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LOCALIZATION OF NASH-TYPE EQUILIBRIA FOR SYSTEMS WITH A PARTIAL VARIATIONAL STRUCTURE

ANDREI STAN*

Abstract. In this paper, we aim to generalize an existing result by obtaining localized solutions within bounded convex sets, while also relaxing specific initial assumptions. To achieve this, we employ an iterative scheme that combines a fixed-point argument based on the Minty-Browder Theorem with a modified version of the Ekeland variational principle for bounded sets. An application to a system of second-order differential equations with Dirichlet boundary conditions is presented.

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1. INTRODUCTION

Numerous problems can be reduced to a fixed point equation $N(u) = u$, where N is some operator. It is said that the equation has a variational form (or admits a variational structure) if it is equivalent with a critical point equation $E'(u) = 0$. Also, in real word processes, one may need the solution to be positive and finite or limited in some sense. From a mathematical perspective, it means that the solution lies in a bounded subset of the positive cone of all states.

We aim to generalize a result from [1], where a system of three equations

$$\begin{cases} N_1(u, v, w) = u \\ N_2(u, v, w) = v \\ N_3(u, v, w) = w, \end{cases}$$

was studied, so that one of the equations lacked the variational structure. In [1], sufficient conditions have been established for the system to admit a solution throughout the whole space, so that the second and third components of the solution represent a Nash equilibrium for the associated energy functionals.

*Faculty of Mathematics and Computer Science, Babeş-Bolyai University, 400084 Cluj-Napoca, Romania and Tiberiu Popoviciu Institute of Numerical Analysis, Romanian Academy, P.O. Box 68-1, 400110 Cluj-Napoca, Romania, e-mail: stan@ictp.acad.ro.

In the present paper, a system with n equations is considered,

$$
\begin{cases}
N_1(u^1,\ldots,u^n) = u^1\\
\ldots\\
N_p(u^1,\ldots,u^p,\ldots,u^n) = u^p\\
\ldots\\
N_n(u^1,\ldots,u^n) = u^n
\end{cases}
\tag{1}
$$

where only the last n-p equations admit a variational structure. Our goal is to find a solution $(u_*^1,\ldots,u_*^p,\ldots,u_*^n)$, such that $(u_*^{p+1},\ldots,u_*^n)$ is a Nash equilibrium for the corresponding energy functionals. Furthermore, this solution is intended to be localized in a bounded set of a positive cone. Localization is obtained in the Cartesian product of n-p bounded sets, as defined below. Specifically,

$$(u_*^{p+1},\ldots,u_*^n) \in K_{p+1}\times\cdots\times K_n,$$

where K_i is a positive cone of some Hilbert space X_i, with the corresponding inner product $(\cdot,\cdot)_i$ and norm $|\cdot|_i$. Additionally, we require that

$$r_q \le l_q(u_*^q)\ ;\ |u_*^q|_q \le R_q \quad (q=\{p+1,\ldots,n\})\,,$$

for some positive real numbers $r_q < R_q$, and a concave upper semicontinous functional $l_q\colon K_q \to \mathbb{R}_+$.

The study of systems with two equations, whose solutions represent a Nash-equilibrium localized in some conical sets, was previously made in [2] or [3]. This was achieved imposing a Perov contraction condition and making use of Ekeland variational principle. In this paper, we use a monotony condition instead of Lipschitz one, inspired by [4] and [5].

The idea of Nash equilibrium dates back to 1838, in a paper of Cournot [6], where the best output of a firm depending on the output of other firms was studied. Later, in 1951, the existence of a such equilibrium for any finite game was proved by John Forbes Nash [7]. From the physical point of view, a Nash equilibrium of a interconnected system with several entities, is that state in which the energy of each entity is minimal one with respect to the others.

The Nash equilibrium regards non-cooperative games, specifically those in which each player is unaware of how its change affects the output of the others. Heed that there are some other types of equilibrium, such as Pareto equilibrium for a cooperative game in which all parts are equal and collaborate to attain some (Pareto) optimal solution. If we are taking about a leading game, in which an individual (leader) takes the first step and all the others (followers) move accordingly, then we were dealing with a Stackelberg model and its equilibrium is called Stackelberg equilibrium.

The outline of this paper is as follows: We commence with Section 2, where several auxiliary results are presented. Section 3 provides the main theoretical result while Section 4 is dedicated to an application for a system of three second order differential equations subject to Dirichlet boundary conditions. Finally, we illustrate the application through a specific system.

2. PRELIMINARIES

Throughout this paper, $\mathbb{R}^n$ is endowed with the usual scalar product denoted by $x \cdot y = \sum_{i=1}^n x_i y_i$ and the Euclidian norm $|x| = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$. Also we consider the Hadamard product

$$\circ\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n;\ x \circ y = (x_1 y_1, \dots, x_n y_n)^T.$$

Unless otherwise noted, all vectors from $\mathbb{R}^n$ are considered to be column.

Proposition 1. *Let $M = (m_{ij})_{1 \le i,j \le n} \in \mathbb{M}_{n,n}(\mathbb{R}_+)$ be a matrix with positive entries and let $x = (x_i), y = (y_i), z = (z_i) \in \mathbb{R}_+^n$. If*

$$Mx \circ y \le z$$

then

$$Mx \cdot y \le \sqrt{n}|z|.$$

Proof. Note that

$$\sum_{j=1}^n m_{ij} x_j y_i \le z_i, \text{ for } i = 1, \dots, n.$$

Thus,

$$Mx \cdot y = \sum_{i=1}^n \sum_{j=1}^n m_{ij} x_j y_i \le \sum_{i=1}^n z_i.$$

The conclusion immediately follows from the Cauchy-Schwartz inequality, as we have $\sum_{i=1}^n z_i \le \sqrt{n}|z|$.

□

A square matrix of non-negative numbers $A = [a_{i,j}]_{1 \le i,j \le n} \in \mathbb{M}_{n,n}(\mathbb{R})$ is said to be convergent to zero if $A^k \to O_n$ as $k \to \infty$, where O_n is the zero matrix. In the following, we outline a few characterizations of such matrices.

Theorem 2 (see, *e.g.*, [5], [8], [9]). *The following statements are equivalent:*

(i) *A is convergent to zero.*
(ii) *$\rho(A) < 1$, where $\rho(A)$ represent the spectral radius of matrix A.*
(iii) *$I - A$ is non-singular, and its inverse has nonnegative entries.*
(iv) *There exist a positive diagonal matrix $D = (d_{ii})_{1 \le i \le n}$ such that*

$$D(I - A)x \cdot x > 0,\ \text{for all } x \in \mathbb{R}^n \setminus \{0\}.$$

For any $r \in \{1, \dots, n\}$, denote with $A_r := [a_{i,j}]_{1 \le i,j \le r}$ a submatrix of A.

Lemma 3. *If matrix A is convergent to zero, then A_r is also convergent to zero.*

Proof. Note that
$$A = \begin{bmatrix} A_r & B \\ C & D \end{bmatrix},$$
where $B \in \mathbb{M}_{r,n-r}(\mathbb{R})$, $C \in \mathbb{M}_{n-r,r}(\mathbb{R})$ and $D \in \mathbb{M}_{n-r,n-r}(\mathbb{R})$ are block matrices. Thus,
$$A^k = \begin{bmatrix} A_r^k + P_{k-1}^1 & P_k^2 \\ P_k^3 & P_k^4 \end{bmatrix}$$
where $P_m^1, P_m^2, P_m^3, P_m^4$ are some matrix polynomials of order m evaluated at A_r, B, C, D. Now, since $A^k \to O_n$, clearly $A_r^k + P_{n-1}^1 \to O_n$. Thus A_r is convergent to zero. □

For the proof of our main result (Theorem 7), we need the following convergence lemma.

Lemma 4 (see, *e.g.*, [1, Lemma 2.2]). *Let $(A_{k,p})_{k\geq 1}$, $(B_{k,p})_{k\geq 1}$ be two sequences of vectors in $\mathbb{R}_+^n$ (column vectors) depending on a parameter p, such that*
$$A_{k,p} \leq M A_{k-1,p} + B_{k,p}$$
for all k and p, where $M \in \mathbb{M}_n(\mathbb{R}_+)$ is a matrix with spectral radius less than one. If the sequence $(A_{k,p})_{k\geq 1}$ is bounded uniformly with respect to p and $B_{k,p} \to O_n$ as $k \to \infty$ uniformly with respect to p, then $A_{k,p} \to O_n$ as $k \to \infty$ uniformly with respect to p.

Let $(X, |\cdot|_X)$ be a Hilbert space together with the scalar product $(\cdot\,,\cdot)_X$, X^* its dual and let $\langle\cdot\,,\cdot\rangle^*$ be the dual paring between X^* and X.

For the convenience of the reader, we recall a theorem of Minty-Browder for operators from a Hilbert space to its dual. An operator $T\colon X \to X^*$ is called *strongly monotone* if there exists a real constant $c > 0$ such that
$$\langle T(u) - T(v)\,,\, u - v\rangle^* \geq c|u - v|_X^2, \text{ for all } u, v \in X.$$

Theorem 5 (Minty-Browder, see, *e.g.*, [10, Theorem 9.14]). *Let $T\colon X \to X^*$ be a continuous and strongly monotone operator. Then T is bijective.*

The following result is a variation of the Ekeland variational principle within a convex conical set (see, *e.g.*, [3]). Let $K \subset X$ be a cone, and let $l\colon K \to \mathbb{R}$ be an upper semicontinuous concave functional. Additionally, suppose we have an operator $N\colon X \to X$ and a functional $E\colon X \to \mathbb{R}$, such that $E'(u) = u - N(u)$ for all $u \in X$. Consider two real numbers $0 < r < R$, and let the convex conical set $K_{r,R}$ be defined by
$$K_{r,R} := \{u \in K \mid r \leq l(u),\ |u|_X \leq R\}.$$

Lemma 6 (see, *e.g.*, [3, Lemma 2.1]). *Assume the following conditions are satisfied:*

(i) *The functional E is bounded from below on $K_{r,R}$, i.e.,*
$$m := \inf_{K_{r,R}} E(\cdot) > -\infty.$$

(ii) *There exists* $\varepsilon > 0$ *such that for all* $u \in K_{r,R}$ *satisfying both* $|u| = R$ *and* $l(u) = r$*, we have* $E(u) \geq m + \varepsilon$*.*

(iii) $l(N(u)) \geq r$*, for all* $u \in K_{r,R}$*.*

Then, there exists a sequence $(u_n) \in K_{r,R}$ *such that*

$$E(u_n) \leq m + \tfrac{1}{n},$$

and

$$|E'(u_n) + \lambda_n u_n|_X \leq \tfrac{1}{n},$$

where

$$\lambda_n = \begin{cases} -\frac{1}{R^2}(E'(u_n), u_n)_X, & \text{when } |u_n|_X = R \text{ and } (E'(u_n), u_n)_X < 0 \\ 0, & \text{otherwise.} \end{cases}$$

3. MAIN RESULT

Let $(X_i, |\cdot|_i)$ be Hilbert spaces identified with their duals $(i \in \{1, \ldots, n\})$. Denote $X := X_1 \times \cdots \times X_n$ and $X_{1,q} := X_1 \times \cdots \times X_q$ $(q \in \{1, \ldots, n\})$ together with the norms $|u|_X = |u_1|_1 + \cdots + |u_n|_n$ and $|u|_{X_{1,q}} := |u_1|_1 + \cdots + |u_q|_q$, respectively. Similarly, $\overline{X}q$ denotes the space obtained from X by excluding X_q, *i.e.*,

$$\overline{X}_q := X_1 \times \cdots \times X_{q-1} \times X_{q+1} \times \cdots \times X_n.$$

For simplicity, for any $q \in \{1, \ldots, n\}$, we refer to

$$(u^1, \ldots, u^q)^T \text{ as } u^{1,q},\ (u^{q+1}, \ldots, u^n)^T \text{ as } u^{q+1,n}$$

and

$$(N_1(u), \ldots, N_q(u))^T \text{ as } N_{1,q}(u),\ (N_{q+1}(u), \ldots, N_n(u))^T \text{ as } N_{q+1,n}(u).$$

With these notations, we have

$$u = \left(u^{1,p}, u^{p+1,n}\right)^T$$

and

$$(N_1(u), \ldots, N_n(u))^T = (N_{1,q}(u), N_{q+1,n}(u))^T.$$

On $X_{1,q}$, we consider the vector-valued inner product

$$\langle u, v \rangle := \left((u^1, v^1)_1, \ldots, (u^q, v^q)_q\right)^T \in \mathbb{R}^q,$$

and vector valued norm

$$\|u\| := \left(|u^1|_1, \ldots, |u^q|_q\right)^T \in \mathbb{R}^q,$$

for any $u = (u^1, \ldots, u^q)$, $v = (v^1, \ldots, v^q) \in X_{1,q}$. Note that these notations are consistent with respect to Hadamard product since $\langle u\,, \,u \rangle = \|u\| \circ \|u\|$.

In the following, we assume that the operators $N_1, \ldots, N_n$ are continuous and that $N_{p+1}, \ldots, N_n$ admit a variational structure, *i.e.*, for each $q \in$

$\{p+1,\dots,n\}$, there exists a functional $E_q\colon X \to \mathbb{R}$ such that for every $(u^1,\cdots,u^{q-1},u^{q+1},\cdots,u^n) \in \overline{X}_q$, the functional

$$E_q(u^1,\cdots,u^{q-1},\cdot,u^{q+1},\cdots,u^n)$$

has a Fréchet derivative E_{qq}. Furthermore, this derivative is given by

$$E_{qq}(u) = u^q - N_q(u). \tag{2}$$

For each $q \in \{p+1,\dots,p\}$, let $K_q \subset X_q$ be a cone and $l_q\colon K_q \to \mathbb{R}_+$ be an upper semicontinous and concave functional such that $l_q(0)=0$. Additionally, consider $r_q, R_q \in \mathbb{R}_+$, and define $(K_q)_{r_q,R_q} \subset K_q$ as

$$(K_q)_{r_q,R_q} := \{u^q \in K_q \,:\, r^q \le l_q(u^q),\, |u^q|_q \le R_q\}.$$

Denote

$$K := (K_{p+1})_{r_{p+1},R_{p+1}} \times \cdots \times (K_n)_{r_n,R_n}$$

and

$$\overline{K}_q := (K_{p+1})_{r_{p+1},R_{p+1}} \times\cdots\times (K_{q-1})_{r_{q-1},R_{q-1}} \times (K_{q+1})_{r_{q+1},R_{q+1}} \times\cdots\times (K_n)_{r_n,R_n}.$$

Theorem 7. *Under the above notations, let us assume the following:*

(h1) *There exists a matrix $A = [a_{ij}]_{1\le i,j\le n}$ convergent to zero such that*

$$\langle N_{1,n}(u) - N_{1,n}(v)\ ,\ u-v\rangle \le A\|u-v\| \circ \|u-v\|, \tag{3}$$

i.e.,

$$\left\langle N_i(u) - N_i(v), u^i - v^i \right\rangle \le \sum_{j=1}^{n} \left|u^i - v^i\right|_i \sum_{j=1}^{n} \left|u^j - v^j\right|_j,\ (i \in \{1,\dots,n\}), \tag{4}$$

for all $u = (u^1,\dots,u^n), v = (v^1,\dots,v^n) \in X$.

For each $q \in \{p+1,\dots,n\}$, one has

(h2)

$l_q\left(N_q(u)\right) \ge r_q$ *for all* $u \in X_{1,p} \times K$,

$N_q(u) - u^q - \lambda\, u^q \ne 0$ *for all* $\lambda > 0$ *and* $u \in X_{1,p} \times K$ *with* $|u^q|_q = R_q$.

(h3) *There exist $m := \inf_{u\in X_{1,p}\times K} E_q(u) > -\infty$ and $\varepsilon > 0$ such that*

$$E_q(u) \ge \inf_{(K_q)_{r_q,R_q}} E_q(u^1,\dots,u^{q-1},\cdot,u^{q+1},\dots,u^n) + \varepsilon,$$

for all $(u^1,\dots,u^{q-1},u^{q+1},u^n) \in X_{1,p} \times \overline{K}_q$ which satisfies $l_q(u^q) = r_q$ and $|u^q|_q = R_q$, simultaneously.

(h4) *The operator $N_q\left(0_{X_1},\dots,0_{X_p},\cdots\right)$ is bounded on K.*

Then there exists two points $u_*^{1,p} \in X_{1,p}$ and $u_*^{p+1,n} \in K$ such that

$$u_* = (u_*^{1,p}, u_*^{p+1,n})$$

is a solution of (1) *and* $u_*^{p+1,n}$ *is a Nash equilibrium in* K *for the functionals* $(E_{p+1}, \ldots, E_n)$*, i.e.,*

$$E_q(u_*) = \inf_{(K_q)_{r_q,R_q}} E_q(u_*^{1,q-1}, \cdot, u_*^{q+1,n}) \quad (q = p+1, \ldots, n).$$

Proof. **Step 1: Construction of the approximation sequence.** Note that the submatrix $A_p = [a_{ij}]_{1\le i,j\le p}$ converges to zero as a consequence of Lemma 3. Thus, Theorem 2 guarantees the existence of a diagonal matrix $D = (d_{ii})_{1\le i\le p}$ with $d_{ii} > 0$ $(i = 1, \ldots, n)$ such that $D(I - A_p)$ is positive definite, *i.e.*,

$$c := \inf_{x\in R^p\setminus\{0\}} \frac{D(I-A_p)x \cdot x}{|x|^2} > 0. \tag{5}$$

Let $u_0^{p+1,n} \in K$ be arbitrarily chosen and let $T\colon X_{1,p} \to X_{1,p}^*$ be defined by

$$T(u^{1,p}) = D\left(u^{1,p} - N_{1,p}\left(u^{1,p}, u_0^{p+1,n}\right)\right), \text{ for all } u^{1,p} = (u^1, \ldots, u^p) \in X_{1,p}.$$

For any $u^{1,p}$, $\overline{u}^{1,p} \in X_{1,p}$, relations (4) yields

$$\begin{aligned} &\left\langle T(u^{1,p}) - T(\overline{u}^{1,p}), u^{1,p} - \overline{u}^{1,p}\right\rangle^* = \\ &= \sum_{i=1}^p d_{ii}|u^i - \overline{u}^i|_i^2 - \left(N_i(u^{1,p}, u_0^{p+1,n}) - N_i(\overline{u}^{1,p}, u_0^{p+1,n}), u^i - \overline{u}^i\right)_i \\ &\ge \sum_{i=1}^p d_{ii}|u^i - \overline{u}^i|_i^2 - d_{ii}\sum_{j=1}^p a_{ij}|u^i - \overline{u}^i|_i|u^j - \overline{u}^j|_j. \end{aligned} \tag{6}$$

Thus, since

$$\begin{aligned} &\sum_{i=1}^p d_{ii}|u^i - \overline{u}^i|_i^2 - d_{ii}\sum_{j=1}^p a_{ij}|u^q - \overline{u}^i|_i|u^j - \overline{u}^j|_j = \\ &= D(I_p - A_p)\|u^{1,p} - \overline{u}^{1,p}\| \cdot \|u^{1,p} - \overline{u}^{1,p}\|, \end{aligned}$$

from relation (5) we deduce

$$\left\langle T(u^{1,p}) - T(\overline{u}^{1,p}), u^{1,p} - \overline{u}^{1,p}\right\rangle \ge c|u^{1,p} - \overline{u}^{1,p}|_{X_{1,p}}^2.$$

Consequently, T is strongly monotone. As T is clearly continuous, Theorem 5 guarantees that it is a bijection from X to X^*. Therefore, there exist a unique $u_1^{1,p} \in X_{1,p}$ such that $T(u_1^{1,p}) = 0$, *i.e.*,

$$u_1^{1,p} = N_{1,p}\left(u_1^{1,p}, u_0^{p+1,n}\right).$$

For each $q \in \{p+1,\dots,n\}$, we fix $u_1^{1,q-1}$ previously determined and $u_0^{q+1,n}$ initially set. Following Lemma 6, we may find $u_1^q \in (K_q)_{r_q,R_q}$ such that

$$E_q\left(u_1^{1,q},\, u_0^{q+1,n}\right) \le \inf_{(K_q)_{r_q,R_q}} E_q\left(u_1^{1,q-1},\,\cdot,\, u_0^{q+1,n}\right)+1\,,$$

$$\left|E_{qq}\left(u_1^{1,q},\, u_0^{q+1,n}\right)+\lambda_1^q u_1^q\right|_q \le 1\,,$$

where

$$\lambda_1^q := \begin{cases} -\frac{1}{R_q^2}\left(E_{qq}\left(u_1^{1,q}, u_0^{q+1,n}\right), u_1^q\right)_q, & \text{if } |u_1^q|_q = R_q \text{ and } \left(E_{qq}\left(u_1^{1,q}, u_0^{q+1,n}\right), u_1^q\right)_q < 0\\ 0, & \text{otherwise}\end{cases}$$

Repeating the process for each step k, we construct recursively a sequence

$$u_k = (u_k^1,\dots,u_k^p,u_k^{p+1},\dots,u_k^n)^T \in X_{1,p}\times K$$

such that

$$u_k^{1,p} = N_{1,p}\left(u_k^{1,p}, u_{k-1}^{p+1,n}\right). \tag{7}$$

and

$$\begin{aligned} &E_q\left(u_k^{1,q},\, u_{k-1}^{q+1,n}\right) \le \inf_{(K_q)_{r_q,R_q}} E_q\left(u_k^{1,q-1},\,\cdot,\, u_{k-1}^{q+1,n}\right)+\tfrac{1}{k}\,,\\ &\left|E_{qq}\left(u_k^{1,q},\, u_{k-1}^{q+1,n}\right)+\lambda_k^q u_k^q\right|_q \le \tfrac{1}{k}\,,\end{aligned} \tag{8}$$

where

$$\lambda_k^q := \begin{cases} -\frac{1}{R_q^2}\left(E_{qq}\left(u_k^{1,q}, u_{k-1}^{q+1,n}\right), u_k^q\right)_q, & \text{if } |u_k^q|_q = R_q \text{ and } \left(E_{qq}\left(u_k^{1,q}, u_{k-1}^{q+1,n}\right), u_k^q\right)_q < 0\\ 0, & \text{otherwise.}\end{cases}$$

for each $q \in \{p+1,\dots,n\}$.

Step 2: Boundedness of the sequence $(u_k^1,\dots,u_k^p)$ **.**

Condition $(h4)$ guarantees that the sequence $N_{1,p}(0,u_{k-1}^{p+1,n})$ is uniformly bounded, *i.e.*, there exists $M>0$ such that

$$M_0 = \sup_{k\in\mathbb{N}}\left|N_{1,p}(0,u_{k-1}^{p+1,n})\right|_X.$$

Since $u_k^{1,p} = N_{1,p}\left(u_k^{1,p}, u_{k-1}^{p+1,n}\right)$, one has

$$\begin{aligned}\|u_k^{1,p}\|\circ\|u_k^{1,p}\| &= \left\langle u_k^{1,p}, N_{1,p}\left(u_k^{1,p}, u_{k-1}^{p+1,n}\right)\right\rangle\\ &\le \left\langle u_k^{1,p}, N_{1,p}\left(u_k^{1,p}, u_{k-1}^{p+1,n}\right) - N_{1,p}\left(0, u_{k-1}^{p+1,n}\right)\right\rangle + M_0\|u_k^{1,p}\|\\ &\le A_p\|u_k^{1,p}\|\circ\|u_k^{1,p}\| + M_0\|u_k^{1,p}\|.\end{aligned} \tag{9}$$

Multiplying both sides of (9) with the diagonal matrix D, we obtain

$$D(I_p - A_p)\|u_k^{1,p}\|\circ\|u_k^{1,p}\| \le M_0 D\|u_k^{1,p}\|.$$

Therefore, relation (5) together with Proposition 1 leads to

$$\sqrt{n}M_0|u^{1,p}|_{X_{1,p}} \geq D(I_p - A_p)\|u_k^{1,p}\| \cdot \|u_k^{1,p}\| \geq c|u^{1,p}|^2_{X_{1,p}},$$

which guarantees the boundedness of the sequence $u_k^{1,p}$.

Step 3: Convergence to zero of the real sequence (λ_k^q).

Let $q \in \{p+1, \ldots, n\}$ and assume

$$\left(E_{qq}\left(u_k^{1,p}, u_{k-1}^{p+1,n}\right) \,,\, u_k^q\right)_q < 0.$$

Observe that

$$\begin{aligned}\lambda_k^q &\leq -\tfrac{1}{R_q^2}\left(E_{qq}\left(u_k^{1,p}, u_{k-1}^{p+1,n}\right) \,,\, u_k^q\right)_q \\ &= -1 + \tfrac{1}{R_q^2}\left(u_k^q, N_q\left(u_k^{1,p}, u_{k-1}^{p+1,n}\right)\right).\end{aligned}$$

Thus, the integration of elementary computations with the monotonicity of N_q ensures that

$$\lambda_k^q \leq \tfrac{1}{R_q}M_0 + \tfrac{1}{R_q^2}\left|u_k^q\right|_q \sum_{i=1}^{p} a_{qi}\left|u_k^p\right|_q$$

Now, the sequence (λ_k^q) is bounded as a consequence of *Step 2*. Hence, eventually passing to a subsequence, (λ_k^q) converges to a non-negative real number λ^q. Further, since $\left(u_k^{1,q}\right)$ and $\left(u_{k-1}^{q+1,n}\right)$ are bounded, they have convergent subsequences whose limits are denoted by $u^{1,p}$ and $\underline{u}^{q+1,n}$, respectively. Now, if we take the limit in (8), we obtain

$$E_{11}(u^{1,q}, \underline{u}^{q+1,p}) + \lambda^q\left(u^{1,q}, \underline{u}^{q+1,p}\right) = 0,$$

where $|u^q|_q = R_q$ if $\lambda^q \geq 0$. In this case, the Leray-Schauder boundary condition in the second relation of (h2) is contradicted, leading to the conclusion that $\lambda_k^q \to 0$ as $k \to \infty$.

Step 4: Convergence of the sequence $(u_k^1, \ldots, u_k^n)$.

For any $q \in \{1, \ldots, n\}$, let us denote

$$x^q(k,m) := \left|u_{k+m}^q - u_k^q\right|_q,$$

$$c^q(k,m) := \begin{cases} 0, & \text{if } q \leq p \\ \left|E_{qq}\left(u_{k+m}^{1,q}, u_{k+m-1}^{q+1,n}\right) - E_{qq}\left(u_k^{1,q}, u_{k-1}^{q+1,n}\right)\right|_q, & \text{if } p+1 \leq q, \end{cases}$$

and

$$x(k,m) := \left(x^1(k,m), \ldots, x^n(k,m)\right)^T,$$

$$c(k,m) := \left(c^1(k,m), \ldots, c^n(k,m)\right)^T.$$

Hence,

$$x^q(k,m)^2 = \left(u^q_{k+m} - u^q_k\,,\, N_q(u^{1,q}_{k+m}, u^{q+1,n}_{k+m-1}) - N_q(u^{1,q}_k, u^{q+1,n}_{k-1})\right) - \left(u^q_{k+m} - u^q_k\,,\, E_{qq}(u^{1,q}_{k+m}, u^{q+1,n}_{k+m-1}) - E_{qq}(u^{1,q}_k, u^{q+1,n}_{k-1})\right)_q.$$

Using the monotonicity assumption of N_q, we derive the subsequent inequality

$$x^q(k,m)^2 \leq x^q(k,m)\left(\sum_{j=1}^{q} a_{q,j}\, x^j(k,m) + \sum_{j=q+1}^{n} a_{q,j}\, x^j(k-1,m)\right) + x^q(k,m)c(k,m). \tag{10}$$

We can put relations (10) in vector form as

$$x(k,m)\circ x(k,m) \leq A'\, x(k,m)\circ x(k,m) + A''\, x(k-1,m)\circ x(k,m) + c(k,m)\circ x(k,m), \tag{11}$$

where

$$A' = \left[a'_{ij}\right]_{1\leq i,j\leq n} \quad \text{with} \quad a'_{ij} = \begin{cases} 0, & \text{if } 0 \leq i \leq j \\ a_{ij}, & \text{otherwise} \end{cases},$$
$$A'' = A - A'.$$

Observe that (11) is equivalent with

$$x(k,m) \leq A'x(k,m) + A''x(k-1,m) + c(k,m),$$

and consequently,

$$x(k,m) \leq \left(I - A'\right)^{-1} A''x(k-1,m) + \left(I - A'\right)^{-1} c(k,m). \tag{12}$$

In order to continue our proof, we need to show that $c^q(k,m) \to 0$ uniformly with respect to m. If $q \leq p$ then there is nothing to prove. Let $p+1 \leq q$. Then

$$\begin{aligned} c(k,m) &= \left|E_{qq}\left(u^{1,q}_{k+m}, u^{q+1,n}_{k+m-1}\right) - E_{qq}\left(u^{1,q}_k, u^{q+1,n}_{k-1}\right)\right|_q \\ &\leq \left|E_{qq}\left(u^{1,q}_{k+m}, u^{q+1,n}_{k+m-1}\right) + \lambda^q_{k+m}\, u^q_{k+m}\right|_q \\ &\quad + \left|E_{qq}\left(u^{1,q}_k, u^{q+1,n}_{k-1}\right) + \lambda^q_k\, u^q_k\right|_q \\ &\quad + \left|\lambda^q_{k+m}\, u^q_{k+m} - \lambda^q_k\, u^q_k\right|_q. \end{aligned} \tag{13}$$

Now, we conclude that $c^q(k,m) \to 0$ uniformly with respect to m since each term of the right-hand side of (13) converges to zero as a consequence of (8) and *Step 3*.

Following [1], the matrix $\left(I - A'\right)^{-1} A''$ is convergent to zero. Consequently, from Lemma 4 we deduce that $x(k,m) \to 0$ uniformly with respect to m, *i.e.*, the sequences $\left(u^1_k\right), \ldots, \left(u^n_k\right)$ are Cauchy. Denote with $u^1_*, \ldots, u^n_*$ their limits.

Step 5: Passing to the limit.

Passing to limit in (7) and (8) we deduce that $u_* = (u_*^1, \ldots, u_*^n)$ solves the system (1), *i.e.*,

$$u_* = N_{1,n}(u_*),$$

and

$$E_q(u_*) = \inf_{(K_q)_{r_q,R_q}} E_q(u_*^1, \ldots, u_*^{q-1}, \cdot, u_*^{q+1}, \ldots, u_*^n),$$

for each $q \in \{p+1, \ldots, n\}$. □

Remark 8 (Perov contraction condition). *If instead of condition (h1), we assume that $N = (N_1, \ldots, N_n)$ is a Perov contraction, then we can relax the spaces $X_1, \ldots, X_p$ to complete metric spaces. In this particular case, the fixed point $u_k^{1,p}$ from (7) can be derived directly since the operator*

$$N_q(u^1, \ldots, u^{q-1}, \cdot, u^{q+1}, \ldots, u^n)$$

is a Lipschitz contraction, for each $(u^1, \ldots, u^{q-1}, u^{q+1}, \ldots, u^n) \in \overline{X}_q$. The above result generalizes the one provided in [1], since the monotony condition (3) is weaker than a Perov contraction condition. For example, if $n = 1$, any decreasing function satisfies (3), but obviously not all of them are Lipschitz.

Remark 9 (Multiplicity solutions). *Note that if we choose*

$$r_1^q, r_2^q, \ldots, r_1^n, r_2^n > 0 \text{ and } R_1^q, R_2^q, \ldots, R_1^n, R_2^n > 0$$

such that

$$R_1^q < l_q(r_2^q), \quad (q = p+1, \ldots, n) \tag{14}$$

then $(K_q)_{r_1^q, R_1^q} \cap (K_q)_{r_2^q, R_2^q} = \emptyset$. Thus, multiple solutions localized in different conical sets can be found, if different pairs (r_i^q, R_i^q) with property (14) are chosen. This localized solution may have the same components for the equations without a variational structure, but different for the others equations.

Remark 10 (Limit cases). *In our theory, the constants r_q and R_q can approach their limit values, meaning $r_q = 0$ when we seek solutions within a ball, and $R_q = \infty$ when we aim to find upper unbounded solutions.*

4. APPLICATION

We apply the results from Section 3 to the Dirichlet problem

$$\begin{cases} -u''(t) = f_1(t, u(t), v(t), w(t), u'(t)) \\ -v''(t) = f_2(t, u(t), v(t), w(t)) \\ -w''(t) = f_3(t, u(t), v(t), w(t)) \quad \text{on } (0,T) \\ u(0) = u(T) = 0 \\ v(0) = v(T) = 0 \\ w(0) = w(T) = 0 \end{cases}, \tag{15}$$

where the functions $f_1\colon (0,T)\times\mathbb{R}^4\to\mathbb{R}_+$, $f_2, f_3\colon (0,T)\times\mathbb{R}^3\to\mathbb{R}_+$ are of Carathéodory type. We emphasize that the presence of u' on the right side of the first equation, as opposed to equations 2 and 3, makes it to lose the variational structure. Moreover, the Hilbert spaces X_1, X_2, X_3 from Theorem 7, are the Sobolev space $H_0^1(0,T)$ endowed with the inner product $(u,v)_{H_0^1}=\displaystyle\int_0^T u'v'$ and the norm $|u|_{H_0^1}=\Big(\displaystyle\int_0^T (u')^2\Big)^{\frac{1}{2}}$.

Due to the continuous embeddings

$$H_0^1\left(0,T\right)\subset C[0,T]\subset L^2\left(0,T\right)\subset \Big(H_0^1(0,T)\Big)',$$

the Poincaré inequality (see, *e.g.*, [11], [12, Remark 3.3]) holds with $\lambda_1=\frac{\pi^2}{T^2}$ (see, [12, p. 72]), *i.e.*,

$$|u|_{L^2}\le \tfrac{1}{\sqrt{\lambda_1}}|u|_{H_0^1}=\tfrac{T}{\pi}|u|_{H_0^1}, \Big(u\in H_0^1(0,T)\Big),$$

where λ_1 is the first eigenvalue of the Dirichlet problem $-u''=\lambda u$, $u(0)=u(T)=0$. Additionally, a positive constant $c>0$ exists such that for all $t\in(0,T)$ and $u\in H_0^1(0,T)$, the following inequality holds true

$$|u(t)|\le c|u|_{H_0^1}.$$

Notably, c can be chosen as $\sqrt{T}$. Indeed, from Cauchy-Schwartz inequality we immediately have

$$\begin{aligned}|u(t)|=|u(t)-u(0)|&\le \int_0^t |u'(x)|\,dx\\ &\le \int_0^T |u'(x)|dx\le |u'|_{L^2}|1|_{L^2}=\sqrt{T}|u|_{H_0^1},\text{ for all } t\in(0,T).\end{aligned}$$

Let $\big(H_0^1(0,T)\big)'$ be the dual space of $H_0^1(0,T)$ and let $(\cdot\,,\cdot)'$ be the dual pairing between $\big(H_0^1(0,T)\big)'$ and $H_0^1(0,T)$, *i.e.*, for any $h\in\big(H_0^1(0,T)\big)'$ and $\phi\in H_0^1(0,T)$, $(h,\phi)'$ stands for the value of the functional h evaluated at the element ϕ. From Riesz's representation theorem, for each $h\in\big(H_0^1(0,T)\big)'$, there exists a unique $u_h\in H_0^1(0,T)$ such that

$$(h,\phi)'=(u_h,\phi)_{H_0^1},\text{ for every } \phi\in H_0^1(0,T).$$

Therefore, we can define the solution operator $S\colon \big(H_0^1(0,T)\big)'\to H_0^1(0,T)$, where $S(h)=u_h$. If $h\in L^2(0,T)$, then $S(h)$ admits a representation via Green function $G(t,s)\colon (0,T)^2\to\mathbb{R}_+$ (see, *e.g.*, [13, Example 1.8.18]),

$$G(t,s)=\begin{cases} s\left(1-\frac{t}{T}\right), & s\le t\\ t\left(1-\frac{s}{T}\right), & s\ge t,\end{cases}$$

specifically $S(h)(t)=\int_0^T G(t,s)h(s)ds$.

Let $K := K_2 = K_3$ be the cone of nonegative functions from $H_0^1(0,T)$ and $[a,b]$ be a fixed compact subinterval of $(0,T)$. Additionally, we consider the concave upper semicontinous functionals $l_2, l_3 \colon K \to \mathbb{R}_+$,

$$l_1(u) = l_2(u) = \min_{t\in[a,b]} u(t) \quad (u \in K),$$

and the conical sets

$$(K)_{r_j,R_j} = \{u \in K_j \,|\, r_j \le l_j(u),\, |u|_{H_0^1} \le R_j\} \quad (j \in \{2,3\}),$$

where $0 < r_j < R_j$ are real numbers.

Note that the second and third equations from (15) admit a variational form given by the energy functionals $E_2, E_3 \colon H_0^1(0,T) \times K \times K \to \mathbb{R}$,

$$E_2(u,v,w) := \tfrac{1}{2}|v|^2_{H_0^1} - \int_0^T F_2(\cdot,u,v,w)$$
$$E_3(u,v,w) := \tfrac{1}{2}|w|^2_{H_0^1} - \int_0^T F_3(\cdot,u,v,w)$$

where

$$F_2(x,u(x),v(x),w(x)) := \int_0^{v(x)} f_2(x,u(x),s,w(x))ds$$
$$F_3(x,u(x),v(x),w(x)) := \int_0^{w(x)} f_2(x,u(x),v(x),s)ds.$$

If we identify $H_0^1(0,T)$ with its dual, we deduce

$$E_{22}(u,v,w) = v - Sf_2(u,v,w),$$
$$E_{33}(u,v,w) = w - Sf_3(u,v,w).$$

Therefore, the system (15) is equivalent with the following fixed point equation

$$\begin{cases} N_1(u,v,w) = u, \\ N_2(u,v,w) = v, \\ N_3(u,v,w) = w, \end{cases}$$

where

$$\begin{cases} N_1(u,v,w) = Sf_1(\cdot,u,v,w,u') \\ N_2(u,v,w) = Sf_2(\cdot,u,v,w) \\ N_3(u,v,w) = Sf_3(\cdot,u,v,w). \end{cases}$$

Let us denote

$$m := \min_{t\in[a,b]} \int_0^T G(t,s)ds = \min_{t\in[a,b]} \tfrac{t(T-t)}{2} = \min\left\{\tfrac{a(T-a)}{2}, \tfrac{b(T-b)}{2}\right\}.$$

THEOREM 11. *Let the above assumptions be satisfied. Furthermore, assume the following*

(H1) *There exist* $a_{ij}, a_{14} > 0$ $(i, j \in \{1,3\})$ *such that for all real numbers* $x_1, \dots, x_4$ *and* $\overline{x}_1, \dots, \overline{x}_4$ *we have*

$$(16)\quad \begin{aligned} &(x_1 - \overline{x}_1)\,(f_1(t, x_1, ..., x_4) - f_1(t, \overline{x}_1, ..., \overline{x}_4)) \le |x_1 - \overline{x}_1| \sum_{j=1}^{4} a_{1j}|x_j - \overline{x}_j| \\ &(x_i - \overline{x}_i)\,(f_i(t, x_1, x_2, x_3) - f_i(t, \overline{x}_1, \overline{x}_2, \overline{x}_3)) \le |x_i - \overline{x}_i| \sum_{j=1}^{3} a_{ij}|x_j - \overline{x}_j|, \end{aligned}$$

where $i \in \{2, 3\}$.

(H2) *The functions* $f_i(t, x, y, z)$, *for* $i = 2, 3$, *satisfy:*

(i) *they are monotonically increasing with respect to the variables* y *and* z.

(ii)

$$(17)\quad f_i(t, \cdot, r_2, r_3) \ge \tfrac{r_i}{m(b-a)}$$

and

$$(18)\quad |f_i(t, \cdot, 0, 0)|_{L^2} \le \tfrac{\pi}{T} R_2 - \tfrac{T}{\pi}\,(a_{i2}R_2 + a_{i3}R_3)$$

for all $t \in (0, T)$.

(iii) *there are real numbers* $M_1, M_2, M_3, M_4 > 0$ *such that*

$$\begin{aligned} &f_2(t, \cdot, cR_2, cR_3) \le M_1 \ ;\ f_2(t, \cdot, 0, r_3) \ge M_2 \\ &f_3(t, \cdot, cR_2, cR_3) \le M_3 \ ;\ f_3(t, \cdot, r_2, 0) \ge M_4 \end{aligned}$$

for every $t \in (0, T)$ *and*

$$\begin{aligned} TcR_2M_1 - \tfrac{R_2^2}{2} &< r_2(b-a)M_2, \\ TcR_3M_3 - \tfrac{R_3^2}{2} &< r_3(b-a)M_4. \end{aligned}$$

Then there exists $(u^*, v^*, w^*) \in H_0^1(0, T) \times (K_2)_{r_2,R_2} \times (K_3)_{r_3,R_3}$ *a solution of the system* (15) *such that* (v^*, w^*) *is a Nash equilibrium for the energy functionals of the second and third equations.*

Proof. The proof entails checking that all conditionTheorem 7 are fulfilled.

Verification of the condition (h1). Let $u, \overline{u}, v, \overline{v}, w, \overline{w} \in H_0^1(0, T)$. Then, from (16) we have

$$\begin{aligned} &(N_1(u, v, w) - N_1(\overline{u}, \overline{v}, \overline{w})\,,\, u - \overline{u})_{H_0^1} = \\ &= (S\,f_1(\cdot, u, v, w, u') - S\,f_1(\cdot, \overline{u}, \overline{v}, \overline{w}, \overline{u}'))\,,\, u - \overline{u})_{H_0^1} \\ &= (f_1(\cdot, u, v, w, u') - f_1(\cdot, \overline{u}, \overline{v}, \overline{w}, \overline{u}')\,,\, u - \overline{u})_{L^2} \\ &\le a_{11}|u - \overline{u}|_{L^2}^2 + a_{12}|v - \overline{v}|_{L^2}|u - \overline{u}|_{L^2} + a_{13}|w - \overline{w}|_{L^2}|u - \overline{u}|_{L^2} \\ &\quad + a_{14}|u' - \overline{u}'|_{L^2}|u - \overline{u}|_{L^2} \\ &\le \left(a_{11}\tfrac{T^2}{\pi^2} + a_{14}\tfrac{T}{\pi}\right)|u - \overline{u}|_{H_0^1}^2 + a_{12}\tfrac{T^2}{\pi^2}|u - \overline{u}|_{H_0^1}|v - \overline{v}|_{H_0^1} \end{aligned}$$

$$+ a_{13}\tfrac{T^2}{\pi^2}|u-\overline{u}|_{H_0^1}|w-\overline{w}|_{H_0^1}.$$

Similarly

$$\begin{aligned}&(N_2(u,v,w)-N_2(\overline{u},\overline{v},\overline{w})\,,\,v-\overline{v})_{H_0^1}\le\\&\le \tfrac{T^2}{\pi^2}|v-\overline{v}|_{H_0^1}\left[a_{21}|u-\overline{u}|_{H_0^1}+a_{22}|v-\overline{v}|_{H_0^1}+a_{23}|w-\overline{w}|_{H_0^1}\right],\end{aligned}$$

and

$$\begin{aligned}&(N_3(u,v,w)-N_3(\overline{u},\overline{v},\overline{w})\,,\,w-\overline{w})_{H_0^1}\le\\&\le \tfrac{T^2}{\pi^2}|w-\overline{w}|_{H_0^1}\left[a_{31}|u-\overline{u}|_{H_0^1}+a_{23}|v-\overline{v}|_{H_0^1}+a_{33}|w-\overline{w}|_{H_0^1}\right].\end{aligned}$$

If the matrix

$$A=\tfrac{T^2}{\pi^2}\begin{bmatrix}(a_{11}+\frac{\pi}{T}a_{41}) & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix} \tag{19}$$

is convergent to zero, then (h1) is fulfilled.

Let $u\in H_0^1(0,T)$, $v\in(K_2)_{r_2,R_2}$, $w\in(K_3)_{r_3,R_3}$. Since f_2 has positive values, using assumption (H2) we deduce

$$\begin{aligned}N_2(u(t),v(t),w(t))&=\int_0^T G(t,s)f_2(s,u(s),v(s),w(s))ds\\&\ge\int_a^b G(t,s)f_2(s,u(s),v(s),w(s))ds\\&\ge\int_a^b G(t,s)f_2(s,u(s),r_2,r_3)ds\\&\ge m(b-a)f_2(\xi,u(\xi),r_2,r_3)\ge r_2,\end{aligned}$$

for all $t\in(0,T)$, where some $\xi\in(a,b)$. This guarantees that $l_2(N_2(u,v,w))\ge r_2$. Moreover, if $|v|_{H_0^1}=R_2$, it follows that $N_2(u,v,w)\ne(1+\lambda)v$ for every $\lambda>0$. Indeed, if we assume the contrary, then

$$\begin{aligned}(1+\lambda)|v|^2_{H_0^1}&=(1+\lambda)R_2^2=(N_2(u,v,w),v)_{H_0^1}\\&=(f_2(u,v,w),v)_{L^2}=\int_0^T v(s)f_2(s,u(s),v(s),w(s))ds\\&=\int_0^T v(s)\left[f_2(s,u(s),v(s),w(s))-f_2(s,u(s),0,0)\right]ds\\&\quad+\int_0^T v(s)f_2(s,u(s),0,0)ds.\end{aligned}$$

Using the monotonicity conditions of f_2, we easily deduce

$$\begin{aligned}(1+\lambda)|v|^2_{H_0^1}&\le\int_0^T v(s)\left[a_{22}v(s)+a_{23}w(s)\right]ds+|v|_{L^2}|f_2(\cdot,u(\cdot),0,0)|_{L^2}\\&\le\tfrac{T^2}{\pi^2}\left[a_{22}R_2^2+a_{23}R_2R_3\right]+\tfrac{T}{\pi}R_2|f_2(\cdot,u(\cdot),0,0)|_{L^2}.\end{aligned}$$

Hence, relation (18) yields

$$(1+\lambda)|v|^2_{H^1_0} \le R_2^2,$$

which is a contradiction. Applying the same reasoning, a similar relation also holds for f_3, and therefore condition (h2) is verified.

Verification of conditions (h3), (h4). One easily sees that for any $v \in (K_2)_{r_2,R_2}$ and $w \in (K_3)_{r_3,R_3}$, one has

$$0 \le v(t) \le cR_2 \ ; \ 0 \le w(t) \le cR_3. \tag{20}$$

Consequently,

$$\begin{aligned} |N_1(0,v,w)| &= \int_0^T G(t,s) f_1(s,0,v(s),w(s),0)ds \\ &\le \int_0^T G(t,s) f_1(s,0,cR_2,cR_3,0)ds \\ &\le T|G|_\infty |f_1(\theta,0,cR_2,cR_3,0)|_\infty, \end{aligned}$$

for some $\theta \in (0,T)$. Hence, condition (h4) holds true.

Since

$$F_2(t,u(t),v(t),w(t)) \le cR_2 f_2\left(t,u(t),cR_2,cR_3\right), \text{ for all } t \in (0,T),$$

we can deduce

$$\begin{aligned} E_2(u,v,w) &\ge -\int_0^T cR_2 f_2(x,u(x),cR_2,cR_3)dx \\ &\ge -cR_2 \int_0^T \left[f_2(x,u(x),cR_2,cR_3) - f_2(x,u(x),0,0)\right]dx \\ &\quad - cR_2|f(\cdot,u(\cdot),0,0)|_{L^1} \\ &\ge -cR_2^2 T\left(a_{22}cR_2 + a_{23}cR_3\right) - cR_2|f(\cdot,u(\cdot),0,0)|_{L^1} > -\infty, \end{aligned}$$

which guarantees that E_2 is lower bounded on $H^1_0(0,T)\times(K_2)_{r_2,R_2}\times(K_3)_{r_3,R_3}$. Furthermore, when $l_2(v) = r_2$ and $|v|_{H^1_0} = R_2$ simultaneously, it follows that

$$\begin{aligned} E_2(u,v,w) &= \frac{|v|^2_{H^1_0}}{2} - \int_0^T \int_0^{v(x)} f_2(x,u(x),v(x),w(x))dsdx \\ &\ge \frac{R_2^2}{2} - \int_0^T \int_0^{cR_2} f_2(x,u(x),s,cR_3)dsdx \\ &\ge \frac{R_2^2}{2} - \int_0^T cR_2 f_2(x,u(x),cR_2,cR_3)dx \\ &\ge \frac{R_2^2}{2} - cR_2TM_1 \ge -r_2(b-a)M_2 + \varepsilon. \end{aligned} \tag{21}$$

On the other hand,

$$E_2(u,r_2,w) = -\int_0^T \int_0^r f_2(x,u(x),s,w(x))dsdx$$

$$\le -\int_a^b \int_0^{r_2} f_2(x,u(x),0,r_3)dsdx$$

$$= -\int_a^b r_2 f_2(x,u(x),0,r_3)dx \le -r_2(b-a)M_2. \tag{22}$$

From (21) and (22), we easily deduce

$$E_2(u,v,w) \ge -r_2(b-a)M_2 + \varepsilon \ge E_2(u,r_2,w) + \varepsilon.$$

Following the same reasoning for E_3 and f_3 we obtain

$$\inf_{H_0^1(0,T)\times(K)_{r_2,R_2}\times(K)_{r_3,R_3}} E_3(\cdot) > -\infty$$

and $E_3(u,v,w) \ge E_3(u,v,w)+\varepsilon$ whenever $l_3(w)=r_3$ and $|w|_{H_0^1}=R_3$ simultaneously. Thus conditions $(h3)$ is satisfied.

As all the assumptions of Theorem 7 are satisfied, there exists

$$(u^*,v^*,w^*) \in H_0^1(0,T)\times(K_2)_{r_2,R_2}\times(K_3)_{r_3,R_3}$$

a solution to system (15), such that (v^*,w^*) is a Nash equilibrium for the energy functionals (E_2,E_3). □

Example 12. *Let the system*

$$\begin{cases} -u''(t) = \bar{a}_1\left(e^{-u^2(t)} + e^{-(u'(t))^2} + e^{-v^2(t)} + e^{-w^2(t)}\right) \\ -v''(t) = \bar{a}_2\left(e^{-u^2(t)} + \arctan(v(t)+2w(t)) + \frac{\pi}{2}\right) \\ -w''(t) = \bar{a}_2\left(e^{-u^2(t)} + \arctan(2v(t)+w(t)) + \frac{\pi}{2}\right)\end{cases} \quad \text{on } (0,3), \tag{23}$$

with Dirichlet boundary conditions

$$\begin{cases} u(0)=u(3)=0 \\ v(0)=v(3)=0 \\ w(0)=w(3)=0,\end{cases}$$

where $\bar{a}_i\ (i=1,3)$ are positive real numbers.

We apply the results from Theorem 11 with,

$$f_1(x_1,x_2,x_3,x_4) = \bar{a}_1\left(e^{-x_1^2}+e^{-x_2^2}+e^{-x_3^2}+e^{-x_4^2}\right)$$

$$f_2(x_1,x_2,x_3) = \bar{a}_2\left(e^{-x_1^2}+\arctan(x_2+x_3)+\frac{\pi}{2}\right)$$

$$f_3(x_1,x_2,x_3) = \bar{a}_3\left(e^{-x_2^2}+\arctan(x_2+x_3)+\frac{\pi}{2}\right)$$

Here, we choose $c=\sqrt{3}$ and set $R_1=R_2=\infty$. The value of r is determined in such a way that, for each $i=2,3$, there exist suitable constants $\bar{a}_2$ and $\bar{a}_3$ such that the following inequality has solutions:

$$\bar{a}_i\left(\arctan 2r + \frac{\pi}{2}\right) \ge r. \tag{24}$$

The closed interval $[a,b]$ is selected to be $[1,2]$. As a result

$$m = \min\left\{\frac{1(3-1)}{2}, \frac{2(3-2)}{2}\right\} = 1.$$

If the matrix

$$A = \frac{9}{\pi^2}\begin{bmatrix} \bar{a}_1\left(\frac{\pi}{3}+1\right) & \bar{a}_1 & \bar{a}_1 \\ \bar{a}_2 & \bar{a}_2 & \bar{a}_2 \\ \bar{a}_2 & \bar{a}_3 & \bar{a}_3 \end{bmatrix}$$

is convergent to zero, then the system (23) *has a solution* (u^*, v^*, w^*) *such that* (v^*, w^*) *is a Nash equilibrium on* $(K)_{r,R} \times (K)_{r,R}$ *for the energy functionals associated with the second and third equations.*

Proof. We will demonstrate that all the conditions outlined in Theorem 11 are satisfied. It is clear that the functions f_1, f_2, f_3 are all nonegative. Since these functions are constructed from Lipschitz functions, each with a Lipschitz constant not exceeding 1, the coefficients a_{ij} are:

$$a_{ij} = \bar{a}_i \quad (i = 1, 3) \text{ and } a_{14} = \bar{a}_1.$$

Therefore, condition (H1) is satisfied, given that the matrix A converges to zero. On the condition (H_2), the first relation (i) follows immediately, as the function arctan is increasing.

Check of (ii). For each $i \in \{2, 3\}$, one has:

$$f_i(x_1, r, r) \geqslant \bar{a}_i \arctan 2r \geq r$$

As we sought solutions without upper bounds, it is not necessary to verify the second conditions from (ii) and condition (iii). Thus, all the assumptions of Theorem 11 are satisfied, which concludes our proof. □

In order the straighten our theory, below we present a numerical example for a particular choice of constants $\bar{a}_i$.

Example 13. *Let the system*

$$\begin{cases} -u''(t) = 0.1\left(e^{-u^2(t)} + e^{-(u'(t))^2} + e^{-v^2(t)} + e^{-w^2(t)}\right) \\ -v''(t) = 0.40\left(e^{-u^2(t)} + \arctan(v(t) + w(t))\right) \\ -w''(t) = 0.45\left(e^{-u^2(t)} + \arctan(v(t) + w(t))\right) \end{cases} \tag{25}$$

together with the boundary conditions

$$\begin{cases} u(0) = u(3) = 0 \\ v(0) = v(3) = 0 \\ w(0) = w(3) = 0 \end{cases}.$$

Here, we chose $r := \min\{r_2, r_3\}$ *where* r_2 *and* r_3 *are the solutions of the equations*

$$0.4 \arctan\left(2x + \tfrac{\pi}{2}\right) = x \text{ and } 0.45 \arctan\left(2x + \tfrac{\pi}{2}\right) = x.$$

After straightforward calculations, we obtain an approximate solution of $r \approx 0.478$.

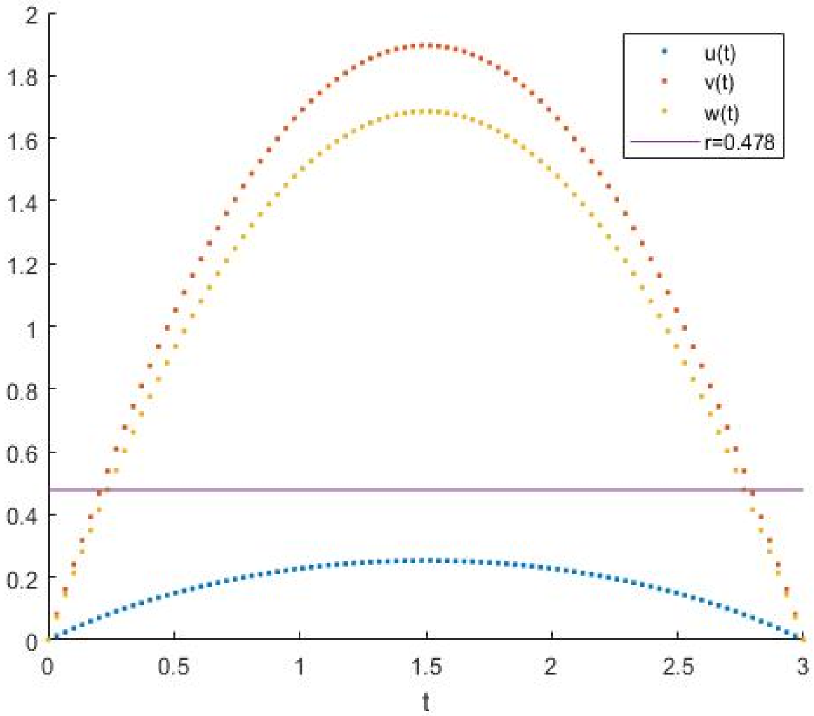

Fig. 1. Plot of the solutions

In Fig. 1, you can observe the approximated solutions of the system (25). It is clear that the minimum values of the second and third solutions over the interval $[1, 2]$ *surpass the threshold of* $r = 0.478$*, that is,*

$$\min_{t\in[1,2]} \{v(t), w(t)\} \geq r.$$

However, the first equation does not satisfy this requirement since we did not request localization.

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