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NONLINEAR RANDOM EXTRAPOLATION ESTIMATES OF π UNDER DIRICHLET DISTRIBUTIONS

SHASHA WANG[†], ZECHENG LI[‡] and WEN-QING XU[*]

Abstract. We construct optimal nonlinear extrapolation estimates of π based on random cyclic polygons generated from symmetric Dirichlet distributions. While the semiperimeter S_n and the area A_n of such random inscribed polygons and the semiperimeter (and area) S'_n of the corresponding random circumscribing polygons are known to converge to π w.p.1 and their distributions are also asymptotically normal as $n \to \infty$, we study in this paper nonlinear extrapolations of the forms $\mathcal{W}_n = S_n^\alpha A_n^\beta {S'_n}^\gamma$ and $\mathcal{W}_n(p) = (\alpha S_n^p + \beta A_n^p + \gamma {S'_n}^p)^{1/p}$ where $\alpha + \beta + \gamma = 1$ and $p \neq 0$. By deriving probabilistic asymptotic expansions with carefully controlled error estimates, we show that $\mathcal{W}_n$ and $\mathcal{W}_n(p)$ also converge to π w.p.1 and are asymptotically normal. Furthermore, to minimize the approximation error associated with $\mathcal{W}_n$ and $\mathcal{W}_n(p)$, the parameters must satisfy the optimality condition $\alpha + 4\beta - 2\gamma = 0$. Our results generalize previous work on nonlinear extrapolations of π which employ inscribed polygons only and the vertices are also assumed to be independently and uniformly distributed on the unit circle.

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1. INTRODUCTION

Given a convex set $K \subseteq \mathbb{R}^d$, the stochastic properties of the convex hull K_n generated by n independent random points on K, such as the area, volume and number of vertices of K_n, their probability distributions and asymptotic behavior have attracted extensive attention (see, *e.g.*, [7, 10, 11, 16, 17, 20]). In the case of n points randomly selected on a unit circle in $\mathbb{R}^2$, the resulting convex hull is a random n-gon inscribed in the circle which is obtained by connecting all adjacent vertices on the circle. Using the same set of random points,

[†]Department of Mathematics and Physics, Shijiazhuang Tiedao University, Shijiazhuang, Hebei 050043, China, e-mail: wshasha@stdu.edu.cn.

[‡]Questrom School of Business, Boston University, Boston, MA 02215, USA, e-mail: lizc0816@bu.edu.

[*]Beijing Institute for Scientific and Engineering Computing, Beijing University of Technology, Beijing 100124, China. Department of Mathematics and Statistics, California State University Long Beach, CA 90840, USA, e-mail: wxu@csulb.edu (corresponding author).

one may also construct (w.p.1) a random circumscribing n-gon which is tangent to the circle at each of the n random points. In the simplest case when the vertices are independent and uniformly distributed on the circle, it is known that the semiperimeter S_n and area A_n of such random inscribed polygons, and the semiperimeter (or area) S_n' of the random circumscribing polygons all converge to π w.p.1 as $n \to \infty$ and their distributions are also asymptotically Gaussian [4, 24]. Furthermore, by using extrapolation techniques [12, 15] originating exactly from the famous Archimedean approximations of π based on regular polygons [3, 13, 19], it has been shown [22, 23, 25] that simple weighted averages such as $\frac{4}{3}S_n - \frac{1}{3}A_n$, $\frac{2}{3}S_n + \frac{1}{3}S_n'$ and $\frac{16}{15}S_n - \frac{1}{5}A_n + \frac{2}{15}S_n'$, etc., provide much more accurate approximations of π, and at the same time also satisfy similar central limit theorems as $n \to \infty$. We note that extrapolation methods are useful in many important applications such as numerical evaluation of integrals, numerical solution of differential equations, and polynomial interpolations, etc. To accelerate the convergence associated with existing low-precision approximations, extrapolation seeks to combine them in a way such that the leading order error terms are cancelled out as much as possible. For example, in the simpler case of the Archimedean approximation of π, while both $S_n = n\sin(\pi/n)$ and $A_n = \frac{1}{2}n\sin(2\pi/n) = S_{n/2}$ converge to π with errors of order $\mathcal{O}(n^{-2})$, a closer look reveals that $S_n - \pi \approx -\frac{\pi^3}{6n^2}$ and $A_n - \pi \approx -\frac{2\pi^3}{3n^2}$. This implies the error of A_n roughly quadruples that of S_n. Consequently, the weighted average $\frac{4}{3}S_n - \frac{1}{3}A_n$, or equivalently, $\frac{4}{3}S_n - \frac{1}{3}S_{n/2}$, exactly cancels out the leading order error terms in S_n and A_n to yield an improved estimate of π with a reduced error now of order $\mathcal{O}(n^{-4})$.

More recently, in [26], the authors have initiated the study of novel nonlinear extrapolation estimates of π in the forms $\mathcal{X}_n = S_n^\alpha A_n^\beta$ and $\mathcal{Y}_n(p) = (\alpha S_n^p + \beta A_n^p)^{1/p}$ where $\alpha + \beta = 1$ and $p \neq 0$. By deriving probabilistic asymptotic expansions with carefully controlled error estimates, it is shown that, for both $\mathcal{X}_n$ and $\mathcal{Y}_n(p)$, the same choice $\alpha = 4/3$, $\beta = -1/3$ minimizes the approximation error with $\mathcal{X}_n = \pi + n^{-3+\delta}o(1)$, $\mathcal{Y}_n(p) = \pi + n^{-3+\delta}o(1)$ where $\delta > 0$ is any positive number and $o(1)$ represents a random variable which converges to 0 w.p.1 as $n \to \infty$. Furthermore, $\mathcal{X}_n$ and $\mathcal{Y}_n(p)$ are also asymptotically normal with $\mathcal{X}_n \sim \mathrm{AN}(\pi - 2\pi^5/n^4, 2496\pi^{10}/n^9)$, $\mathcal{Y}_n(p) \sim \mathrm{AN}(\pi - 2(p+1)\pi^5/n^4, (160p^2 + 960p + 2496)\pi^{10}/n^9)$ where for a sequence of random variables $\{Z_n\}$ and $\mu_n \in \mathbb{R}$, $\sigma_n > 0$, the notation $Z_n \sim \mathrm{AN}(\mu_n, \sigma_n^2)$ means $(Z_n - \mu_n)/\sigma_n \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$. In particular, for $p = 1$, $\mathcal{Y}_n(1)$ reduces to the optimal *linear* extrapolation estimate $\frac{4}{3}S_n - \frac{1}{3}A_n$. Moreover, $\mathcal{X}_n$ may be viewed as the limit of $\mathcal{Y}_n(p)$ when $p \to 0$, a reflection of the relation $\lim_{p\to 0}(\alpha x^p + \beta y^p)^{1/p} = x^\alpha y^\beta$ for any $x, y > 0$ and $\alpha + \beta = 1$.

In this paper, we aim to further develop nonlinear random extrapolation methods for approximating π. On the one hand, it would be natural to include also random circumscribing polygons in the approximation process. Motivated by the work in [26], we study nonlinear functions of S_n, A_n and S'_n in the forms $\mathcal{W}_n = S_n^{\alpha} A_n^{\beta} S_n'^{\gamma}$, $\mathcal{W}_n(p) = (\alpha S_n^p + \beta A_n^p + \gamma S_n'^p)^{1/p}$ where $\alpha + \beta + \gamma = 1$ and $p \neq 0$. On the other hand, we are also interested in extending the theory to more general random cyclic polygons whose vertices are not independently and uniformly distributed on the circle. While this is a very challenging problem in general, as a first step, we focus on the particular case of random cyclic polygons generated from symmetric Dirichlet distributions with an arbitrary concentration parameter $a > 0$. We note that in such cases, it has been proved [21] that the respective semiperimeters and areas, again denoted by S_n, A_n and S'_n, satisfy similar convergence estimates and central limit theorems as in [4, 24] for the uniform case, which in fact corresponds to the special case $a = 1$ of the Dirichlet distribution.

Clearly, as in [26], the case $p = 1$ reduces to linear extrapolations based on S_n, A_n and S'_n, and due to the relation $\lim_{p\to 0} (\alpha x^p + \beta y^p + \gamma z^p)^{1/p} = x^{\alpha} y^{\beta} z^{\gamma}$, we expect to recover $\mathcal{W}_n$ from $\mathcal{W}_n(p)$ in the limit $p \to 0$. More importantly, based on similar asymptotic expansion results established in [21] for S_n, A_n and S'_n in terms of various power sums of the underlying Dirichlet distribution, we derive rigorous probabilistic asymptotic expansions with carefully controlled error estimates for various nonlinear functions of S_n, A_n and S'_n, particularly $\mathcal{W}_n$ and $\mathcal{W}_n(p)$. Such probabilistic asymptotic expansions resemble the well-known Taylor series expansions in many deterministic approximation problems and provide a cornerstone for establishing the corresponding probability convergence estimates and central limit theorems.

It turns out that, for both $\mathcal{W}_n$ and $\mathcal{W}_n(p)$, the optimal approximation occurs when $\alpha + 4\beta - 2\gamma = 0$ with $\mathcal{W}_n = \pi + n^{-3+\delta} o(1)$ and $\mathcal{W}_n(p) = \pi + n^{-3+\delta} o(1)$. Such results are comparable with those obtained in [26] for the case $\gamma = 0$ and are actually weaker than the corresponding optimal linear extrapolation estimate $\mathcal{W}_n(1) = \frac{16}{15} S_n - \frac{1}{5} A_n + \frac{2}{15} S'_n = \pi + n^{-5+\delta} o(1)$, see Theorems 11–13 below for details. Note that together with $\alpha+\beta+\gamma = 1$, the condition $\alpha+4\beta-2\gamma = 0$ implies that $\alpha = 4/3 - 2\gamma$, $\beta = -1/3 + \gamma$ where γ is an arbitrary constant. Due to complicated nonlinear effects, however, the extra "free" parameter γ can no longer be used to further improve the approximation associated with $\mathcal{W}_n$ and $\mathcal{W}_n(p)$. Nevertheless, by further combining such nonlinear extrapolation estimates with *different* values of γ, it is possible achieve additional improvements better than the linear case.

Finally, it is interesting to note that in the case of the classical Archimedean polygons, for both $\mathcal{W}_n$ and $\mathcal{W}_n(p)$, the optimal estimates also occur when $\alpha+4\beta-2\gamma = 0$ with $\mathcal{W}_n = \pi + \frac{1}{45} \frac{\pi^5}{n^4} + \mathcal{O}(n^{-6})$, $\mathcal{W}_n(p) = \pi + \frac{1}{180} \frac{\pi^5}{n^4} [45\gamma p - 10p + 4] + \mathcal{O}(n^{-6})$. In fact, for $\mathcal{W}_n(p)$, by choosing $\gamma = \frac{10p-4}{45p}$ and $p = p_{\pm} = \frac{-21 \pm \sqrt{721}}{70}$, we can further obtain $\mathcal{W}_n(p_{\pm}) = \pi - \frac{119 \pm \sqrt{721}}{661500} \frac{\pi^9}{n^8} + \mathcal{O}(n^{-10})$, which is two orders of magnitude higher than the optimal linear extrapolation estimate $\mathcal{W}_n(1) = \frac{16}{15} S_n - \frac{1}{5} A_n + \frac{2}{15} S'_n = \pi + \frac{\pi^7}{105 n^6} + \mathcal{O}(n^{-8})$. However, for $\mathcal{W}_n$, the result turns out to be completely independent of γ. This is due to the relation $A_n S'_n = S_n^2$, a variant of Archimedes's celebrated geometric mean relation, which implies $\mathcal{W}_n = S_n^\alpha A_n^\beta S_n'^\gamma = S_n^{\alpha+2\gamma} A_n^{\beta-\gamma}$.

The remainder of the paper is organized as follows. In Section 2, we present some useful preliminary results related to the Dirichlet distribution and its various power sums. Section 3 is devoted to the study of nonlinear extrapolation estimates of π based on random inscribed and circumscribing polygons generated from symmetric Dirichlet distributions. Finally, we offer several additional remarks in Section 4 and some concrete numerical simulation results in Section 5 to conclude our study on nonlinear random extrapolation approximations.

2. PRELIMINARIES

2.1. Basic properties of Dirichlet distributions. Recall that a random vector $\mathbf{Y}' = (Y_1, \cdots, Y_{n-1}) \in \mathbb{R}^{n-1}$, $n \geq 2$, is said to have Dirichlet distribution [2] with parameters $\mathbf{a} = (a_1, \cdots, a_{n-1}; a_n) \in \mathbb{R}^{n+}$ if it has joint probability density function

$$f_{Y_1,\cdots,Y_{n-1}}(y_1, \cdots, y_{n-1}) = \tfrac{\Gamma(a_1+\cdots+a_n)}{\Gamma(a_1)\cdots\Gamma(a_n)} y_1^{a_1-1} \cdots y_{n-1}^{a_{n-1}-1} y_n^{a_n-1}$$

where $y_i > 0$, $\sum_{i=1}^{n-1} y_i < 1$, $y_n = 1 - \sum_{i=1}^{n-1} y_i$, and $\Gamma(a) = \int_0^\infty v^{a-1} e^{-v}\, dv$ is the gamma function defined for all $a > 0$. Let $Y_n = 1 - \sum_{i=1}^{n-1} Y_i$. With slight abuse of notation, we also refer to $\mathbf{Y} = (\mathbf{Y}', Y_n) \in \mathbb{R}^n$ as Dirichlet distribution and write $\mathbf{Y} \sim \mathbb{D}\mathrm{ir}(\mathbf{a})$, $\mathbf{Y}' \sim \mathbb{D}\mathrm{ir}'(\mathbf{a})$.

In this paper, we focus on symmetric Dirichlet distributions, that is, $a_i = a > 0$ for all $1 \leq i \leq n$. In such cases, all $Y_i \sim \mathbb{B}\mathrm{eta}(a, (n-1)a)$ have identical Beta distribution.

Lemma 1 (Tail probability, [21]). *Let* $\Delta_n = \max_{1\leq i\leq n} Y_i$ *and* Z_n *any measurable function of* $Y_1, Y_2, \cdots, Y_n$. *Then for any* $t \in (0, 1)$,

(1) $\Pr(\Delta_n \geq t)$ *decays exponentially as* $n \to \infty$.
(2) $Z_n \cdot 1_{\{\Delta_n \geq t\}} \to 0$ *w.p.1 as* $n \to \infty$.

Lemma 2 (Dirichlet integrals, [2]). *Let* $\mathbf{k} = (k_1, \cdots, k_n) \in \mathbb{R}^n$ *such that* $a + k_j > 0$ *for all* $1 \leq j \leq n$. *Then*

$$\mathbb{E}\left(\prod_{j=1}^n Y_j^{k_j}\right) = \tfrac{\Gamma(|\mathbf{a}|)}{\Gamma(|\mathbf{a}+\mathbf{k}|)} \prod_{j=1}^n \tfrac{\Gamma(a+k_j)}{\Gamma(a)} = \tfrac{\Gamma(na)}{\Gamma(na+\sum_{j=1}^n k_j)} \prod_{j=1}^n \tfrac{\Gamma(a+k_j)}{\Gamma(a)}. \tag{1}$$

Let $\mathcal{D}_{n,k} = \sum_{i=1}^n Y_i^k$, $k \in \mathbb{N}$. Then $n^{-(k-1)} \leq \mathcal{D}_{n,k} \leq 1$ w.p.1. Furthermore, for large n, by using the above Dirichlet integrals, it is easy to compute $\mathbb{E}(\mathcal{D}_{n,k}) = n\gamma_k(a)/\gamma_k(na) \approx n^{-(k-1)} m_k$ and $\mathbb{V}\mathrm{ar}(\mathcal{D}_{n,k}) \approx n^{-(2k-1)} \sigma_k^2$ where $\gamma_k(a) = \Gamma(a)^{-1}\Gamma(a + k) = a(a + 1)\cdots(a + k - 1)$, $m_k = a^{-k}\gamma_k(a)$, and $\sigma_k^2 = m_{2k} - (1 + k^2/a)m_k^2$.

Lemma 3 (Asymptotic convergence of $\mathcal{D}_{n,k}$, [21]). *Let $k \in \mathbb{N}$. Then*

(1) $n^{k-1-\delta}\mathcal{D}_{n,k} \to 0$ *in probability for all* $\delta > 0$ *as* $n \to \infty$.

(2) $n^{k-2-\delta}\mathcal{D}_{n,k} \to 0$ *w.p.1 for all* $\delta > 0$ *as* $n \to \infty$.

(3) $\mathcal{D}_{n,k} \sim \mathrm{AN}\left(n^{-(k-1)}m_k, n^{-(2k-1)}\sigma_k^2\right)$, *that is,* $\sqrt{n}\left(n^{k-1}\mathcal{D}_{n,k} - m_k\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \sigma_k^2\right)$ *as* $n \to \infty$.

The following two lemmas provide additional asymptotic convergence results for various nonlinear expressions of $\mathcal{D}_{n,k}$. Their proofs are slightly lengthy and are deferred to the Appendix.

Lemma 4. *For any $l \geq 1$, $k_1, k_2, \cdots, k_l$ and $p_1, p_2, \cdots, p_l \in \mathbb{N}$, then for any $\delta > 0$, it holds that*

(1) $n^{\left(\sum_{i=1}^{l}(k_i-1)p_i\right)-\delta}\prod_{j=1}^{l}\mathcal{D}_{n,k_j}^{p_j} \to 0$ *in probability.*

(2) $n^{\left(\sum_{i=1}^{l}(k_i-1)p_i\right)-1-\delta}\prod_{j=1}^{l}\mathcal{D}_{n,k_j}^{p_j} \to 0$ *w.p.1 as* $n \to \infty$.

Lemma 5. *Let $\alpha, \beta \in \mathbb{R}$, $\alpha^2 + \beta^2 \neq 0$ and $T_n = \alpha\mathcal{D}_{n,3}^2 + \beta\mathcal{D}_{n,5}$. Then $T_n \sim \mathrm{AN}\left(n^{-4}\mu_T(\alpha,\beta), n^{-9}\sigma_T^2(\alpha,\beta)\right)$ where $\mu_T(\alpha,\beta) = \alpha m_3^2 + \beta m_5 = a^{-4}(a+1)(a+2)\left[\alpha(a+1)(a+2) + \beta(a+3)(a+4)\right]$ and*

$$\begin{aligned}
\sigma_T^2(\alpha,\beta) &= 4\alpha^2 m_3^2\sigma_3^2 + \beta^2\sigma_5^2 + 4\alpha\beta(m_3m_8 - (1+15/a)m_3^2m_5) \\
&= 8a^{-9}(a+1)(a+2)\big[3\alpha^2(a+1)^2(a+2)^2(3a+7) \\
&\quad + 30\alpha\beta(a+1)(a+2)(a+3)^2(a+4) \\
&\quad + 5\beta^2(a+3)(a+4)(5a^3+60a^2+250a+363)\big].
\end{aligned}$$

Remark 6. *The underlying matrix $A = (a_{ij})$ associated with the quadratic form in $\sigma_T^2(\alpha,\beta)$ is strictly positive definite with*

$$\begin{aligned}
a_{11} &= 3(a+1)^2(a+2)^2(3a+7), \\
a_{12} &= a_{21} = 15(a+1)(a+2)(a+3)^2(a+4), \\
a_{22} &= 5(a+3)(a+4)(5a^3+60a^2+250a+363), \\
\det A &= 15(a+1)^2(a+2)^2(a+3)(a+4)\left\{20a^3+225a^2+814a+921\right\} > 0.
\end{aligned}$$

This implies that $\sigma_T^2(\alpha,\beta)$ is non-degenerate unless $\alpha = \beta = 0$.

2.2. Random cyclic polygons under symmetric Dirichlet distributions. The Dirichlet distribution $\mathbf{Y} \sim \mathbb{D}\mathrm{ir}(\mathbf{a})$ is naturally associated with the (non-uniform) random division $0 = X_0 < X_1 < \cdots < X_{n-1} < X_n = 1$ of the unit interval where $X_0 = 0$ and $X_i = \sum_{j=1}^{i} Y_j$ for $1 \leq i \leq n$. In the special case $a = 1$, this corresponds to the classical uniform random division [6, 14] generated by $n-1$ independent and uniformly distributed random points on $(0,1)$. With the rescaling $X_i \mapsto \theta_i = 2\pi X_i$, this can be further mapped to a random division of the unit circle, separated by points $P_i(\cos\theta_i, \sin\theta_i)$, $0 \leq i \leq n$, in counterclockwise direction where P_n represents the same point as P_0. By connecting these points consecutively, we obtain an inscribed random n-gon with its semiperimeter S_n and area A_n given by $S_n = \sum_{i=1}^n \sin \pi(X_i - X_{i-1}) = \sum_{i=1}^n \sin \pi Y_i$, $A_n = \frac{1}{2}\sum_{i=1}^n \sin 2\pi(X_i - X_{i-1}) = \frac{1}{2}\sum_{i=1}^n \sin 2\pi Y_i$. Similarly, using the same random vertices P_i, we can also construct w.p.1 a circumscribing random n-gon which is tangent to the circle at each point P_i with its semiperimeter and area both given by $S'_n = \sum_{i=1}^n \tan \pi(X_i - X_{i-1}) = \sum_{i=1}^n \tan \pi Y_i$. Note that in the event (which has probability 0) all vertices are equally spaced, that is, $Y_i = 1/n$ for all $1 \le i \le n$, these random n-gons happen to be regular n-gons inscribed in or circumscribed about the circle with $S_n = n \sin(\pi/n)$, $A_n = \frac{1}{2} n \sin(2\pi/n) = S_{n/2}$ and $S'_n = n \tan(\pi/n)$. Additionally, such random cyclic polygons generated from the symmetric Dirichlet distribution $\mathbf{Y} \sim \mathbb{D}\mathrm{ir}(\mathbf{a})$ also degenerate to regular n-gons in the limit as $a \to \infty$.

By using the Taylor series expansion of the sine and tangent functions, it is easy to obtain, at least formally, the following probabilistic asymptotic expansions for S_n, A_n and S'_n:

$$\begin{aligned} S_n =& \sum_{j=1}^{\infty} \tfrac{(-1)^{j-1}}{(2j-1)!} \pi^{2j-1} \mathcal{D}_{n,2j-1} \\ =& \pi - \tfrac{1}{3!}\pi^3 \mathcal{D}_{n,3} + \tfrac{1}{5!}\pi^5 \mathcal{D}_{n,5} - \tfrac{1}{7!}\pi^7 \mathcal{D}_{n,7} + \cdots , \end{aligned} \tag{2}$$

$$\begin{aligned} A_n =& \sum_{j=1}^{\infty} \tfrac{(-1)^{j-1} 2^{2j-2}}{(2j-1)!} \pi^{2j-1} \mathcal{D}_{n,2j-1} \\ =& \pi - \tfrac{4}{3!}\pi^3 \mathcal{D}_{n,3} + \tfrac{16}{5!}\pi^5 \mathcal{D}_{n,5} - \tfrac{64}{7!}\pi^7 \mathcal{D}_{n,7} + \cdots , \end{aligned} \tag{3}$$

$$\begin{aligned} S'_n =& \sum_{j=1}^{\infty} \tfrac{B_{2j}(-4)^j (1-4^j)}{(2j)!} \pi^{2j-1} \mathcal{D}_{n,2j-1} \\ =& \pi + \tfrac{1}{3}\pi^3 \mathcal{D}_{n,3} + \tfrac{2}{15}\pi^5 \mathcal{D}_{n,5} + \tfrac{17}{315}\pi^7 \mathcal{D}_{n,7} + \cdots . \end{aligned} \tag{4}$$

where B_j is the jth Bernoulli number. Note that by Lemma 3, the random infinitesimal terms $\mathcal{D}_{n,k}$ in the above expansions decrease progressively in order of magnitude. The validity of these asymptotic expansions is rigorously justified by the following lemma.

Lemma 7 ([21]). *Let m be any positive integer and $\delta > 0$. Then*

$$\begin{aligned} S_n &= \sum_{j=1}^{m} \tfrac{(-1)^{j-1}}{(2j-1)!} \pi^{2j-1} \mathcal{D}_{n,2j-1} + n^{-(2m-1)+\delta} o(1), \\ A_n &= \sum_{j=1}^{m} \tfrac{(-1)^{j-1} 2^{2j-2}}{(2j-1)!} \pi^{2j-1} \mathcal{D}_{n,2j-1} + n^{-(2m-1)+\delta} o(1), \\ S'_n &= \sum_{j=1}^{m} \tfrac{B_{2j}(-4)^j (1-4^j)}{(2j)!} \pi^{2j-1} \mathcal{D}_{n,2j-1} + n^{-(2m-1)+\delta} o(1). \end{aligned}$$

In particular, this implies that S_n, A_n and S'_n all converge to π w.p.1 and their distributions are also asymptotically normal with

$$S_n \sim \mathrm{AN}(\pi - \tfrac{1}{6}n^{-2}m_3\pi^3, \tfrac{1}{36}n^{-5}\sigma_3^2\pi^6),$$
$$A_n \sim \mathrm{AN}\left(\pi - \tfrac{2}{3}n^{-2}m_3\pi^3, \tfrac{4}{9}n^{-5}\sigma_3^2\pi^6\right),$$
$$S'_n \sim \mathrm{AN}(\pi + \tfrac{1}{3}n^{-2}m_3\pi^3, \tfrac{1}{9}n^{-5}\sigma_3^2\pi^6)$$

where $m_3 = a^{-2}(a+1)(a+2)$ and $\sigma_3^2 = 6a^{-5}(a+1)(a+2)(3a+7)$.

3. NONLINEAR EXTRAPOLATION ESTIMATES

3.1. Probabilistic asymptotic expansions for nonlinear functions of S_n, A_n and S'_n. In this section, we study nonlinear random extrapolation estimates of π based on the semiperimeters and areas of both inscribed and circumscribed random polygons with an aim to construct more accurate nonlinear extrapolation estimates than in [26]. To facilitate the derivation of asymptotic expansions for nonlinear functions of S_n, A_n and S'_n in the forms $\mathcal{W}_n = S_n^\alpha A_n^\beta S_n'^\gamma$ and $\mathcal{W}_n(p) = (\alpha S_n^p + \beta A_n^p + \gamma S_n'^p)^{1/p}$ where $\alpha+\beta+\gamma = 1$ and $p \neq 0$, we follow the development in [26] and write

$$\mathcal{W}_{n,1} = \tfrac{1}{3!}\mathcal{D}_{n,3}, \quad \mathcal{W}_{n,2} = \tfrac{1}{2(3!)^2}\mathcal{D}_{n,3}^2 - \tfrac{1}{5!}\mathcal{D}_{n,5},$$
$$\mathcal{W}_{n,3} = \tfrac{1}{3(3!)^3}\mathcal{D}_{n,3}^3 - \tfrac{1}{3!5!}\mathcal{D}_{n,3}\mathcal{D}_{n,5} + \tfrac{1}{7!}\mathcal{D}_{n,7},$$
$$\mathcal{U}_{n,p,1} = \tfrac{1}{3!}\tbinom{p}{1}\mathcal{D}_{n,3}, \qquad \mathcal{U}_{n,p,2} = \tfrac{1}{(3!)^2}\tbinom{p}{2}\mathcal{D}_{n,3}^2 + \tfrac{1}{5!}\tbinom{p}{1}\mathcal{D}_{n,5},$$
$$\mathcal{U}_{n,p,3} = \tfrac{1}{(3!)^3}\tbinom{p}{3}\mathcal{D}_{n,3}^3 + \tfrac{2}{3!5!}\tbinom{p}{2}\mathcal{D}_{n,3}\mathcal{D}_{n,5} + \tfrac{1}{7!}\tbinom{p}{1}\mathcal{D}_{n,7}.$$

LEMMA 8. *Let $\delta > 0$. Then it holds that*

(5) $$\log(S_n/\pi) = -\pi^2\mathcal{W}_{n,1} - \pi^4\mathcal{W}_{n,2} - \pi^6\mathcal{W}_{n,3} + n^{-7+\delta}o(1),$$
(6) $$\log(A_n/\pi) = -4\pi^2\mathcal{W}_{n,1} - 16\pi^4\mathcal{W}_{n,2} - 64\pi^6\mathcal{W}_{n,3} + n^{-7+\delta}o(1),$$
(7) $$(S_n/\pi)^p = 1 - \mathcal{U}_{n,p,1}\pi^2 + \mathcal{U}_{n,p,2}\pi^4 - \mathcal{U}_{n,p,3}\pi^6 + n^{-7+\delta}o(1),$$
(8) $$(A_n/\pi)^p = 1 - 4\mathcal{U}_{n,p,1}\pi^2 + 16\mathcal{U}_{n,p,2}\pi^4 - 64\mathcal{U}_{n,p,3}\pi^6 + n^{-7+\delta}o(1).$$

We mention that while the analysis in [26] is carried out for uniform random divisions only, with Lemmas 1, 3 and 4, it is straightforward to verify that exactly the same asymptotic expansion results in fact extend to the case of symmetric Dirichlet distributions $\mathbf{Y} \sim \mathbb{D}\mathrm{ir}(\mathbf{a})$ for arbitrary $a > 0$. Note also that, in view of Lemmas 3 and 4, we have, for any $\delta > 0$, $\mathcal{W}_{n,1} = n^{-1+\delta}o(1)$, $\mathcal{U}_{n,p,1} = n^{-1+\delta}o(1)$, $\mathcal{W}_{n,2} = n^{-3+\delta}o(1)$, $\mathcal{U}_{n,p,2} = n^{-3+\delta}o(1)$, $\mathcal{W}_{n,3} = n^{-5+\delta}o(1)$, $\mathcal{U}_{n,p,3} = n^{-5+\delta}o(1)$. Thus, the above probabilistic asymptotic expansions in Lemma 8 imply, $\log(S_n/\pi) = -\pi^2\mathcal{W}_{n,1} + n^{-3+\delta}o(1) = -\pi^2\mathcal{W}_{n,1} - \pi^4\mathcal{W}_{n,2} + n^{-5+\delta}o(1)$ and $(S_n/\pi)^p = 1 - \mathcal{U}_{n,p,1}\pi^2 + n^{-3+\delta}o(1) = 1 - \mathcal{U}_{n,p,1}\pi^2 + \mathcal{U}_{n,p,2}\pi^4 + n^{-5+\delta}o(1)$.

Next, to derive asymptotic expansions for nonlinear functions of S'_n such as $\log(S'_n/\pi)$ and $S_n'^p$, we apply the same Taylor series expansions of $\log(1+x)$

and $(1+x)^p$ on $S'_n/\pi - 1 = \frac{1}{3}\pi^2\mathcal{D}_{n,3} + \frac{2}{15}\pi^4\mathcal{D}_{n,5} + \frac{17}{315}\pi^6\mathcal{D}_{n,7} + n^{-7+\delta}o(1)$. Similar to Lemma 8, we now obtain

Lemma 9. *As $n \to \infty$, it holds that, for any $\delta > 0$,*

$$\log(S'_n/\pi) = \mathcal{M}_{n,1}\pi^2 + \mathcal{M}_{n,2}\pi^4 + \mathcal{M}_{n,3}\pi^6 + n^{-7+\delta}o(1), \tag{9}$$

$$(S'_n/\pi)^p = 1 + \mathcal{V}_{n,p,1}\pi^2 + \mathcal{V}_{n,p,2}\pi^4 + \mathcal{V}_{n,p,3}\pi^6 + n^{-7+\delta}o(1) \tag{10}$$

where

$$\mathcal{M}_{n,1} = \tfrac{1}{3}\mathcal{D}_{n,3}, \quad \mathcal{M}_{n,2} = -\tfrac{1}{18}\mathcal{D}_{n,3}^2 + \tfrac{2}{15}\mathcal{D}_{n,5},$$
$$\mathcal{M}_{n,3} = \tfrac{1}{81}\mathcal{D}_{n,3}^3 - \tfrac{2}{45}\mathcal{D}_{n,3}\mathcal{D}_{n,5} + \tfrac{17}{315}\mathcal{D}_{n,7},$$
$$\mathcal{V}_{n,p,1} = \tfrac{1}{3}\tbinom{p}{1}\mathcal{D}_{n,3}, \quad \mathcal{V}_{n,p,2} = \tfrac{1}{9}\tbinom{p}{2}\mathcal{D}_{n,3}^2 + \tfrac{2}{15}\tbinom{p}{1}\mathcal{D}_{n,5},$$
$$\mathcal{V}_{n,p,3} = \tfrac{1}{27}\tbinom{p}{3}\mathcal{D}_{n,3}^3 + \tfrac{4}{45}\tbinom{p}{2}\mathcal{D}_{n,3}\mathcal{D}_{n,5} + \tfrac{17}{315}\tbinom{p}{1}\mathcal{D}_{n,7}.$$

Proof. Let $0 < t < 1/2$ and $\tau = \pi t$. From Lemma 1, it is clear that $\log(S'_n/\pi)\, 1_{\{\Delta_n > t\}} = n^{-k}o(1)$ for all $k \geq 0$. Next we consider $\log(S'_n/\pi)\, 1_{\{\Delta_n \leq t\}}$. For $\Delta_n \leq t$, since $S'_n = \sum_{i=1}^{n} \tan \pi Y_i$, by using the uniform estimate $T_{2m-1} \leq \tan x \leq T_{2m-1} + C_{m,\tau}x^{2m+1}$ for $0 \leq x \leq \tau$ where $m \geq 1$ and $T_{2m-1} = \sum_{j=1}^{m} \frac{(-1)^{j-1}4^j(4^j-1)B_{2j}}{(2j)!}x^{2j-1}$ is the mth Taylor polynomial of the tangent function and $C_{m,\tau}$ is some positive constant which depends on m and τ, we obtain

$$\mathcal{T}_{n,2m-1} \leq S'_n \leq \mathcal{T}_{n,2m-1} + C_{m,\tau}\mathcal{D}_{n,2m+1}$$

where

$$\mathcal{T}_{n,2m-1} = \sum_{j=1}^{m} \tfrac{(-1)^{j-1}4^j(4^j-1)B_{2j}}{(2j)!}\mathcal{D}_{n,2j-1}.$$

With $\mathcal{T}_{n,1} = \pi$ and $\mathcal{D}_{n,k} \leq \Delta_n^{k-1} \leq t^{k-1}$, it is clear that we may choose t suitably small such that $0 < S'_n/\pi - 1 \leq 1/2$ and $0 \leq \mathcal{T}_{n,2m-1}/\pi - 1 \leq 1/2$. By the mean value theorem, we thus obtain

$$\begin{aligned}&\log(S'_n/\pi)1_{\{\Delta_n \leq t\}} - \log(\mathcal{T}_{n,2m-1}/\pi)1_{\{\Delta_n \leq t\}}\\ &\leq S'_n/\pi - \mathcal{T}_{n,2m-1}/\pi \ = n^{-(2m-1)+\delta}o(1).\end{aligned}$$

We now take $m = 4$. By inserting $\mathcal{T}_{n,7}/\pi - 1 = \frac{1}{3}\pi^2\mathcal{D}_{n,3} + \frac{2}{15}\pi^4\mathcal{D}_{n,5} + \frac{17}{315}\pi^6\mathcal{D}_{n,7} = n^{-1+\delta}o(1)$ into the Taylor series approximation $\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \mathcal{O}(1)x^4$ for $|x| \leq 1/2$ and keeping only terms at order $n^{-7+\delta}$ and below, we obtain $\log(\mathcal{T}_{n,7}/\pi)1_{\{\Delta_n \leq t\}} = \frac{1}{3}\pi^2\mathcal{D}_{n,3} + \left(-\frac{1}{18}\mathcal{D}_{n,3}^2 + \frac{2}{15}\mathcal{D}_{n,5}\right)\pi^4 + \left(\frac{1}{81}\mathcal{D}_{n,3}^3 - \frac{2}{45}\mathcal{D}_{n,3}\mathcal{D}_{n,5} + \frac{17}{315}\mathcal{D}_{n,7}\right)\pi^6 + n^{-7+\delta}o(1) = \mathcal{M}_{n,1}\pi^2 + \mathcal{M}_{n,2}\pi^4 + \mathcal{M}_{n,3}\pi^6 + n^{-7+\delta}o(1)$. Consequently, we have $\log(S'_n/\pi) = \mathcal{M}_{n,1}\pi^2 + \mathcal{M}_{n,2}\pi^4 + \mathcal{M}_{n,3}\pi^6 + n^{-7+\delta}o(1)$. Similarly, we can verify $(S'_n/\pi)^p = 1 + \mathcal{V}_{n,p,1}\pi^2 + \mathcal{V}_{n,p,2}\pi^4 + \mathcal{V}_{n,p,3}\pi^6 + n^{-7+\delta}o(1)$. □

3.2. Nonlinear extrapolations of the form $\mathcal{W}_n = S_n^\alpha A_n^\beta S_n'^\gamma$. With the above preparations, we are now ready to derive probabilistic asymptotic expansions for the nonlinear extrapolation estimates $\mathcal{W}_n = S_n^\alpha A_n^\beta S_n'^\gamma$ and $\mathcal{W}_n(p) = (\alpha S_n^p + \beta A_n^p + \gamma S_n'^p)^{1/p}$ where $\alpha+\beta+\gamma = 1$ and $p \neq 0$. By taking the exponential of the linear combination of $\log(S_n/\pi)$, $\log(A_n/\pi)$ and $\log(S_n'/\pi)$ in (5), (6) and (9), or by multiplying S_n^α, A_n^β and $S_n'^\gamma$ directly from (7), (8), and (10), we obtain

LEMMA 10. *For any $\delta > 0$, it holds that*

$$\begin{aligned}\log(\mathcal{W}_n/\pi) =& -\left[(\alpha+4\beta)\mathcal{W}_{n,1} - \gamma\mathcal{M}_{n,1}\right]\pi^2 - \left[(\alpha+16\beta)\mathcal{W}_{n,2} - \gamma\mathcal{M}_{n,2}\right]\pi^4 \\ & - \left[(\alpha+64\beta)\mathcal{W}_{n,3} - \gamma\mathcal{M}_{n,3}\right]\pi^6 + n^{-7+\delta}o(1),\end{aligned}$$

$$\begin{aligned}\mathcal{W}_n =& \pi - \pi^3[(\alpha+4\beta)\mathcal{W}_{n,1} - \gamma\mathcal{M}_{n,1}] \\ & + \pi^5\left\{\tfrac{1}{2}[(\alpha+4\beta)\mathcal{W}_{n,1} - \gamma\mathcal{M}_{n,1}]^2 - (\alpha+16\beta)\mathcal{W}_{n,2} + \gamma\mathcal{M}_{n,2}\right\} \\ & + \pi^7\Big\{ - \tfrac{1}{3!}[(\alpha+4\beta)\mathcal{W}_{n,1} - \gamma\mathcal{M}_{n,1}]^3 \\ & \quad + [(\alpha+4\beta)\mathcal{W}_{n,1} - \gamma\mathcal{M}_{n,1}]\,[(\alpha+16\beta)\mathcal{W}_{n,2} - \gamma\mathcal{M}_{n,2}] \\ & \quad - [(\alpha+64\beta)\mathcal{W}_{n,3} - \gamma\mathcal{M}_{n,3}]\Big\} + n^{-7+\delta}o(1), \end{aligned} \tag{11}$$

where

$$\begin{aligned}(\alpha+4\beta)\mathcal{W}_{n,1} - \gamma\mathcal{M}_{n,1} =& \tfrac{1}{6}(\alpha+4\beta-2\gamma)\mathcal{D}_{n,3}, \\ (\alpha+16\beta)\mathcal{W}_{n,2} - \gamma\mathcal{M}_{n,2} =& \tfrac{1}{72}(\alpha+16\beta+4\gamma)\mathcal{D}_{n,3}^2 - \tfrac{1}{120}(\alpha+16\beta+16\gamma)\mathcal{D}_{n,5}, \\ (\alpha+64\beta)\mathcal{W}_{n,3} - \gamma\mathcal{M}_{n,3} =& \tfrac{1}{648}(\alpha+64\beta-8\gamma)\mathcal{D}_{n,3}^3 \\ & - \tfrac{1}{720}(\alpha+64\beta-32\gamma)\mathcal{D}_{n,3}\mathcal{D}_{n,5} \\ & + \tfrac{1}{5040}(\alpha+64\beta-272\gamma)\mathcal{D}_{n,7}.\end{aligned}$$

Let $\eta = \alpha+4\beta-2\gamma$. Then with $\alpha+\beta+\gamma = 1$, we may write $\alpha = \frac{4}{3} - 2\gamma - \frac{1}{3}\eta$, $\beta = -\frac{1}{3} + \gamma + \frac{1}{3}\eta$. Clearly if $\eta \neq 0$, we have $\mathcal{W}_n = \pi + n^{-1+\delta}o(1) = \pi - \frac{1}{6}\eta\pi^3\mathcal{D}_{n,3} + n^{-3+\delta}o(1)$. Then by Slutsky's theorem, it follows that $\mathcal{W}_n(p) \sim \mathrm{AN}(\pi - \frac{1}{6}n^{-2}\eta\pi^3 m_3, \frac{1}{36}n^{-5}\eta^2\pi^6\sigma_3^2)$.

However, if $\eta = 0$, that is, $\alpha = 4/3 - 2\gamma$, $\beta = -1/3 + \gamma$ and γ is an arbitrary constant, it is then possible to eliminate the leading error term involving $\mathcal{D}_{n,3}$ in (11) to obtain

$$\begin{aligned}\mathcal{W}_n =& \pi - \tfrac{1}{4}\pi^5\left[(\gamma - 2/9)\,\mathcal{D}_{n,3}^2 + (2/15 - \gamma)\,\mathcal{D}_{n,5}\right] \\ & - \tfrac{1}{4536}\pi^7\left[14(27\gamma - 10)\mathcal{D}_{n,3}^3 - 63(3\gamma-2)\mathcal{D}_{n,3}\mathcal{D}_{n,5} - 9(21\gamma+2)\mathcal{D}_{n,7}\right] \\ & + n^{-7+\delta}o(1).\end{aligned} \tag{12}$$

In particular, this implies $\mathcal{W}_n = \pi - \frac{1}{4}\pi^5\left[\left(\gamma - \frac{2}{9}\right)\mathcal{D}_{n,3}^2 + \left(\frac{2}{15} - \gamma\right)\mathcal{D}_{n,5}\right] + n^{-5+\delta}o(1)$. By Lemma 5 and Remark 6, it is clear that $T_n = T_n(\gamma - \frac{2}{9}, \frac{2}{15} - \gamma) =$

$\left(\gamma - \frac{2}{9}\right)\mathcal{D}_{n,3}^2 + \left(\frac{2}{15} - \gamma\right)\mathcal{D}_{n,5}$ is nondegenerate for all $\gamma \in \mathbb{R}$ and is asymptotically normal with $T_n \sim \mathrm{AN}\left(n^{-4}\mu_T(\gamma - \frac{2}{9}, \frac{2}{15} - \gamma), n^{-9}\sigma_T^2(\gamma - \frac{2}{9}, \frac{2}{15} - \gamma)\right)$.

By further applying Slutsky's theorem, a related central limit theorem can be established for the optimal nonlinear extrapolation estimate $\mathcal{W}_n = S_n^{4/3-2\gamma} A_n^{-1/3+\gamma} S_n'^{\gamma}$. Note that in such cases, it is impossible to take advantage of γ to further eliminate the leading order error term in (12) to achieve $\mathcal{W}_n = \pi + n^{-5+\delta}o(1)$.

Theorem 11. *(1) If $\eta = 0$, then $\mathcal{W}_n = \pi + n^{-3+\delta}o(1)$ for any $\delta > 0$, $\mathcal{W}_n \sim \mathrm{AN}\left(\pi - \frac{\pi^5}{4n^4}\mu_T(\gamma - \frac{2}{9}, \frac{2}{15} - \gamma), \frac{\pi^{10}}{16n^9}\sigma_T^2(\gamma - \frac{2}{9}, \frac{2}{15} - \gamma)\right)$.*

(2) If $\eta \neq 0$, then $\mathcal{W}_n = \pi + n^{-1+\delta}o(1)$ for any $\delta > 0$ and $\mathcal{W}_n \sim \mathrm{AN}\left(\pi - \frac{1}{6}n^{-2}\eta\pi^3 m_3, \frac{1}{36}n^{-5}\eta^2\pi^6\sigma_3^2\right)$.

3.3. Nonlinear extrapolations of the form $\mathcal{W}_n(p) = (\alpha S_n^p + \beta A_n^p + \gamma S_n'^p)^{1/p}$. Next, we consider $\mathcal{W}_n(p) = (\alpha S_n^p + \beta A_n^p + \gamma S_n'^p)^{1/p}$. By taking the linear combination of $(S_n/\pi)^p$, $(A_n/\pi)^p$ and $(S_n'/\pi)^p$ in (7), (8) and (10), and applying Newton's generalized binomial formula for $(1+x)^{1/p}$, we derive

Lemma 12. *For any $\delta > 0$, it holds that*

$$\begin{aligned}(\mathcal{W}_n(p)/\pi)^p = 1 &- [(\alpha + 4\beta)\mathcal{U}_{n,p,1} - \gamma\mathcal{V}_{n,p,1}]\pi^2 \\ &+ [(\alpha + 16\beta)\mathcal{U}_{n,p,2} + \gamma\mathcal{V}_{n,p,2}]\pi^4 \\ &- [(\alpha + 64\beta)\mathcal{U}_{n,p,3} - \gamma\mathcal{V}_{n,p,3}]\pi^6 + n^{-7+\delta}o(1),\end{aligned}$$

$$\begin{aligned}\mathcal{W}_n(p) = \pi &- \pi^3\tbinom{1/p}{1}[(\alpha + 4\beta)\mathcal{U}_{n,p,1} - \gamma\mathcal{V}_{n,p,1}] \\ &+ \pi^5\Big\{\tbinom{1/p}{2}[(\alpha + 4\beta)\mathcal{U}_{n,p,1} - \gamma\mathcal{V}_{n,p,1}]^2 \\ &\qquad + \tbinom{1/p}{1}[(\alpha + 16\beta)\mathcal{U}_{n,p,2} + \gamma\mathcal{V}_{n,p,2}]\Big\} \\ &+ \pi^7\Big\{\tbinom{1/p}{3}[-(\alpha + 4\beta)\mathcal{U}_{n,p,1} + \gamma\mathcal{V}_{n,p,1}]^3 \\ &\qquad + 2\tbinom{1/p}{2}[-(\alpha + 4\beta)\mathcal{U}_{n,p,1} + \gamma\mathcal{V}_{n,p,1}][(\alpha + 16\beta)\mathcal{U}_{n,p,2} + \gamma\mathcal{V}_{n,p,2}] \\ &\qquad - \tbinom{1/p}{1}[(\alpha + 64\beta)\mathcal{U}_{n,p,3} - \gamma\mathcal{V}_{n,p,3}]\Big\} + n^{-7+\delta}o(1),\end{aligned}$$

where

$$\begin{aligned}(\alpha + 4\beta)\mathcal{U}_{n,p,1} - \gamma\mathcal{V}_{n,p,1} &= \tfrac{1}{6}\tbinom{p}{1}(\alpha + 4\beta - 2\gamma)\mathcal{D}_{n,3}, \\ (\alpha + 16\beta)\mathcal{U}_{n,p,2} + \gamma\mathcal{V}_{n,p,2} &= \tfrac{1}{36}\tbinom{p}{2}(\alpha + 16\beta + 4\gamma)\mathcal{D}_{n,3}^2 + \tfrac{1}{120}\tbinom{p}{1}(\alpha + 16\beta + 16\gamma)\mathcal{D}_{n,5}, \\ (\alpha + 64\beta)\mathcal{U}_{n,p,3} - \gamma\mathcal{V}_{n,p,3} &= \tfrac{1}{216}\tbinom{p}{3}(\alpha + 64\beta - 8\gamma)\mathcal{D}_{n,3}^3 \\ &\quad + \tfrac{1}{360}\tbinom{p}{2}(\alpha + 64\beta - 32\gamma)\mathcal{D}_{n,3}\mathcal{D}_{n,5} \\ &\quad + \tfrac{1}{5040}\tbinom{p}{1}(\alpha + 64\beta - 272\gamma)\mathcal{D}_{n,7}.\end{aligned}$$

Again, let $\eta = \alpha + 4\beta - 2\gamma$ so that $\alpha = \frac{4}{3} - 2\gamma - \frac{1}{3}\eta$, $\beta = -\frac{1}{3} + \gamma + \frac{1}{3}\eta$. Thus as in the case of $\mathcal{W}_n$, if $\eta \neq 0$, then $\mathcal{W}_n(p) = \pi - \frac{1}{6}\eta\pi^3\mathcal{D}_{n,3} + n^{-3+\delta}o(1)$

and $\mathcal{W}_n(p) \sim \mathrm{AN}(\pi - \frac{1}{6}n^{-2}\eta\pi^3 m_3, \frac{1}{36}n^{-5}\eta^2\pi^6\sigma_3^2)$. However, if $\eta = 0$, that is, $\alpha = 4/3 - 2\gamma$, $\beta = -1/3 + \gamma$, a further improvement as in Theorem 11 is possible. In such cases, we have

$$\begin{aligned}\mathcal{W}_n(p) &= \pi + \tfrac{\pi^5}{4}\left[(p-1)(\gamma - 2/9)\mathcal{D}_{n,3}^2 + (\gamma - 2/15)\mathcal{D}_{n,5}\right] \\ &\quad - \tfrac{\pi^7}{4536}[7(p-1)(p-2)(27\gamma - 10)\mathcal{D}_{n,3}^3 \\ &\qquad + 63(p-1)(3\gamma-2)\mathcal{D}_{n,3}\mathcal{D}_{n,5} - 9(21\gamma+2)\mathcal{D}_{n,7}] + n^{-9+\delta}o(1). \end{aligned} \tag{13}$$

THEOREM 13. (1) *If $\eta = 0$, then $\mathcal{W}_n(p) = \pi + n^{-3+\delta}o(1)$ for any $\delta > 0$ and $\mathcal{W}_n(p) \sim \mathrm{AN}\Big(\pi + \frac{1}{4}n^{-4}\pi^5\mu_T\big((p-1)(\gamma - \frac{2}{9}), \gamma - \frac{2}{15}\big),$ $\frac{1}{16}n^{-9}\pi^{10}\sigma_T^2((p-1)(\gamma - \frac{2}{9}), \gamma - \frac{2}{15})\Big)$.*

(2) *If $\eta \neq 0$, then $\mathcal{W}_n(p) = \pi + n^{-1+\delta}o(1)$ for any $\delta > 0$ and $\mathcal{W}_n(p) \sim \mathrm{AN}\left(\pi - \frac{1}{6}n^{-2}\eta\pi^3 m_3, \frac{1}{36}n^{-5}\eta^2\pi^6\sigma_3^2\right)$.*

REMARK 14. *The asymptotic estimates for $\mathcal{W}_n$ in Theorem 11 can be recovered from those for $\mathcal{W}_n(p)$ in Theorem 13 by setting $p = 0$.*

REMARK 15. *When $p = 1$, we may choose $\gamma = \frac{2}{15}$ to further eliminate the leading order error term in (13). This yields $\alpha = \frac{16}{15}$, $\beta = -\frac{1}{5}$ and the optimal* linear *extrapolation estimate $\mathcal{W}_n(1) = \frac{16}{15}S_n - \frac{1}{5}A_n + \frac{2}{15}S'_n = \pi + n^{-5+\delta}o(1)$ in [21], which satisfies $\mathcal{W}_n(1) \sim \mathrm{AN}(\pi + \frac{1}{105}n^{-6}\pi^7 m_7, \frac{1}{11025}n^{-13}\pi^{14}\sigma_7^2)$. However this is not possible if $p \neq 1$ since, as in the case of $\mathcal{W}_n$, that would require $\gamma = \frac{2}{9}$ and $\gamma = \frac{2}{15}$ simultaneously.*

4. ADDITIONAL REMARKS

We offer some additional remarks to conclude our study on the nonlinear random extrapolation estimates $\mathcal{W}_n = S_n^\alpha A_n^\beta S_n'^\gamma$ and $\mathcal{W}_n(p) = (\alpha S_n^p + \beta A_n^p + \gamma S_n'^p)^{1/p}$. For brevity, we address for both $\mathcal{W}_n$ and $\mathcal{W}_n(p)$ the optimal case only with $\alpha + 4\beta - 2\gamma = 0$, that is, $\alpha = 4/3 - 2\gamma$, $\beta = -1/3 + \gamma$.

4.1. Special cases of $\alpha = 0$, $\beta = 0$, or $\gamma = 0$.

Note that for $\gamma = 0$, $\gamma = 1/3$ and $\gamma = 2/3$, $\mathcal{W}_n$ reduces to $\mathcal{X}_n = S_n^{4/3}A_n^{-1/3}$, $\mathcal{Y}_n = S_n^{2/3}S_n'^{1/3}$ and $\mathcal{Z}_n = A_n^{1/3}S_n'^{2/3}$ respectively with $\mathcal{X}_n = \pi + \frac{\pi^5}{90}\left(5\mathcal{D}_{n,3}^2 - 3\mathcal{D}_{n,5}\right) + n^{-5+\delta}o(1)$, $\mathcal{Y}_n = \pi - \frac{\pi^5}{180}\left(5\mathcal{D}_{n,3}^2 - 9\mathcal{D}_{n,5}\right) + n^{-5+\delta}o(1)$, $\mathcal{Z}_n = \pi - \frac{\pi^5}{45}\left(5\mathcal{D}_{n,3}^2 - 6\mathcal{D}_{n,5}\right) + n^{-5+\delta}o(1)$.

COROLLARY 16. *For any $\delta > 0$, it holds that*

(1) $\mathcal{X}_n = \pi + n^{-3+\delta}o(1)$,
$\mathcal{X}_n \sim \mathrm{AN}\left(\pi + \frac{1}{90}n^{-4}\pi^5\mu_T(5,-3), \frac{1}{8100}n^{-9}\pi^{10}\sigma_T^2(5,-3)\right)$.

(2) $\mathcal{Y}_n = \pi + n^{-3+\delta}o(1)$,
$\mathcal{Y}_n \sim \mathrm{AN}\left(\pi - \frac{1}{180}n^{-4}\pi^5\mu_T(5,-9), \frac{1}{32400}n^{-9}\pi^{10}\sigma_T^2(5,-9)\right)$.

(3) $\mathcal{Z}_n = \pi + n^{-3+\delta} o(1)$,
$\mathcal{Z}_n \sim \mathrm{AN}\left(\pi - \frac{1}{45} n^{-4}\pi^5 \mu_T(5,-6), \frac{1}{2025} n^{-9}\pi^{10}\sigma_T^2(5,-6)\right)$.

Similarly, for $\mathcal{W}_n(p)$, the same choices $\gamma = 0$, $\gamma = 1/3$, $\gamma = 2/3$ yield

$$\mathcal{X}_n(p) = \left(\tfrac{4}{3}S_n^p - \tfrac{1}{3}A_n^p\right)^{1/p} = \pi - \tfrac{1}{90}\pi^5 \left[5(p-1)\mathcal{D}_{n,3}^2 + 3\mathcal{D}_{n,5}\right] + n^{-5+\delta}o(1),$$

$$\mathcal{Y}_n(p) = \left(\tfrac{2}{3}S_n^p + \tfrac{1}{3}S_n'^p\right)^{1/p} = \pi + \tfrac{1}{180}\pi^5 \left[5(p-1)\mathcal{D}_{n,3}^2 + 9\mathcal{D}_{n,5}\right] + n^{-5+\delta}o(1),$$

$$\mathcal{Z}_n(p) = \left(\tfrac{1}{3}A_n^p + \tfrac{2}{3}S_n'^p\right)^{1/p} = \pi + \tfrac{1}{45}\pi^5 \left[5(p-1)\mathcal{D}_{n,3}^2 + 6\mathcal{D}_{n,5}\right] + n^{-5+\delta}o(1).$$

Corollary 17. *For any $\delta > 0$, it holds that*

(1) $\mathcal{X}_n(p) = \pi + n^{-3+\delta}o(1)$,
$\mathcal{X}_n(p) \sim \mathrm{AN}\left(\pi - \frac{\pi^5}{90n^4}\mu_T(5(p-1),3), \frac{\pi^{10}}{8100n^9}\sigma_T^2(5(p-1),3)\right)$.

(2) $\mathcal{Y}_n(p) = \pi + n^{-3+\delta}o(1)$,
$\mathcal{Y}_n(p) \sim \mathrm{AN}\left(\pi + \frac{\pi^5}{180n^4}\mu_T\left(5(p-1),9\right), \frac{\pi^{10}}{32400n^9}\sigma_T^2\left(5(p-1),9\right)\right)$.

(3) $\mathcal{Z}_n(p) = \pi + n^{-3+\delta}o(1)$,
$\mathcal{Z}_n(p) \sim \mathrm{AN}\left(\pi + \frac{\pi^5}{45n^4}\mu_T\left(5(p-1),6\right), \frac{\pi^{10}}{2025n^9}\sigma_T^2\left(5(p-1),6\right)\right)$.

4.2. Uniform spacings. In the special case of $a = 1$, the symmetric Dirichlet distribution corresponds to the uniform spacings generated by $n-1$ independent and uniformly distributed random points on the unit interval. Thus, by setting $a = 1$, we immediately obtain optimal nonlinear extrapolation estimates for random polygons generated by independent and uniformly distributed random points on the unit circle.

Theorem 18 (Uniform spacings). (1) *$\mathcal{W}_n = \pi + n^{-3+\delta}o(1)$, $\mathcal{W}_n(p) = \pi + n^{-3+\delta}o(1)$ for any $\delta > 0$.*

(2) $\mathcal{W}_n \sim \mathrm{AN}\left(\pi + n^{-4}\pi^5(21\gamma - 2), 48n^{-9}\pi^{10}(3405\gamma^2 - 840\gamma + 52)\right)$.

(3) *$\mathcal{W}_n(p) \sim \mathrm{AN}\left(\pi + n^{-4}\pi^5\mu_{p,\gamma}, n^{-9}\pi^{10}\sigma_{p,\gamma}^2\right)$ where $\mu_{p,\gamma} = (9p+21)\gamma - 2(p+1)$, $\sigma_{p,\gamma}^2 = \frac{1}{16}\sigma_T^2((p-1)(\gamma - 2/9), \gamma - 2/15) = 360(9p^2 + 102p + 454)\gamma^2 - 480(3p^2 + 26p + 84)\gamma + 32(5p^2 + 30p + 78)$.*

(4) *For $p = 1$ and $\gamma = 2/15$, $\mathcal{W}_n(1) = \frac{16}{15}S_n - \frac{1}{5}A_n + \frac{2}{15}S_n'$ satisfies $\mathcal{W}_n(1) = \pi + \frac{1}{105}\pi^7\mathcal{D}_{n,7} + n^{-7+\delta}o(1) = \pi + n^{-5+\delta}o(1)$, $\mathcal{W}_n(1) \sim \mathrm{AN}(\pi + 48n^{-6}\pi^7, 7792128n^{-13}\pi^{14})$.*

4.3. Regular polygons. Finally, we remark that in the case of regular polygons, with $Y_i = 1/n$ and $\mathcal{D}_{n,k} = n^{-(k-1)}$, it is straightforward to check that the optimal estimate for both $\mathcal{W}_n$ and $\mathcal{W}_n(p)$ occurs also when $\alpha + 4\beta - 2\gamma = 0$ with

$$\mathcal{W}_n = \pi + \frac{1}{45}\frac{\pi^5}{n^4} + \frac{4}{567}\frac{\pi^7}{n^6} + \frac{1}{405}\frac{\pi^9}{n^8} + \mathcal{O}(n^{-10}),$$

$$\mathcal{W}_n(p) = \pi + \frac{\pi^5}{180n^4}\left[45\gamma p - 10p + 4\right] - \frac{\pi^7}{4536n^6}\left[189p(p-2)\gamma - 70p^2 + 84p - 32\right]$$

$$-\frac{\pi^9}{25920n^8}\big[810p^2(p-1)\gamma^2-45(13p^2-20p+20)\gamma + 2(55p^3-120p^2+100p-32)\big]+\mathcal{O}(n^{-10}).$$

Note that the result for $\mathcal{W}_n$ actually does not depend on the parameter γ at all. This is due to the fact $A_nS_n'=S_n^2$, a variant of Archimedes's celebrated geometric mean relation, which implies $\mathcal{W}_n=S_n^\alpha A_n^\beta S_n'^\gamma=S_n^{\alpha+2\gamma}A_n^{\beta-\gamma}$. In particular, in such cases, $\mathcal{X}_n=S_n^{4/3}A_n^{-1/3}$, $\mathcal{Y}_n=A_n^{2/3}S_n'^{1/3}$ and $\mathcal{Z}_n=A_n^{1/3}S_n'^{2/3}$ all yield exactly the same result. While this relation no longer holds for random polygons, it helps explain why throwing in an extra term does not always increase the accuracy of the extrapolation estimates.

For $\mathcal{W}_n(p)$, however, by taking $\gamma=\frac{10p-4}{45p}$, it is possible to eliminate the leading error term $\mathcal{O}(n^{-4})$ to obtain $\mathcal{W}_n(p)=\pi+\frac{1}{5670}\frac{\pi^7}{n^6}(35p^2+21p-2)-\frac{1}{32400}\frac{\pi^9}{n^8}(25p^3-75p^2-52p+12)+\mathcal{O}(n^{-10})$. Thus if we choose $p_\pm=\frac{-21\pm\sqrt{721}}{70}$, we can further obtain $\mathcal{W}_n(p_\pm)=\pi-\frac{119\pm\sqrt{721}}{661500}\frac{\pi^9}{n^8}+\mathcal{O}(n^{-10})$, which is two orders of magnitude higher than the optimal linear estimate $\mathcal{W}_n(1)=\frac{16}{15}S_n-\frac{1}{5}A_n+\frac{2}{15}S_n'=\pi+\frac{1}{105}\frac{\pi^7}{n^6}+\frac{1}{360}\frac{\pi^9}{n^8}+\mathcal{O}(n^{-10})$.

5. NUMERICAL SIMULATIONS

In this section, we present numerical simulation results to confirm the main probabilistic convergence estimates obtained in this paper. For this purpose, we use the MATLAB command `gamrnd(a,1)` to first generate n independent gamma random variables $\mathbf{V}=(V_1,V_2,\cdots V_n)$ with shape parameter $a>0$. The symmetric Dirichlet random vector $\mathbf{Y}\sim\mathbb{D}\text{ir}(a\mathbf{1})$ is then obtained by normalization $\mathbf{Y}=\mathbf{V}/\|\mathbf{V}\|_1$. Next, we compute $S_n=\sum_{i=1}^n\sin\pi Y_i$, $A_n=\frac{1}{2}\sum_{i=1}^n\sin 2\pi Y_i$, $S_n'=\sum_{i=1}^n\tan\pi Y_i$, and subsequently, $\mathcal{W}_n=S_n^\alpha A_n^\beta S_n'^\gamma$, $\mathcal{W}_n(p)=(\alpha S_n^p+\beta A_n^p+\gamma S_n'^p)^{1/p}$ for $n=96\times 2^k$, $k=4,5,6,7$. For simplicity, we consider two p values: $p=2$, $p=-1$ and choose $\alpha=16/15$, $\beta=-1/5$, $\gamma=2/15$ which clearly satisfies the optimality condition $\eta=\alpha+4\beta-2\gamma=0$. We repeat these simulations for $m=100,000$ times. For each of these random samples, we compute its empirical mean $\hat{\mu}(X_n)$ and empirical standard deviation $\hat{\sigma}(X_n)$ with normalization $\tilde{X}_n=(X_n-\hat{\mu}(X_n))/\hat{\sigma}(X_n)$. The histograms (with bin size 400) for the empirical PDFs of normalized S_n, A_n, S_n', W_n, $W_n(p)$ are displayed in Fig. 1 and Fig. 2 below for specified parameter values.

To effectively compare the empirical data with their theoretical asymptotic values, suitable scaling factors are used for $\hat{\mu}(X_n)$ and $\hat{\sigma}(X_n)$ in the tables below. For $\mathcal{W}_n$ and $\mathcal{W}_n(p)$, by Theorems 11 and 13, it is clear that $\hat{\mu}(X_n)=\pi+\mathcal{O}(1)n^{-4}\pi^5$, and $\hat{\sigma}(X_n)=\mathcal{O}(1)n^{-9/2}\pi^5$. Thus, for easy numerical comparison, we display $\hat{\mu}_n=n^4\pi^{-5}(\hat{\mu}(X_n)-\pi)$ and $\hat{\sigma}_n=n^{9/2}\pi^{-5}\hat{\sigma}(X_n)$ instead, together with similarly scaled limiting values. For S_n, A_n and S_n', the scaling factors for the mean and standard deviation are $n^2\pi^{-3}$ and $n^{5/2}\pi^{-3}$ respectively.

Finally, we note that for $p=1$, the optimal linear extrapolation $\mathcal{W}_n(1)=\frac{16}{15}S_n-\frac{1}{5}A_n+\frac{2}{15}S_n'$ converges most rapidly with an asymptotical mean $\hat{\mu}(\mathcal{W}_n(1))$

$= \pi + \mathcal{O}(1)n^{-6}$ and standard deviation $\hat{\sigma}(\mathcal{W}_n(1)) = \mathcal{O}(1)n^{-13/2}$. At such "atomic" scales, however, for n in the range $10^3 \sim 10^4$, the usual double precision computation may not be enough to prevent severe loss of significant digits. In such cases, the distribution of the *rescaled* simulated data would appear to be more "discrete" with unusually large variance. As a compromise, partial numerical evidence of the convergence results may be witnessed by using relatively smaller values of n instead. See Fig. 3 and Table 7. Such phenomena also occur to $\mathcal{W}_n$ and $\mathcal{W}_n(p)$, but to a much lesser extent.

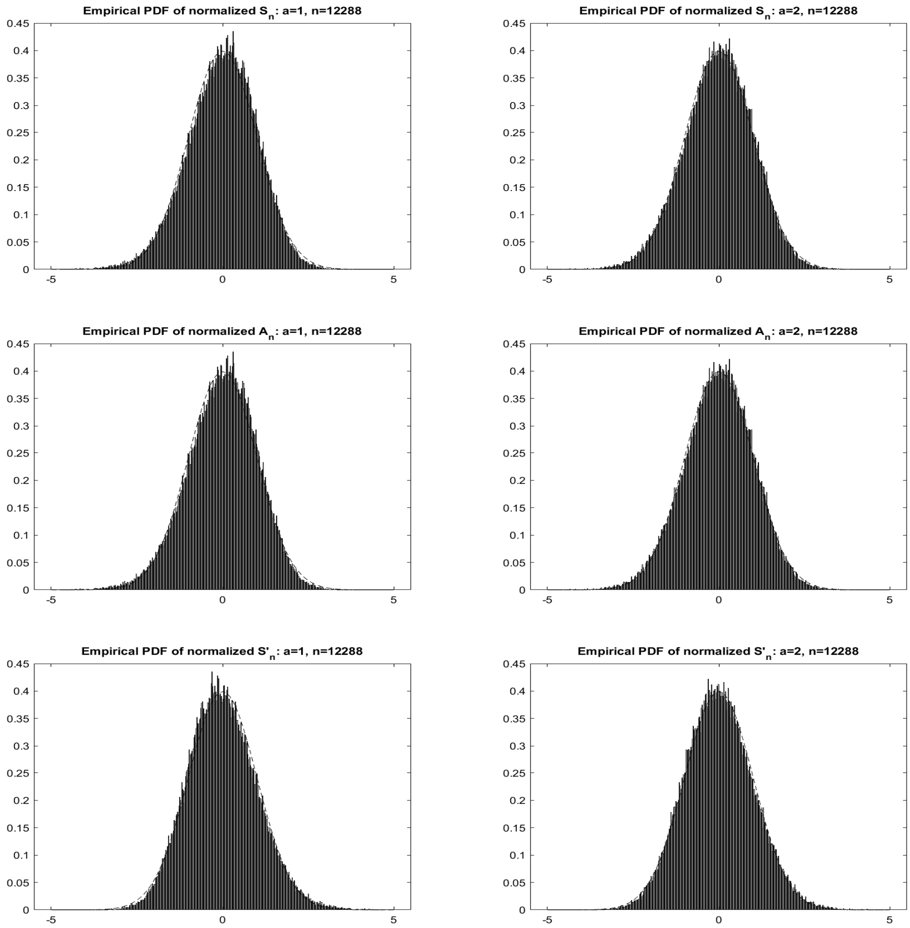

Fig. 1. Empirical PDF of normalized S_n, A_n, S'_n: $a = 1$, $a = 2$ and $n = 12288$.

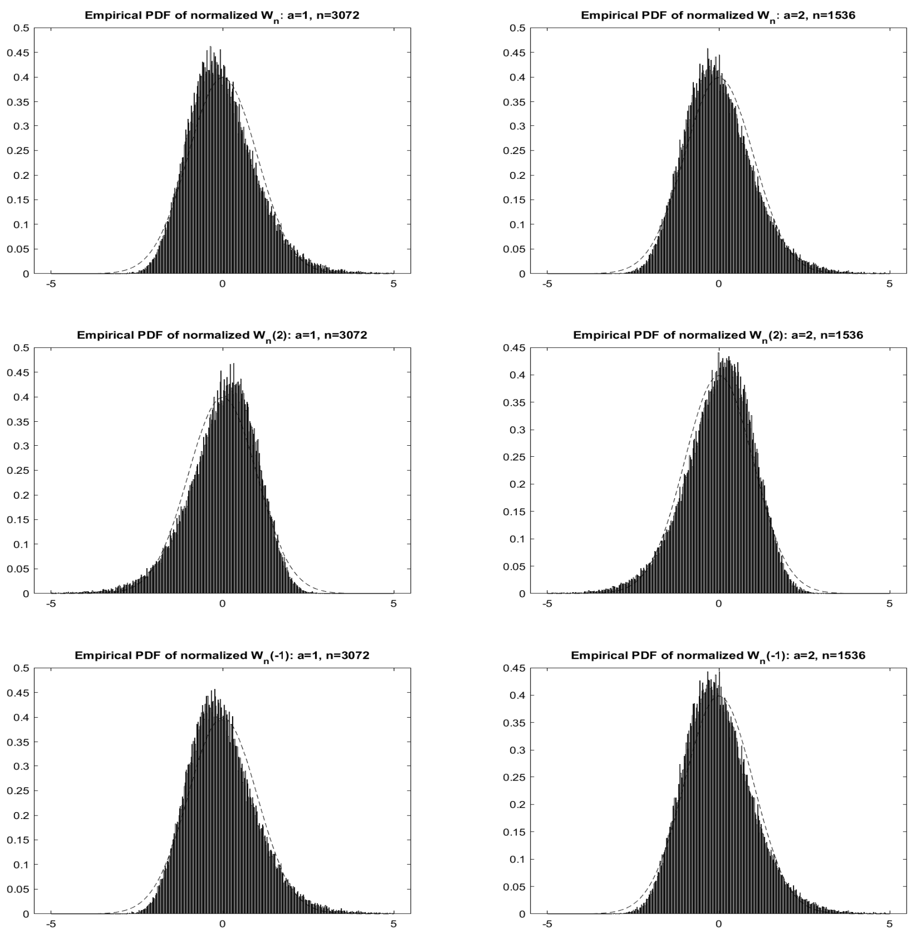

Fig. 2. Empirical PDF of normalized W_n, $W_n(2)$, $W_n(-1)$: $a = 1$, $n = 3072$ and $a = 2$, $n = 1536$.

	n	$\hat{\mu}_n$	μ_∞	$\hat{\mu}_n/\mu_\infty$	$\hat{\sigma}_n$	σ_∞	$\hat{\sigma}_n/\sigma_\infty$
	1536	-0.9980	-1.0000	0.9980	3.1522	3.1623	0.9968
$a = 1$	3072	-0.9989	-1.0000	0.9989	3.1388	3.1623	0.9926
	6144	-0.9996	-1.0000	0.9996	3.1692	3.1623	1.0022
	12288	-0.9998	-1.0000	0.9998	3.1627	3.1623	1.0001
	1536	-0.4997	-0.5000	0.9993	0.8996	0.9014	0.9980
$a = 2$	3072	-0.4998	-0.5000	0.9996	0.9018	0.9014	1.0005
	6144	-0.4999	-0.5000	0.9997	0.9021	0.9014	1.0008
	12288	-0.5000	-0.5000	0.9999	0.9026	0.9014	1.0013

Table 1. Adjusted means and standard deviations of S_n: $\hat{\mu}(S_n) = \pi + \pi^3 n^{-2}\hat{\mu}_n$, $\hat{\sigma}(S_n) = \pi^3 n^{-5/2}\hat{\sigma}_n$. The first four rows are for $a = 1$, and the next four rows for $a = 2$.

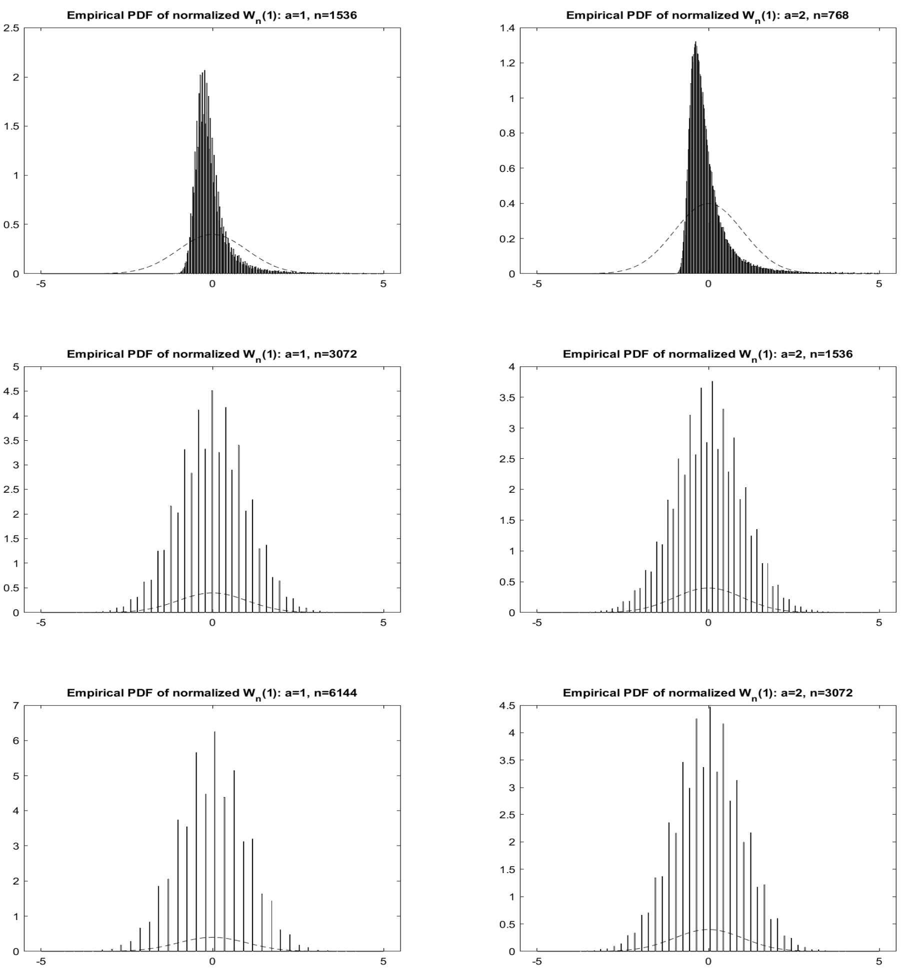

Fig. 3. Empirical PDF of normalized $W_n(1) = \frac{16}{15}S_n - \frac{1}{5}A_n + \frac{2}{15}S'_n$: $a = 1$, $n = 1536, 3072, 6144$ and $a = 2$, $n = 768, 1536, 3072$.

	n	$\hat{\mu}_n$	μ_∞	$\hat{\mu}_n/\mu_\infty$	$\hat{\sigma}_n$	σ_∞	$\hat{\sigma}_n/\sigma_\infty$
$a = 1$	1536	-3.9921	-4.0000	0.9980	12.6082	12.6491	0.9968
	3072	-3.9955	-4.0000	0.9989	12.5550	12.6491	0.9926
	6144	-3.9984	-4.0000	0.9996	12.6769	12.6491	1.0022
	12288	-3.9992	-4.0000	0.9998	12.6510	12.6491	1.0001
$a = 2$	1536	-1.9987	-2.0000	0.9993	3.5983	3.6056	0.9980
	3072	-1.9993	-2.0000	0.9996	3.6072	3.6056	1.0004
	6144	-1.9995	-2.0000	0.9997	3.6085	3.6056	1.0008
	12288	-1.9998	-2.0000	0.9999	3.6102	3.6056	1.0013

Table 2. Adjusted means and standard deviations of A_n: $\hat{\mu}(A_n) = \pi + \pi^3 n^{-2}\hat{\mu}_n$, $\hat{\sigma}(A_n) = \pi^3 n^{-5/2}\hat{\sigma}_n$ for $a = 1$ and $a = 2$.

	n	$\hat{\mu}_n$	μ_∞	$\hat{\mu}_n/\mu_\infty$	$\hat{\sigma}_n$	σ_∞	$\hat{\sigma}_n/\sigma_\infty$
$a=1$	1536	1.9962	2.0000	0.9981	6.3053	6.3246	0.9970
	3072	1.9978	2.0000	0.9989	6.2778	6.3246	0.9926
	6144	1.9992	2.0000	0.9996	6.3385	6.3246	1.0022
	12288	1.9996	2.0000	0.9998	6.3255	6.3246	1.0002
$a=2$	1536	0.9994	1.0000	0.9994	1.7993	1.8028	0.9981
	3072	0.9996	1.0000	0.9996	1.8036	1.8028	1.0005
	6144	0.9997	1.0000	0.9997	1.8042	1.8028	1.0008
	12288	0.9999	1.0000	0.9999	1.8051	1.8028	1.0013

Table 3. Adjusted means and standard deviations of S'_n: $\hat{\mu}(S'_n) = \pi + \pi^3 n^{-2}\hat{\mu}_n$, $\hat{\sigma}(S'_n) = \pi^3 n^{-5/2}\hat{\sigma}_n$ for $a=1$ and $a=2$.

	n	$\hat{\mu}_n$	μ_∞	$\hat{\mu}_n/\mu_\infty$	$\hat{\sigma}_n$	σ_∞	$\hat{\sigma}_n/\sigma_\infty$
$a=1$	1536	0.8022	0.8000	1.0027	5.1913	5.0596	1.0260
	3072	0.8008	0.8000	1.0009	5.0906	5.0596	1.0061
	6144	0.8001	0.8000	1.0001	5.1441	5.0596	1.0167
	12288	0.7905	0.8000	0.9881	11.5189	5.0596	2.2766
$a=2$	1536	0.2002	0.2000	1.0008	0.7266	0.7211	1.0076
	3072	0.2000	0.2000	1.0002	0.7258	0.7211	1.0065
	6144	0.1995	0.2000	0.9976	0.9367	0.7211	1.2990
	12288	0.1926	0.2000	0.9631	10.2946	0.7211	14.2760

Table 4. Adjusted means and standard deviations of W_n: $\hat{\mu}(W_n) = \pi + \pi^5 n^{-4}\hat{\mu}_n$, $\hat{\sigma}(W_n) = \pi^5 n^{-9/2}\hat{\sigma}_n$ for $a=1$ and $a=2$.

	n	$\hat{\mu}_n$	μ_∞	$\hat{\mu}_n/\mu_\infty$	$\hat{\sigma}_n$	σ_∞	$\hat{\sigma}_n/\sigma_\infty$
$a=1$	1536	-0.8018	-0.8000	1.0022	5.1782	5.0596	1.0234
	3072	-0.8007	-0.8000	1.0009	5.0872	5.0596	1.0055
	6144	-0.8010	-0.8000	1.0012	5.1421	5.0596	1.0163
	12288	-0.8075	-0.8000	1.0094	11.2584	5.0596	2.2251
$a=2$	1536	-0.2001	-0.2000	1.0007	0.7261	0.7211	1.0070
	3072	-0.2001	-0.2000	1.0004	0.7258	0.7211	1.0065
	6144	-0.2003	-0.2000	1.0013	0.9326	0.7211	1.2932
	12288	-0.2025	-0.2000	1.0123	10.1136	0.7211	14.0251

Table 5. Adjusted means and standard deviations of $W_n(2)$: $\hat{\mu}(W_n(2)) = \pi + \pi^5 n^{-4}\hat{\mu}_n$, $\hat{\sigma}(W_n(2)) = \pi^5 n^{-9/2}\hat{\sigma}_n$ for $a=1$ and $a=2$.

APPENDIX

Proof for Lemma 4. We use similar ideas in [21] to prove Lemmas 4 and 5. First, by using the multinomial expansion formula

$$(x_1 + x_2 + \cdots + x_n)^p = \sum_{\mathbf{m}} \binom{p}{\mathbf{m}} x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} = \sum_{\mathbf{m}} \binom{p}{\mathbf{m}} \prod_{j=1}^{n} x_j^{m_j}$$

	n	$\hat{\mu}_n$	μ_∞	$\hat{\mu}_n/\mu_\infty$	$\hat{\sigma}_n$	σ_∞	$\hat{\sigma}_n/\sigma_\infty$
$a=1$	1536	1.6042	1.6000	1.0026	10.3763	10.1193	1.0254
	3072	1.6016	1.6000	1.0010	10.1794	10.1193	1.0059
	6144	1.6016	1.6000	1.0010	10.2350	10.1193	1.0114
	12288	1.6028	1.6000	1.0017	14.5945	10.1193	1.4422
$a=2$	1536	0.4003	0.4000	1.0008	1.4529	1.4422	1.0074
	3072	0.4002	0.4000	1.0004	1.4503	1.4422	1.0056
	6144	0.4004	0.4000	1.0011	1.5646	1.4422	1.0849
	12288	0.4046	0.4000	1.0115	10.5186	1.4422	7.2933

Table 6. Adjusted means and standard deviations of $W_n(-1)$: $\hat{\mu}(W_n(-1)) = \pi + \pi^5 n^{-4}\hat{\mu}_n$, $\hat{\sigma}(W_n(-1)) = \pi^5 n^{-9/2}\hat{\sigma}_n$ for $a=1$ and $a=2$.

	n	$\hat{\mu}_n$	μ_∞	$\hat{\mu}_n/\mu_\infty$	$\hat{\sigma}_n$	σ_∞	$\hat{\sigma}_n/\sigma_\infty$
$a=1$	768	46.6014	48	0.9709	2330.9783	2791.4383	0.8351
	1536	46.8429	48	0.9759	2704.3826	2791.4383	0.9688
	3072	9.3958	48	0.1957	34597.5684	2791.4383	12.3942
$a=2$	768	2.9572	3	0.9857	82.3636	83.7250	0.98374
	1536	2.5341	3	0.8447	467.2516	83.7250	5.5808
	3072	25.8888	3	-8.6296	34440.6661	83.7250	411.3546

Table 7. Adjusted means and standard deviations of $W_n(1) = \frac{16}{15}S_n - \frac{1}{5}A_n + \frac{2}{15}S_n'$: $\hat{\mu}(W_n(1)) = \pi + \pi^7 n^{-6}\hat{\mu}_n$, $\hat{\sigma}(W_n(1)) = \pi^7 n^{-13/2}\hat{\sigma}_n$ for $a=1$ and $a=2$.

where p is a positive integer, $\mathbf{m} = (m_1, m_2, \cdots, m_n)$ is a multi-index with each $m_j \geq 0$ and $|\mathbf{m}| = \sum_{j=1}^n m_j = p$, and $\binom{p}{\mathbf{m}}$ is given by

$$\binom{p}{\mathbf{m}} = \binom{p}{m_1, m_2, \cdots, m_n} = \frac{p!}{m_1! m_2! \cdots m_n!},$$

we obtain for each $1 \leq i \leq l$

$$\mathcal{D}_{n,k_i}^{p_i} = \left(\sum_{j=1}^n Y_j^{k_i}\right)^{p_i} = \sum_{\mathbf{m}_i} \binom{p_i}{\mathbf{m}_i} \prod_{j=1}^n Y_j^{k_i m_{i,j}}$$

where $\mathbf{m}_i = (m_{i,1}, m_{i,2}, \cdots, m_{i,n})$ such that $|\mathbf{m}_i| = \sum_{j=1}^n m_{i,j} = p_i$. Then for all $1 \leq i \leq l$, we obtain

$$\prod_{i=1}^l \mathcal{D}_{n,k_i}^{p_i} = \sum_{\mathbf{m}_i} \left[\prod_{i=1}^l \binom{p_i}{\mathbf{m}_i}\right] \prod_{j=1}^n Y_j^{\sum_{i=1}^l k_i m_{i,j}}.$$

By using (1) and $\sum_{j=1}^n \sum_{i=1}^l k_i m_{i,j} = \sum_{i=1}^l \sum_{j=1}^n k_i m_{i,j} = \sum_{i=1}^l k_i p_i$, we obtain

$$\mathbb{E}\left(\prod_{i=1}^l \mathcal{D}_{n,k_i}^{p_i}\right) = \sum_{\mathbf{m}_i} \left[\prod_{i=1}^l \binom{p_i}{\mathbf{m}_i}\right] \frac{\Gamma(na)}{\Gamma(na + \sum_{j=1}^n \sum_{i=1}^l k_i m_{i,j})} \prod_{j=1}^n \frac{\Gamma(a + \sum_{i=1}^l k_i m_{i,j})}{\Gamma(a)}$$

$$= \sum_{\mathbf{m}_i} \left[\prod_{i=1}^{l} \binom{p_i}{\mathbf{m}_i}\right] \frac{\Gamma(na)}{\Gamma(na+\sum_{i=1}^{l} k_i p_i)} \prod_{j=1}^{n} \frac{\Gamma(a+\sum_{i=1}^{l} k_i m_{i,j})}{\Gamma(a)}. \tag{14}$$

The key is to estimate $\prod_{j=1}^{n} \frac{\Gamma(a+q_j)}{\Gamma(a)}$ with $q_j = \sum_{i=1}^{l} k_i m_{i,j}$. Note that

$$\prod_{j=1}^{n} \frac{\Gamma(a+q_j)}{\Gamma(a)} = \begin{cases} \prod\limits_{j:\, q_j \geq 1} \gamma_{q_j}(a), & \text{if } q_j \geq 1, \\ 1, & \text{if } q_j = 0. \end{cases}$$

Since

$$\sum_{j=1}^{n} q_j = \sum_{j=1}^{n} \sum_{i=1}^{l} k_i m_{i,j} = \sum_{i=1}^{l} \sum_{j=1}^{n} k_i m_{i,j} = \sum_{i=1}^{l} k_i p_i = m_*,$$

the number r of indices $\#\{1 \leq j \leq n : q_j \geq 1\}$ is at most m_*. Then if $q_j \geq 1$, we have

$$\prod_{j:\, q_j \geq 1} \frac{\Gamma(a+q_j)}{\Gamma(a)} = \prod_{j:\, q_j \geq 1} \gamma_{q_j}(a)$$

$$\leq \prod_{j:\, q_j \geq 1} \gamma_{q_j}(\widehat{a}) = (\widehat{a} + q_{i_1} - 1)!\, (\widehat{a} + q_{i_2} - 1)! \cdots (\widehat{a} + q_{i_r} - 1)!$$

$$\leq \left(m_*(\widehat{a} - 1) + \sum_{s=1}^{r} q_{i_s}\right)! \leq \left(m_*(\widehat{a} - 1) + \sum_{j=1}^{n} q_j\right)! = (m_* \widehat{a})! = \mathcal{O}(1), \tag{15}$$

where $\widehat{a} = \lceil a \rceil$ is the smallest integer that is greater than or equal to a, $\mathcal{O}(1)$ is some positive constant independent of n. By using Stirling's formula [1]

$$\Gamma(z) \sim e^{-z} z^{z-1/2} (2\pi)^{1/2} \left[1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} - \frac{571}{2488320z^4} + \cdots\right]$$

as $z \to \infty$, we obtain

$$\frac{\Gamma(na)}{\Gamma(na+\sum_{i=1}^{l} k_i p_i)} \sim \frac{e^{-na} (na)^{na-1/2}}{e^{-(na+\sum_{i=1}^{l} k_i p_i)} (na+\sum_{i=1}^{l} k_i p_i)^{(na+\sum_{i=1}^{l} k_i p_i)-1/2}}$$

$$= e^{\sum_{i=1}^{l} k_i p_i} \left(na + \sum_{i=1}^{l} k_i p_i\right)^{-\sum_{i=1}^{l} k_i p_i} \left(1 - \frac{\sum_{i=1}^{l} k_i p_i}{na+\sum_{i=1}^{l} k_i p_i}\right)^{na-1/2}$$

$$\sim n^{-\sum_{i=1}^{l} k_i p_i}. \tag{16}$$

Substituting (15) and (16) into (14), and using

$$\sum_{\mathbf{m}_i} \prod_{i=1}^{l} \binom{p_i}{\mathbf{m}_i} = \prod_{i=1}^{l} \sum_{\mathbf{m}_i} \binom{p_i}{\mathbf{m}_i} = \prod_{i=1}^{l} n^{p_i} = n^{\sum_{i=1}^{l} p_i},$$

we have

$$\mathbb{E}\left(\prod_{i=1}^{l} \mathcal{D}_{n,k_i}^{p_i}\right) = \mathcal{O}(1)\, n^{-\sum_{i=1}^{l} (k_i-1) p_i}, \quad \text{for large } n.$$

By applying Markov inequality, for any $\delta > 0$ and $\varepsilon > 0$, we obtain

$$\begin{aligned}
&\Pr\left(n^{\left(\sum_{i=1}^{l}(k_i-1)p_i\right)-\delta}\prod_{i=1}^{l}\mathcal{D}_{n,k_i}^{p_i} > \varepsilon\right) \\
&\le \varepsilon^{-1}\mathbb{E}\left[n^{\left(\sum_{i=1}^{l}(k_i-1)p_i\right)-\delta}\prod_{i=1}^{l}\mathcal{D}_{n,k_i}^{p_i}\right] = \mathcal{O}(1)\varepsilon^{-1}n^{-\delta} \to 0
\end{aligned}$$

as $n \to \infty$, which implies $n^{\left(\sum_{i=1}^{l}(k_i-1)p_i\right)-\delta}\prod_{j=1}^{l}\mathcal{D}_{n,k_j}^{p_j} \to 0$ in probability as $n \to \infty$. In addition, we have

$$\sum_{n\ge 3}\Pr\left(n^{\left(\sum_{i=1}^{l}(k_i-1)p_i\right)-1-\delta}\prod_{i=1}^{l}\mathcal{D}_{n,k_i}^{p_i} > \varepsilon\right) \le \mathcal{O}(1)\varepsilon^{-1}\sum_{n\ge 3}n^{-1-\delta} < \infty.$$

By Borel-Cantelli Lemma, it follows that $n^{\left(\sum_{i=1}^{l}(k_i-1)p_i\right)-1-\delta}\prod_{i=1}^{l}\mathcal{D}_{n,k_i}^{p_i} \to 0$ with probability 1 as $n \to \infty$. This completes the proof of Lemma 4.

Proof for Lemma 5. By applying the equivalent representation [2]

$$(X_1 - X_0, X_2 - X_1, \cdots, X_n - X_{n-1}) \stackrel{\mathcal{L}}{=} \left(\frac{V_1}{\sum_{i=1}^{n}V_i}, \frac{V_2}{\sum_{i=1}^{n}V_i}, \cdots, \frac{V_n}{\sum_{i=1}^{n}V_i}\right),$$

where $V_1, V_2, \cdots, V_n$ are independent gamma random variables with $V_i \sim \Gamma(a, 0, 1)$, $i = 1, 2, \cdots, n$, for $a > 0$, we may rewrite

$$T_n = \alpha\mathcal{D}_{n,3}^2 + \beta\mathcal{D}_{n,5} = \alpha\frac{(\sum_{i=1}^{n}V_i^3)^2}{(\sum_{i=1}^{n}V_i)^6} + \beta\frac{\sum_{i=1}^{n}V_i^5}{(\sum_{i=1}^{n}V_i)^5}.$$

With the above reformulation of T_n, we may consider the joint asymptotic distribution of the three sums $\sum_{i=1}^{n}V_i, \sum_{i=1}^{n}V_i^3, \sum_{i=1}^{n}V_i^5$. By using the multivariate central limit theorem, we obtain, as $n \to \infty$,

$$\sqrt{n}\left(\begin{pmatrix}\frac{1}{n}\sum_{i=1}^{n}V_i \\ \frac{1}{n}\sum_{i=1}^{n}V_i^3 \\ \frac{1}{n}\sum_{i=1}^{n}V_i^5\end{pmatrix} - \begin{pmatrix}\mu_1 \\ \mu_2 \\ \mu_3\end{pmatrix}\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \Sigma)$$

where $\mu_1 = \mathbb{E}(V_i) = a$, $\mu_2 = \mathbb{E}(V_i^3) = \gamma_3(a)$, $\mu_3 = \mathbb{E}(V_i^5) = \gamma_5(a)$, and Σ is the covariance matrix of the random vector (V_i, V_i^3, V_i^5) with

$$\Sigma = \begin{pmatrix} a & 3\gamma_3(a) & 5\gamma_5(a) \\ 3\gamma_3(a) & \gamma_6(a) - \gamma_3^2(a) & \gamma_8(a) - \gamma_3(a)\gamma_5(a) \\ 5\gamma_5(a) & \gamma_8(a) - \gamma_3(a)\gamma_5(a) & \gamma_{10}(a) - \gamma_5^2(a)\end{pmatrix}.$$

Next, we apply Cramér's theorem [9] to obtain

$$\sqrt{n}\left(f\left(\frac{1}{n}\sum_{i=1}^{n}V_i, \frac{1}{n}\sum_{i=1}^{n}V_i^3, \frac{1}{n}\sum_{i=1}^{n}V_i^5\right) - f(\mu)\right) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma_T^2)$$

where f is a mapping: $\mathbb{R}^3 \to \mathbb{R}$ such that $\nabla f(\mu)$ is continuous in a neighborhood of $\mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3$, and $\sigma_F^2 = \nabla f(\mu)\Sigma(\nabla f(\mu))^T$. To do so, we choose $f(x,y,z) = \alpha x^{-6}y^2 + \beta x^{-5}z$ with $\frac{\partial f}{\partial x} = -6\alpha x^{-7}y^2 - 5\beta x^{-6}z$, $\frac{\partial f}{\partial y} = 2\alpha x^{-6}y$, $\frac{\partial f}{\partial z} = \beta x^{-5}$. Then we have

$$f\left(\frac{1}{n}\sum_{i=1}^{n} V_i, \frac{1}{n}\sum_{i=1}^{n} V_i^3, \frac{1}{n}\sum_{i=1}^{n} V_i^5\right) = n^4 T_n$$

with $f(\mu) = \alpha m_3^2 + \beta m_5 = a^{-4}(a+1)(a+2)[\alpha(a+1)(a+2) + \beta(a+3)(a+4)]$, $\nabla f(\mu) = (-6\alpha a^{-1}m_3^2 - 5\beta a^{-1}m_5, 2\alpha a^{-6}\gamma_3(a), \beta a^{-5})$. Then $\sigma_T^2 = \nabla f(\mu)\Sigma(\nabla f(\mu))^T = 4\alpha^2 m_3^2\sigma_3^2 + 4\alpha\beta(m_3 m_8 - (1+15/a)m_3^2 m_5) + \beta^2\sigma_5^2 = 8a^{-9}(a+1)(a+2)[3\alpha^2(a+1)^2(a+2)^2(3a+7) + 5\beta^2(a+3)(a+4)(5a^3+60a^2+250a+363) + 30\alpha\beta(a+1)(a+2)(a+3)^2(a+4)]$. This completes the proof of Lemma 5.

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