JOURNAL OF NUMERICAL ANALYSIS AND APPROXIMATION THEORY

J. Numer. Anal. Approx. Theory, vol. 52 (2023) no. 2, pp. 128–138, doi.org/10.33993/jnaat522-1350

ictp.acad.ro/jnaat

FUZZY KOROVKIN TYPE THEOREMS VIA DEFERRED CESÁRO AND DEFERRED EULER EQUI-STATISTICAL CONVERGENCE

PURSHOTTAM N. AGRAWAL[†] and BEHAR BAXHAKU[*]

Abstract. We establish a fuzzy Korovkin type approximation theorem by using $eq-stat^D_{CE}$(deferred Cesáro and deferred Euler equi-statistical) convergence proposed by Saini *et al.* [31] for continuous functions over $J^b_a := [a, b] \subset \mathbb{R}$. Further, we determine the rate of convergence via fuzzy modulus of continuity.

MSC. $41A36, 41A25, 26A15$.

Keywords. Korovkin theorem, fuzzy number, $eq-stat^D_{CE}$, fuzzy positive linear operators, fuzzy modulus of continuity.

1. INTRODUCTION

The fuzziness is used to deal with imprecise or uncertain information. It measures the imperfection of an example. As a result, the computing by Fuzzy logic approach is based on “degrees of truth” rather than the usual “true or false” (1 or 0) Boolean logic used by the present day computers. Fuzzy logic is used by various types of AI systems and technologies, for instance vehicle intelligence, consumer electronics, medical diagnosis, software, chemicals, aerospace and environment control systems etc. The concept of fuzzy logic was put forward by Zadeh [38] in 1965, while working on the computer understanding of natural languages. A modified definition of fuzzy numbers was given by Goetschel Jr. and Voxman [20]. The concept of a sequence of fuzzy numbers was proposed by Matloka [25]. Nanda [27] established that the set of convergent sequences of fuzzy numbers is complete. Subrahmanyam [34] introduced the Cesáro summability of fuzzy numbers. Gal [19] generalized the classical results of approximation theory to the fuzzy setting. Motivated by this work, Anastassiou [6] established the fuzzy analogues of many approximation theorems.

Anastassiou [8] established the basic fuzzy Korovkin type theorem for fuzzy positive linear operators by means of fuzzy Shisha–Mond inequality and also

[†]Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee-247667, India, e-mail: pnappfma@gmail.com.

[*]Department of Mathematics, University of Prishtina “Hasan Prishtina”, Prishtina, Kosovo, e-mail: behar.baxhaku@uni-pr.edu, corresponding author.

presented the rate of convergence with the aid of fuzzy modulus of continuity. Anastassiou *et al.* [9] investigated a Korovkin-type theorem in the fuzzy setting by using a matrix summability method and also examined the rate of convergence with the aid of fuzzy modulus of continuity. Yavuz [36] presented a fuzzy trigonometric Korovkin type theorem via power series summability method and also established another related approximation theorem with the aid of fuzzy modulus of continuity for functions belonging to $C_{2\pi}^{\mathcal{F}}(\mathbb{R})$, the space of 2π-periodic fuzzy continuous functions on $\mathbb{R}$. Baxhaku *et al.* [12] derived fuzzy Korovkin type theorem by means of power series method for $C^{\mathcal{F}}(J_a^b)$, the space of fuzzy continuous functions on J_a^b and also examined the fuzzy rate of convergence via fuzzy modulus of continuity. Yavuz [37] investigated a trigonometric Korovkin type result for fuzzy valued functions of two variables.

In the past two decades, statistical convergence and its various generalizations have been an active area of research in approximation theory. In the year 1951, Steinhaus [33] and Fast [16] independently introduced the notion of statistical convergence to assign a limit to the sequences which are not convergent in the usual sense. Gadjiev and Orhan [18] established a Korovkin-type approximation theorem for the first time via statistical convergence. Duman and Orhan [15] derived the Korovkin-type result using the concept of A-statistical convergence defined by Freedman and Sember [17], where A is a non-negative regular infinite summability matrix. Karakaya and Chishti [23] introduced the concept of weighted statistical convergence. Mohiuddine [26] discussed the relationship of statistical weighted A-summability with the weighted A-statistical convergence of a sequence and derived a Korovkin type result via statistical weighted A-summability. Agrawal *et al.* [2] studied the weighted A-statistical convergence for an exponential type operator defined by Ismail and May [22]. Srivastava *et al.* [32] introduced the notion of deferred weighted A-statistical convergence and established a Korovkin type approximation theorem for continuous functions on J_0^1. Patro *et al.* [29] defined the notions of deferred Euler statistical convergence and statistical deferred Euler summability means and derived some inclusion relations between them. Furthermore, the authors [29] proved a Korovokin-type approximation theorem based on the proposed mean. Agrawal *et al.* [3] derived a Korovkin type theorem for a sequence of bivariate generalized Bernstein-Kantorovich type operators on a triangle by means of deferred weighted A-statistical convergence. Later, Agrawal *et al.* [4] established a general Korovkin type theorem for the deferred weighted A-statistical convergence of a sequence of positive linear operators. Demirci *et al.* [14] proved a Korovkin type theorem by using equi-statistical convergence in the sense of power series method. Saini *et al.* [31] presented the notions of equi-statistical convergence, pointwise statistical convergence and uniform statistical convergence for a sequence of real-valued functions by using deferred Cesáro and deferred Euler statistical convergence. Furthermore, the authors [31] established a Korovkin type theorem and the rate of convergence by using the notion of deferred Cesáro and deferred Euler equi-statistical convergence.

Nuray and Savaş [28] proposed the fuzzy analogue of statistical convergence of a fuzzy number valued sequence. Anastassiou and Duman [10] extended the results obtained in [8] by using the notion of A-statistical convergence. Dass *et al.* [13] investigated a fuzzy Korovkin type theorem by using statistical $(C,1)(E,\mu)$ product summability method and also determined the associated fuzzy rate of convergence. Aiyub *et al.* [5] proved Korovkin type theorem via lacunary equi-statistical convergence in the fuzzy space $C^{\mathcal{F}}(J_a^b)$ and also determined the associated rate of convergence via fuzzy modulus of continuity. Very recently, by using fractional difference operator Raj *et al.* [30] presented a fuzzy Korovkin type result via statistical Euler summability and also studied the corresponding fuzzy convergence rate.

The purpose of the present paper is to extend the study carried out in [31] in the fuzzy environment. We establish the fuzzy Korovkin type approximation theorem for functions in $C^{\mathcal{F}}(J_a^b)$ by means of $eq-stat_{CE}^{D}$ convergence and also determine the associated order of convergence via fuzzy modulus of continuity.

2. PRELIMINARIES

In our study, we shall need the following definitions.

A fuzzy real number is a function $\nu:\mathbb{R}\to J_0^1$ satisfying:

1) ν is normal, *i.e.*, we can find a number $z_0\in\mathbb{R}$ such that $\nu(z_0)=1$;
2) ν is a fuzzy convex subset, *i.e.*, $\nu(\xi z_1+(1-\xi)z_2)\geq\min(\nu(z_1),\nu(z_2))$, $\forall\ z_1,z_2\in\mathbb{R}$ and $\xi\in J_0^1$;
3) for a given $\epsilon>0$ and for any $z_0\in\mathbb{R}$ $\exists$ a neighbourhood W of z_0 such that $\nu(z)\leq\nu(z_0)+\epsilon$, $\forall z\in W$, *i.e.*, ν is upper semi-continuous on $\mathbb{R}$;
4) the closure of $\operatorname{supp}(\nu)$ is compact, where $\operatorname{supp}(\nu):=\{z\in\mathbb{R}:\nu(z)>0\}$.

Let $C^{\mathcal{F}}(\mathbb{R}):=\{\varphi:\varphi$ is fuzzy continuous over $\mathbb{R}\}$, then an operator $\mathcal{M}: C^{\mathcal{F}}(\mathbb{R})\to C^{\mathcal{F}}(\mathbb{R})$ is called fuzzy linear if

$$\mathcal{M}(\alpha\bigodot\phi_1\bigoplus\beta\bigodot\phi_2)=\alpha\bigodot\mathcal{M}(\phi_1)\bigoplus\beta\bigodot\mathcal{M}(\phi_2),$$

for all $\alpha,\beta\in\mathbb{R}$ and $\phi_1,\phi_2\in C^{\mathcal{F}}(\mathbb{R})$. In addition, the fuzzy linear operator $\mathcal{M}$ is said to be positive if for any $\phi_1,\phi_2\in C_{\mathcal{F}}(\mathbb{R})$, with $\phi_1(z)\preceq\phi_2(z)$, $\forall\ z\in\mathbb{R}$, we have $\mathcal{M}(\phi_1;z)\preceq\mathcal{M}(\phi_2;z)$, $\forall\ z\in\mathbb{R}$.

A sequence $\langle\tilde{w}_\nu\in\mathbb{R}_{\mathcal{F}}\rangle_{\nu\in\mathbb{N}}$ is called statistically convergent (see [28]) to $\tilde{w}_0\in\mathbb{R}_{\mathcal{F}}$, if for a given $\epsilon>0$, the natural density

$$\delta(\{\nu\leq n: D(\tilde{w}_\nu,\tilde{w}_0)\geq\epsilon\})=0$$

We denote this convergence by writing $stat_{\mathcal{F}}-\lim_{\nu\to\infty}\tilde{w}_\nu=\tilde{w}_0$.

Following [1], let $p=\langle p_n\rangle$ and $q=\langle q_n\rangle$ be sequences in $\mathbb{N}^0=\mathbb{N}\cup\{0\}$ satisfying

(i) $p_n<q_n$, $\forall n\in\mathbb{N}$;

(ii) $\lim_{n\to\infty} q_n = \infty$.

Then the deferred Cesáro mean of $\langle \tilde{w}_\nu \in \mathbb{R}_{\mathcal{F}} \rangle$ is defined as

$$\sigma_n = \tfrac{1}{q_n - p_n} \sum_{\nu=p_n+1}^{q_n} \tilde{w}_\nu, \quad n \in \mathbb{N}.$$

The deferred Euler mean of order μ [29] is given by

$$\xi_n = \tfrac{1}{(\mu+1)^{q_n}} \sum_{\nu=p_n+1}^{q_n} \tbinom{q_n}{m} \mu^{q_n-m} \tilde{w}_\nu, \; for \; \mu > 0.$$

Following [31], we call the sequence $\langle \tilde{w}_m \rangle$ to be deferred Euler statistically convergent to $\tilde{w}_0$, if for each $\epsilon > 0$,

$$G_n(\epsilon) = \{m \in \mathbb{N}; m \leq (1+\mu)^{q_n} \text{ and } \mu^{q_n-m} D(\tilde{w}_m, \tilde{w}_0) \geq \epsilon\},$$

has asymptotic density 0, *i.e.*,

$$\lim_{n\to\infty} \tfrac{|G_n(\epsilon)|}{(\mu+1)^{q_n}} = 0.$$

The sequence $\langle \tilde{w}_m \rangle$ is called $stat^D_{CE}$ (deferred Cesáro and deferred Euler statistical) convergent to $\tilde{w}_0 \in \mathbb{R}_{\mathcal{F}}$ (see [31]), if for any $\epsilon > 0$,

$$V_n(\epsilon) = \left\{m \in \mathbb{N}; m \leq (q_n - p_n)(1+\mu)^{q_n} \text{ and } \mu^{q_n-m} D(\tilde{w}_m, \tilde{w}_0) \geq \epsilon\right\},$$

has natural density 0, *i.e.*,

$$\lim_{n\to\infty} \tfrac{|V_n(\epsilon)|}{(q_n-p_n)(\mu+1)^{q_n}} = 0.$$

Let $X \subset \mathbb{R}$ be a compact set. Then $C(X) := \{\varphi : X \to \mathbb{R} |\ \varphi \text{ is continuous on } X\}$ is a Banach space with the sup-norm $\|.\|_{C(X)}$. Following [31], a sequence $g_n \in C(X)$, $n \in \mathbb{N}$ is said to be $eq - stat^D_{CE}$ convergent to g, if for any $\epsilon > 0$, we have

$$\lim_{n\to\infty} \tfrac{\|U_n(z,\epsilon)\|_{C(X)}}{(q_n-p_n)(\mu+1)^{q_n}} = 0,$$

where

$$U_n(z,\epsilon) = \left|\{m; m \leq (q_n - p_n)(1+\mu)^{q_n} \text{ and } \mu^{q_n-m} D(g_m(z), g(z)) \geq \epsilon\}\right|. \tag{1}$$

We denote this convergence by $g_n \twoheadrightarrow g$ $(eq - stat^D_{CE})$.

3. FUZZY KOROVKIN THEOREM

Let $C(J^b_a) := \left\{\varphi : \varphi \text{ is a continuous function on } J^b_a\right\}$ with the sup-norm $\|\cdot\|$. Further, let $w_\zeta(z) = z^\zeta$, $\zeta = 0, 1, 2$, $\forall\, z \in J^b_a$. Anastassiou and Gal [11] extended the Korovkin theorem (algebraic case) [24] to the fuzzy setting as given below:

THEOREM 1 ([11], Th. 4). *Let $\mathcal{K}_n : C^{\mathcal{F}}(J_a^b) \to C^{\mathcal{F}}(J_a^b)$ be a sequence of fuzzy p.l.o. (positive linear operators). Suppose that $\exists$ a sequence $\tilde{\mathcal{K}}_n : C(J_a^b) \to C(J_a^b)$ of p.l.o. such that*

$$\{\mathcal{K}_n(\varphi; z)\}_{\pm}^{(t)} = \tilde{\mathcal{K}}_n(\varphi_{\pm}^{(t)}; z) \tag{2}$$

for all $t \in J_0^1$, $n \in \mathbb{N}$, and $\varphi \in C^{\mathcal{F}}(J_a^b)$. Further, let

$$\lim_{n\to\infty} \|\tilde{\mathcal{K}}_n(w_\zeta) - w_\zeta\| = 0,$$

for $\zeta = 0, 1, 2$. Then, for all $\varphi \in C^{\mathcal{F}}(J_a^b)$, we have

$$\lim_{n\to\infty} D^* \left(\mathcal{K}_n(\varphi), \varphi\right) = 0.$$

In the following result, we prove the fuzzy Korovkin Theorem via $eq-stat_{CE}^{D}$ convergence.

THEOREM 2. *Let $\mathcal{K}_n : C^{\mathcal{F}}(J_a^b) \to C^{\mathcal{F}}(J_a^b)$ be a sequence of fuzzy p.l.o. Suppose that $\exists$ a sequence $\tilde{\mathcal{K}}_n : C(J_a^b) \to C(J_a^b)$ of p.l.o. satisfying (2). Further, let*

$$\lim_{n\to\infty} \tfrac{\|W_{n,k}(z,\epsilon)\|}{(q_n-p_n)(1+\mu)^{q_n}} = 0, \tag{3}$$

where

$$W_{n,k}(z,\epsilon) = \Big\{ m \in \mathbb{N} : m \le (q_n - p_n)(1+\mu)^{q_n}$$

$$\text{and } \mu^{q_n - m} \left|\tilde{\mathcal{K}}_m(w_k; z) - w_k(z)\right| \ge \tfrac{\epsilon'-\epsilon}{3K(\epsilon)} \Big\}; \tag{4}$$

i.e., $\tilde{\mathcal{K}}_n(w_k) \twoheadrightarrow w_k$ $(eq - stat_{CE}^{D})$, for all $k = 0, 1, 2$. Then, for every $\varphi \in C^{\mathcal{F}}(J_a^b)$, we have

$$\lim_{n\to\infty} \tfrac{\|W_n(z.\epsilon)\|}{(q_n-p_n)(1+\mu)^{q_n}} = 0,$$

i.e., where,

$$W_n(z,\epsilon) = \left\{ m \in \mathbb{N} : m \le (q_n - p_n)(1+\mu)^{q_n} \text{ and } \mu^{q_n - m} D\left(\mathcal{K}_m(\varphi; z), \varphi(z)\right) \ge \epsilon \right\},$$

i.e., $\mathcal{K}_n(\varphi) \twoheadrightarrow \varphi$ $(eq - stat_{CE}^{D})$.

Proof. Let $\varphi \in C^{\mathcal{F}}(J_a^b)$, $t \in J_0^1$ and $z \in J_a^b$. Then, $\varphi_{\pm}^{(t)} \in C(J_a^b)$. Hence for any $\epsilon > 0$, we can find $\delta > 0$ such that $\left|\varphi_{\pm}^{(t)}(v) - \varphi_{\pm}^{(t)}(z)\right| < \epsilon$, holds for $\forall\, v \in J_a^b$ with $|v - z| < \delta$. Then, for all $v \in J_a^b$, it follows that

$$\left|\varphi_{\pm}^{(t)}(v) - \varphi_{\pm}^{(t)}(z)\right| < \epsilon + 2M_{\pm}^{(t)} \tfrac{(v-z)^2}{\delta^2}, \tag{5}$$

where $M_{\pm}^{(t)} = \|\varphi_{\pm}^{(t)}\|$ (see [24]). Due to the positivity and linearity of the operator $\tilde{\mathcal{K}}_n$, and using (5) one may write

$$\left|\tilde{\mathcal{K}}_n(\varphi_{\pm}^{(t)}; z) - \varphi_{\pm}^{(t)}(z)\right| \le$$

$$\leq \tilde{\mathcal{K}}_n\Big(|\varphi_{\pm}^{(t)}(v)-\varphi_{\pm}^{(t)}(z)|;z\Big)+M_{\pm}^{(t)}|\tilde{\mathcal{K}}_n(w_0;z)-w_0(z)|$$
$$<\tilde{\mathcal{K}}_n\left(\epsilon+2M_{\pm}^{(t)}\tfrac{(v-z)^2}{\delta^2};z\right)+M_{\pm}^{(t)}|\tilde{\mathcal{K}}_n(w_0;z)-w_0(z)|$$
$$<\epsilon\tilde{\mathcal{K}}_n(w_0;z)+M_{\pm}^{(t)}|\tilde{\mathcal{K}}_n(w_0;z)-w_0(z)|+\tfrac{2M_{\pm}^{(t)}}{\delta^2}\tilde{\mathcal{K}}_n\left((v-z)^2;z\right),$$

which yields

$$\left|\tilde{\mathcal{K}}_n(\varphi_{\pm}^{(t)};z)-\varphi_{\pm}^{(t)}(z)\right|<\epsilon+(M_{\pm}^{(t)}+\epsilon)|\tilde{\mathcal{K}}_n(w_0;z)-w_0(z)|$$
$$+\tfrac{2M_{\pm}^{(t)}}{\delta^2}\Big[z^2\{\tilde{\mathcal{K}}_n(w_0;z)-w_0(z)\}$$
$$+2|z|\{\tilde{\mathcal{K}}_n(w_1;z)-w_1(z)\}+\{\tilde{\mathcal{K}}_n(w_2;z)-w_2(z)\}\Big]$$
$$<\epsilon+K_{\pm}^{(t)}(\epsilon)\Big[|\tilde{\mathcal{K}}_n(w_0;z)-w_0(z)|$$
$$+|\tilde{\mathcal{K}}_n(\omega_1;z)-\omega_1(z)|+|\tilde{\mathcal{K}}_n(\omega_2;z)-\omega_2(z)|\Big], \tag{6}$$

where $K_{\pm}^{(t)}(\epsilon)=\left\{\frac{2M_{\pm}^{(t)}}{\delta^2},\frac{4\eta M_{\pm}^{(t)}}{\delta^2},\epsilon+M_{\pm}^{(t)}+\frac{2\eta^2M_{\pm}^{(t)}}{\delta^2}\right\}$, with $\eta=\max\{|a|,|b|\}$. Hence using (2), we get

$$D\left(\mathcal{K}_n(\varphi;z),\varphi(z)\right)<\epsilon+K\left\{\sum_{k=0}^{2}\left|\tilde{\mathcal{K}}_n(w_k;z)-w_k(z)\right|\right\},$$

where $K=K(\epsilon)=\sup\limits_{t\in[0,1]}\max\left\{K_{+}^{(t)}(\epsilon),K_{-}^{(t)}(\epsilon)\right\}$.

For any $\epsilon'>0$, take ϵ, satisfying $0<\epsilon<\epsilon'$, and define:

$$W_n(z,\epsilon)=$$
$$=\left\{m\in\mathbb{N}:m\leq(q_n-p_n)(1+\mu)^{q_n}\text{ and }\mu^{q_n-m}D\left(\mathcal{K}_m(\varphi;z),\varphi(z)\right)\geq\epsilon\right\};$$

$$W_{n,0}(z,\epsilon)=$$
$$=\left\{m\in\mathbb{N}:m\leq(q_n-p_n)(1+\mu)^{q_n}\text{ and }\mu^{q_n-m}\left|\tilde{\mathcal{K}}_m(w_0;z)-w_0(z)\right|\geq\tfrac{\epsilon'-\epsilon}{3K(\epsilon)}\right\};$$

$$W_{n,1}(z,\epsilon)=$$
$$=\left\{m\in\mathbb{N}:m\leq(q_n-p_n)(1+\mu)^{q_n}\text{ and }\mu^{q_n-m}\left|\tilde{\mathcal{K}}_m(w_1;z)-w_1(z)\right|\geq\tfrac{\epsilon'-\epsilon}{3K(\epsilon)}\right\};$$

and

$$W_{n,2}(z,\epsilon)=$$
$$=\left\{m\in\mathbb{N}:m\leq(q_n-p_n)(1+\mu)^{q_n}\text{ and }\mu^{q_n-m}\left|\tilde{\mathcal{K}}_m(w_2;z)-w_2(z)\right|\geq\tfrac{\epsilon'-\epsilon}{3K(\epsilon)}\right\}.$$

Then,

$$\frac{\|W_n(z,\epsilon)\|}{(q_n-p_n)(1+\mu)^{q_n}} \leq \sum_{k=0}^{2} \frac{\left\|W_{n,k}(z,\epsilon)\right\|}{(q_n-p_n)(1+\mu)^{q_n}}.$$

Hence using hypothesis (3), the proof is completed. □

4. STATISTICAL FUZZY CONVERGENCE RATE

We discuss fuzzy rate of approximation by the operators $\mathcal{K}_n$ for any function $\varphi \in C^{\mathcal{F}}(J_a^b)$ by means of the fuzzy modulus of continuity via $eq - stat_{CE}^{D}$ convergence.

Let $\langle \phi_n \rangle_{n\in\mathbb{N}}$ be a monotonically decreasing positive sequence. Then, we call a sequence $\langle g_n \in C^{\mathcal{F}}(J_a^b)\rangle_{n\in\mathbb{N}}$ to be $eq - stat_{CE}^{D}$ convergent to g with the fuzzy rate $o(\phi_n)$, provided for any $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{\|U_n(z,\epsilon)\|}{\phi_n(q_n-p_n)(\mu+1)^{q_n}} = 0, \tag{7}$$

where $U_n(z,\epsilon)$ is same as defined by (1). Symbolically, we write it as $g_n - g = o(\phi_n)\ (eq - stat_{CE}^{D})$.

From [8], the fuzzy modulus of continuity is given by:

$$\omega_1^{\mathcal{F}}(\varphi;\delta) = \sup_{v,z\in J_a^b;|v-z|\leq\delta} D(\varphi(v),\varphi(z)), \quad \text{for any } \delta > 0, \varphi \in C^{\mathcal{F}}(J_a^b).$$

LEMMA 3 ([7]). *Let $\varphi \in C^{\mathcal{F}}(J_a^b)$. Then,*

$$\omega_1^{\mathcal{F}}(\varphi;\delta) = \sup_{t\in J_0^1} \max\{\omega_1(\varphi_-^{(t)};\delta), \omega_1(\varphi_+^{(t)};\delta)\},\ \delta > 0.$$

THEOREM 4. *Let $\mathcal{K}_n : C^{\mathcal{F}}(J_a^b) \to C^{\mathcal{F}}(J_a^b)$ be any sequence of fuzzy p.l.o. Suppose that ∃ a sequence $\tilde{\mathcal{K}}_n : C(J_a^b) \to C(J_a^b)$ of p.l.o. satisfying the property Eq. (2). Further, let $\langle \phi_n \rangle_{n\in\mathbb{N}}$ and $\langle \psi_n \rangle_{n\in\mathbb{N}}$ be the monotonically decreasing positive sequences such that the following conditions hold:*

(i) $\left|\tilde{\mathcal{K}}_n(w_0;z) - w_0(z)\right| = o(\phi_n), (eq - stat_{CE}^{D}),\ \forall\, z \in J_a^b$;

(ii) $\omega_1^{\mathcal{F}}(\varphi;\gamma_n) = o(\psi_n), (eq - stat_{CE}^{D})$, *where* $\gamma_n = \sqrt{\|\tilde{\mathcal{K}}_n(\psi)\|}$ *with* $\psi(v) = (v-z)^2$, $\forall\, z \in J_a^b$.

Then, for all $\varphi \in C^{\mathcal{F}}(J_a^b)$, the sequence $\mathcal{K}_n(\varphi) - \varphi = o(\xi_n), (eq - stat_{CE}^{D})$, where $\xi_n = \max\{\phi_n, \psi_n\}$, $\forall n \in \mathbb{N}$.

Proof. Let $\varphi \in C^{\mathcal{F}}(J_a^b)$ and $z \in J_a^b$. Since $\varphi_\pm^{(t)} \in C(J_a^b)$, for a given $\epsilon > 0$, ∃ $\delta > 0$ such that $|\varphi_\pm^{(t)}(v) - \varphi_\pm^{(t)}(z)| < \epsilon$, whenever $|v - z| < \delta$. Hence from [24], for all $v \in J_a^b$, we obtain

$$|\varphi_\pm^{(t)}(v) - \varphi_\pm^{(t)}(z)| \leq \left(1 + \frac{(v-z)^2}{\delta^2}\right)\omega_1(\varphi_\pm^{(t)};\delta)$$

and hence we obtain

$$\left|\tilde{\mathcal{K}}_n(\varphi_\pm^{(t)}(v);z)/ - \varphi_\pm^{(t)}(z\right| \leq$$

$$\leq \tilde{\mathcal{K}}_n\left(|\varphi_\pm^{(t)}(v)-\varphi_\pm^{(t)}(z)|;z\right)+M_\pm^{(t)}|\tilde{\mathcal{K}}_n(w_0;z)-w_0(z)|$$
$$\leq \omega_1(\varphi_\pm^{(t)};\delta)\tilde{\mathcal{K}}_n\left(1+\tfrac{(v-z)^2}{\delta^2};z\right)+M_\pm^{(t)}|\tilde{\mathcal{K}}_n(w_0;z)-w_0(z)|$$
$$\leq \omega_1(\varphi_\pm^{(t)};\delta)|\tilde{\mathcal{K}}_n(w_0;z)-w_0(z)|+\omega_1(\varphi_\pm^{(t)};\delta)+M_\pm^{(t)}|\tilde{\mathcal{K}}_n(w_0;z)-w_0(z)|$$
$$+\tfrac{\omega_1(\varphi_\pm^{(t)};\delta)}{\delta^2}\tilde{\mathcal{K}}_n\left(\phi(v);z\right),$$

where $M_\pm^{(t)}=\|\varphi_\pm^{(t)}\|$. Then using (2) and Lemma 3, we obtain

$$\sup_{t\in J_0^1}\left|\tilde{\mathcal{K}}_n(\varphi_\pm^{(t)}(v);z)-\varphi_\pm^{(t)}(z)\right|\leq\omega_1^{\mathcal{F}}(\varphi;\delta)|\tilde{\mathcal{K}}_n(w_0;z)-w_0(z)|+\omega_1^{\mathcal{F}}(\varphi;\delta)$$
$$+M|\tilde{\mathcal{K}}_n(w_0;z)-w_0(z)|+\tfrac{\omega_1^{\mathcal{F}}(\varphi;\delta)}{\delta^2}\{\tilde{\mathcal{K}}_n\left(\phi(v);z\right)\},$$

where $M:=\sup_{t\in J_0^1}\max\{M_+^{(t)},M_-^{(t)}\}$. Now, choosing $\delta=\gamma_n$, $\forall z\in J_a^b$ we have

$$D\left(\mathcal{K}_n(\varphi;z),\varphi(z)\right)\leq$$
$$\leq\omega_1^{\mathcal{F}}(\varphi;\gamma_n)|\tilde{\mathcal{K}}_n(w_0;z)-w_0(z)|+2\omega_1^{\mathcal{F}}(f;\gamma_n)+M|\tilde{\mathcal{K}}_n(w_0;z)-w_0(z)|$$
$$\leq K\Big\{|\tilde{\mathcal{K}}_n(w_0;z)-w_0(z)|\omega_1^{\mathcal{F}}(\varphi;\gamma_n)+\omega_1^{\mathcal{F}}(\varphi;\gamma_n)+|\tilde{\mathcal{K}}_n(w_0;z)-w_0(z)|\Big\},$$

where $K=\max\{M,2\}$.

For any $\epsilon>0$, define:

$$U_n(z,\epsilon)=\{m\in\mathbb{N}:m\leq(q_n-p_n)(1+\mu)^{q_n}\text{ and }\mu^{q_n-m}D\left(\mathcal{K}_m(\varphi;z),\varphi(z)\right)\geq\epsilon\};$$

$$U_{n,1}(z,\epsilon)=$$
$$=\left\{m\in\mathbb{N}:m\leq(q_n-p_n)(1+\mu)^{q_n}\text{ and }\mu^{q_n-m}\left|\tilde{\mathcal{K}}_m(w_0;z)-w_0(z)\right|\omega_1^{\mathcal{F}}(\varphi;\gamma_m)\geq\tfrac{\epsilon}{3K}\right\}$$

$$U_{n,2}(z,\epsilon)=\left\{m\in\mathbb{N}:m\leq(q_n-p_n)(1+\mu)^{q_n}\text{ and }\mu^{q_n-m}\omega_1^{\mathcal{F}}(\varphi;\gamma_m)\geq\tfrac{\epsilon}{3K}\right\};$$

and

$$U_{n,3}(z,\epsilon)=\left\{m\in\mathbb{N}:m\leq(q_n-p_n)(1+\mu)^{q_n}\text{ and }\mu^{q_n-m}|\tilde{\mathcal{K}}_m(w_0;z)-w_0(z)|\geq\tfrac{\epsilon}{3K}\right\}.$$

Then,

$$\tfrac{\|U_n(z,\epsilon)\|}{\xi_n(q_n-p_n)(1+\mu)^{q_n}}\leq\tfrac{\|U_{n,1}(z,\epsilon)\|}{\phi_n\psi_n(q_n-p_n)(1+\mu)^{q_n}}+\tfrac{\|U_{n,2}(z,\epsilon)\|}{\psi_n(q_n-p_n)(1+\mu)^{q_n}}+\tfrac{\|U_{n,3}(z,\epsilon)\|}{\phi_n(q_n-p_n)(1+\mu)^{q_n}}.$$

Finally, using (7) and the assumptions (i) and (ii), we reach the desired assertion. □

Acknowledgements. The authors are extremely grateful to the learned reviewer for a very careful reading of the manuscript and making invaluable suggestions and comments leading to an overall improvement in the presentation of the paper.

REFERENCES

[1] R.P. Agnew, *On deferred Cesáro means*, Ann. Math., **33** (3) (1932), pp. 413–421, https://doi.org/10.2307/1968524.

[2] P.N. Agrawal, A.M. Acu, R. Chauhan, *Weighted A-statistical convergence and Bögel approximation by operators of exponential type*, J. Math. Inequal., **16**(3) (2022), pp. 827–850, http://dx.doi.org/10.7153/jmi-2022-16-57.

[3] P.N. Agrawal, A.M. Acu, T. Garg, *Approximation of Bögel continuous functions and deferred weighted A-statistical convergence by Bernstein-Kantorovich operators on a triangle*, J. Math. Inequal., **15**(4) (2021), pp. 1695–1711, http://dx.doi.org/10.7153/jmi-2021-15-116.

[4] P.N. Agrawal, B. Baxhaku, R. Shukla, *Characterization of deferred type statistical convergence and P-summability method for operators: Applications to q-Lagrange–Hermite operator*, Math. Methods Appl. Sci., **46**(4) (2023), pp. 4449–4465, https://doi.org/10.1002/mma.8770.

[5] M. Aiyub, K. Saini, K. Raj, *Korovkin type approximation theorem via lacunary equi-statistical convergence in fuzzy spaces*, J. Math. Comput. Sci., **25** (2022), pp. 312–321, http://dx.doi.org/10.22436/jmcs.025.04.02.

[6] G.A. Anastassiou, *Fuzzy Mathematics: Approximation Theory*, Studies in Fuzziness and Soft Computing, Springer-Verlag Berlin, Heidelberg, 2010, https://doi.org/10.1007/978-3-642-11220-1.

[7] G.A. Anastassiou, *Fuzzy approximation by fuzzy convolution type operators*, Comput. Math. Appl., **48** (2004), pp. 1369–1386, http://doi.org/10.1016/j.camwa.2004.10.027.

[8] G.A. Anastassiou, *On basic fuzzy Korovkin theory*, Studia Univ. Babes-Bolyai Math., **L** (2005), pp. 3–10.

[9] G.A. Anastassiou, K. Demirci, S. Karakuş, *A-summability and fuzzy trigonometric Korovkin-type approximation*, J. Fuzzy Math., **19** (2011), pp. 453–462.

[10] G.A. Anastassiou, O. Duman, *Statistical fuzzy approximation by fuzzy positive linear operators*, Comput. Math. Appl., **55** (2008), pp. 573–580, https://doi.org/10.1016/j.camwa.2007.05.007.

[11] G.A. Anastassiou, S.G. Gal, *On fuzzy trigonometric Korovkin theory*, Nonlinear Funct. Anal. Appl., **11** (2006), pp. 385–395.

[12] B. Baxhaku, P.N. Agrawal, R. Shukla, *Some fuzzy Korovkin type approximation theorems via power series summability method*, Soft Computing, **26**(21) (2022), pp. 11373–11379, https://doi.org/10.1007/s00500-022-07429-6.

[13] A.A. Das, S.K. Paikray, T. Pradhan, H. Dutta, *Statistical $(C,1)(E,\mu)$-summability and associated fuzzy approximation theorems with statistical fuzzy rates*, Soft Computing, **24** (2020), pp. 10883–10892, https://doi.org/10.1007/s00500-019-04591-2.

[14] K. Demirci, F. Dirik, S. Yildiz, *Approximation via equi-statistical convergence in the sense of power series method*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat., **116** 65 (2022), https://doi.org/10.1007/s13398-021-01191-4.

[15] O. Duman, C. Orhan, *Rates of A-statistical convergence of positive linear operators*, Appl. Math. Lett., **18** (2005), pp. 1339–1344, https://doi.org/10.1016/j.aml.2005.02.029.

[16] H. Fast, *Sur la convergence statistique*, Colloq. Math., **2** (1951), pp. 241–244, https://doi.org/10.4064/cm-2-3-4-241-244.

[17] A.R. Freedman, J.J. Sember, *Densities and summability*, Pacific J. Math., **95** (1981), pp. 293–305.

[18] A.D. Gadjiev, C. Orhan, *Some approximation theorems via statistical convergence*, Rocky Mountain J. Math., **32** (2002), pp. 129–138, https://doi.org/10.1216/rmjm/1030539612.

[19] S. Gal, *Approximation theory in Fuzzy setting*, Handbook of Analytic-Computational Methods in Applied Mathematics, Chapman & Hall/CRC, Boca Raton, 2000.

[20] R. Goetschel Jr., W. Voxman, *Topological properties of fuzzy numbers*, Fuzzy Set Syst., **10** (1983), pp. 87–99, https://doi.org/10.1016/S0165-0114(83)80107-9.

[21] R. Goetschel Jr., W. Voxman, *Elementary fuzzy calculus*, Fuzzy Set Syst., **18** (1986), pp. 31–43, https://doi.org/10.1016/0165-0114(86)90026-6.

[22] E.H. Ismail, C.P. May, *On a family of approximation operators*, J. Math. Anal. Appl., **63** (1978), pp. 446–462, https://doi.org/10.1016/0022-247x(78)90090-2,

[23] V. Karakaya, T.A. Chishti, *Weighted statistical convergence*, Iran. J. Sci. Technol. Trans. A. Sci., **33** (2009), pp. 219–223.

[24] P.P. Korovkin, *Linear Operators and Approximation Theory*, Hindustan Publ. Co., Delhi, 1960.

[25] M. Matloka, *Sequence of fuzzy numbers*, BUSEFAL, **28** (1986), pp. 28–37.

[26] S.A. Mohiuddine, *Statistical weighted A-summability with application to Korovkin's type approximation theorem.*, J. Inequal. Appl., **101** (2016), https://doi.org/10.1186/s13660-016-1040-1.

[27] S. Nanda, *On sequence of fuzzy numbers*, Fuzzy sets Syst., **33** (1989), pp. 123–126, https://doi.org/10.1016/0165-0114(89)90222-4.

[28] F. Nuray, E. Savaş, *Statistical convergence of sequences of fuzzy numbers*, Math. Slovaca, **45** (1995), pp. 269–273, http://dml.cz/dmlcz/129143.

[29] M. Patro, S.K. Paikray, H. Dutta, *Statistical deferred Euler summability mean and associated Korovkin type approximation theorem*, Science & Technology Asia, **25**(1) (2020), pp. 31–37, https://doi.org/10.14456/scitechasia.2020.4.

[30] K. Raj, K. Saini, M. Mursaleen, *Applications of the fractional difference operator for studying Euler statistical convergence of sequences of fuzzy real numbers and associated Korovkin-type theorems*, Probl. Anal. Issues Anal., **11**(29) (2022) no. 3, pp. 91–108, http://doi.org/10.15393/j3.art.2022.11770.

[31] K. Saini, K. Raj, M. Mursaleen, *Deferred Cesáro and deferred Euler equi-statistical convergence and its implications to Korovkin-type approximation theorem*, Int. J. General Syst., 50(5) (2021), pp. 567–579, https://doi.org/10.1080/03081079.2021.1942867.

[32] H.M. Srivastava, B.B. Jena, S.K. Paikray, U.K. Mishra, *Deferred weighted A-statistical convergence based upon the (p, q)-Lagrange polynomials and its applications to approximation theorems*, J. Appl. Anal., **24**(1) (2018), pp. 1–16, https://doi.org/10.1515/jaa-2018-0001.

[33] H. Steinhaus, *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math., **2** (1951), pp. 73–74.

[34] P.V. Subrahmanyam, *Cesáro summability for fuzzy real numbers. p-adic analysis*, Summability theory, fuzzy analysis and applications, INCOPASFA, Chennai, 1998, J. Anal., **7** (1999), pp. 159–168.

[35] Wu Cong-Xin, Ma Ming, *On embedding problem of fuzzy number space: Part 1*, Fuzzy Set Syst., **44** (1991), pp. 33–38, https://doi.org/10.1016/0165-0114(91)90030-T.

[36] E. Yavuz, *Fuzzy trigonometric Korovkin type approximation via power series methods of summability*, U.P.B. Sci. Bull. Series A, **80**(3) (2018), pp. 123–132.

[37] E. Yavuz, *Basic trigonometric Korovkin approximation for fuzzy valued functions of two variables*, C. R. Acad. Bulg. Sci., **74**(9) (2021), pp. 1276–1288, https://doi.org/10.48550/arXiv.1901.06682.

[38] L.A. Zadeh, *Fuzzy sets*, Inform. Control, **8** (1965), pp. 338–353, https://doi.org/10.1016/S0019-9958(65)90241-X.

Received by the editors: June 23, 2023; accepted: October 5, 2023; published online: December 28, 2023.