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CONVERGENCE OF λ-BERNSTEIN-KANTOROVICH OPERATORS IN THE L_p-NORM

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Abstract. We show the convergence of λ-Bernstein-Kantorovich operators defined by Acu et al. [J. Ineq. Appl. 2018], for functions in $L_p[0,1]$, $p \geq 1$. We also determine the convergence rate via integral modulus of smoothness.

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1. INTRODUCTION

Bernstein [4] gave a marvellous proof of the Weierstrass approximation theorem by defining a sequence of polynomials as follows:

$$\mathcal{B}_m(\varphi; z) = \sum_{\nu=0}^{m} q_{m,\nu}(z)\varphi\Big(\frac{\nu}{m}\Big), \ \ \forall\, z \in I, \ m \in \mathbb{N}, \tag{1}$$

where $q_{m,\nu}(z) = \binom{m}{\nu} z^{\nu}(1-z)^{m-\nu}$, $0 \leq \nu \leq m$, $I = [0,1]$ and $\varphi \in \mathcal{C}(I)$, $\mathcal{C}(I) := \{\phi : \ \phi \text{ is continuous on } I\}$ with the sup-norm $\|.\|_{C(I)}$. Later, many researchers [8, 9, 17, 12, 11], etc. introduced new sequences of operators based on (1) and studied their approximation behaviour for functions in several function spaces. Ye et al. [16] proposed the following Bezier basis through a parameter $\lambda \in [-1,1]$:

$$\begin{aligned}
\tilde{q}_{m,0}(z) &= q_{m,0}(z) - \tfrac{\lambda}{m+1} q_{m+1,1}(z)\\
\tilde{q}_{m,\nu}(z) &= q_{m,\nu}(z) + \lambda\Big(\tfrac{m-2\nu+1}{m^2-1} q_{m+1,\nu}(z) - \tfrac{m-2\nu-1}{m^2-1} q_{m+1,\nu+1}(z)\Big),\\
&\quad 1 \leq \nu \leq m-1\\
\tilde{q}_{m,m}(z) &= q_{m,m}(z) - \tfrac{\lambda}{m+1} q_{m+1,m}(z).
\end{aligned} \tag{2}$$

In the particular case $\lambda = 0$, it is evident that (2) leads us to the Bernstein basis $q_{m,\nu}(z)$, $\forall\, 0 \leq \nu \leq m$. We remark here that addition of the parameter λ

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provides better modeling flexibility to the basis (2). Cai et al. [6] generalized the operators (1) by involving the Bezier basis (2) in the following manner:

$$\mathfrak{B}_m(\varphi; z) = \sum_{\nu=0}^{m} \tilde{q}_{m,\nu}(z)\varphi\Big(\frac{\nu}{m}\Big), \ \ \forall\, z \in I, \ m \in \mathbb{N} \tag{3}$$

and studied some direct approximation results.

For $\varphi \in L_p(I)$, $p \geq 1$, Acu et al. [1] presented a Kantorovich variant of the operators (3) as

$$\mathcal{L}_{m,\lambda}(\varphi; z) = (m+1)\sum_{\nu=0}^{m} \tilde{q}_{m,\nu}(z) \int_{\frac{\nu}{m+1}}^{\frac{\nu+1}{m+1}} \varphi(u)du, \tag{4}$$

and studied a quantitative Voronovskaja type theorem by means of the Ditzian-Totik modulus of smoothness and a Grüss-Voronovskaja type theorem. Rahman et al. [14] investigated the convergence properties of a generalized case of (3) by shifting the nodes, for functions in $C(I)$. Agrawal et al. [2] considered a two dimensional version of the operators defined in [14] and obtained the degree of approximation. Further, the authors [2] also examined the approximation behavior of the associated generalized boolean sum operators. Kumar [10] considered another generalization of (3) along the lines of [13] and discussed some direct theorems in the continuous functions space $C(I)$. Aslan [3] derived the approximation properties for a new class of (3) in the univariate and the bivariate cases. Bodur et al. [5] discussed a Stancu type variant of (3) and established some results in local approximation.

Sucu and Ibikli [15] investigated the convergence of the Bernstein-Stancu-Kantorovich type operators in the spaces $C(I)$ and $L_p(I)$, $p \geq 1$. Encouraged by their work, our objective in this paper is to prove that $\mathcal{L}_{m,\lambda}(\varphi; z)$ converge to $\varphi(z)$ in the L_p-norm, as $m \to \infty$, $\forall\, \varphi \in L_p(I)$, $p \geq 1$ and also estimate the approximation error via integral modulus of smoothness.

2. PRELIMINARIES

In our study, the following results are needed:

LEMMA 1 ([1]). *For all $m > 2$, there hold the inequalities:*

$$|\mathcal{L}_{m,\lambda}(u - z; z)| \leq \Big(\frac{1}{2(m+1)} + \frac{|\lambda|}{m^2-1}\Big);$$

and

$$\Big|\mathcal{L}_{m,\lambda}((u-z)^2; z)\Big| \leq \Big(\frac{3m+4}{12(m+1)^2} + \frac{|\lambda|}{2(m^2-1)}\Big) = \zeta(m, \lambda). \tag{5}$$

THEOREM 2 ([1]). *For $\varphi \in C(I)$, the operators (4) verify*

$$\lim_{m\to\infty} \|\mathcal{L}_{m,\lambda}(\varphi) - \varphi\|_{C(I)} = 0.$$

For $\varphi \in L_p(I)$, $1 \leq p < \infty$, the integral modulus of smoothness is given by

$$\omega_p^1(\varphi; \delta) = \sup_{0<\eta\leq\delta} \|\varphi(.+\eta) - \varphi(.)\|_{L_p(\Omega_\eta)},$$

where $\|.\|_{L_p(\Omega_\eta)}$ is the L_p-norm over the interval $\Omega_\eta = [0, 1-\eta]$.
For $\varphi \in L_p(I)$, $1 \le p < \infty$, the Peetre's K- functional is defined as

$$\mathfrak{K}_p(\varphi;\delta) = \inf_{g \in W_p^1(I)} \left(\|\varphi - g\|_{L_p(I)} + \delta \|g'\|_{L_p(I)} \right),$$

where $W_p^1(I) = \{\psi \in L_p(I) : \psi \text{ is absolutely continuous and } \psi' \in L_p(I)\}$.
It is well known [7, Thm. 2.4, p. 177] that for some positive constants c_1 and c_2, there holds the following relation:

$$c_1 \omega_p^1(\varphi;\delta) \le \mathfrak{K}_p(\varphi;\delta) \le c_2 \omega_p^1(\varphi;\delta). \tag{6}$$

Lemma 3. *For $\psi \in W_p^1(I)$, $p > 1$, we have*

$$\|\mathcal{L}_{m,\lambda}(\psi) - \psi\|_{L_p(I)} \le 2^{\frac{1}{p}} \left(\tfrac{p}{p-1}\right) \sqrt{\zeta(m,\lambda)} \|\psi'\|_{L_p(I)}.$$

where $\zeta(m,\lambda)$ is given by (5).

Proof. For any $z \in I$, we have

$$\begin{aligned} |\mathcal{L}_{m,\lambda}(\psi;z) - \psi(z)| &= (m+1) \left| \sum_{\nu=0}^{m} \tilde{q}_{m,\nu}(z) \int_{\frac{\nu}{m+1}}^{\frac{\nu+1}{m+1}} (\psi(u) - \psi(z))\, dz \right| \\ &\le (m+1) \sum_{\nu=0}^{m} \tilde{q}_{m,\nu}(z) \int_{\frac{\nu}{m+1}}^{\frac{\nu+1}{m+1}} \left| \int_z^u \psi'(t)dt \right| du \\ &\le \theta_{\psi'}(z)(m+1) \sum_{\nu=0}^{m} \tilde{q}_{m,\nu}(z) \int_{\frac{\nu}{m+1}}^{\frac{\nu+1}{m+1}} |u-z| dz, \end{aligned} \tag{7}$$

where

$$\theta_{\psi'}(z) = \sup_{u \in I, u \ne z} \tfrac{1}{|u-z|} \left| \int_z^u \psi'(t)dt \right|,$$

is the Hardy-Littlewood majorant of ψ'. Now, applying the Cauchy-Schwarz inequality to (7) and Lemma 2.1 of [6], one gets

$$\begin{aligned} &|\mathcal{L}_{m,\lambda}(\psi;z) - \psi(z)| \le \\ &\le \theta_{\psi'}(z)\sqrt{(m+1)} \left(\sqrt{\sum_{\nu=0}^{m} \tilde{q}_{m,\nu}(z)} \right) \times \left(\sqrt{\sum_{\nu=0}^{m} \tilde{q}_{m,\nu}(z) \int_{\frac{\nu}{m+1}}^{\frac{\nu+1}{m+1}} (u-z)^2 du} \right) \\ &\le \theta_{\psi'}(z) \max_{z \in I} \left(\sqrt{\mathcal{L}_{m,\lambda}((u-z)^2;z)} \right) \le \theta_{\psi'}(z)\sqrt{\zeta(m,\lambda)}. \end{aligned} \tag{8}$$

Using Hardy-Littlewood theorem (see [18]), one has

$$\int_0^1 (\theta_{\psi'}(z))^p\, dz \le 2 \left(\tfrac{p}{p-1}\right)^p \int_0^1 |\psi'(z)|^p dz, \; p > 1. \tag{9}$$

From (8) and (9), we get

$$\int_0^1 |\mathcal{L}_{m,\lambda}(\psi;z) - \psi(z)|^p\, dz \le \left(\sqrt{\zeta(m,\lambda)} \right)^p \left\{ 2 \left(\tfrac{p}{p-1}\right)^p \int_0^1 |\psi'(z)|^p dz \right\}.$$

Hence,

$$\|\mathcal{L}_{m,\lambda}(\psi)-\psi\|_{L_p(I)} \leq 2^{\frac{1}{p}}\left(\frac{p}{p-1}\right)\sqrt{\zeta(m,\lambda)}\|\psi'\|_{L_p(I)}.$$

□

3. MAIN RESULTS

The following result shows that the operator (4) is an approximation method for functions in $L_p(I)$.

THEOREM 4. *For $\varphi \in L_p(I)$, $1 \leq p < \infty$, the operators (4), verify*

$$\lim_{m\to\infty}\|\mathcal{L}_{m,\lambda}(\varphi)-\varphi\|_{L_p(I)} = 0.$$

Proof. By Luzin theorem, we know that for a given $\epsilon > 0$, $\exists$ a function $g \in C(I)$ satisfying

$$\|\varphi - g\|_{L_p(I)} < \epsilon. \tag{10}$$

From Theorem 2, we have

$$\lim_{m\to\infty}\|\mathcal{L}_{m,\lambda}(g)-g\|_{C(I)} = 0,$$

hence for $\epsilon > 0$, $\exists$ an integer $m_0 \in \mathbb{N}$ in such a way that

$$\|\mathcal{L}_{m,\lambda}(g)-g\|_{C(I)} < \epsilon,\ \forall\ m \geq m_0. \tag{11}$$

Next, we show that $\exists$ a constant $M > 0$ satisfying $\|\mathcal{L}_{m,\lambda}\| \leq M$, for all $m \geq 2$. By Jensen's inequality

$$\begin{aligned}|\mathcal{L}_{m,\lambda}(\varphi;z)|^p &\leq \left\{(m+1)\sum_{\nu=0}^{m}\tilde{q}_{m,\nu}(z)\int_{\frac{\nu}{m+1}}^{\frac{\nu+1}{m+1}}\varphi(u)du\right\}^p\\ &\leq (m+1)\sum_{\nu=0}^{m}\tilde{q}_{m,\nu}(z)\int_{\frac{\nu}{m+1}}^{\frac{\nu+1}{m+1}}|\varphi(u)|^p du.\end{aligned}$$

Hence,

$$\begin{aligned}\int_0^1|\mathcal{L}_{m,\lambda}(\varphi;z)|^p dz &\leq (m+1)\sum_{\nu=0}^{m}\left(\int_0^1\tilde{q}_{m,\nu}(z)dz\right)\int_{\frac{\nu}{m+1}}^{\frac{\nu+1}{m+1}}|\varphi(u)|^p du\\ &\leq 2\|\varphi\|^p_{L_p(I)}, \forall m \geq 2.\end{aligned}$$

Hence,

$$\|\mathcal{L}_{m,\lambda}(\varphi)\|_{L_p(I)} \leq 2^{1/p}\|\varphi\|_{L_p(I)}.$$

Consequently, $\exists$ a constant $M > 0$ satisfying

$$\|\mathcal{L}_{m,\lambda}\| \leq M,\ \forall m \geq 2. \tag{12}$$

Let us define $m' = \max(m_0, 2)$. Then in view of (10)–(12), we have

$$\begin{aligned}\|\mathcal{L}_{m,\lambda}(\varphi)-\varphi\|_{L_p(I)} &\leq \|\mathcal{L}_{m,\lambda}(\varphi)-\mathcal{L}_{m,\lambda}(g)\|_{L_p(I)}+|\mathcal{L}_{m,\lambda}(g)-g\|_{C(I)}+\|\varphi-g\|_{L_p(I)}\\ &\leq (\|\mathcal{L}_{m,\lambda}\|+1)\|\varphi-g\|_{L_p(I)}+|\mathcal{L}_{m,\lambda}(g)-g\|_{C(I)}\\ &\leq (M+2)\epsilon,\ \forall\ m \geq m'.\end{aligned}$$

Due to the arbitrariness of $\epsilon > 0$, the result follows. □

The next result yields the approximation degree for the operators (4) via integral modulus of smoothness.

THEOREM 5. *For $\varphi \in L_p(I)$, $p > 1$, the operators $\mathcal{L}_{m,\lambda}$ verify the following inequality*

$$\|\mathcal{L}_{m,\lambda}(\varphi) - \varphi\|_{L_p(I)} \leq C\omega_p^1(\varphi; \sqrt{\zeta(m,\lambda)}),$$

where $\zeta(m,\lambda)$ is given by (5) and the constant C is independent of φ and m.

Proof. From the proof of Theorem 4 and Lemma 3, we have

$$\|\mathcal{L}_{m,\lambda}(g) - g\|_{L_p(I)} \leq \begin{cases} 3\|g\|_{L_p(I)}, & g \in L_p(I) \\ 2^{\frac{1}{p}}\left(\frac{p}{p-1}\right)\sqrt{\zeta(m,\lambda)}\|g'\|_{L_p(I)}, & g \in W_p^1(I) \end{cases}.$$

Then for $\varphi \in L_p(I)$ and any $\psi \in W_p^1(I)$, we may write

$$\begin{aligned} \|\mathcal{L}_{m,\lambda}(\varphi) - \varphi\|_{L_p(I)} &\leq \|\mathcal{L}_{m,\lambda}(\varphi - \psi) - (\varphi - \psi)\|_{L_p(I)} + \|\mathcal{L}_{m,\lambda}(\psi) - \psi\|_{L_p(I)} \\ &\leq 3\left(\|\varphi - \psi\|_{L_p(I)} + 2^{\frac{1}{p}}\left(\frac{p}{p-1}\right)\sqrt{\zeta(m,\lambda)}\|\psi'\|_{L_p(I)}\right). \end{aligned}$$

Hence due to the arbitrariness of $\psi \in W_p^1(I)$, we get

$$\|\mathcal{L}_{m,\lambda}(\varphi) - \varphi\|_{L_p(I)} \leq 3\mathfrak{K}\left(\varphi; 2^{\frac{1}{p}}\left(\frac{p}{p-1}\right)\sqrt{\zeta(m,\lambda)}\right).$$

Finally using (6), we have

$$\begin{aligned} \|\mathcal{L}_{m,\lambda}(\varphi) - \varphi\|_{L_p(I)} &\leq 3c_2\omega_p^1\left(\varphi; 2^{\frac{1}{p}}\left(\frac{p}{p-1}\right)\sqrt{\zeta(m,\lambda)}\right) \\ &\leq 3c_2\left(1 + 2^{\frac{1}{p}}\left(\frac{p}{p-1}\right)\right)\omega_p^1(\varphi; \sqrt{\zeta(m,\lambda)}) \\ &\leq C\omega_p^1(\varphi; \sqrt{\zeta(m,\lambda)}), \end{aligned}$$

whence the result follows. □

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