(\cdot,f)$ is continuous at $t=0$, and such that $K(0,f)=0$.

(2) *For each fixed $f \in E$, $K(\cdot,f):\mathbb{R}_+\to\mathbb{R}_+$ is a continuous, monotonically increasing, and concave function, i.e., for $t_1,t_2\in\mathbb{R}_+$ and $0\le\lambda\le 1$ one has*

$$K((1-\lambda)t_1+\lambda t_2, f) \ge (1-\lambda)K(t_1,f)+\lambda K(t_2,f).$$

(3) *For arbitrary* $\lambda, t \in \mathbb{R}_+$, *and fixed* $f \in E$, *the inequality*
$$K(\lambda t, f) \leq \max\{1, \lambda\} \cdot K(t, f)$$
holds.

(4) *For each* $t \in \mathbb{R}_+$ *fixed,* $K(t, \cdot)$ *is a seminorm on* E, *and such that* $K(t, f) \leq p(f)$ *for all* $f \in E$.

2.2. K-functionals and least concave majorants of moduli of continuity. In the following we shall treat the approximation of continuous functions, and thus shall use only special cases of the K-functionals introduced in Section 2.1. In this section we consider again a compact metric space (X, d) with diameter $d(X) > 0$. If further properties of X are needed, these will be stated explicitly.

For $0 < r \leq 1$ let Lip r denote the set of all functions g in $C(X)$ such that
$$|g|_{\text{Lip } r} := \sup\{|g(x) - g(y)|/d^r(x, y) : d(x, y) > 0\} < \infty.$$
Then Lip r is a dense subspace of $C(X)$, and $|\cdot|_{\text{Lip } r}$ is a seminorm on Lip r. Thus it makes sense to use the K-functional with respect to $(\text{Lip } r, |\cdot|_{\text{Lip } r})$ in order to prove quantitative assertions. As indicated above, this functional is given by
$$K(t, f; C(X), \text{Lip } r) := \inf\Big\{\|f - g\|_X + t \cdot |g|_{\text{Lip } r} : g \in \text{Lip } r\Big\},$$
where $f \in C(X)$ and $t \geq 0$.

Another tool for our proofs will be the following lemma of Yu.A. Brudnyĭ relating the above K-functional for $r = 1$ to the least concave majorant of the metric modulus of continuity as introduced in Section 1. For a sketch of proof see B.S. Mitjagin and E.M. Semenov [38].

Lemma 2.3. *Let* (X, d) *be a compact metric space with* $d(X) > 0$. *Every continuous function* f *on* X *satisfies the equality*
$$K\Big(\varepsilon/2, f; C(X), \text{Lip } 1\Big) = \tfrac{1}{2} \cdot \widetilde{\omega}_d(f, \varepsilon), \quad 0 \leq \varepsilon.$$
Here $\widetilde{\omega}_d(f, \cdot)$ *denotes the least concave majorant of* $\omega_d(f, \cdot)$ *given by*
$$\widetilde{\omega}_d(f, \varepsilon) = \begin{cases} \sup\limits_{\substack{0 \leq x \leq \varepsilon \leq y \leq d(X) \\ x \neq y}} \frac{(\varepsilon - x)\omega_d(f, y) + (y - \varepsilon)\omega_d(f, x)}{y - x}, & \textit{for } 0 \leq \varepsilon \leq d(X), \\ \widetilde{\omega}_d(f, d(X)) = \omega_d(f, d(X)), & \textit{if } \varepsilon > d(X). \end{cases}$$

The definition of $\widetilde{\omega}_d(f, \cdot)$ shows that $\omega_d(f, \cdot) \leq \widetilde{\omega}_d(f, \cdot)$. However, an estimate from above by some multiple of $\omega_d(f, \cdot)$ is not possible for all metric spaces (X, d), as will be seen below. For some further properties of $\widetilde{\omega}_d(f, \cdot)$ see *e.g.*, V.K. Dzjadyk [12, p. 153ff].

If (X, d) is such that the modulus of continuity $\omega_d(f, \cdot)$ of some function $f \in C(X)$ is continuous, nondecreasing and subadditive on $[0, \infty)$, then it was shown by N.P. Korneičuk [30, p. 670] that for any $\varepsilon \geq 0$ and $\xi > 0$ the

functions $\omega_d(f,\cdot)$ and its least concave majorant $\widetilde{\omega}_d(f,\cdot)$ are related by the inequality
$$\widetilde{\omega}_d(f,\xi\cdot\varepsilon)\le(1+\xi)\cdot\omega_d(f,\varepsilon),$$
and that this inequality cannot be improved for each $\varepsilon>0$ and $\xi=1,2,\ldots$ In Lemma 2.4 a generalization of this lemma to the setting of certain metric spaces including those with a finite coefficient of convex deformation will be accomplished.

The generalization just mentioned will be for metric spaces (X,d) of the following type.

There exists a fixed constant $\eta>0$ such that for all $\xi,\varepsilon>0$ and all $f\in C(X)$, the inequality
$$\omega_d(f,\xi\cdot\varepsilon)\le(1+\eta\cdot\xi)\cdot\omega_d(f,\varepsilon)$$
holds.

Examples of such spaces are *e.g.*, compact metric spaces being convex in the sense of K. Menger [37], or compact convex subsets (X,d) of a metric linear space (Y,d) with translation invariant metric and starshaped $d(\cdot,0)$ (see T. Nishishiraho [43]). In both cases the above inequality holds for $\eta=1$. Other examples are given by spaces (X,d) having a coefficient of convex deformation $\varrho=\varrho(X)$. In this case the above inequality holds for $\eta=\varrho$ (see Corollary 1.7). Provided an inequality of the above type holds, $\widetilde{\omega}_d(f,\cdot)$ and $\omega_d(f,\cdot)$ are related as described in the following lemma; for the case $\eta=1$ it was obtained by N.P. Korneičuk [30, p. 670].

Lemma 2.4. *If (X,d) is a compact metric space such that for all $\xi,\varepsilon>0$, all $f\in C(X)$, and some fixed $\eta>0$ the inequality $\omega_d(f,\xi\cdot\varepsilon)\le(1+\eta\cdot\xi)\cdot\omega_d(f,\varepsilon)$ holds, then for $f\in C(X)$, and any real number $\xi,\varepsilon\ge0$ we have*

(1) $$\omega_d(f,\xi\cdot\varepsilon)\le\widetilde{\omega}_d(f,\xi\cdot\varepsilon)\le(1+\eta\cdot\xi)\cdot\omega_d(f,\varepsilon).$$

In particular, for $\xi=1$ this reduces to the inequalities

(2) $$\omega_d(f,\varepsilon)\le\widetilde{\omega}_d(f,\varepsilon)\le(1+\eta)\cdot\omega_d(f,\varepsilon),$$

and for $\eta=1$ to

(3) $$\omega_d(f,\xi\cdot\varepsilon)\le\widetilde{\omega}_d(f,\xi\cdot\varepsilon)\le(1+\xi)\cdot\omega_d(f,\varepsilon).$$

Proof. Obviously (1) is true if at least one of the numbers ξ and ε is equal to zero. So let $\xi,\varepsilon>0$ and such that $\xi\cdot\varepsilon\le d(X)$. As mentioned earlier, for $0\le\xi\cdot\varepsilon\le d(X)$ one has
$$\widetilde{\omega}_d(f,\xi\cdot\varepsilon)=\sup_{\substack{0\le x\le\xi\cdot\varepsilon\le y\le d(X)\\ x\ne y}}\tfrac{(\xi\cdot\varepsilon-x)\omega_d(f,y)+(y-\xi\cdot\varepsilon)\omega_d(f,x)}{y-x}.$$
Putting for instance $x=\xi\cdot\varepsilon$ shows that $\omega_d(f,\xi\cdot\varepsilon)\le\widetilde{\omega}_d(f,\xi\cdot\varepsilon)$.
For the proof of the second inequality in (1) let $0\le x\le\xi\cdot\varepsilon\le y\le d(X)$, $x\ne y$, be arbitrarily given, and write $\omega(\cdot)=\omega_d(f,\cdot)$ for the sake of brevity. Then
$$\tfrac{\xi\varepsilon-x}{y-x}\cdot\omega(y)+\tfrac{y-\xi\varepsilon}{y-x}\cdot\omega(x)=\tfrac{\xi\varepsilon-x}{y-x}\cdot\omega(\tfrac{y}{\varepsilon}\varepsilon)+\tfrac{y-\xi\varepsilon}{y-x}\cdot\omega(\tfrac{x}{\varepsilon}\varepsilon)$$

$$\begin{aligned}
&\leq \tfrac{\xi\varepsilon - x}{y-x} \cdot (1+\eta\tfrac{y}{\varepsilon}) \cdot \omega(\varepsilon) + \tfrac{y-\xi\varepsilon}{y-x} \cdot (1+\eta\tfrac{x}{\varepsilon}) \cdot \omega(\varepsilon) \\
&= \Big(\tfrac{\xi\varepsilon - x}{y-x} \cdot (1+\eta\tfrac{y}{\varepsilon}) + \tfrac{y-\xi\varepsilon}{y-x} \cdot (1+\eta\tfrac{x}{\varepsilon})\Big) \cdot \omega(\varepsilon) \\
&= (1+\eta\xi) \cdot \omega(\varepsilon).
\end{aligned}$$

This shows that inequality (1) holds for $\xi\varepsilon \leq d(X)$. For $\xi\varepsilon > d(X)$ we have

$$\begin{aligned}
\widetilde{\omega}_d(f,\xi\varepsilon) = \widetilde{\omega}_d(f,d(X)) &= \omega_d(f, d(X)\cdot\varepsilon^{-1}\cdot\varepsilon) \\
&\leq (1+\eta d(X)/\varepsilon)\cdot\omega_d(f,\varepsilon) \\
&\leq (1+\eta\xi)\cdot\omega_d(f,\varepsilon).
\end{aligned}$$

Thus the lemma is proved for all possible choices of $\xi\varepsilon$. □

However, it is not always possible to estimate $\widetilde{\omega}_d(f,\cdot)$ from above by some multiple of $\omega_d(f,\cdot)$. This can be seen from

REMARK 2.5. *The second inequality in Lemma 2.4 (2) does not hold for an arbitrary compact metric space. This can be seen from the following example. Let* $X = [0; 0.25] \cup [0.75; 1]$ *and* $d(x,y) = |x-y|$ *for* $x, y \in X$. *The function* f *given by*

$$f(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 0.25 \\ 2 & \text{for } 0.75 \leq x \leq 1 \end{cases}$$

is continuous on X. *Its (metric) modulus of continuity is*

$$\omega_d(f,\varepsilon) = \begin{cases} 0 & \text{for } 0 \leq \varepsilon < 0.5 \\ 1 & \text{for } 0.5 \leq \varepsilon \leq 1\,. \end{cases}$$

Thus the least concave majorant $\widetilde{\omega}_d(f,\cdot)$ *is the function* $\widetilde{\omega}_d(f,\varepsilon) = \min\{2\varepsilon, 1\}$. *Hence* $\widetilde{\omega}_d(f,\varepsilon) \leq c\cdot\omega_d(f,\varepsilon)$ cannot *hold for any* $c > 0$ *and all* $\varepsilon \geq 0$. *In fact, it is easily verified that there is no* $\eta > 0$ *such that for all* $\xi, \varepsilon > 0$ *one has* $\omega_d(f,\xi\varepsilon) \leq (1+\eta\xi)\cdot\omega_d(f,\varepsilon)$.

REMARK 2.6. *As is well-known, there are intimate relationships between certain* K*-functionals and moduli of smoothness of various orders in both the univariate and the multivariate setting. These relationships will not be needed below. The interested reader is referred to L.L. Schumaker's book* [52], *and to the references cited there for both cases mentioned.*

CHAPTER II: APPROXIMATION IN C(X)

This chapter deals with quantitative Korovkin-type theorems for approximation by bounded linear operators defined on $C(X)$, and in particular by positive ones. Here $C(X) = C_{\mathbb{R}}((X,d))$ denotes the Banach lattice of real-valued continuous functions defined on the compact metric space (X,d) equipped with the canonical ordering, and with norm given by $\|f\| = \|f\|_X = \max\{|f(x)| : x \in X\}$. We also assume that X has diameter $d(X) > 0$.

The first such theorem for general positive linear operators and $X = [a, b]$ equipped with the euclidian distance is apparently due to R.G. Mamedov [34]. For spaces (X, d) being *metrically convex* in the sense of K. Menger [37], D.J. Newman and H.S. Shapiro [41] proved the following theorem similar to that of Mamedov (see also H.S. Shapiro [55, Chapter 8.8.2], and papers by M.W. Müller and H. Walk [40, p. 225] and G. Mastroianni [36]).

Theorem A (D.J. Newman–H.S. Shapiro [41, Lemma 4]). *Suppose the compact metric space (X, d) has the property that, whenever $d(x, y) = a + b$, where $a > 0, b > 0$, there exists a point $z \in X$ such that $d(x, z) = a$ and $d(z, y) = b$. Let L denote a positive linear operator from $C(X) \to C(X)$, such that $L(1_X) = 1_X$. Then for any $f \in C(X)$, any $x \in X$ and any $\varepsilon > 0$ we have*

$$|L(f, x) - f(x)| \leq \left(1 + \varepsilon^{-1} L(d^2(\cdot, x); x)^{1/2}\right) \cdot \omega(f, \varepsilon).$$

Here $1_X : X \ni x \mapsto 1 \in \mathbb{R}$, and the modulus of continuity $\omega(f, \cdot)$ is given as in Definition 1.1, i.e., $\omega(f, \varepsilon) = \sup\{|f(x) - f(y)| : x, y \in X,\ d(x, y) \leq \varepsilon\}$.

This direct approach (*i.e.*, one which avoids the intermediate use of K-functionals) was further developed in papers of T. Nishishiraho [42, 43, 44], where additional references can be found. For compact spaces (X, d) having a coefficient of convex deformation $\varrho < \infty$ (see Definition 1.4), M.A. Jiménez Pozo [23] published a generalization of the result of Newman and Shapiro involving mainly the modulus of continuity of f. See Jiménez Pozo [26] and Jiménez Pozo–Baile Baldet [27] for a discussion of several earlier results.

Furthermore, for *arbitrary* compact (X, d), M.A. Jiménez Pozo [21, 26] proved a certain generalization of the following

Theorem B (cf. M.A. Jiménez Pozo [26, Th. 1]). *Let $L : C(X) \to C(X)$ be a positive linear operator. Then for all $f \in C(X)$, all $x \in X$ and all $\varepsilon > 0$, the following inequality holds:*

$$\begin{aligned}&|L(f, x) - f(x)| \leq \\ &\leq \omega(f, \varepsilon) \cdot L(1_X, x) + |f(x)| \cdot |L(1_X, x) - 1| + \varepsilon^{-1} \cdot 2\|f\|_X \cdot L(d(\cdot, x); x).\end{aligned}$$

A disadvantage of this latter type of estimate is the fact that, for operators satisfying $L(1_X) = 1_X$, the upper bound is not given solely in terms of a modulus of continuity. This observation is the main motivation for our investigations in Section 3, where we shall prove several inequalites in terms of the least concave majorant of the modulus of continuity (see Section 2.2), thus avoiding the problems mentioned.

As will be discussed in detail in Section 4, there are basically two ways to obtain estimates in terms of the metric modulus of continuity from Definition 1.1. One way is to use the results from Section 3 and the relationship between $\tilde{\omega}$ and ω as given in Lemma 2.4. The other method is the direct approach mentioned above. Section 4 contains a refined version of this approach which is more sensitive than the technique employed by T. Nishishiraho. See the introductory remarks to Section 4 for further details.

3. ESTIMATES ON APPROXIMATION BY BOUNDED LINEAR OPERATORS IN TERMS OF LEAST CONCAVE MAJORANTS OF MODULI OF CONTINUITY

This section contains several results most of which were presented in the author's papers [16] and [17]. The main point is to show that the assumption of metric convexity or of existence of a finite coefficient of convex deformation can be completely dropped when using the least concave majorant of $\omega(f,\cdot)$ instead of $\omega(f,\cdot)$ itself. As will be seen below, our general results imply estimates similar to (and sometimes even better than) those of T. Nishishiraho and M.A. Jiménez Pozo provided (X,d) satisfies the additional assumptions they imposed.

It was observed by H. Berens and G.G. Lorentz [5] among others that the results on approximation of lattice homomorphisms $A : C(X) \to E$, E a Banach lattice, by positive linear operators are similar to those for approximation of the injection $i : C(X) \to B(X)$. Here $B(X)$ is the space of bounded real-valued functions on X. This is our motivation for proving estimates on approximation of mappings $A : C(X) \to B(Y)$, $Y \neq \emptyset$, given by $A(f,y) = \psi_A(y) \cdot f(g_A(y))$; here ψ_A is a bounded real-valued function on Y and g_A maps Y into X. For the relationship between such mappings and lattice homomorphisms see *e.g.*, M. Wolf [62].

It is possible to generalize the operators to be approximated. Some results in this direction are due to M.A. Jiménez Pozo [22, 25] and to the author [15]. However, this problem will not be discussed here.

One of the key results of this section is the following Theorem 3.1. Its simple proof reveals the smoothing technique in spaces of real-valued continuous functions defined on compact metric spaces mentioned above, and uses some of the results of Section 2.

THEOREM 3.1 (H. Gonska, [17, Theorem 2.1]). *Let $Y \neq \emptyset$ be some set, and let $B(Y)$ denote the space of real-valued and bounded functions on Y equipped with the norm $\|f\|_Y = \sup\{|f(y)| : y \in Y\}$. If $C(X)$ is given as above, and if Δ is a bounded linear operator mapping $C(X)$ into $B(Y)$ such that for some $y \in Y$ one has*

$$|\Delta(g,y)| \leq \varphi(y) \cdot |g|_{\mathrm{Lip}\ 1} \quad \text{for } \varphi(y) \geq 0 \text{ and all } g \in \mathrm{Lip}\ 1,$$

then for all $f \in C(X)$ and $\varepsilon > 0$ the inequality

$$|\Delta(f,y)| \leq \max\left\{\tfrac{1}{2}\|\Delta\|, \varphi(y) \cdot \varepsilon^{-1}\right\} \cdot \widetilde{\omega}(f,\varepsilon)$$

holds.

Proof. The inequality is obviously correct for $\Delta = 0$; so let $\Delta \neq 0$. Let g be arbitrarily given in Lip 1. For any $f \in C(X)$ we have

$$\begin{aligned}|\Delta(f,y)| &\le |\Delta(f-g,y)| + |\Delta(g,y)| \le \|\Delta\| \cdot \|f-g\| + \varphi(y) \cdot |g|_{\text{Lip }1} \\ &= \|\Delta\| \cdot \left(\|f-g\| + \|\Delta\|^{-1} \cdot \varphi(y) \cdot |g|_{\text{Lip }1}\right).\end{aligned}$$

Passing to the inf in Lip 1 implies for each $\varepsilon > 0$ (see Lemma 2.2 (3)):

$$\begin{aligned}|\Delta(f,y)| &\le \|\Delta\| \cdot K\left(\|\Delta\|^{-1} \cdot \varphi(y) \cdot \tfrac{1}{2}\varepsilon \cdot (\tfrac{1}{2}\varepsilon)^{-1}, f; C(X), \text{Lip } 1\right) \\ &\le \|\Delta\| \max\left\{1, \|\Delta\|^{-1} \cdot \varphi(y) \cdot (\tfrac{1}{2}\varepsilon)^{-1}\right\} \cdot K\left(\tfrac{1}{2}\varepsilon, f; C(X), \text{Lip } 1\right) \\ &= \max\left\{\|\Delta\|, \varphi(y) \cdot (\tfrac{1}{2}\varepsilon)^{-1}\right\} \cdot K\left(\tfrac{1}{2}\varepsilon, f; C(X), \text{Lip } 1\right).\end{aligned}$$

Brudnyĭ's Lemma 2.3 implies

$$|\Delta(f,y)| \le \max\left\{\tfrac{1}{2}\|\Delta\|, \varphi(y) \cdot \varepsilon^{-1}\right\} \cdot \widetilde{\omega}(f,\varepsilon).$$

□

The next theorem deals with approximation of operators A given for $f \in C(X)$ and $y \in Y$ by $A(f,y) = \psi_A(y) \cdot f(g_A(y))$ where $\psi_A \in B(Y)$ and g_A maps Y into X. It contains a general inequality for the approximation of operators A by bounded linear operators, and was announced in the author's paper [17, Theorem 2.2].

THEOREM 3.2. *Let A be of the form $A(f,y) = \psi_A(y) \cdot f(g_A(y))$, and let L be a bounded linear operator, both mapping $C(X)$ into $B(Y)$. Then for $f \in C(X)$, $y \in Y$ and $0 < \varepsilon$ we have*

$$\begin{aligned}&|(L-A)(f,y)| \le \\ &\le \max\left\{\tfrac{1}{2}(\|L\| + \|L(1_X)\|_Y), \varepsilon^{-1}\left[d(x)(\|\varepsilon_y \circ L\| - L(1_X,y)) + |L(d(\cdot, g_A(y)); y)|\right]\right\} \\ &\quad \times \widetilde{\omega}(f,\varepsilon) + |(L-A)(1_X,y)| \cdot |f(g_A(y))|.\end{aligned}$$

Here 1_X denotes the function $X \ni x \mapsto 1 \in \mathbb{R}$.

Proof. If A is given as above, then

$$A(f,y) = \psi_A(y) \cdot f(g_A(y)) = A(1_X,y) \cdot f(g_A(y)).$$

Thus for all $f \in C(X)$ and all $y \in Y$

$$|(L-A)(f,y)| \le |L(f,y) - L(1_X,y) \cdot f(g_A(y))| + |(L-A)(1_X,y)| \cdot |f(g_A(y))|.$$

Defining $\widetilde{A}(f,y) := L(1_X,y) \cdot f(g_A(y))$ we have to consider $|(L-\widetilde{A})(f,y)|$. First observe that $L - \widetilde{A}$ is a bounded operator. Moreover, if $g \in \text{Lip } 1$ is arbitrarily given, then

$$|(L-\widetilde{A})(g,y)| = |L(g - g(g_A(y)) \cdot 1_X; y)|.$$

In order to estimate this quantity we introduce the functions

$$h_1 \quad := \quad g(g_A(y)) - |g|_{\text{Lip }1} \cdot d(\cdot, g_A(y)), \quad \text{and}$$

$$h_2 \quad := \quad g(g_A(y)) + |g|_{\mathrm{Lip}\ 1} \cdot d(\cdot, g_A(y)).$$

Note that $h_1 \le g \le h_2$. Now

$$\begin{aligned} L(g - g(g_A(y)) \cdot 1_X; y) &= L(g - h_1; y) - |g|_{\mathrm{Lip}\ 1} \cdot L\big(d(\cdot, g_A(y)); y\big) \\ &\ge L(g - h_1; y) - |g|_{\mathrm{Lip}\ 1} \cdot \big|L\big(d(\cdot, g_A(y)); y\big)\big|, \end{aligned}$$

and

$$\begin{aligned} -L(g - g(g_A(y)) \cdot 1_X; y) &= L(h_2 - g; y) - |g|_{\mathrm{Lip}\ 1} \cdot L\big(d(\cdot, g_A(y)); y\big) \\ &\ge L(h_2 - g; y) - |g|_{\mathrm{Lip}\ 1} \cdot \big|L\big(d(\cdot, g_A(y)); y\big)\big|. \end{aligned}$$

As mentioned above, both $g - h_1$ and $h_2 - g$ are nonnegative functions. We prove lower estimates for the quantities $L(g - h_1; y)$ and $L(h_2 - g; y)$.

For y fixed, the continuous linear functional

$$\varepsilon_y \circ L : C(X) \ni f \mapsto L(f, y) \in \mathbb{R}$$

can be identified with some real measure μ on the compact space X. Hence $L(f, y) = \int_X f d\mu$ for $f \in C_{\mathbb{R}}(X)$ (see, *e.g.*, J. Dieudonné [11, Ch. 13]). It is also known that μ can be represented as $\mu = \mu^+ - \mu^-$ with both μ^+ and μ^- being positive measures. Because of

$$L(h, y) = \int_X h\, d\mu^+ - \int_X h\, d\mu^-,$$

for every nonnegative function $h \in C(X)$ we have

$$L(h, y) + \int_X h\, d\mu^- = \int_X h\, d\mu^+ \ge 0.$$

We estimate $\int_X h\, d\mu^-$ from above to obtain

$$\begin{aligned} \int_X h\, d\mu^- &\le \|h\|_X \cdot \int_X 1_X\, d\mu^- \\ &= \|h\|_X \cdot \int_X 1_X\, d\left(\sup(-\mu, 0)\right) \\ &= \|h\|_X \cdot \int_X 1_X\, d\left(\tfrac{1}{2}(-\mu + |\mu|)\right) \\ &= \|h\|_X \cdot \tfrac{1}{2} \cdot \left(\int_X 1_X\, d(-\mu) + \int_X 1_X\, d|\mu|\right) \\ &= \|h\|_X \cdot \tfrac{1}{2} \cdot \left(-L(1_X, y) + \|\mu\|\right). \end{aligned}$$

(For the definition and existence of $\sup(-\mu, 0)$, $|\mu|$, and $\|\mu\|$ see J. Dieudonné [11, Chapts. 13.3, 13.9, 13.15, and 13.20]). Moreover,

$$\begin{aligned} \|\mu\| &= \sup\left\{\left|\int_X f\, d\mu\right| : f \in C(X),\ \|f\|_X \le 1\right\} \\ &= \sup\left\{|L(f, y)| : f \in C(X),\ \|f\|_X \le 1\right\} \\ &= \|\varepsilon_y \circ L\|. \end{aligned}$$

Thus

$$0 \le L(h,y) + \int_X h\,d\mu^- \le L(h,y) + \|h\|_X \cdot \tfrac{1}{2} \cdot (-L(1_X;y) + \|\varepsilon_y \circ L\|)\,,$$

or

$$L(h,y) \ge -\|h\|_X \cdot \tfrac{1}{2} \cdot (-L(1_X;y) + \|\varepsilon_y \circ L\|)\,.$$

This implies

$$\begin{aligned} L(g-h_1;y) &\ge -\|g-h_1\|_X \cdot \tfrac{1}{2} \cdot (-L(1_X;y) + \|\varepsilon_y \circ L\|)\,, \quad \text{and} \\ L(h_2-g;y) &\ge -\|h_2-g\|_X \cdot \tfrac{1}{2} \cdot (-L(1_X;y) + \|\varepsilon_y \circ L\|)\,; \end{aligned}$$

or

$$\begin{aligned} &L\big(g - g(g_A(y)) \cdot 1_X; y\big) \ge \\ &\ge -\|g-h_1\|_X \cdot \tfrac{1}{2} \cdot (-L(1_X;y) + \|\varepsilon_y \circ L\|) - |g|_{\text{Lip } 1} \cdot |L(d(\cdot, g_A(y)); y)|, \end{aligned}$$

and

$$\begin{aligned} &-L\big(g - g(g_A(y)) \cdot 1_X; y\big) \ge \\ &\ge -\|h_2-g\|_X \cdot \tfrac{1}{2} \cdot (-L(1_X;y) + \|\varepsilon_y \circ L\|) - |g|_{\text{Lip } 1} \cdot |L(d(\cdot, g_A(y)); y)|. \end{aligned}$$

These inequalities yield

$$\begin{aligned} |L\big(g - g(g_A(y)) \cdot 1_X; y\big)| \le{}& \max\{\|g-h_1\|_X, \|h_2-g\|_X\} \cdot \tfrac{1}{2} \cdot (\|\varepsilon_y \circ L\| \\ &- L(1_X;y)) + |g|_{\text{Lip } 1} \cdot |L(d(\cdot, g_A(y)); y)|. \end{aligned}$$

Now

$$\begin{aligned} \|g-h_1\|_X &= \sup\{|g(x) - g(g_A(y)) + |g|_{\text{Lip } 1} \cdot d(x, g_A(y))| : x \in X\} \\ &\le 2 \cdot |g|_{\text{Lip } 1} \cdot \sup\{|d(x, g_A(y))| : x \in X\} \\ &\le 2 \cdot d(X) \cdot |g|_{\text{Lip } 1}, \end{aligned}$$

and

$$\begin{aligned} \|h_2-g\|_X &= \sup\{|g(g_A(y)) + |g|_{\text{Lip } 1} \cdot d(x, g_A(y)) - g(x)| : x \in X\} \\ &\le 2 \cdot d(X) \cdot |g|_{\text{Lip } 1}. \end{aligned}$$

Thus

$$\begin{aligned} |(L - \widetilde{A})(g,y)| &= |L(g - g(g_A(y)) \cdot 1_X; y)| \\ &\le \Big[d(X) \cdot (\|\varepsilon_y \circ L\| - L(1_X, y)) + |L(d(\cdot, g_A(y)); y)|\Big] \cdot |g|_{\text{Lip } 1}. \end{aligned}$$

This settles the estimate for Lipschitz functions g. Applying Theorem 3.1 to $\Delta = L - \widetilde{A}$ gives

$$\begin{aligned} |(L - \widetilde{A})(f,y)| \le \max\Big\{&\tfrac{1}{2}\,(\|L\| + \|L(1_X)\|_Y)\,, \varepsilon^{-1}\Big[d(X) \cdot (\|\varepsilon_y \circ L\| - L(1_X, y)) \\ &+ |L(d(\cdot, g_A(y)); y)|\Big]\Big\}\widetilde{\omega}(f, \varepsilon). \end{aligned}$$

Together with the decomposition for $(L - A)(f; y)$ from the beginning of the proof, we arrive at the estimate of Theorem 3.2. □

Remark 3.3. *For estimates on approximation by bounded linear operators similar to the one of Theorem 3.2 using $\omega(f,\cdot)$ instead of $\widetilde{\omega}(f,\cdot)$ see Theorem 4.4.*

For positive linear operators we have the simpler estimate of

Theorem 3.4 (H. Gonska, [16, Theorem 3.1]). *Let A be of the form $A(f,y) = \psi_A(y)\cdot f(g_A(y))$, and let L be a positive linear operator, both mapping $C(X)$ into $B(Y)$. Then the following inequality holds for all $f \in C(X)$, $y \in Y$, and $\varepsilon > 0$:*

$$|(L-A)(f,y)| \leq$$
$$\leq \max\left\{\|L\|,\, \varepsilon^{-1}\cdot L(d(\cdot,g_A(y));y)\right\}\cdot\widetilde{\omega}(f,\varepsilon) + |(L-A)(1_X,y)|\cdot|f(g_A(y))|.$$

Proof. Let $0<\varepsilon$. Since L is a positive linear operator, $\varepsilon_y \circ L = L(\cdot,y)$ given by $L(\cdot,y) : C(X) \ni f \mapsto L(f,y) \in \mathbb{R}$ is a positive linear functional such that $\|\varepsilon_y \circ L\| = \|L(\cdot,y)\| = |L(1_X,y)| = L(1_X,y)$. Moreover, $\|L\| = \|L(1_X)\|_Y$, so that the estimate from Theorem 3.2 recudes to the one in Theorem 3.4. □

However, for positive linear operators a simpler proof of Theorem 3.4 is available which is given below.

Second proof of Theorem 3.4. First recall that $\widetilde{A}(f,y) = L(1_X,y)\cdot f(g_A(y))$. The estimate of $|(L-\widetilde{A})(g,y)|$ for functions $g \in \mathrm{Lip}\ 1$ may now be obtained in the following way.

$$\begin{aligned}|(L-\widetilde{A})(g,y)| &= |L(g-g(g_A(y))\cdot 1_X,y)| \leq L(|g-g(g_A(y))\cdot 1_X|,y)\\ &\leq L(d(\cdot,(g_A(y));y)\cdot|g|_{\mathrm{Lip}\ 1}.\end{aligned}$$

Applying Theorem 3.1 to the bounded operator $\Delta = L-\widetilde{A}$ implies for any $f \in C(X)$, and any $\varepsilon > 0$

$$\begin{aligned}|(L-\widetilde{A})(f,y)| &\leq \max\left\{\tfrac{1}{2}\cdot\|L-\widetilde{A}\|,\, \varepsilon^{-1}\cdot L(d(\cdot,(g_A(y));y)\right\}\cdot\widetilde{\omega}(f,\varepsilon)\\ &\leq \max\left\{\|L\|,\, \varepsilon^{-1}\cdot L(d(\cdot,(g_A(y));y)\right\}\cdot\widetilde{\omega}(f,\varepsilon),\end{aligned}$$

and the remaining part of the proof may be carried out as that of Theorem 3.2.
□

In the example to follow we illustrate Theorem 3.4 by discussing two univariate cases.

Example 3.5. (1) *If $X = K$ is the unit circle in $\mathbb{R}^2$ consisting of all complex numbers of the form e^{ix}, $0 \leq x < 2\pi$, and equipped with the metric*

$$d(e^{ix},e^{iy}) := \min\{|x-y|, 2\pi-|x-y|),$$

and if A is the canonical injection of $C_{\mathbb{R}}(K)$ into $B_{\mathbb{R}}(K)$, then for any positive linear operator L the estimate of Theorem 3.4 reads

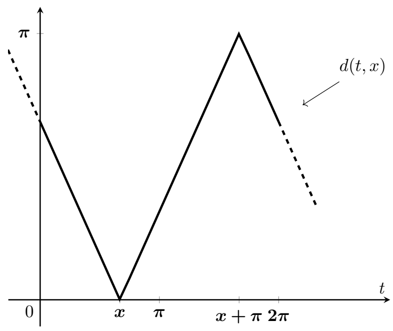
$$|L(f,e^{ix})-f(e^{ix})| \leq$$

$$\leq \max\left\{\|L\|,\, \varepsilon^{-1}\cdot L(d(\cdot,e^{ix});e^{ix})\right\}\cdot\widetilde{\omega}(f,\varepsilon)+|L(1_K;e^{ix})-1|\cdot|f(e^{ix})|.$$

Since $C_{\mathbb{R}}$ $(B_{\mathbb{R}}(K))$ may be identified with $C_{2\pi}$ $(B_{2\pi})$, the space of real-valued, 2π-periodic and continuous (bounded) functions defined on $\mathbb{R}$, the approximation problem sketched above is equivalent to considering the differences $|L(f,x)-f(x)|$ for positive linear operators L mapping $C_{2\pi}$ into $B_{2\pi}$, $f\in C_{2\pi}$, $x\in\mathbb{R}$. The above inequality then translates into

$$\begin{aligned}&|L(f,x)-f(x)|\leq\\&\leq \max\left\{\|L\|,\,\varepsilon^{-1}\cdot L(d(\cdot,x);x)\right\}\cdot\widetilde{\omega}(f,\varepsilon)+|L(1_{\mathbb{R}};x)-1|\cdot|f(x)|.\end{aligned}$$

Here, for x fixed, $d(\cdot,x)$ is the "saw tooth function" given below.



If L is a singular integral of the form

$$L(f;x)=(2\pi)^{-1}\cdot\int_{-\pi}^{\pi}f(x-u)\cdot\chi(u)\,du,$$

with a positive and even kernel $\chi\in L_{2\pi}$, and such that $\int_{-\pi}^{\pi}\chi(u)\,du=2\pi$, then it is well known that $L(d(\cdot,x);x)=(2\pi)^{-1}\cdot\int_{-\pi}^{\pi}|u|\cdot\chi(u)\,du$ and $L1_{\mathbb{R}}=1_{\mathbb{R}}$. Hence $\|L\|=1$, and for any $f\in C_{2\pi}$ and $\varepsilon>0$ we have

$$|L(f,x)-f(x)|\leq\max\left\{1,\,\varepsilon^{-1}\cdot(2\pi)^{-1}\cdot\int_{-\pi}^{\pi}|u|\cdot\chi(u)\,du\right\}\cdot\widetilde{\omega}(f,\varepsilon).$$

If, moreover, $\chi=\chi_n$ is a trigonometric polynomial of degree $\leq n$, then this is also the case for $Lf=L_nf$. Thus in this case any estimate obtained for $|L_n(f,x)-f(x)|$ in terms of $\widetilde{\omega}(f,\cdot)$ has to be compared to N.P. Korneičuk's estimate for the approximation constant E_nf, namely $E_n(f)\leq\frac{1}{2}\cdot\widetilde{\omega}(f,\pi/(n+1))$. Here the constant $\frac{1}{2}$ is best possible; see his book [29, p. 231f.] *for details. Moreover, using Jensen's inequality, S.B. Stečkin* [59] *showed that there is a* linear *operator $A_n:C_{2\pi}\to T_n$ (the space of trigonometric polynomials of degree $\leq n$) such that for all f in $C_{2\pi}$ one has*

$$\|A_nf-f\|\leq\tfrac{1}{4}\cdot(3+\sqrt{3})\cdot\widetilde{\omega}(f,\pi/(n+1))\leq 1.19\cdot\widetilde{\omega}(f,\pi/(n+1)).$$

It is thus natural to consider what can be obtained using positive linear *operators. Putting* $\varepsilon = \pi/(n+1)$ *in the next to the last inequality we arrive at*

$$\begin{aligned}&|L_n(f,x) - f(x)| \le \\ &\le \max\left\{1,\ (n+1)\cdot\pi^{-1}\cdot(2\pi)^{-1}\cdot\int_{-\pi}^{\pi}|u|\cdot\chi_n(u)\,du\right\}\cdot\widetilde{\omega}(f,\pi/(n+1)) \\ &= \max\left\{1,\ (n+1)\cdot\pi^{-2}\cdot\int_0^{\pi}u\cdot\chi_n(u)\,du\right\}\cdot\widetilde{\omega}(f,\pi/(n+1))\end{aligned}$$

for any positive and even kernel χ_n *of the form*

$$\chi_n(t) = 1 + 2\cdot\sum_{k=1}^{n}\varrho_k^{(n)}\cdot\cos kt.$$

Choosing $\varepsilon = \pi^{-1}\cdot\int_0^{\pi}u\cdot\chi_n(u)\,du$ *implies the inequality*

$$|L_n(f,x) - f(x)| \le \widetilde{\omega}\left(f, \pi^{-1}\cdot\int_0^{\pi}u\cdot\chi_n(u)\,du\right).$$

For an application using this choice of ε *see the paper of V.V. Žuk and G.I. Natanson* [65, Theor. 2]*; they investigated trigonometric polynomial operators* $Q_n : C_{2\pi}\to T_{n-1}$ *satisfying*

$$\|Q_nf - f\| \le \widetilde{\omega}(f, 2.4307/n).$$

See Example 3.11 (1) for a continuation of this discussion.

(2) *If* $X = [0,1]$ *and* $d(x,y) = |x-y|$*, then the general estimate implies*

$$|L(f,x) - f(x)| \le \max\left\{\|L\|,\ \varepsilon^{-1}L(|\cdot - x|;x)\right\}\widetilde{\omega}(f,\varepsilon) + |L(1_X;x) - 1|\cdot|f(x)|.$$

For $L1_X = 1_X$ *and* $\varepsilon = L(|\cdot - x|;x)$ *this inequality was proved in a paper of J. Meier and the author* [20] *using the* K*-functional approach (see e.g., Theorem 3.1). If* L *is a discrete operator of the form*

$$L(f,x) = \sum_{k=0}^{n} f(x_k^{(n)})\cdot p_k^{(n)}(x) \quad \text{with } p_k^{(n)}(x)\ge 0 \text{ for } x\in[0,1],$$

then the second quantity in the above max is equal to

$$\varepsilon^{-1}\sum_{k=0}^{n}|x_k^{(n)} - x|\cdot p_k^{(n)}(x).$$

For this special case and $\sum_{k=0}^{n}p_k^{(n)} = 1$*, V. Žuk and G. Natanson* [64] *used Jensen's inequality to derive the upper bound* $1\cdot\widetilde{\omega}\left(f, \sum_{k=0}^{n}|x_k^{(n)} - x|\cdot p_k^{(n)}(x)\right)$.

However, an explicit representation of $\sum_{k=0}^{n}|x_k^{(n)} - x|\cdot p_k^{(n)}(x)$ *has been given so far only for special operators, e.g., Bernstein's classical example (see F. Schurer and F.W. Steutel* [53]*) and – for a corresponding quantity – for the operators introduced by Meyer-König and Zeller (see F. Schurer and F.W. Steutel* [54, Lemma 2.3]*).*

For further univariate results in terms of $\widetilde{\omega}(f,\cdot)$ see Example 3.11. Several estimates for the univariate case in terms of $\omega(f,\cdot)$ instead of $\widetilde{\omega}(f,\cdot)$ will be classified in Discussion 4.13.

The right hand side in the estimate of Theorem 3.4 contains two test functions, namely

$$1_X : X \ni x \mapsto 1 \in \mathbb{R},$$

and

$$d(\cdot, g_A(y)) : X \ni x \mapsto d(x, g_A(y)) \in \mathbb{R}.$$

It is possible to replace both of them by other test functions as will be seen below. We do this first for 1_X. The estimate of the next theorem resembles one obtained by T. Nishishiraho [42, Lemma 4].

THEOREM 3.6. *If A is of the form $A(f,y) = \psi_A(y)\cdot f(g_A(y))$, and L is a positive linear operator, both mapping $C(X)$ into $B(Y)$, then for all $f \in C(X)$, $y \in Y$, $\varepsilon > 0$ and $h \in C(X)$ such that $(f/h)(g_A(y))$ is defined we have*

$$\begin{aligned}|(L-A)(f,y)| \le \max\Big\{\|L\|,\ \varepsilon^{-1}\cdot L(d(\cdot,g_A(y));y)\Big\}\cdot\\ \cdot\Big(\widetilde{\omega}(f,\varepsilon) + |(f/h)(g_A(y))|\cdot\widetilde{\omega}(h,\varepsilon)\Big)\\ + |(f/h)(g_A(y))|\cdot|(L-A)(h,y)|.\end{aligned}$$

Proof. If A is given as above, and if $h \in C(X)$ is such that for the fixed y the expression $(f/h)(g_A(y))$ is defined, then

$$\begin{aligned}|(L-A)(f,y)| &\le |L(f,y) - (f/h)(g_A(y))\cdot L(h,y)| + |(f/h)(g_A(y))L(h,y) -\\ &\quad - \psi_A(y)\cdot f(g_A(y))|\\ &= |L(f,y) - (f/h)(g_A(y))\cdot L(h,y)|\\ &\quad + |(f/h)(g_A(y))L(h,y) - (f/h)(g_A(y))\psi_A(y)\cdot h(g_A(y))|\\ &= |L(f,y) - (f/h)(g_A(y))\cdot L(h,y)| + |(f/h)(g_A(y))|\cdot|(L-A)(h,y)|.\end{aligned}$$

The first term of the last sum can be estimated as follows:

$$\begin{aligned}&|L(f,y) - (f/h)(g_A(y))\cdot L(h,y)| \le\\ &\le |L(f,y) - L(1_X,y)\cdot f(g_A(y))| + |L(1_X,y)\cdot f(g_A(y)) - (f/h)(g_A(y))\cdot L(h,y)|\\ &= |(L-\widetilde{A})(f,y)| + |(f/h)(g_A(y))|\cdot|(\widetilde{A}-L)(h,y)|.\end{aligned}$$

Here $\widetilde{A}$ is the operator from the proofs of Theorem 3.2, Theorem 3.4, for which

$$|(L-\widetilde{A})(f,y)| \le \max\Big\{\|L\|,\ \varepsilon^{-1}\cdot L(d(\cdot,g_A(y));y)\Big\}\cdot\widetilde{\omega}(f,\varepsilon).$$

Using this estimate for both $|(L-\widetilde{A})(f,y)|$ and $|(\widetilde{A}-L)(h,y)|$ gives

$$\begin{aligned}&|L(f,y) - (f/h)(g_A(y))\cdot L(h,y)| \le\\ &\le \max\Big\{\|L\|,\ \varepsilon^{-1}\cdot L(d(\cdot,g_A(y));y)\Big\}\cdot\Big(\widetilde{\omega}(f,\varepsilon) + |(f/h)(g_A(y))|\cdot\widetilde{\omega}(h,\varepsilon)\Big),\end{aligned}$$

which yields the claim of Theorem 3.6. □

Obviously, for $h = 1_X$ our last estimate is again the one from Theorem 3.4. It is also possible to replace 1_X in a different way:

THEOREM 3.7. *Let $A : C(X) \to B(Y)$ be of the form $A(f, y) = \psi_A(y) \cdot f(g_A(y))$, let $L : C(X) \to B(Y)$ be a positive linear operator, $f \in C(X)$, $y \in Y$ and $\varepsilon > 0$. Moreover, let $h \in C(X)$ be such that $f/h \in C(X)$ and $\|L\| \cdot \|h\|_X + |L(h, y)| \neq 0$. Then*

$$\begin{aligned} &|(L - A)(f, y)| \leq \\ &\leq \max\left\{\tfrac{1}{2}\Big(\|L\| \cdot \|h\| + |L(h, y)|\Big), \|h\| \cdot \varepsilon^{-1} \cdot L(d(\cdot, g_A(y)); y)\right\} \cdot \widetilde{\omega}(f/h, \varepsilon) \\ &\quad + |(f/h)(g_A(y))| \cdot |(L - A)(h, y)|. \end{aligned}$$

Proof. As in the proof of Theorem 3.6, we first have

$$|(L-A)(f, y)| \leq |L(f, y)-(f/h)(g_A(y))\cdot L(h, y)|+|(f/h)(g_A(y))|\cdot|(L-A)(h, y)|.$$

The second term is already part of the right hand side in the above estimate. For the first term we have

$$\begin{aligned} |L(f, y) - (f/h)(g_A(y)) \cdot L(h, y)| &= |L((f/h) \cdot h, y) - (f/h)(g_A(y)) \cdot L(h, y)| \\ &= \Big|L\Big([(f/h) - (f/h)(g_A(y))] \cdot h; y\Big)\Big|. \end{aligned}$$

Now let $g \in \text{Lip } 1$ be arbitrarily given. Then the last quantity is equal to

$$\begin{aligned} &\Big|L\Big([(f/h) - g + g - ((f/h) - g + g)(g_A(y))] \cdot h; y\Big)\Big| \leq \\ &\leq \Big|L\Big([(f/h) - g - ((f/h) - g)(g_A(y))] \cdot h; y\Big)\Big| + \Big|L\Big([g - g(g_A(y))] \cdot h; y\Big)\Big| \\ &\leq \|L\|\cdot\|h\|\cdot\|(f/h) - g\|+|L(h, y)|\cdot\|(f/h) - g\|+\|h\| \cdot |g|_{\text{Lip } 1} \cdot L(d(\cdot, g_A(y)); y) \\ &= \Big(\|L\| \cdot \|h\| + |L(h, y)|\Big)\cdot \\ &\quad \cdot\left\{\|(f/h) - g\| + \|h\| \cdot \Big(\|L\| \cdot \|h\| + |L(h, y)|\Big)^{-1} \cdot L(d(\cdot, g_A(y)); y) \cdot |g|_{\text{Lip } 1}\right\}. \end{aligned}$$

Thus

$$\begin{aligned} &|L(f, y) - (f/h)(g_A(y)) \cdot L(h, y)| \leq \\ &\leq \Big(\|L\| \cdot \|h\| + |L(h, y)|\Big)\cdot \\ &\quad K\Big(\|h\|\Big(\|L\|\cdot\|h\|+|L(h, y)|\Big)^{-1}\cdot L(d(\cdot, g_A(y)); y)\cdot(2/\varepsilon)\cdot(\varepsilon/2), f/h; C(X), \text{Lip } 1\Big) \\ &\leq \Big(\|L\| \cdot \|h\| + |L(h, y)|\Big)\cdot \\ &\quad \cdot \max\left\{1, \|h\|\Big(\|L\| \cdot \|h\| + |L(h, y)|\Big)^{-1} \cdot L(d(\cdot, g_A(y)); y) \cdot 2/\varepsilon\right\}\cdot \\ &\quad \cdot K\Big(\varepsilon/2, f/h; C(X), \text{Lip } 1\Big) \\ &= \max\left\{\tfrac{1}{2}\Big(\|L\| \cdot \|h\| + |L(h, y)|\Big), \|h\| \cdot \varepsilon^{-1} \cdot L(d(\cdot, g_A(y)); y)\right\} \cdot \widetilde{\omega}(f/h, \varepsilon). \end{aligned}$$

This concludes the proof. □

Corollary 3.8. (1) *If $h = 1_X$ we obtain the inequality*

$$|(L-A)(f,y)| \le \max\left\{\tfrac{1}{2}\Big(\|L\| + |L(1_X,y)|\Big),\ \varepsilon^{-1}\cdot L(d(\cdot,g_A(y));y)\right\}\cdot\widetilde{\omega}(f,\varepsilon) + |f(g_A(y))|\cdot|(L-A)(1_X,y)|,$$

which is slightly better than that of Theorem 3.4.

(2) *If $Y = X$ and $A(f,x) = f(x)$ for $x \in X$, then the estimate of Theorem 3.7 becomes*

$$|L(f,x) - f(x)| \le$$
$$\le \max\left\{\tfrac{1}{2}\Big(\|L\|\cdot\|h\| + |L(h,y)|\Big),\ \|h\|\cdot\varepsilon^{-1}\cdot L(d(\cdot,x);x)\right\}\cdot\widetilde{\omega}(f/h,\varepsilon)$$
$$+ |(f/h)(x)|\cdot|L(h,x)-h(x)| \le \|h\|\cdot\max\left\{\|L\|,\ \varepsilon^{-1}\cdot L(d(\cdot,x);x)\right\}\cdot\widetilde{\omega}(f/h,\varepsilon)$$
$$+ |(f/h)(x)|\cdot|L(h,x)-h(x)|.$$

Next we show how the test functions

$$d(\cdot,g_A(y))\ :\ X \ni x \mapsto d(x,g_A(y)) \in \mathbb{R}$$

can be replaced. To this end let Φ be a function on X^2 such that $\Phi(\cdot,y) \in C(X)$ for all $y \in X$, and that for some fixed $q > 0$ the condition

$$d(x,y)^q \le \Phi(x,y) \quad \text{for all } x,y \in X$$

holds. For possible choices of Φ see the examples of T. Nishishiraho [43, p. 445f]. The following theorem is an extension of Theorem 3.2 in our paper [17].

Theorem 3.9. *Let $A : C(X) \to B(Y)$ be of the form $A(f,y) = \psi_A(y)\cdot f(g_A(y))$, let $L : C(X) \to B(Y)$ be a positive operator, $f \in C(X)$, $y \in Y$, and $\varepsilon > 0$. Let $h \in C(X)$ be such that $f/h \in C(X)$. If Φ and q are given as above, then the following estimates hold:*

$$|(L-A)(f,y)| \le$$
$$\le \max\left\{\|L\|,\ \varepsilon^{-1}\cdot C(L,\Phi,A,y)\right\}\cdot\left\{\widetilde{\omega}(f,\varepsilon) + |(f/h)(g_A(y))|\cdot\widetilde{\omega}(h,\varepsilon)\right\}$$
$$+ |(f/h)(g_A(y))|\cdot|(L-A)(h,y)|,$$
$$|(L-A)(f,y)| \le$$
$$\le \max\left\{\tfrac{1}{2}\cdot\Big(\|L\|\cdot\|h\| + |L(h,y)|\Big),\ \|h\|/\varepsilon\cdot C(L,\Phi,A,y)\right\}\cdot\widetilde{\omega}(f/h,\varepsilon)$$
$$+ |(f/h)(g_A(y))|\cdot|(L-A)(h,y)|,$$

where

$$C(L,\Phi,A,y) := \inf\left\{\Big(L(\Phi(\cdot,g_A(y))^{p/q};y)^{1/p}\cdot L(1_X;y)^{1-1/p}\Big)\ :\ p \ge 1\right\}.$$

Proof. (1) is obtained by using Theorem 3.6, estimate (2) with the aid of Theorem 3.7. In both cases it is only necessary to estimate $L(d(\cdot, g_A(y)); y)$ in order to obtain the claim of Theorem 3.9.

For y fixed, the functional $L(*, y)$ is a positive linear form on $C(X)$. Thus if $p \geq 1$, Hölder's inequality implies

$$L(d(\cdot, g_A(y)); y) \leq L(d(\cdot, g_A(y))^p; y)^{1/p} \cdot L(1_X; y)^{1-1/p}.$$

If $q > 0$ is fixed and such that

$$d(x, y)^q \leq \Phi(x, y) \quad \text{for all } x, y \in X,$$

we have

$$\begin{aligned} L(d(\cdot, g_A(y)); y) &\leq L\Big(d(\cdot, g_A(y))^p; y\Big)^{1/p} \cdot L(1_X; y)^{1-1/p} \\ &\leq L\Big(\Phi(\cdot, g_A(y))^{p/q}; y\Big)^{1/p} \cdot L(1_X; y)^{1-1/p}. \end{aligned}$$

Hence

$$L(d(\cdot, g_A(y)); y) \leq \inf\left\{\Big(L(\Phi(\cdot, g_A(y))^{p/q}; y)^{1/p} \cdot L(1_X; y)^{1-1/p}\Big) \;:\; p \geq 1\right\}.$$

Combining this with the inequalties from Theorem 3.6, Theorem 3.7, respectively, shows the validity of our claim. □

Corollary 3.10 (H. Gonska [17, Corollary 3.3]). *For the special case $Y = X$, $A(f, y) = f(x)$, $h = 1_X$, $L1_X = 1_X$ both inequalities of Theorem 3.9 imply*

$$|L(f, x) - f(x)| \leq \max\left[1,\, \varepsilon^{-1} \cdot \inf\left\{\Big(L(\Phi(\cdot, x)^{p/q}; x)^{1/p}\Big) \;:\; p \geq 1\right\}\right] \cdot \widetilde{\omega}(f, \varepsilon).$$

Our next example continues the discussion of Example 3.5.

Example 3.11. 1) *If $C_{2\pi}$ is given as in Example 3.5 (1), then the computation of $L(d(\cdot, x); x)$ may be cumbersome. From Theorem 3.9 with $h = 1_{\mathbb{R}}$, $q = 1$, $\Phi(x, y) = \pi \cdot \sin \frac{1}{2}|x - y|$, there follows for any $p \geq 1$*

$$\begin{aligned} &|L(f, x) - f(x)| \leq \\ &\leq \max\left\{\|L\|,\, \varepsilon^{-1}\pi \cdot L\left(\sin^p(\tfrac{1}{2}|\cdot - x|); x\right)^{1/p} \cdot L(1_{\mathbb{R}}; x)^{1-1/p}\right\} \cdot \widetilde{\omega}(f, \varepsilon) \\ &\quad + |L(1_{\mathbb{R}}; x) - 1| \cdot |f(x)|. \end{aligned}$$

For $p = 2$ we obtain

$$\begin{aligned} &|L(f, x) - f(x)| \leq \\ &\leq \max\left\{\|L\|,\, \varepsilon^{-1}\pi \cdot L\left(\sin^2 \tfrac{1}{2}(\cdot - x); x\right)^{1/2} \cdot L(1_{\mathbb{R}}; x)^{1/2}\right\} \cdot \widetilde{\omega}(f, \varepsilon) \\ &\quad + |L(1_{\mathbb{R}}; x) - 1| \cdot |f(x)|. \end{aligned}$$

If, as in Example 3.5 (2), L_n is a singular integral with kernel χ_n being an even and positive trigonometric polynomial of degree $\le n$ and given by

$$\chi_n(t) = 1 + 2 \cdot \sum_{k=1}^{n} \varrho_k^{(n)} \cos kt,$$

then $L_n 1_{\mathbb{R}} = 1_{\mathbb{R}}$ and

$$L_n\left(\sin^2 \tfrac{1}{2}(\cdot - x); x\right) = \tfrac{1}{2}(1 - \varrho_1^{(n)}).$$

Hence $\|L_n\| = 1$, and the last inequality becomes

$$|L_n(f,x) - f(x)| \le \max\left\{1,\, \varepsilon^{-1} \cdot \pi \cdot \left(\tfrac{1}{2}(1 - \varrho_1^{(n)})\right)^{1/2}\right\} \cdot \widetilde{\omega}(f, \varepsilon).$$

The crucial quantity $\left(\frac{1}{2}(1 - \varrho_1^{(n)})\right)^{1/2}$ was already determined for quite a number of kernels; its importance seems to have been first observed by P.P. Korovkin [31]. In order to make $\varrho_1^{(n)}$ as big as possible among all χ_n's of the above form, one has to choose the Fejér-Korovkin kernel in which case

$$\varrho_1^{(n)} = \cos \tfrac{\pi}{n+2}.$$

Because $\left(\frac{1}{2}(1 - \varrho_1^{(n)})\right)^{1/2} = \sin \frac{\pi}{2(n+2)}$, for the Fejér-Korovkin operators the general inequality mentioned at the beginning of this example implies

$$|L_n(f,x) - f(x)| \le \tfrac{1}{2}\pi \cdot \widetilde{\omega}\left(f, \tfrac{\pi}{n+1}\right), \quad n \ge 0.$$

Note that this result is somewhat worse than the one for the operators Q_n of V.V. Žuk and G.I. Natanson [65] mentioned in Example 3.5 (1). However, note as well that the above upper bound for approximation by the Fejér-Korovkin operators was obtained after *applying the Cauchy-Schwarz inequality (which in many cases causes quite bad results).*

2) *For $X = [0,1]$, $d(x,y) = |x-y|$, and for $h = 1_X$, $q = 1$, $\Phi(x,y) = d(x,y)$ we arrive at ($p \ge 1$ arbitrary)*

$$\begin{aligned} |L(f,x) - f(x)| \le{}& \max\left\{\|L\|,\, \varepsilon^{-1} \cdot L\left(|\cdot - x|^p; x\right)^{1/p} \cdot L(1_X; x)^{1-1/p}\right\} \cdot \widetilde{\omega}(f, \varepsilon) \\ &+ |L(1_X; x) - 1| \cdot |f(x)|. \end{aligned}$$

If $p = 2$ one has

$$\begin{aligned} |L(f,x) - f(x)| \le{}& \max\left\{\|L\|,\, \varepsilon^{-1} \cdot L\left((\cdot - x)^2; x\right)^{1/2} \cdot L(1_X; x)^{1/2}\right\} \cdot \widetilde{\omega}(f, \varepsilon) \\ &+ |L(1_X; x) - 1| \cdot |f(x)|, \end{aligned}$$

and for $L1_X = 1_X$ and $\varepsilon = L\left((\cdot - x)^2; x\right)^{1/2}$ (if greater than 0) it reduces to

$$|L(f,x) - f(x)| \le \widetilde{\omega}\left(f; L\left((\cdot - x)^2; x\right)^{1/2}\right).$$

This was first proved by J. Peetre [45].

For a discussion of estimates for the univariate case involving $\omega(f,\cdot)$ *instead of* $\widetilde{\omega}(f,\cdot)$ *the reader is referred to Discussion 4.13.*

Our next theorem generalizes Theorem 3.4 in another direction. It shows that this theorem implies a variety of estimates including uniform ones and estimates in L_1 spaces.

THEOREM 3.12 (H. Gonska [16, Theorem 3.4]). *Let* (X,d) *be a compact metric space and* $Y \neq 0$ *be some set. Let* L *be a positive linear operator and* A *be given by* $Af = \psi_A(f \circ g_A)$*, both mapping* $C(X)$ *into* $B(Y)$*. Moreover, let* M *be a set of positive linear functionals* μ *defined on* $B(Y)$ *such that* $p_M[h] := \sup\{\mu(|h|) : \mu \in M\} < \infty$ *for all* $h \in B(Y)$*, and* $p_M[1_Y] > 0$*. Then for all* $f \in C(X)$ *and all* $\varepsilon > 0$ *the following inequality holds:*

$$\begin{aligned} p_M[Lf - Af] \leq\ & p_M[(L1_X - A1_X)\cdot(f\circ g_A)]+ \\ & + \max\left\{\|L\|\cdot p_M[1_Y], \varepsilon^{-1}\cdot p_M\left[L(d(\cdot, g_A(*)); *)\right]\right\}\cdot\widetilde{\omega}(f,\varepsilon). \end{aligned}$$

Here L *is applied with respect to the variable indicated by "$\cdot$", and* $\mu \in M$ *is applied with respect to "$*$".*

Proof. If $g \in \mathrm{Lip}\ 1$ is arbitrarily given, then, as in the proof of Theorem 3.2 and in the second proof of Theorem 3.4, the following holds:

$$\begin{aligned} &|(L-A)(f,y)| \leq \\ &\leq |(L-A)(1_X,y)|\cdot|f(g_A(y))| + |(L-\widetilde{A})(f-g+g;y)| \\ &\leq |(L-A)(1_X,y)|\cdot|f(g_A(y))| + |(L-\widetilde{A})(f-g;y)| + |(L-\widetilde{A})(g;y)| \\ &\leq |(L-A)(1_X,y)|\cdot|f(g_A(y))| + 2\cdot\|L\|\cdot\|f-g\|_X + L(d(\cdot,g_A(y));y)\cdot|g|_{\mathrm{Lip}\ 1}. \end{aligned}$$

Applying the positive functional $\mu \in M$ to this inequality means

$$\begin{aligned} (*)\qquad \mu(|Lf-Af|) \leq\ & \mu(|L1_X - A1_X|\cdot|f\circ g_A|) + 2\cdot\|L\|\cdot\|f-g\|_X\cdot\mu(1_Y) \\ & +\mu(L(d(\cdot,g_A(*));*))\cdot|g|_{\mathrm{Lip}\ 1}. \end{aligned}$$

If $L = 0$, then it reads

$$\mu(|Lf-Af|) \leq \mu(|L1_X - A1_X|\cdot|f\circ g_A|),$$

implying

$$p_M[Lf-Af] \leq p_M[(L1_X - A1_X)\cdot(f\circ g_A)].$$

Hence in this case our claim is true.

If $L \neq 0$, then the right hand side of $(*)$ may be estimated from above by

$$\begin{aligned} & p_M[(L1_X - A1_X)\cdot(f\circ g_A)]+2\cdot\|L\|\cdot p_M[1_Y]\cdot \\ & \cdot\left\{\|f-g\|_X + (2\cdot\|L\|\cdot p_M[1_Y])^{-1}\cdot p_M[L(d(\cdot,g_A(*));*)]\cdot|g|_{\mathrm{Lip}\ 1}\right\}. \end{aligned}$$

Thus in this case we arrive at

$$\mu(|Lf-Af|) \leq p_M[(L1_X - A1_X)\cdot(f\circ g_A)] + 2\cdot\|L\|\cdot p_M[1_Y]\cdot$$

$$\cdot K\left((2\|L\|\cdot p_M[1_Y])^{-1}\cdot p_M[L(d(\cdot,g_A(*));*)],\ f;\ C(X),\ \mathrm{Lip}\ 1\right).$$

For $\varepsilon > 0$ the second term on the right hand side may be written as

$$2\|L\|p_M[1_Y]\cdot K\left((2\|L\|p_M[1_Y])^{-1}p_M[L(d(\cdot,g_A(*));*)]\cdot\tfrac{\varepsilon}{2}\cdot\tfrac{2}{\varepsilon},\ f;\ C(X),\mathrm{Lip}\ 1\right)\le$$
$$\le\max\left\{\|L\|\cdot p_M[1_Y],\varepsilon^{-1}\cdot p_M[L(d(\cdot,g_A(*));*)]\right\}\cdot\widetilde{\omega}(f,\varepsilon),$$

and thus we have

$$\mu(|Lf-Af|)\le p_M[(L1_X-A1_X)\cdot(f\circ g_A)]+$$
$$+\max\left\{\|L\|\cdot p_M[1_Y],\varepsilon^{-1}\cdot p_M[L(d(\cdot,g_A(*));*)]\right\}\cdot\widetilde{\omega}(f,\varepsilon).$$

Passing to the supremum in M now gives the estimate of Theorem 3.12. $\square$

Note that if in Theorem 3.12 one has $M=\{\varepsilon_y\}$ for some point evaluation functional ε_y, $y\in Y$, then the estimate given reduces to that given in Theorem 3.4. The following Corollary 3.13 establishes a similar generalization of Theorem 3.9 (1) (for the case $h=1_X$).

Corollary 3.13 (H. Gonska [16, Corollary 3.5]). *Let the assumptions of Theorem 3.12 be fulfilled. If Φ is given on X^2 such that $\Phi(\cdot,y)\in C(X)$ for all $y\in X$, and that for some $q>0$ the condition*

$$d(x,y)^q\le\Phi(x,y)\quad\text{for all } x,y\in X$$

holds, then

$$p_M[Lf-Af]\le p_M[(L1_X-A1_X)\cdot(f\circ g_A)]+$$
$$+\max\left\{\|L\|\cdot p_M[1_Y],\varepsilon^{-1}\cdot C(L,\Phi,A,M)\right\}\cdot\widetilde{\omega}(f,\varepsilon),$$

where

$$C(L,\Phi,A,M):=$$
$$:=\inf\left\{\left(p_M[L(\Phi(\cdot,g_A(*))^{p/q},*)]\right)^{1/p}\cdot p_M[L(1_X;*)]^{1-1/p}\ :\ p\ge1\right\}.$$

Proof. The assertion in Corollary 3.13 results from an estimate of $p_M[L(d(\cdot,g_A(*));*)]$ in Theorem 3.12. For any $p\ge1$ we have

$$p_M[L(d(\cdot,g_A(*));*)]\le p_M\left[L(\Phi(\cdot,g_A(*))^{p/q},*)^{1/p}\cdot L(1_X;*)^{1-1/p}\right].$$

Applying Hölder's inequality again, we conclude as in the proof of Theorem 3.12 that

$$p_M\left[L(\Phi(\cdot,g_A(*))^{p/q},*)^{1/p}\cdot L(1_X;*)^{1-1/p}\right]\le$$
$$\le\left(p_M\left[L(\Phi(\cdot,g_A(*))^{p/q},*)\right]\right)^{1/p}\cdot p_M\left[L(1_X;*)\right]^{1-1/p}.$$

Passing to the inf over all $p\ge1$ implies

$$p_M[L(d(\cdot,g_A(*));*)]\le$$

$$\le \inf\left\{\left(p_M\left[L(\Phi(\cdot, g_A(*))^{p/q}, *)\right]\right)^{1/p} \cdot p_M\left[L(1_X; *)\right]^{1-1/p} \;:\; p \ge 1\right\}.$$

This yields the claim of Corollary 3.13. □

REMARK 3.14. *The estimates given in Theorems 3.4 and 3.12 are best possible in a certain sense. To show this, let* $Y = \{x_0\}$, $A(f, x_0) = f(x_0)$, *and* $L(1_X, x_0) = 1$ *for some fixed point* x_0 *in* X. *In this case the inequality in Theorem 3.4 (or the one in Theorem 3.12 for* $M = \{\varepsilon_{x_0}\}$*) reduces to*

$$|L(f, x_0) - f(x_0)| \le \max\left\{1, \varepsilon^{-1} \cdot L(d(\cdot, x_0); x_0)\right\} \cdot \tilde{\omega}(f, \varepsilon).$$

For $L(d(\cdot, x_0); x_0) = 0$ *we have* $|L(f, x_0) - f(x_0)| \le \tilde{\omega}(f, \varepsilon)$ *for all* $\varepsilon > 0$, *and thus*

$$|L(f, x_0) - f(x_0)| \le \tilde{\omega}\Big(f, L(d(\cdot, x_0); x_0)\Big).$$

If $L(d(\cdot, x_0); x_0) > 0$, *then choose* $\varepsilon = L(d(\cdot, x_0); x_0)$, *and this gives the same inequality.*

Now take $f(\cdot) = d(\cdot, x_0)$. *Hence*

$$(*) \quad |L(d(\cdot, x_0); x_0) - d(x_0, x_0)| = L(d(\cdot, x_0); x_0) \le \tilde{\omega}\Big(d(\cdot, x_0), L(d(\cdot, x_0); x_0)\Big).$$

If, for instance, $X = [a, b]$ *and* $d(x, y) = |y - x|$, *then*

$$\omega(|\cdot - x_0|, h) = \min\{h, \max\{b - x_0, x_0 - a\}\}$$

is a concave function, and hence

$$\tilde{\omega}(|\cdot - x_0|, h) = \omega(|\cdot - x_0|, h).$$

In particular, $\tilde{\omega}(|\cdot - x_0|, L(|\cdot - x_0|; x_0)) = L(|\cdot - x_0|; x_0)$, *so that inequality* $(*)$ *becomes an equality which shows that the constant 1 in*

$$|L(f, x_0) - f(x_0)| \le 1 \cdot \tilde{\omega}\Big(f, L(d(\cdot, x_0); x_0)\Big)$$

cannot be improved in general.

As a further consequence of Theorem 3.4 we mention

THEOREM 3.15 (cf. T. Nishishiraho [43, Th. 4]). *Let* X *be a compact subset of a real pre-Hilbert space with inner product* $\langle \cdot, * \rangle$. *Let* $\varepsilon > 0$. *If* $L : C(X) \to C(X)$ *is a positive linear operator, then for all* $f \in C(X)$ *and* $x \in X$ *there holds*

$$\begin{aligned} |L(f, x) - f(x)| &\le |f(x)| \cdot |L(1_X, x) - 1| + \\ &\quad + \max\left\{\|L\|, \varepsilon^{-1} \cdot L(d(\cdot, x)^2; x)^{1/2} \cdot L(1_X, x)^{1/2}\right\} \cdot \tilde{\omega}(f, \varepsilon). \end{aligned}$$

Here $d(x, y) = \langle x - y, x - y \rangle^{1/2}$.

Proof. Use Theorem 3.9 (1) with $Y = X$, $A(f, x) = f(x)$, $h = 1_X$, $\Phi(x, y) = d(x, y)$ and $p = 2$. □

Under the assumptions of Theorem 3.15 it is of course also possible to give estimates similar to those in Theorem 3.12 or Corollary 3.13.

4. ESTIMATES ON BOUNDED AND POSITIVE LINEAR OPERATOR APPROXIMATION IN TERMS OF MODULI OF CONTINUITY

In this section we shall prove inequalities in terms of the modulus of continuity. In Section 4.1 we shall give some consequences of Theorem 3.12, thus employing the relationship between $\widetilde{\omega}$ and ω as given in Lemma 2.4. As will be seen in Remark 4.3, in many important cases our technique yields results as good as those obtained by T. Nishishiraho. Thus even for the special compact spaces (X, d) considered by Nishishiraho (who used the direct approach), our general approach via least concave majorants may be more powerful.

Another main point of Section 4 is motivated by an example in Nishishiraho's paper [43, p. 453f.] dealing with Bernstein operators. The estimates given there are consequences of one of his more general theorems. As will be discussed in Section 4.2, however, his version of the direct technique is not sensitive enough to imply a certain best possible result obtained by P.C. Sikkema [58]. Thus the second purpose of this section is to refine the direct technique in a way such that best possible constants can be obtained when evaluating the general upper bound.

Finally, in Section 4.3 it will be shown that the indirect and the direct approach are of equal value in the sense that each of them may give better results than the other one.

4.1. Estimates in terms of ω via Least Concave Majorants.

We first discuss some of the consequences of Theorem 3.12.

THEOREM 4.1 (cf. H. Gonska [16, Theorem 3.9]). *Let (X, d) be a compact metric space such that for all ξ, $\varepsilon > 0$ and all $f \in C(X)$ the inequality*

$$\omega(f, \xi \cdot \varepsilon) \leq (1 + \eta\xi) \cdot \omega(f, \varepsilon)$$

holds, where $\eta > 0$ is a fixed constant. If the assumptions of Theorem 3.12 are satisfied, then the following hold.

(1) *If the function $f \in C(X)$ has a concave modulus of continuity, then for each $\varepsilon > 0$ we have*

$$p_M[Lf - Af] \leq p_M[(L1_X - A1_X) \cdot (f \circ g_A)] + \\ + \max\left\{\|L\| \cdot p_M[1_Y],\, \varepsilon^{-1} \cdot p_M[L(d(\cdot, g_A(*)); *)]\right\} \cdot \omega(f, \varepsilon).$$

(2) *Otherwise we have for any $h, \varepsilon > 0$ the inequality*

$$p_M[Lf - Af] \leq p_M[(L1_X - A1_X) \cdot (f \circ g_A)] + \\ + (1 + \eta h \varepsilon^{-1}) \max\left\{\|L\| \cdot p_M[1_Y],\, h^{-1} \cdot p_M[L(d(\cdot, g_A(*)); *)]\right\} \cdot \omega(f, \varepsilon).$$

Proof. The inequality of (1) is an immediate consequence of $\widetilde{\omega}(f, \varepsilon) = \omega(f, \varepsilon)$. That of (2) is obtained by using the estimate of Theorem 3.12 (with h instead of ε) and then observing that because of Lemma 2.4 (1) one has

$$\widetilde{\omega}(f, h) = \widetilde{\omega}(f, h \cdot \varepsilon^{-1} \cdot \varepsilon) \leq (1 + \eta \cdot h \cdot \varepsilon^{-1}) \cdot \omega(f, \varepsilon),$$

where $\varepsilon > 0$ again is arbitrarily given (and independent of h). □

COROLLARY 4.2. *If $M = \{\varepsilon_y\}$ for $y \in Y$ then the estimates of Theorem 4.1 reduce to*

$$(i') \qquad |(L-A)(f,y)| \leq |(L-A)(1_X,y)| \cdot |f(g_A(y))| + \max\left\{\|L\|,\, \varepsilon^{-1} \cdot p_M[L(d(\cdot,g_A(y));y)\right\} \cdot \omega(f,\varepsilon),$$

and

(ii')

$$|(L-A)(f,y)| \leq |(L-A)(1_X,y)| \cdot |f(g_A(y))| + (1+\eta h\varepsilon^{-1}) \cdot \max\left\{\|L\|,\, h^{-1} \cdot L(d(\cdot,g_A(y));y)\right\} \cdot \omega(f,\varepsilon),$$

respectively.

REMARK 4.3. 1) *For $h = \varepsilon$, estimate (2) of Theorem 4.1 was given in the author's paper* [16]. *It is however of advantage to have two independent parameters h and ε available, as can be seen in (2), for example.*

2) *In order to compare the result in Theorem 4.1 to one obtained by T. Nishishiraho* [42], *we consider again the following situation:*

$Y = X$, $M = \{\varepsilon_x\}$ for some fixed point evaluation functional ε_x, $A(f,x) = f(x)$, $L1_X = 1_X$. Then, for any $f \in C(X)$ having a concave modulus of continuity, the above inequality yields for any $\varepsilon > 0$ Î

$$|L(f,x) - f(x)| \leq \max\left\{1,\, \varepsilon^{-1} \cdot L(d(\cdot,x);x)\right\} \cdot \omega(f,\varepsilon).$$

If in Lemma 4 of Nishishiraho's paper [42] *we take $g = 1_X$ and $\Phi(x,y) = d(x,y)$, then the lemma implies*

$$|L(f,x) - f(x)| \leq \max\left(1 + \varepsilon^{-1} \cdot \eta \cdot L(d(\cdot,x);x)\right) \cdot \omega(f,\varepsilon).$$

Thus, for $\eta \geq 1$ (which is the case for every coefficient of convex deformation) and for functions f with the property mentioned above, we can apply Corollary 4.2 (ii′) to find for any $h > 0$ the inequality

$$|L(f,x) - f(x)| \leq (1 + \eta \cdot h \cdot \varepsilon^{-1}) \cdot \max\left\{1,\, h^{-1} \cdot L(d(\cdot,x);x)\right\} \cdot \omega(f,\varepsilon).$$

Thus if $L(d(\cdot,x);x) > 0$ we can choose $h = L(d(\cdot,x);x)$ to arrive at

$$|L(f,x) - f(x)| \leq \left(1 + \varepsilon^{-1} \cdot \eta \cdot L(d(\cdot,x);x)\right) \cdot \omega(f,\varepsilon).$$

This is exactly the estimate given by T. Nishishiraho [42]. *Observe that it remains true if $L(d(\cdot,x);x) = 0$.*

3) *A statement akin to (2) holds with respect to Theorem 4 in M.A. Jiménez Pozo's paper* [26].

4) *The choice* $h = L(d(\cdot,x);x)$ *in (2) of this remark is best possible in the sense that it yields best possible constants. To see this assume that we choose* $h = r \cdot L(d(\cdot,x);x)$ *for* $r < 1$. *Hence*

$$\begin{aligned}(1+\eta\cdot h\cdot\varepsilon^{-1})\cdot\max\Big\{1,\, h^{-1}\cdot L(d(\cdot,x);x)\Big\} &= \Big(1+\eta\cdot\varepsilon^{-1}\cdot r\cdot L(d(\cdot,x);x)\Big)\cdot r^{-1}\\ &= r^{-1}+\eta\cdot\varepsilon^{-1}\cdot L(d(\cdot,x);x)\\ &> 1+\eta\cdot\varepsilon^{-1}\cdot L(d(\cdot,x);x).\end{aligned}$$

If we choose $h = r \cdot L(d(\cdot,x);x)$ *for* $r > 1$, *then*

$$\begin{aligned}(1+\eta\cdot h\cdot\varepsilon^{-1})\cdot\max\Big\{1,\, h^{-1}\cdot L(d(\cdot,x);x)\Big\} &= 1+\eta\cdot\varepsilon^{-1}\cdot r\cdot L(d(\cdot,x);x)\\ &\geq 1+\eta\cdot\varepsilon^{-1}\cdot L(d(\cdot,x);x).\end{aligned}$$

Thus the choice $h = L(d(\cdot,x);x)$ *gives the best constant in front of* $\omega(f,\varepsilon)$.

4.2. Refined estimates in terms of ω using the direct technique. All estimates of Section 4.1 in terms of ω were consequences of a single estimate being true for arbitrary metric spaces. The assumption $\omega(f,\xi\cdot\varepsilon)\leq(1+\eta\xi)\cdot\omega(f,\varepsilon)$ was made in order to enable us to apply Lemma 2.4 and thus to achieve inequalities in terms of ω instead of $\widetilde{\omega}$.

As a first result of this section we shall show how this assumption can be used together with the so-called direct technique for the case of certain bounded operators. The result given below in Theorem 4.4 resembles that of Theorem 3.2; the technique of proof is due to M.A. Jiménez Pozo [22].

Theorem 4.4 (cf. M.A. Jiménez Pozo [22, Lemma 2]). *Let* (X,d) *be a compact metric space such that, for the modulus of continuity of any function* $f \in C_{\mathbb{R}}(X)$, *one has* $\omega(f,\xi\cdot\varepsilon)\leq(1+\eta\xi)\cdot\omega(f,\varepsilon)$ *for all* $\xi,\varepsilon>0$ *and some fixed constant* $\eta>0$. *Let* A *be of the form* $A(f,y)=\psi_A(y)\cdot f(g_A(y))$, *and let* L *be a bounded linear operator, both mapping* $C(X)$ *into* $B(Y)$. *If* $y\in Y$ *is such that* $L(1_X;y)\neq 0$, *then for all* $f\in C(X)$ *and all* $\varepsilon>0$ *we have*

$$\begin{aligned}&|(L-A)(f,y)|\leq\\ &\leq\Bigg[\Big(\|\varepsilon_y\circ L\|-|L(1_X,y)|\Big)\cdot\Big\{1+\eta\cdot\varepsilon^{-1}\cdot d(X)\Big\}\\ &\quad+|L(1_X,y)|+\Big|1+\eta\cdot\varepsilon^{-1}\cdot L(1_X,y)^{-1}\cdot L(d(\cdot,g_A(y));y)\Big|\Bigg]\cdot\omega(f,\varepsilon)\\ &\quad+|(L-A)(1_X,y)|\cdot|f(g_A(y))|.\end{aligned}$$

Here $d(X)$ *is the diameter of* X, *and* 1_X *denotes the function* $X\ni x\mapsto 1\in\mathbb{R}$.

Proof. Similar to the proof of Theorem 3.2, we first construct two auxiliary functions h_1 and h_2. Their definition is based upon the following observations. If $f\in C(X)$ and $x\in X$, then for all $t\in X$

$$|f(t)-f(g_A(y))|\leq\Big\{1+\eta\cdot\varepsilon^{-1}\cdot d(t,g_A(y))\Big\}\cdot\omega(f,\varepsilon),\quad \varepsilon>0.$$

For $t \in X$ (and $y \in Y$ fixed) we now define

$$h_1(t) := f(g_A(y)) - \left\{1 + \eta \cdot \varepsilon^{-1} \cdot d(t, g_A(y))\right\} \cdot \omega(f, \varepsilon), \quad \text{and}$$

$$h_2(t) := f(g_A(y)) + \left\{1 + \eta \cdot \varepsilon^{-1} \cdot d(t, g_A(y))\right\} \cdot \omega(f, \varepsilon).$$

Due to the above observations for the differences $f(t) - f(g_A(y))$ it is clear that the *continuous* functions h_i, $i = 1, 2$, satisfy

$$h_1(t) \le f(t) \le h_2(t), \quad t \in X.$$

Furthermore,

$$\begin{aligned} |f(t) - h_1(t)| &= f(t) - h_1(t) \\ &= f(t) - f(g_A(y)) + \left\{1 + \eta \cdot \varepsilon^{-1} \cdot d(t, g_A(y))\right\} \cdot \omega(f, \varepsilon) \\ &\le |f(t) - f(g_A(y))| + \left\{1 + \eta \cdot \varepsilon^{-1} \cdot d(t, g_A(y))\right\} \cdot \omega(f, \varepsilon) \\ &\le 2 \cdot \left\{1 + \eta \cdot \varepsilon^{-1} \cdot d(t, g_A(y))\right\} \cdot \omega(f, \varepsilon). \end{aligned}$$

Also,

$$|h_2(t) - f(t)| \le 2 \cdot \left\{1 + \eta \cdot \varepsilon^{-1} \cdot d(t, g_A(y))\right\} \cdot \omega(f, \varepsilon).$$

Hence,

$$\max\{\|f - h_i\| \ : \ i = 1, 2\} \le 2 \cdot \left\{1 + \eta \cdot \varepsilon^{-1} \cdot d(X)\right\} \cdot \omega(f, \varepsilon),$$

where $d(X)$ is the diameter of X.
The assumption that $L(1_X, y) \neq 0$ (so that $\|\varepsilon_y \circ L\| \neq 0$, too) allows us to introduce the auxiliary functional T given by

$$T(f) := T_y(f) := \frac{|L(1_X, y)|}{L(1_X, y) \cdot \|\varepsilon_y \circ L\|} \cdot L(f, y).$$

For fixed $y \in Y$ this is a continuous functional on $C_{\mathbb{R}}(X)$. Using the same technique as in the proof of Theorem 3.2 (representing measures), it may be seen that for each $f \in C_{\mathbb{R}}(X)$, $f \ge 0$, the inequality

$$T(f) + \|f\| \cdot \tfrac{1}{2}(1 - M) \ge 0$$

holds; here

$$M := \frac{|L(1_X, y)|}{\|\varepsilon_y \circ L\|} \le 1.$$

Applying the latter inequality to $(f - h_1)$ and $(h_2 - f)$ shows that

$$T(f - h_1) + \|f - h_1\| \cdot \tfrac{1}{2}(1 - M) \ge 0, \quad \text{and}$$
$$T(h_2 - f) + \|h_2 - f\| \cdot \tfrac{1}{2}(1 - M) \ge 0.$$

Consequently,

$$\begin{aligned} &T(f) - f(g_A(y)) \cdot T(1_X) \ge \\ &\ge -\|f - h_1\| \cdot \tfrac{1}{2}(1 - M) - \left\{T(1_X) + \eta \cdot \varepsilon^{-1} \cdot T(d(\cdot, g_A(y)))\right\} \cdot \omega(f, \varepsilon) \end{aligned}$$

$$\geq -\max\{\|f-h_1\| : i=1,2\} \cdot \tfrac{1}{2}(1-M) - \Big\{T(1_X)+\eta \cdot \varepsilon^{-1} \cdot T(d(\cdot, g_A(y)))\Big\} \cdot \omega(f,\varepsilon)$$

$$\geq -\max\{\|f-h_1\| : i=1,2\} \cdot \tfrac{1}{2}(1-M) - \Big|\Big\{T(1_X)+\eta \cdot \varepsilon^{-1} \cdot T(d(\cdot, g_A(y)))\Big\}\Big| \cdot \omega(f,\varepsilon).$$

Similarly, from

$$T(h_2 - f) + \|f - h_2\| \cdot \tfrac{1}{2}(1-M) \geq 0$$

it follows that

$$T(f) - f(g_A(y)) \cdot T(1_X) \leq$$

$$\leq \max\{\|f-h_i\| : i=1,2\} \cdot \tfrac{1}{2}(1-M) + \Big|\Big\{T(1_X)+\eta \cdot \varepsilon^{-1} \cdot T(d(\cdot, g_A(y)))\Big\}\Big| \cdot \omega(f,\varepsilon).$$

Thus

$$|T(f) - f(g_A(y)) \cdot T(1_X)| \leq$$

$$\leq \max\{\|f-h_i\| : i=1,2\} \cdot \tfrac{1}{2}(1-M)|1 + \Big|T(1_X)+\eta \cdot \varepsilon^{-1} \cdot T(d(\cdot, g_A(y)))\Big| \cdot \omega(f,\varepsilon)$$

$$\leq (1-M) \cdot \Big\{1+\eta \cdot \varepsilon^{-1} \cdot d(X)\Big\} \cdot \omega(f,\varepsilon) + \Big|T(1_X)+\eta \cdot \varepsilon^{-1} \cdot T(d(\cdot, g_A(y)))\Big| \cdot \omega(f,\varepsilon)$$

$$= \Big[(1-M) \cdot \Big\{1+\eta \cdot \varepsilon^{-1} \cdot d(X)\Big\} + \Big|T(1_X)+\eta \cdot \varepsilon^{-1} \cdot T(d(\cdot, g_A(y)))\Big|\Big] \cdot \omega(f,\varepsilon).$$

Recalling the definition of T, the final estimate is now obtained by observing first that

$$|(L-A)(f,y)| \leq |L(f,y) - L(1_X,y) \cdot f(g_A(y))| + |(L-A)(1_X,y)| \cdot |f(g_A(y))|,$$

and then estimating the first term of the upper bound as follows:

$$|L(f,y) - L(1_X,y) \cdot f(g_A(y))| = \|\varepsilon_y \circ L\| \cdot |T(f) - f(g_A(y)) \cdot T(1_X)| \leq$$

$$\leq |\varepsilon_y \circ L\| \cdot \Big[(1-M) \cdot \Big\{1+\eta \cdot \varepsilon^{-1} \cdot d(X)\Big\} + \Big|T(1_X)+\eta \cdot \varepsilon^{-1} \cdot T(d(\cdot, g_A(y)))\Big|\Big] \cdot \omega(f,\varepsilon)$$

$$= \|\varepsilon_y \circ L\| \cdot \Bigg[\Big(1 - \tfrac{|L(1_X,y)|}{\|\varepsilon_y \circ L\|}\Big) \cdot \Big\{1+\eta \cdot \varepsilon^{-1} \cdot d(X)\Big\}$$

$$+ \Big|\tfrac{|L(1_X,y)|}{\|\varepsilon_y \circ L\|} + \eta \cdot \varepsilon^{-1} \cdot \tfrac{|L(1_X,y)|}{L(1_X,y) \cdot \|\varepsilon_y \circ L\|} L(d(\cdot, g_A(y)); y)\Big|\Bigg] \cdot \omega(f,\varepsilon)$$

$$= \Bigg[(\|\varepsilon_y \circ L\| - |L(1_X,y)|) \cdot \Big\{1+\eta \cdot \varepsilon^{-1} \cdot d(X)\Big\}$$

$$+ \Big||L(1_X,y)| + \eta \cdot \varepsilon^{-1} \cdot \tfrac{|L(1_X,y)|}{L(1_X,y)} L(d(\cdot, g_A(y)); y)\Big|\Bigg] \cdot \omega(f,\varepsilon).$$

This yields the inequality of Theorem 4.4. □

Corollary 4.5.

(1) *As is immediately seen from the inequality of Theorem 4.4, we also have that*

$$|(L-A)(f,y)| \leq \Big[(\|\varepsilon_y \circ L\| - |L(1_X,y)|) \cdot \Big\{1+\eta \cdot \varepsilon^{-1} \cdot d(X)\Big\} +$$

$$+ |L(1_X,y)| + \eta \cdot \varepsilon^{-1} \cdot |L(d(\cdot, g_A(y)); y)|\Big] \cdot \omega(f,\varepsilon)$$

$$+ \ |(L-A)(1_X,y)| \cdot |f(g_A(y))|.$$

(2) *If L is a positive linear operator, and hence $\|\varepsilon_y \circ L\| = |L(1_X,y)|$, then the above reduces to*

$$|(L-A)(f,y)| \leq$$
$$\leq \Big\{L(1_X,y)+\eta\cdot\varepsilon^{-1}\cdot L(d(\cdot,g_A(y));y)\Big\}\cdot\omega(f,\varepsilon)+|(L-A)(1_X,y)|\cdot|f(g_A(y))|.$$

Remark 4.6. *For a special case the inequality given in Corollary 4.5 (2) will appear again in Discussion 4.13 (3), but as a consequence of a different inequality. The most important expression in it is the term $\eta\cdot\varepsilon^{-1}\cdot L(d(\cdot,g_A(y));y)$, the occurrence of which is a consequence of the construction of the* continuous *functions h_1 and h_2 in the proof of Theorem 4.4. These had to be continuous in order to enable us to apply the representation theorem for continuous linear functionals and to derive the crucial inqualities (see the proof of Theorem 4.4)*

$$T(f-h_1)+\|f-h_1\|\cdot\tfrac{1}{2}(1-M)\geq 0, \quad \textit{and}$$
$$T(h_2-f)+\|h_2-f\|\cdot\tfrac{1}{2}(1-M)\geq 0.$$

However, if the above assumption

$$\omega(f,\xi\cdot\varepsilon)\leq(1+\eta\cdot\xi)\cdot\omega(f,\varepsilon)$$

is replaced by the slightly stronger one given below, then it would be possible to replace h_1 and h_2 by two functions h_1^* and h_2^* which are not necessarily continuous and satisfy

$$h_1\leq h_1^*\leq f\leq h_2^*\leq h_2.$$

Clearly, the use of these functions would lead to refined inequalities if L were applicable to, say, bounded functions.

In the remainder of this section we shall assume again that L is a positive linear operator. We give a development which parallels that of Section 3. The slightly stronger assumption mentioned in Remark 4.6 follows.

In the sequel we shall assume the existence of a fixed number $\varrho>0$ such that we have

$$\omega(f,\xi\cdot\varepsilon)\leq(1+]\varrho\xi[)\cdot\omega(f,\varepsilon)\quad\text{for }\xi,\,\varepsilon\geq 0;$$

here $]\varrho\xi[=\max\{z\in\mathbb{Z}\ :\ z<\varrho\cdot\xi\}$. As shown in Lemma 1.6 (3), this type of inequality holds at least for spaces (X,d) having a coefficient of convex deformation $\varrho\geq 1$. In particular, for space (X,d) being metrically convex in the sense of Menger (and thus $\varrho=1$), we have the better inequality

$$\omega(f,\xi\cdot\varepsilon)\leq(1+]\xi[)\cdot\omega(f,\varepsilon)\quad\text{for }\xi,\,\varepsilon\geq 0.$$

For approximation by positive linear operators this stronger inequality (as opposed to the weaker one $\omega(f,\xi\cdot\varepsilon)\leq(1+\xi)\cdot\omega(f,\varepsilon)$) is of great importance when determining optimal constants (see, *e.g.*, P.C. Sikkema [58]). However, the use of the weaker form frequently has the advantage of leading to simpler estimates.

Corollary 4.5 (2) already contains an inequality concerning the approximation by positive linear operators which was obtained via the direct technique. The following theorem is a first step to refine this method in connection with approximation by such operators. It constitutes an analogy of Theorem 3.4.

THEOREM 4.7. *Let (X, d) be a compact metric space with $d(X) > 0$ and such that for each $f \in C(X)$ its modulus of continuity satisfies*

$$\omega(f, \xi \cdot \varepsilon) \leq (1+]\varrho\xi[) \cdot \omega(f, \varepsilon) \quad \textit{for all } \xi, \varepsilon \geq 0$$

and some fixed $\varrho > 0$. Moreover, let $Y \neq \emptyset$ be some set, $\psi_A : Y \to \mathbb{R}$ be bounded, and let $g_A : Y \to X$ be any mapping. If $A : C(X) \to B(Y)$ is given by $A(f, y) = \psi_A(y) \cdot f(g(y))$, and if $L : B(X) \to B(Y)$ is a positive linear operator, then for all $f \in C(X)$, $y \in Y$ and $\varepsilon > 0$ the following inequalities hold:

$$|(L - A)(f, y)| \leq$$
$$\leq \left[L(1_X, y) + L\left(\chi_{\{0\}\cup(1,\infty)}\{\varepsilon^{-1} \cdot d(\cdot, g_A(y))\} \cdot]\varrho \cdot \varepsilon^{-1} \cdot d(\cdot, g_A(y))[\ ; y\right)\right]$$
$$\cdot\, \omega(f, \varepsilon) + |(L - A)(1_X, y)| \cdot |f(g_A(y))|.$$

Here $\chi_{\{0\}\cup(1,\infty)}$ denotes the characteristic function of the set $\{0\} \cup (1, \infty)$.

2. If $\varrho = 1$, then the estimate simplifies to

$$|(L - A)(f, y)| \leq$$
$$\leq \left[L(1_X, y)+L\left(]\varepsilon^{-1} \cdot d(\cdot, g_A(y))[; y\right)\right] \cdot \omega(f, \varepsilon)+|(L-A)(1_X, y)| \cdot |f(g_A(y))|.$$

An immediate consequence of Theorem 4.7 is

COROLLARY 4.8. *Under the assumptions of Theorem 4.7 the following (slightly weaker) inequalities also hold:*

1. $|(L - A)(f, y)| \leq$

$$\left[L(1_X, y)+\varrho\cdot\varepsilon^{-1}\cdot L\left(\chi_{(1,\infty)}\{\varepsilon^{-1} \cdot d(\cdot, g_A(y))\} \cdot d(\cdot, g_A(y)); y\right)\right]\cdot\omega(f, \varepsilon)$$
$$+|(L - A)(1_X, y)| \cdot |f(g_A(y))|.$$

2. *If $L(1_X, y) = A(1_X, y)$ for some $y \in Y$, then inequality (1) reduces to*

$$|(L - A)(f, y)| \leq$$
$$\leq \left[A(1_X, y) + \varrho \cdot \varepsilon^{-1} \cdot L\left(\chi_{(1,\infty)}\left\{\varepsilon^{-1} \cdot d(\cdot, g_A(y))\right\} \cdot d(\cdot, g_A(y)); y\right)\right] \cdot \omega(f, \varepsilon).$$

Proof of Theorem 4.7. As in the proof of Theorem 3.2 first observe that $|(L-A)(f,y)| \leq |L(f,y) - L(1_X,y) \cdot f(g_A(y))| + |(L-A)(1_X,y)| \cdot |f(g_A(y))|$. Hence it remains to estimate the first term on the right hand side of this inequality. Next observe that for $t \in X$

$$|f(t) - f(g_A(y))| \leq \Big(1 +]\varrho \cdot \varepsilon^{-1} \cdot d(t, g_A(y))[\Big) \cdot \omega(f,\varepsilon), \quad \text{if } d(t,g_A(y)) > \varepsilon,$$
$$|f(t) - f(g_A(y))| \leq \omega(f,\varepsilon) \quad \text{for } 0 < d(t,g_A(y)) \leq \varepsilon, \qquad \text{and}$$
$$|f(t) - f(g_A(y))| = 0 \quad \text{if } t = g_A(y).$$

Thus

$$|f(t) - f(g_A(y))| \leq$$
$$\leq \Big\{1 + \chi_{\{0\}\cup(1,\infty)}\{\varepsilon^{-1} \cdot d(t,g_A(y))\} \cdot]\varrho \cdot \varepsilon^{-1} \cdot d(t,g_A(y))[\Big\} \cdot \omega(f,\varepsilon).$$

Application of L yields

$$|L(f,y) - L(1_X,y) \cdot f(g_A(y))| \leq$$
$$\leq \Big[L(1_X,y) + L\Big(\chi_{\{0\}\cup(1,\infty)}\{\varepsilon^{-1} \cdot d(\cdot,g_A(y))\} \cdot]\varrho \cdot \varepsilon^{-1} \cdot d(\cdot,g_A(y))[;y\Big)\Big] \cdot \omega(f,\varepsilon).$$

This gives inequality (1).

Inequality (2) is obtained by observing that for $\varrho = 1$ and all $t \in X$ one has

$$\chi_{\{0\}\cup(1,\infty)}\Big\{\varepsilon^{-1} \cdot d(t,g_A(y))\Big\} \cdot \Big]\varepsilon^{-1} \cdot d(t,g_A(y))\Big[= \Big]\varepsilon^{-1} \cdot d(t,g_A(y))\Big[.$$

Finally, the estimates of the corollary are achieved by observing that

$$\chi_{\{0\}\cup(1,\infty)}\Big\{\varepsilon^{-1} \cdot d(\cdot,g_A(y))\Big\} \cdot \Big]\varrho \cdot \varepsilon^{-1} \cdot d(\cdot,g_A(y))\Big[\leq$$
$$\leq \chi_{(1,\infty)}\Big\{\varepsilon^{-1} \cdot d(\cdot,g_A(y))\Big\} \cdot \varrho \cdot \varepsilon^{-1} \cdot d(\cdot,g_A(y)).$$

Remark 4.9. 1) *The functions*

$$X \ni t \mapsto \chi_{\{0\}\cup(1,\infty)}\Big\{\varepsilon^{-1} \cdot d(t,g_A(y))\Big\} \cdot \Big]\varrho \cdot \varepsilon^{-1} \cdot d(t,g_A(y))\Big[\in \mathbb{R},$$

and

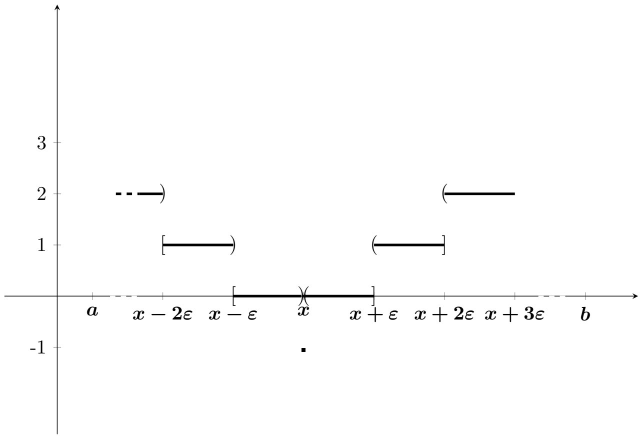
$$X \ni t \mapsto \chi_{(1,\infty)}\Big\{\varepsilon^{-1} \cdot d(t,g_A(y))\Big\} \cdot d(t,g_A(y)) \in \mathbb{R},$$

figuring in Theorem 4.7 and Corollary 4.8 are, at least for small values of ε, not in $C(X)$, but are merely bounded. Thus the assumption that L is defined for bounded functions is essential.

For the special case $X = Y = [a,b]$, $d(x,y) = |x-y|$ (and thus $\varrho = 1$), $A(f,x) = f(x)$, the first of these functions becomes for x fixed in $[a,b]$:

$$f_{\varepsilon,x} : [a,b] \ni t \ \mapsto]\varepsilon^{-1} \cdot |t-x|[\in \mathbb{Z}.$$

This is not *a positive function and – as was seen above – enters the estimates mainly because of the inequality $\omega(f,\xi\cdot\varepsilon) \leq (1+]\xi[) \cdot \omega(f,\varepsilon)$ for the modulus of continuity. Here is a partial view of its graph:*



It is evident that the above function is majorized by

$$[a,b] \ni t \mapsto \chi_{(1,\infty)}\{\varepsilon^{-1} \cdot |t-x|\} \cdot \varepsilon^{-1} \cdot |t-x| \in \mathbb{R}.$$

This is ε^{-1} times the second function mentioned in this remark. Its fundamental importance for the estimates of T. Nishishiraho will become apparent in Discussion 4.13.

2) *The estimate given in Theorem 4.7 is best possible in a certain sense. For a discussion of this fact the reader is referred to Remark 4.16.*

As was the case for the estimate of Theorem 3.4, the direct approach also allows a partial replacement of the function 1_X by suitable functions h. This will be shown in the following two theorems. The first one constitutes an analogy to Theorem 3.6.

THEOREM 4.10. *Let A, L, f, y, ε be given as in Theorem 4.7. If $h \in C(X)$ is such that $h(g_A(y)) \neq 0$, then*

$$|(L-A)(f,y)| \le$$
$$\le \left\{L(1_X,y) + L\left(\chi_{\{0\}\cup(1,\infty)}\left\{\varepsilon^{-1}\cdot d(\cdot, g_A(y))\right\}\cdot\right]\varrho\cdot\varepsilon^{-1}\cdot d(\cdot g_A(y))\left[\,;y\right)\right\}\cdot$$
$$\cdot\left\{\omega(f,\varepsilon) + |(f/h)(g_A(y))|\cdot\omega(h,\varepsilon)\right\} + |(f/h)(g_A(y))|\cdot|(L-A)(h,y)|.$$

Proof. As in the proof of Theorem 3.6 the first observation is that

$$|L(f,y)-(f/h)(g_A(y))\cdot L(h,y)| \le |(L-\widetilde{A})(f,y)| + |(f/h)(g_A(y))|\cdot|(\widetilde{A}-L)(h,y)|.$$

Here $\widetilde{A}$ is again given by $\widetilde{A}(f,y) = L(1_X,y)\cdot f(g_A(y))$. We now estimate $|(L-\widetilde{A})(f,y)|$ and $|(L-\widetilde{A})(h,y)|$ as in the proof of Theorem 4.7 (1) to arrive at

$$|L(f,y)-(f/h)(g_A(y))\cdot L(h,y)| \le$$
$$\le \left\{L(1_X,y) + L\left(\chi_{\{0\}\cup(1,\infty)}\left\{\varepsilon^{-1}\cdot d(\cdot, g_A(y))\right\}\cdot\right]\varrho\cdot\varepsilon^{-1}\cdot d(\cdot g_A(y))\left[\,;y\right)\right\}\cdot$$
$$\cdot\left\{\omega(f,\varepsilon) + |(f/h)(g_A(y))|\cdot\omega(h,\varepsilon)\right\}.$$

Adding this quantity to the term $|(f/h)(g_A(y))| \cdot |(L-A)(h,y)|$ yields the estimate of Theorem 4.10. $\square$

As will be seen in Discussion 4.13 (1) and Discussion 4.13 (9), the estimate of Theorem 4.10 contains all the main results of Nishishiraho's papers cited in this work. Before showing this we are going to partially replace 1_X by the second method already employed in Theorem 3.7.

THEOREM 4.11. *Let A, L, f, y, ε be given as in Theorem 4.7. Moreover, let $h \in C(X)$ be such that $f/h \in C(X)$. Then*

$$|(L-A)(f,y)| \leq$$
$$\leq \|h\| \cdot \Big\{L(1_X,y)+L\Big(\chi_{\{0\}\cup(1,\infty)}\Big\{\varepsilon^{-1}\cdot d(\cdot,g_A(y))\Big\}\cdot\Big]\varrho\cdot\varepsilon^{-1}\cdot d(\cdot,g_A(y))\Big[\,;y\Big)\Big\}$$
$$\cdot\,\omega(f/h,\varepsilon)+|(f/h)(g_A(y))|\cdot|(L-A)(h,y)|.$$

Proof. As in Theorems 3.6 and 4.10 it suffices to estimate

$$\begin{aligned}|L(f,y)-(f/h)(g_A(y))\cdot L(h,y)| &= \Big|L\Big(f/h)\cdot h-(f/h)(g_A(y))\cdot h;y\Big)\Big| \\ &= \Big|L\Big(\Big[(f/h)-(f/h)(g_A(y))\Big]\cdot h;y\Big)\Big| \\ &\leq \|h\|\cdot L\Big(\Big|(f/h)-(f/h)(g_A(y))\Big|;y\Big).\end{aligned}$$

Also, as in the proof of Theorem 4.7 it may be observed that for $t \in X$ one has

$$|(f/h)(t)-(f/h)(g_A(y))| \leq$$
$$\leq \Big\{1+\chi_{\{0\}\cup(1,\infty)}\Big\{\varepsilon^{-1}\cdot d(t,g_A(y))\Big\}\cdot\Big]\varrho\cdot\varepsilon^{-1}\cdot d(t,g_A(y))\Big[\,\Big\}\cdot\omega(f/h,\varepsilon),$$

which implies

$$|L(f,y)-(f/h)(g_A(y))\cdot L(h,y)| \leq$$
$$\leq|h|\cdot\Big\{L(1_X,y)+L\Big(\chi_{\{0\}\cup(1,\infty)}\{\varepsilon^{-1}\cdot d(\cdot,g_A(y))\}\cdot\,]\varrho\cdot\varepsilon^{-1}\cdot(\cdot,g_A(y))[;y\Big)\Big\}$$
$$\cdot\omega(f/h,\varepsilon).$$

Combining this with the remaining term $|(f/h)(g_A(y))|\cdot|(L-A)(h,y)|$ gives the estimate of Theorem 4.11. $\square$

The analogy of Corollary 3.8 is the following

COROLLARY 4.12. (1) *If $h = 1_X$, both Theorems 4.10 and 4.11 imply the estimate of Theorem 4.7 (1).*

(2) *If $Y = X$ and $A(f,x) = f(x)$, then the estimate of Theorem 4.10 becomes*

$$|L(f,x)-f(x)| \leq$$

$$\leq \Big\{L(1_X, y) + L\left(\chi_{\{0\}\cup(1,\infty)}\{\varepsilon^{-1}\cdot d(\cdot,x)\}\cdot\,]\,\varrho\cdot\varepsilon^{-1}\cdot d(\cdot,x)\,[\,;y\right)\Big\}\cdot$$
$$\cdot\Big\{\omega(f,\varepsilon) + |(f/h)(x)|\cdot\omega(h,\varepsilon)\Big\} + |(f/h)(x)|\cdot|L(h,x)-h(x)|.$$

(3) *Under the assumptions of (2) the inequality of Theorem 4.11 reduces to*

$$|L(f,x)-f(x)| \leq$$
$$\leq \|h\|\Big\{L(1_X,x)+L\left(\chi_{\{0\}\cup(1,\infty)}\{\varepsilon^{-1}\cdot d(\cdot,x)\}\cdot\,]\,\varrho\cdot\varepsilon^{-1}\cdot d(\cdot,x)\,[\,;y\right)\Big\}\omega(f/h,\varepsilon) \quad \square$$
$$+ |(f/h)(x)|\cdot|L(h,x)-h(x)|.$$

The following discussion will show how the bounded functions

$$X \ni t \mapsto \chi_{\{0\}\cup(1,\infty)}\left\{\varepsilon^{-1}\cdot d(t,g_A(y))\right\}\cdot\Big]\,\varrho\cdot\varepsilon^{-1}\cdot d(t,g_A(y))\Big[\,\in\mathbb{R}$$

may be replaced in the estimates of Theorems 4.7, 4.10, and 4.11. This discussion will complete the information available from Corollary 4.8 and will yield results analogous to the ones from Theorem 3.9, among others. It is also intended to relate the main results of this section to earlier work. At the same time it will give us an opportunity to briefly review univariate results of several types, and will thus parallel the discussion conducted in Examples 3.5 and 3.11.

Discussion 4.13. *We now discuss some further consequences of Theorem 4.7 and Corollary 4.8 by comparing the estimates given there to such obtained by other authors and us earlier. For this purpose it suffices to consider the case where $Y = X$ and $A : C(X) \to B(X)$ is the canonical imbedding, so that $g_A(x) = x$ for $x \in X$.*

(1) *The inequality (2) of Theorem 4.7 implies for this case and $\varrho = 1$*

$$|L(f,x)-f(x)| \leq \Big\{L(1_X;x)+L\left(\Big]\varepsilon^{-1}\cdot d(\cdot,x)\Big[\,;x\right)\Big\}\cdot\omega(f,\varepsilon)+|(L(1_X;x)-1|\cdot|f(x)|.$$

If $L(1_X;x) = 1$, then this simplifies to

$$|L(f,x)-f(x)| \leq \Big\{1+L\left(\Big]\varepsilon^{-1}\cdot d(\cdot,x)\Big[\,;x\right)\Big\}\cdot\omega(f,\varepsilon).$$

For the special case $X = [0,1]$ and the Bernstein operators this type of estimate was given by P.C. Sikkema [58, p. 108, formula(4)]*; the analogy of the above expression was used to determine a certain best possible constant in approximation by these operators. After giving this estimate, Sikkema passes to the equivalent of the weaker estimate (see* [58, p. 109, formula (5)]*)*

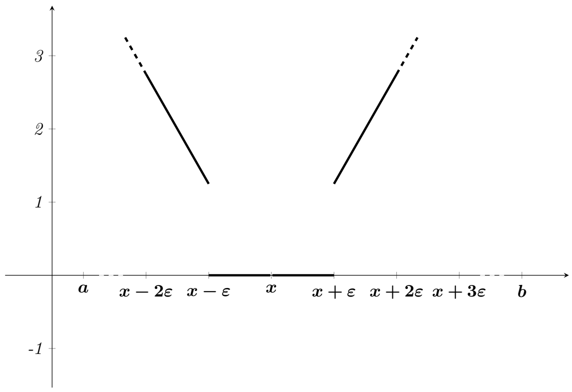
$$|L(f,x)-f(x)| \leq \Big\{1+\varepsilon^{-1}L\left(\chi_{(1,\infty)}\left(\varepsilon^{-1}\cdot d(\cdot,x)\right)\cdot d(\cdot,x);x\right)\Big\}\cdot\omega(f,\varepsilon),$$

which proved to be helpful to determine certain cases where the optimal constant does not occur.

For the case $X = [a, b]$ *and* $d(x, y) = |x - y|$ *the function to which* L *is applied becomes*

$$[a, b] \ni t \; \mapsto \; \chi_{(1,\infty)} \left(\varepsilon^{-1} \cdot |t - x| \right) \cdot \varepsilon^{-1} \cdot |t - x| \in \mathbb{R};$$

its graph is below.



Note that this is a non-negative *function, and thus an estimate based upon the use of this function may be weaker than one based upon the use of* $f_{\varepsilon,x}$ *from* Remark 4.9 *(1). See* Remark 4.16 *with respect to the importance of this statement. Note that T. Nishishiraho's estimates are based upon the use of*

$$\chi_{(1,\infty)} \left(\varepsilon^{-1} \cdot d(\cdot, x) \right) \cdot \varepsilon^{-1} \cdot d(\cdot, x),$$

and thus are necessarily weaker than ours. See (7), (8), and (9) of this discussion for further information in this regard.

(2) *Because of* $]a[\leq [a]$ *for* $a \in \mathbb{R}$ *and the positivity of* L *the second inequality of (1) immediately yields*

$$|L(f, x) - f(x)| \leq \left\{ 1 + L \left([\varepsilon^{-1} \cdot d(\cdot, x)]; x \right) \right\} \cdot \omega(f, \varepsilon).$$

In connection with approximation of univariate and 2π*-periodic functions this type of estimate seems to have been first used by V.V. Žuk and G.I. Natanson* [65, Lemma 1]*; they were also able to show the optimality of their inequality for the convolution type operators considered by them.*

(3) *Because*

$$L \left(\chi_{(1,\infty)}(\varepsilon^{-1} \cdot d(\cdot, x)) \cdot d(\cdot, x); x \right) \leq L(d(\cdot, x); x)$$

the estimate of Corollary 4.8 *(1) implies*

$$|L(f, x) - f(x)| \leq \left\{ L(1_X; x) + \varrho \cdot \varepsilon^{-1} \cdot L(d(\cdot, x); x) \right\} \cdot \omega(f, \varepsilon) + |L(1_X; x) - 1| \cdot |f(x)|.$$

(Note that this also follows from Corollary 4.5 *(2)). If the latter quantity is greater than 0, then choosing* $\varepsilon = L(d(\cdot, x); x)$ *gives*

$$|L(f, x) - f(x)| \leq \{L(1_X; x) + \varrho\} \cdot \omega \left(f, L(d(\cdot, x); x) \right) + |L(1_X; x) - 1| \cdot |f(x)|.$$

This estimate remains true if $L(d(\cdot,x);x)=0$.

For $\varrho=1$, *i.e., for metrically convex spaces, this inequality can be found in a paper of G. Mastroianni* [36, p. 345]. *Similar results for the univariate case can be found in papers of R.G. Mamedov* [34] *(see also his book* [35] *for both the* $C[a,b]$ *and the* $C_{2\pi}$ *case) and of T. Popoviciu* [46].

(4) *Observing further that due to the Cauchy-Schwarz inequality we get*

$$L(d(\cdot,x);x)\le\sqrt{L(d^2(\cdot,x);x)}\cdot\sqrt{L(1_X;x)},$$

it is also true that

$$|L(f,x)-f(x)|\le$$
$$\le\{L(1_X;x)+\varrho\}\cdot\omega\left(f,\sqrt{L(d^2(\cdot,x);x)}\cdot\sqrt{L(1_X;x)}\right)+|L(1_X;x)-1|\cdot|f(x)|.$$

For $\varrho=1$ *this estimate was also used by. G. Mastroianni* [36, p. 345] *(see also H. Shapiro* [55, Sec. 8.8].

(5) *A slightly modified combination of the use of the Cauchy-Schwarz inequality and the particular choice of* ε *implies*

$$|L(f,x)-f(x)|\le$$
$$\le\left\{L(1_X;x)+\varrho\sqrt{L(1_X;x)}\right\}\cdot\omega\left(f,\sqrt{L(d^2(\cdot,x);x)}\right)+|L(1_X;x)-1|\cdot|f(x)|.$$

For $\varrho=1$ *and a univariate setting this type of estimate seems to have been first used by R.A. DeVore* [10, p. 28f.].

(6) *Another observation is that for* $d(t,x)\le\varepsilon$ *we have*

$$\varepsilon^{-1}\cdot\chi_{(1,\infty)}\left(\varepsilon^{-1}\cdot d(t,x)\right)\cdot d(t,x)=0\le\varepsilon^{-q}\cdot d^q(t,x),$$

and for $d(t,x)>\varepsilon$ *and all* $q\ge1$ *the inequality*

$$\varepsilon^{-1}\cdot\chi_{(1,\infty)}\left(\varepsilon^{-1}\cdot d(t,x)\right)\cdot d(t,x)=\varepsilon^{-1}\cdot d(t,x)\le\varepsilon^{-q}\cdot d^q(t,x)$$

holds. Thus it follows that for all $q\ge1$

$$\begin{aligned}&\varepsilon^{-1}\cdot L\left(\chi_{(1,\infty)}\left(\varepsilon^{-1}\cdot d(\cdot,x)\right)\cdot d(\cdot,x);x\right)\le\\&\le\varepsilon^{-q}\cdot L\left(\chi_{(1,\infty)}\left(\varepsilon^{-1}\cdot d(\cdot,x)\right)\cdot d^q(\cdot,x);x\right)\\&\le\varepsilon^{-q}\cdot L\left(d^q(\cdot,x);x\right).\end{aligned}$$

Using this fact in Corollary 4.8 (1) for A *as above shows that for all* $q\ge1$

$$|L(f,x)-f(x)|\le\left\{L(1_X;x)+\varrho\cdot\varepsilon^{-q}\cdot L\left(d^q(\cdot,x);x\right)\right\}\cdot\omega(f,\varepsilon)+|L(1_X;x)-1|\cdot|f(x)|.$$

For the univariate case, i.e., $X=[a,b]$, $\varrho=1$, *and* $d(t,x)=|t-x|$, $L(1_X)=1_X$, $q=2m$ *with* $m\in\mathbb{N}$, *this type of estimate was first proved by A. Lupaş and M.W. Müller* [33, Theorem 2.2].

Choosing ε *such that* $\varepsilon^q=B^{-1}\cdot L\left(d^q(\cdot,x);x\right)$ *for some* $B>0$, *i.e.,*

$$\varepsilon^{-q}=B\cdot\left\{L\left(d^q(\cdot,x);x\right)\right\}^{-1}$$

implies

$$|L(f,x)-f(x)| \le \{L(1_X;x)+\varrho\cdot B\}\cdot\omega\left(f, B^{-1/q}\cdot L\left(d^q(\cdot,x);x\right)^{1/q}\right)+ \\ + L(1_X;x)-1|\cdot|f(x)|.$$

A similiar estimate was obtained by M.A. Jiménez Pozo [26, Theorem 4].

For $\varrho = B = 1$ *and* $q = 2$ *the estimate becomes*

$$\begin{aligned}&|L(f,x)-f(x)| \le \\ &\le \{L(1_X;x)+1\}\cdot\omega\left(f, L\left(d^2(\cdot,x);x\right)^{1/2}\right)+|L(1_X;x)-1|\cdot|f(x)| \\ &\le \|L(1_X)+1_X\|\cdot\omega\left(f,\|L(d^2(\cdot,*);*)\|^{1/2}\right)+\|L(1_X)-1_X\|\cdot\|f\|;\end{aligned}$$

here $\|L(d^2(\cdot,*);*)\| := \sup\left\{L\left(d^2(\cdot,x);x\right) \ : \ x\in X\right\}$.

For compact and convex subsets of $\mathbb{R}^m$ *this uniform estimate was first given by E. Censor* [7, Theorem 1]*; a similar estimate with* $\|L(1_X)-1_X\|$ *replaced by* $|L(1_X;x)-1|$ *was obtained independently by A.G. Kukuš* [32, Lemma 1] *who formulated his result for hypercubes of* $\mathbb{R}^m$*. Both authors were guided by the corresponding univariate result as proved by O. Shisha and B. Mond* [57, Theorem 1].

For $\varrho = 1$, $B > 0$ *and* $q = 2$ *the above estimate reads*

$$\begin{aligned}&|L(f,x)-f(x)| \le \\ &\le \{L(1_X;x)+B\}\cdot\omega\left(f, B^{-1/2}\cdot L\left(d^2(\cdot,x);x\right)^{1/2}\right)+|L(1_X;x)-1|\cdot|f(x)| \\ &\le \|L(1_X)+B\|\cdot\omega\left(f, B^{-1/2}\cdot\|L(d^2(\cdot,*);*)\|^{1/2}\right)+\|L(1_X)-1_X\|\cdot\|f\|.\end{aligned}$$

A univariate analogy (with B *replaced by* B^{-2}*) of this type of estimate was given by B. Mond* [39]*; a version for continuous functions defined on compact and convex subsets of* $\mathbb{R}^m$ *can be found in M.A. Jiménez Pozo's paper* [24, Teorema 1].

(7) *If* $\Phi : X\times X\to\mathbb{R}$ *is a function such that for some* $q\ge 1$

$$d^q(t,x)\le\kappa\cdot\Phi(t,x) \quad \text{for some } \kappa>0 \text{ and } t,x\in X,$$

then the estimate of (6) involving an arbitrary $\varepsilon > 0$ *implies*

$$\begin{aligned}&(*)|L(f,x)-f(x)| \le \\ &\le \left\{L(1_X;x)+\varrho\cdot\varepsilon^{-q}\cdot\kappa\cdot L(\Phi(\cdot,x);x)\right\}\cdot\omega(f,\varepsilon)+|L(1_X;x)-1|\cdot|f(x)|.\end{aligned}$$

For the $C_{2\pi}$ *case this approach was used by O. Shisha and B. Mond* [56, p. 335f]*; they used the special cases* $q = 2$, $\kappa\cdot\Phi(t,x) = \pi^2\cdot\sin^2(\frac{1}{2}(t-x))$, $\varepsilon = \pi\cdot\sqrt{\|L_n(\sin^2(\frac{1}{2}(\cdot-*));*)\|}$, *and formulated a uniform version. B. Mond* [39, p. 305] *indicated the possibility of introducing an arbitrary constant* $B > 0$ *into the estimate and thus to arrive at an inequality analogous to the one at the end of (6).*

(8) *Trivially, as in (6), one also has*

$$\varepsilon^{-1} \cdot \chi_{(1,\infty)} \left(\varepsilon^{-1} \cdot d(t,x)\right) \cdot d(t,x) \leq \varepsilon^{-1} \cdot d(t,x)$$

so that

$$\varepsilon^{-1} \cdot L\left(\chi_{(1,\infty)} \left(\varepsilon^{-1} \cdot d(\cdot,x)\right) \cdot d(\cdot,x); x\right) \leq \varepsilon^{-1} \cdot L(d(\cdot,x);x).$$

If Φ is again given as in (7), then the last quantity is less than or equal to

$$\varepsilon^{-1} \cdot \kappa^{1/q} \cdot L\left(\Phi^{1/q}(\cdot,x);x\right),$$

and applying Hölder's inequality to $L\left(\Phi^{1/q}(\cdot,x);x\right)$ yields as an even larger upper bound for any $p \geq 1$

$$\varepsilon^{-1} \cdot \kappa^{1/q} \cdot \left\{L\left(\Phi(\cdot,x)^{p/q};x\right)\right\}^{1/p} \cdot \{L(1_X;x)\}^{1-1/p}.$$

Altogether we also obtain for $p \geq 1$, and $d^q(t,x) \leq \kappa \cdot \Phi(t,x)$ for $q \geq 1$ fixed and all $t,x \in X$ the following inequality ($\varepsilon > 0$ arbitrarily given):

$$(**)|L(f,x) - f(x)| \leq$$
$$\leq \left[L(1_X;x) + \varrho \cdot \varepsilon^{-1} \cdot \kappa^{1/q} \cdot \left\{L\left(\Phi(\cdot,x)^{p/q};x\right)\right\}^{1/p} \cdot \{L(1_X;x)\}^{1-1/p}\right] \cdot \omega(f,\varepsilon)$$
$$+ |L(1_X;x) - 1| \cdot |f(x)|.$$

This inequality is analogous to the ones of Theorem 3.9. Since $q \geq 1$ in this discussion, we may choose $p = q$ to arrive at a simplified version:

$$|L(f,x) - f(x)| \leq$$
$$\leq \left[L(1_X;x) + \varrho \cdot \varepsilon^{-1} \cdot \kappa^{1/q} \cdot \{L(\Phi(\cdot,x);x)\}^{1/q} \cdot \{L(1_X;x)\}^{1-1/q}\right] \cdot \omega(f,\varepsilon)$$
$$+ |L(1_X;x) - 1| \cdot |f(x)|.$$

For the $C_{2\pi}$ case, $\varrho = 1$, $q = 2$, $\kappa = \pi^2$, $\Phi(t,x) = \sin^2(\frac{1}{2}(t-x))$, and the choice $\varepsilon = L\left(\sin^2(\frac{1}{2}(t-x));x\right)$ the above method was used by R.A. DeVore [10, p. 30].

(9) *Combining inequalites $(*)$ of (7) and $(**)$ of (8) (for $p = q \geq 1$) shows that for any $\varepsilon > 0$ one has*

$$(***) \ \ |L(f,x) - f(x)| \leq [L(1_X;x) + \varrho \cdot m(x)] \cdot \omega(f,\varepsilon) + |L(1_X;x) - 1| \cdot |f(x)|,$$

where $m(x) :=$

$$\min\left\{\varepsilon^{-q} \cdot \kappa \cdot L(\Phi(\cdot,x);x),\ \varepsilon^{-1} \cdot \kappa^{1/q} \cdot \{L(\Phi(\cdot,x);x)\}^{1/q} \cdot \{L(1_X;x)\}^{1-1/q}\right\}.$$

Estimate $(***)$ is the one upon which T. Nishishiraho's paper [42] *is based (see Lemma 3 and the subsequent Lemma 4 in* [42]*).*

(10) *For the special case $\varrho = 1$, $q = 2$, $\Phi(x,y) = d^2(x,y)$, and (consequently) $\kappa = 1$ the inequality of (9) reads*

$$|L(f,x) - f(x)| \leq$$

$$\leq \left[L(1_X;x) + \min\left\{\varepsilon^{-2} \cdot L(d^2(\cdot,x);x),\ \varepsilon^{-1} \cdot \sqrt{L(d^2(\cdot,x);x)} \cdot \sqrt{L(1_X;x)}\right\}\right]$$
$$\cdot\, \omega(f,\varepsilon) + |L(1_X;x) - 1| \cdot |f(x)|.$$

In fact, this is exactly the estimate obtained earlier by the author [14]*. It was the starting point for Nishishiraho's generalizations in* [42, 43]*. The estimate* $(***)$ *was also used as the basic inequality of T. Nishishiraho's subsequent paper* [44, Proposition 1] *where he arrived at generalizations similar to those in* [42, 43]*. For the case* $X = [a,b]$ *the possibility of improving the estimate by using a min was first observed by M.W. Müller and H. Walk* [40, Satz 1].

Remark 4.14. *In summary we point out the following:*

1) *All the estimates in Nishishiraho's papers* [42, 43, 44] *are simple consequences of the estimates given in Theorem 4.7 and Corollary 4.8.*

2) *There seems to be hardly a way around estimating*

$$L\left(\chi_{(1,\infty)}\left(\varepsilon^{-1} \cdot d(\cdot, g_A(y))\right) \cdot d(\cdot, g_A(y)); y\right)$$

in order to obtain good orders of approximation (for sequences L_n*) as well as good constants for single operators* L*. Passing to quantities such as* $L\left(d(\cdot, g_A(y)); y\right)$ *or – even worse – to expresssions such as* $L\left(d^2(\cdot,x);x\right)^{1/2}$ *using the Cauchy-Schwarz inequality in certain instances is simply a means of destroying information being carried in the operator's definition. This statement is confirmed in an impressive way by P.C. Sikkema's* [58] *work and the subsequent failure of virtually every more general estimate to reproduce such a good result for Bernstein polynomials. In an even more impressive way the destructive effect of using upper bounds such as the ones given above may be seen by applying R.A. DeVore's (see Discussion 4.13 (5)) or O. Shisha's and B. Mond's (see Discussion 4.13 (6)) results to classical Hermite-Fejér interpolation operators, for instance.*

3) *In some instances it is even worthwile to evaluate the quantity*

$$L\left(\chi_{\{0\}\cup(1,\infty)}\{\varepsilon^{-1} \cdot d(\cdot, g_A(y))\} \cdot \,]\varrho \cdot \varepsilon^{-1} \cdot d(\cdot, g_A(y))[;y\right),$$

used in Theorem 4.7 (1). Remark 4.16 will elucidate this.

It is possible to obtain analogies of Theorem 3.12 and Corollary 3.13 using the direct approach. For instance, the following is a consequence of Corollary 4.8 (1).

Theorem 4.15. *Let the assumptions of Corollary 4.8 (1) be satisfied. Moreover, let* $M = \{\mu\}$ *be a set of positive linear functionals on* $B(Y)$ *such that* $p_M[h] = \sup\{\mu\{|h|\} : \mu \in M\} < \infty$ *for all* $h \in B(Y)$*. Then for all* $f \in C(X)$ *and all* $\varepsilon > 0$ *the following inequality holds:*

$$p_M[Lf - Af] \leq p_M[(L1_X - A1_X) \cdot (f \circ g_A)] +$$
$$+ \left\{p_M[L1_X] + \varrho\varepsilon^{-1} \cdot p_M\left[L\left(\chi_{(1,\infty)}\left(\varepsilon^{-1} \cdot d(\cdot, g_A(*))\right) \cdot d(\cdot, g_A(*)); *\right)\right]\right\} \cdot \omega(f,\varepsilon).$$

Here L is applied with respect to the variable indicated by "$\cdot$", and $\mu \in M$ is applied with respect to "$*$".

The proof of Theorem 4.15 is a simple consequence of Corollary 4.8 and is similar to that of Theorem 3.12.

It is easily seen that it is also possible to find slightly improved estimates by using Theorem 4.7 or such involving certain upper bounds of $d(\cdot, g_A(*))$ as was for instance done in Corollary 3.13.

As was the case for Theorem 3.12 and Corollary 3.13, the approach of Theorem 4.15 implies, for instance, uniform estimates if M is a set of point evaluation functionals, or estimates in L_1-spaces if the functional μ is an integral operator.

The next remark will discuss the optimality of the estimates given in this section. Its importance is also evident in view of Discussion 4.13 (see Discussion 4.13 (1) in particular.)

Remark 4.16. *As was the case for Theorems 3.4 and 3.12, the estimates given in Theorem 4.7 are best possible in a certain sense. To see this, again let $Y = \{x_0\}$, $A(f, x_0) = f(x_0)$, and $L(1_X, x_0) = 1$ for some fixed $x_0 \in X$. We also assume that (X, d) is metrically convex, so that $\varrho = 1$. For this special case Theorem 4.7 implies for all $\varepsilon > 0$*

$$|L(f, x_0) - f(x_0)| \leq \left\{1 + L\left(\left]\varepsilon^{-1} \cdot d(\cdot, x_0)\right[; x_0\right)\right\} \cdot \omega(f, \varepsilon).$$

Defining $f_{\varepsilon,x_0}(\cdot) := \left]\varepsilon^{-1} \cdot d(\cdot, x_0)\right[$, it is clear that $f_{\varepsilon,x_0}(x_0) = -1$, that f_{ε,x_0} attains only integer values, and that $\omega(f_{\varepsilon,x_0}, \varepsilon) = 1$. Thus

$$\begin{aligned}|L(f_{\varepsilon,x_0}; x_0) - f_{\varepsilon,x_0}(x_0)| &= \left|L\left(\left]\varepsilon^{-1} \cdot d(\cdot, x_0)\right[; x_0\right) - (-1)\right| \\ &= \left\{1 + L\left(\left]\varepsilon^{-1} \cdot d(\cdot, x_0)\right[; x_0\right)\right\} \cdot \omega(f_{\varepsilon,x_0}, \varepsilon).\end{aligned}$$

Here the absolute value bars may be omitted, because the inequality $-1_X \leq f_{\varepsilon,x_0}$ and the positivity of L imply

$$-1 = L(-1_X; x_0) \leq L(f_{\varepsilon,x_0}; x_0),$$

and thus

$$0 \leq 1 + L(f_{\varepsilon,x_0}; x_0).$$

Hence for all $\varepsilon > 0$ and all $x_0 \in X$ the above estimate is locally the best possible, in the sense that for the bounded *function f_{ε,x_0} equality occurs.*

For many special cases there are, however, continuous *functions serving the same purpose. We show this by discussing a univariate example.*

If $X = [a, b]$, $d(x, y) = |x - y|$, and if the operator L is given as

$$L(f, x) = \sum_{k=0}^{n} f(x_k) \cdot q_k(x), \quad n \geq 0,$$

where $a \le x_0 < x_1 < \ldots < x_n \le b$, *and* $q_k(x) \ge 0$ *for* $0 \le k \le n$, *then – after replacing* x_0 *by* x *– for this case we have*

$$L\left(]\varepsilon^{-1} \cdot d(\cdot, x)[; x\right) = \sum_{k=0}^{n}]\varepsilon^{-1}|x_k - x|[\cdot q_k(x).$$

The next step is to construct a continuous function $g_{\varepsilon,x}$ *such that*

$$L(g_{\varepsilon,x}; x) = L\left(]\varepsilon^{-1} \cdot d(\cdot, x)[; x\right) \quad \text{and} \quad \omega(g_{\varepsilon,x}; \varepsilon) = 1.$$

Let $g_{\varepsilon,x}$ *be partially defined by*

$$g_{\varepsilon,x}(x_k) =]\varepsilon^{-1}|x_k - x|[, \quad 0 \le k \le n, \tag{1}$$

$$g_{\varepsilon,x}(x) = -1, \quad \text{if } x \ne x_k \text{ for } 0 \le k \le n. \tag{2}$$

Then it is immediately clear that $g_{\varepsilon,x}$ *coincides with the function* $]\varepsilon^{-1}|\cdot - x|[$ *on the point set* $\{x_k\} \cup \{x\}$. *Hence for an operator of the above type we have* $L(g_{\varepsilon,x}) = L\left(]\varepsilon^{-1}|\cdot - x|[\right)$. *It remains to show that the graph of* $g_{\varepsilon,x}$ *can be completed in a way such that* $g_{\varepsilon,x}$ *is continuous and satisfies* $\omega(g_{\varepsilon,x}, \varepsilon) = 1$.

W.l.o.g. we assume that $x < b$ *and restrict ourselves to the interval* $[x, b]$. *Here* $f_{\varepsilon,x} :=]\varepsilon^{-1}|\cdot - x|[$ *is a step function such that*

$$\begin{aligned} f_{\varepsilon,x}(x) &= -1, \quad \text{and} \\ f_{\varepsilon,x}(y) &= k, \quad \text{if } x + k\varepsilon < y \le x + (k+1)\varepsilon, \quad 0 \le k \le [(b - x)/\varepsilon]. \end{aligned}$$

If $x_n \le x$, *then the values* $f_{\varepsilon,x}(y)$ *for* $y > x$ *do not have any impact on* $L(f_{\varepsilon,x})$. *In this case define*

$$g_{\varepsilon,x}(y) = -1 \quad \text{for } x < y \le b$$

and go on by carrying out a construction for the interval $[a, x]$ *which is 'symmetric' to the one described below.*

We may thus assume that there is at least one point x_λ *such such* $x < x_\lambda$. *Define* $\ell := \min\{\lambda : x < x_\lambda\}$.

For each λ *such that* $\ell \le \lambda \le n$ *let*

$$\begin{aligned} k_\lambda &:= \max\{k : x + k\varepsilon < x_\lambda\}, \quad \text{and} \\ d_\lambda &:= x_\lambda - (x + k_\lambda \varepsilon) > 0. \end{aligned}$$

Let also

$$\begin{aligned} d &:= \min\{d_\lambda : \ell \le \lambda \le n\} > 0, \quad \text{and} \\ d^* &:= \min\{\tfrac{1}{2}\varepsilon, d\} > 0. \end{aligned}$$

Now the graph of $g_{\varepsilon,x}$ *is completed in the following way:*

$$g_{\varepsilon,x}(y) = \begin{cases} -1 & \text{if } y = x, \\ f_{\varepsilon,x}(y) & \text{if } x + k\varepsilon + d^* \le y \le x + (k+1)\varepsilon \text{ for some } k \ge 0, \\ \text{linearly and continuously extended otherwise.} \end{cases}$$

Note that due to our construction, $g_{\varepsilon,x}$ *indeed satisfies* $g_{\varepsilon,x}(x_k) =]\varepsilon^{-1}|x_k - x|[$ *for* $\ell \le k \le n$.

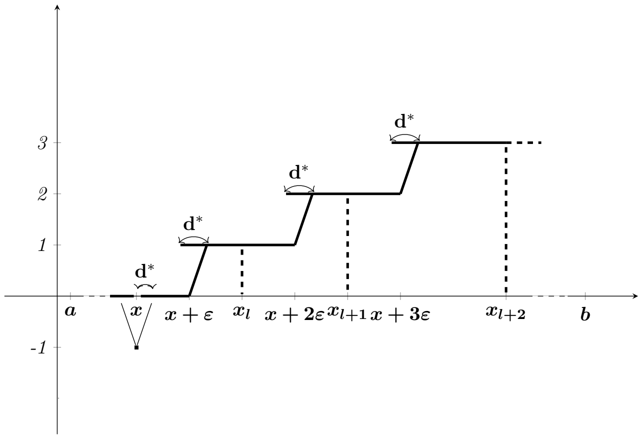


Fig. 4.1. Partial view of the step function $f_{\varepsilon,x}$ and of the continuous function $g_{\varepsilon,x}$.

An analogous construction may be carried out on the interval $[a,x]$ *if* $a < x$. *The complete procedure generates a continuous function* $g_{\varepsilon,x}$ *for which it is easy to see that*

$$|L(g_{\varepsilon,x};x)-g_{\varepsilon,x}(x)|=|L(f_{\varepsilon,x};x)-f_{\varepsilon,x}(x)|=\left\{1+L\left(\left]\varepsilon^{-1}\cdot d(\cdot,x)\right[;x\right)\right\}\cdot\omega(g_{\varepsilon,x};\varepsilon),$$

and hence for the special positive linear operators considered above, the quantity in curly parentheses is also locally optimal if L *is considered as an operator on* $C(X)$.

As was the case in Section 3 (Theorem 3.15), there is also a particular consequence of the direct approach for the pre-Hilbert space setting. For the sake of brevity we restrict ourselves to only giving the following corollary of Discussion 4.13; two of Nishishiraho's papers [43, Theorem 4] and [44, Theorem 5] contain similar estimates. However, it has to be noted that the use of refined estimates of Discussion 4.13 leads to slightly improved inequalities sometimes yielding better results in applications. We decline to cite these at this point since their proofs are straightforward.

Theorem 4.17. *Let* X *be a compact and convex subset of a real pre-Hilbert space with inner product* $\langle\cdot,*\rangle$. *Let* $\varepsilon > 0$. *If* L *is a positive linear operator, then for all* $f \in C(X)$ *and* $x \in X$ *there holds*

$$|L(f,x)-f(x)| \le |f(x)|\cdot|L(1_X,x)-1|+$$
$$+\left[L(1_X,x)+\min\left\{\varepsilon^{-2}L\left(d^2(\cdot,x);x\right),\varepsilon^{-1}\sqrt{L\left(d^2(\cdot,x);x\right)}\sqrt{L(1_X,x)}\right\}\right]\cdot\omega(f,\varepsilon).$$

Here $d(x,y)=\sqrt{\langle x-y,x-y\rangle}$.

Proof. If X is a compact and convex set of a real pre-Hilbert space X, then X equipped with metric $d(x,y)=\sqrt{\langle x-y,x-y\rangle}$ is metrically convex. Thus the inequality of Discussion 4.13 (10) is applicable to this case, so that the error estimate above is an immediate consequence of this observation. □

4.3. Comparison of the results of 4.1 and 4.2. In order to conclude this section we show that both approaches discussed so far in order to obtain estimates in terms of $\widetilde{\omega}$ or ω are useful, *i.e.*, none of them yields a better estimate than the other one in general. Thus the following discussion is of particular importance. To demonstrate the fact mentioned, we consider the special situation $Y = X$, $\varrho = 1$, $A = \mathrm{Id}$, $L(1_X, x) = 1$ for some $x \in X$.

Discussion 4.18. *Theorem 3.4 implies*

$$|L(f,x) - f(x)| \le \max\left\{1, \varepsilon^{-1} \cdot L\left(d(\cdot,x);x\right)\right\} \cdot \widetilde{\omega}(f,\varepsilon) \quad \textit{for } \varepsilon > 0,$$

and it follows from Corollary 4.8 (2) that we also have

$$|L(f,x)-f(x)| \le \left\{1+\varepsilon^{-1}\cdot L\left(\chi_{(1,\infty)}\left(\varepsilon^{-1}\cdot d(\cdot,x)\right)\cdot d(\cdot,x);x\right)\right\}\cdot\omega(f,\varepsilon), \quad \varepsilon > 0.$$

1. *If $f \in C(X)$ is such that $\widetilde{\omega}(f,\cdot) = \omega(f,\cdot)$, and if $\varepsilon \ge L(d(\cdot,x);x)$, then the first estimate reduces to $|L(f,x) - f(x)| \le \omega(f,\varepsilon)$, which is clearly better than or at least as good as the second one. If $\varepsilon < L(d(\cdot,x);x)$, then the first inequality becomes*

$$\begin{aligned}|L(f,x) - f(x)| &\le \varepsilon^{-1} \cdot L\left(d(\cdot,x);x\right) \cdot \omega(f,\varepsilon)\\ &\le \left\{1 + \varepsilon^{-1} \cdot L\left(\chi_{(1,\infty)}\left(\varepsilon^{-1} \cdot d(\cdot,x)\right) \cdot d(\cdot,x);x\right)\right\} \cdot \omega(f,\varepsilon),\end{aligned}$$

the latter inequality following from

$$\begin{aligned}&L\left(\left[1 - \chi_{(1,\infty)}(\varepsilon^{-1} \cdot d(\cdot,x))\right] \cdot d(\cdot,x);x\right) \le\\ &\le \left\|\left[1 - \chi_{(1,\infty)}(\varepsilon^{-1} \cdot d(\cdot,x))\right] \cdot d(\cdot,x)\right\| \cdot L(1_X,x)\\ &= \left\|\left[1 - \chi_{(1,\infty)}(\varepsilon^{-1} \cdot d(\cdot,x))\right] \cdot d(\cdot,x)\right\| = \sup\left\{d(t,x) \;:\; d(t,x) \le \varepsilon\right\} = \varepsilon.\end{aligned}$$

Thus for a function f having a concave modulus of continuity the first inequality is also sharper for $\varepsilon < L(d(\cdot,x);x)$.

2. *If f and ε are given in a way such that $\widetilde{\omega}(f,\varepsilon) = 2 \cdot \omega(f,\varepsilon)$, then the first inequality from Theorem 3.4 reads*

$$|L(f,x) - f(x)| \le 2 \cdot \max\left\{1, \varepsilon^{-1} \cdot L\left(d(\cdot,x);x\right)\right\} \cdot \omega(f,\varepsilon).$$

For $\varepsilon \ge L\left(d(\cdot,x);x\right)$ the right hand side becomes

$$2 \cdot \omega(f,\varepsilon) \ge \left\{1 + \varepsilon^{-1} \cdot L\left(\chi_{(1,\infty)}(\varepsilon^{-1} \cdot d(\cdot,x)) \cdot d(\cdot,x);x\right)\right\} \cdot \omega(f,\varepsilon)$$

which is true because of

$$L\left(\chi_{(1,\infty)}(\varepsilon^{-1} \cdot d(\cdot,x)) \cdot d(\cdot,x);x\right) \le L\left(d(\cdot,x);x\right) \le \varepsilon.$$

Moreover, if $\varepsilon < L\left(d(\cdot,x);x\right)$, then

$$2 \cdot \max\left\{1, \varepsilon^{-1} \cdot L\left(d(\cdot,x);x\right)\right\} \cdot \omega(f,\varepsilon) = 2 \cdot \varepsilon^{-1} \cdot L\left(d(\cdot,x);x\right) \cdot \omega(f,\varepsilon)$$

$$\begin{aligned}
&= \left[\varepsilon^{-1}\cdot L\left(d(\cdot,x);x\right)+\varepsilon^{-1}\cdot L\left(d(\cdot,x);x\right)\right]\cdot\omega(f,\varepsilon)\\
&> \left[1+\varepsilon^{-1}\cdot L\left(d(\cdot,x);x\right)\right]\cdot\omega(f,\varepsilon)\\
&\geq \left\{1+\varepsilon^{-1}\cdot L\left(\chi_{(1,\infty)}(\varepsilon^{-1}\cdot d(\cdot,x))\cdot d(\cdot,x);x\right)\right\}\cdot\omega(f,\varepsilon).
\end{aligned}$$

Hence in case 2. the second inequality is also better for $\varepsilon < L(d(\cdot,x);x)$.

This discussion shows that both quantities in the min of an estimate of the form

$$|L(f,x)-f(x)| \leq \min\Big\{\max\left[1,\varepsilon^{-1}\cdot L\left(d(\cdot,x);x\right)\right]\cdot\widetilde{\omega}(f,\varepsilon),\\
\left[1+\varepsilon^{-1}\cdot L\left(\chi_{(1,\infty)}(\varepsilon^{-1}\cdot d(\cdot,x))\cdot d(\cdot,x);x\right)\right]\cdot\omega(f,\varepsilon)\Big\}$$

may determine the actual value of the min, and so neither of the two approaches taken above yields better results than the one in general.

5. CONCLUDING REMARKS

A) After submission of the author's "Habilitationschrift" in 1985 and its subsequent 1986 publication in "Schriftenreihe des Fachbereichs Mathematik" of the University of Duisburg, several further papers dealing with related questions appeared. For obvious reasons, none of them contained estimates better than the ones given in the above. As a general source for related material we mention here the book by F. Altomare and M. Campiti [1]. Recent contributions influenced by the Romanian school are due to D. Andrica, C. Badea, I. Raşa, and several of their collaborators (see, *e.g.*, [2], [3], [4], [47]). As articles from China papers by Tian-ping Chen and Wen-ge Zhu ([63], [8]) have to be mentioned. Related work was also carried out in the work of Ch. Richter and I. Stephani from the University of Jena (cf. [48], [49], [50], [60], [61]). An excellent survey on Shepard's method, probably the most significant application of the general theory presented here, was recently finished by H. Knauf [28]. In regard to the latter technique, there is also the interesting work of the group around Gh. Coman from Cluj-Napoca (see, *e.g.*, [9] and the references given there).

B) In our recent paper "The second order modulus again: some still (?) open problems" (see [19]) we asked the following question: Is there a generalization of ω_2 to $C(X,d)$, where (X,d) is a compact metric space with at least some suitable geometric structure? Here we would like to suggest the following definition.

Suppose (X,d) is metrically convex in the sense of Menger. Then, for given $x,y\in X$ there is always a point $z\in X$ such that

$$d(x,y)=\tfrac{1}{2}d(x,y)+\tfrac{1}{2}d(x,y)=d(x,z)+d(z,y). \qquad (*)$$

Now consider *all* differences of the form

$$|f(x)-2f(z)+f(y)|, \quad \text{where } d(x,y)\leq h.$$

Then put

$$\omega_2(f;h) := \sup\Big\{|f(x)-2f(z)+f(y)| \ :\ d(x,y) \le h \text{ and } z \text{ such that } (*) \text{ holds}\Big\}.$$

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