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A STANCU TYPE EXTENSION OF THE CHENEY-SHARMA CHLODOVSKY OPERATORS

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Abstract. In this paper we introduce a Stancu type extension of the Cheney-Sharma Chlodovsky operators based on the ideas presented by Cătinaş and Buda, Bostanci and Başcanbaz-Tunca, respectively Söylemez and Taşdelen. For this new operators we study some approximation and convexity properties and the preservation of the Lipschitz constant and order. Finally, we study approximation properties of the new operators with the help of Korovkin type theorems.

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1. INTRODUCTION

In 1932, I. Chlodovsky (see [10] and [13]) introduced the classical Bernstein-Chlodovsky polynomials as a generalization of Bernstein polynomials on unbounded set. For every $n \in \mathbb{N}$, $f \in C[0,\infty)$ continuous and $x \in [0,\infty)$ the polynomials C_n are defined by

$$C_n(f;x) = \begin{cases} \displaystyle\sum_{k=0}^{n} f\left(\tfrac{k}{n}\lambda_n\right)\binom{n}{k}\left(\tfrac{x}{\lambda_n}\right)^k\left(1-\tfrac{x}{\lambda_n}\right)^{n-k}, & 0 \le x \le \lambda_n \\ f(x), & x > \lambda_n, \end{cases} \tag{1}$$

where $(\lambda_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers such that $\lim\limits_{n\to\infty} \lambda_n = \infty$ and $\lim\limits_{n\to\infty} (\lambda_n/n) = 0$. This type of polynomials was recently used by Söylemez and Taşdelen in [14] and [15] in the study of Cheney-Sharma Chlodovsky operators.

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Remark 1. *Let $u, v, \beta \in \mathbb{R}$. Then the Abel-Jensen equalities (see [1], [6] or [14]) are given by the following formulae:*

$$(u+v+m\beta)^m = \sum_{k=0}^{m} \binom{m}{k} u(u+k\beta)^{k-1}[v+(m-k)\beta]^{m-k} \tag{2}$$

and

$$(u+v)(u+v+m\beta)^{m-1} = \sum_{k=0}^{m} \binom{m}{k} u(u+k\beta)^{k-1} v[v+(m-k)\beta]^{m-k-1} \tag{3}$$

for $m, n \in \mathbb{N}$.

Based on the equalities presented above, Cheney and Sharma (see [9] and [15]) introduced the following Bernstein type operators for every $n \in \mathbb{N}$, $f \in C[0,1]$ continuous and $x \in [0,1]$:

Definition 2. *Let $f \in C[0,1]$, $x \in [0,1]$, $n \in \mathbb{N}$ and $\beta \geq 0$. Then*

- *the operator P_n^β is defined by*

$$(P_n^\beta f)(x) = \sum_{k=0}^{n} p_{n,k}^\beta(x) f\left(\frac{k}{n}\right), \tag{4}$$

  where

$$p_{n,k}^\beta(x) = \binom{n}{k} \frac{x(x+k\beta)^{k-1}\left[1-x+(n-k)\beta\right]^{n-k}}{(1+n\beta)^n},$$

  respectively,

- *the operator Q_n^β is defined by*

$$(Q_n^\beta f)(x) = \sum_{k=0}^{n} q_{n,k}^\beta(x) f\left(\frac{k}{n}\right), \tag{5}$$

  where

$$q_{n,k}^\beta(x) = \binom{n}{k} \frac{x(x+k\beta)^{k-1}(1-x)\left[1-x+(n-k)\beta\right]^{n-k-1}}{(1+n\beta)^{n-1}}.$$

  These two operators generalized the well-known Bernstein operator, given by

$$(B_n f)(x) = \sum_{k=0}^{\infty} p_{n,k}(x) f\left(\frac{k}{n}\right),$$

  where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$.

Remark 3. *It is important to mention that, more recent, the operators P_n^β and Q_n^β were also studied on different domains in higher dimensions. In [4] and [5] the authors present some extension of the Cheney-Sharma type operators on a triangle with straight sides, respectively with one curved side. Other properties on such domains are studied in [7] and [8] in terms of iterates of multivariate Cheney-Sharma type operators.*

Taking into account the operators defined by Cheney and Sharma and also the idea introduced by Stancu in [16], Cătinaş and Buda defined in [6] the Stancu type extension of the Cheney-Sharma operator of the first kind $P_{n,r}$ by

$$(L^{\beta}_{P_{n,r}} f)(x) = \sum_{k=0}^{n-r} p^{\beta}_{n-r,k}(x)\Big[(1-x)f\Big(\tfrac{k}{n}\Big) + xf\Big(\tfrac{k+r}{n}\Big)\Big], \tag{6}$$

where $p^{\beta}_{n-r,k}$ is taken as in formula (4). On the other hand, Bostanci and Başcanbaz-Tunca considered in [3] a Stancu type extension of the Cheney-Sharma operator (of the second kind) $Q_{n,r}$, given by

$$(L^{\beta}_{Q_{n,r}} f)(x) = \sum_{k=0}^{n-r} q^{\beta}_{n-r,k}(x)\Big[(1-x)f\Big(\tfrac{k}{n}\Big) + xf\Big(\tfrac{k+r}{n}\Big)\Big], \tag{7}$$

where $q^{\beta}_{n-r,k}$ is defined by (5). Hence, we have a extension of both operators defined by Cheney and Sharma. It is important to mention that these two operators are generalizations of the Stancu operator (see [16]) given by

$$(S_{n,r} f)(x) = \sum_{k=0}^{n-r} b_{n-r,k}(x)\Big[(1-x)f\Big(\tfrac{k}{n}\Big) + xf\Big(\tfrac{k+r}{n}\Big)\Big],$$

where $b_{n-r,k}(x) = \binom{n-r}{k}x^k(1-x)^{n-r-k}$. Indeed, if $\beta = 0$, then $L^{0}_{P_{n,r}} = L^{0}_{Q_{n,r}} = S_{n,r}$, since $p^{0}_{n-r,k} = q^{0}_{n-r,k} = b_{n-r,k}$. On the other hand, if $r = 0$, then the operators $L^{\beta}_{P_{n,r}}$ and $L^{\beta}_{P_{n,r}}$ reduces to the classical Cheney-Sharma operators P^{β}_{n}, respectively Q^{β}_{n} defined above. For more details, one may consult also [3], [6], [9], [15], [16] and [17].

2. THE STANCU TYPE EXTENSION OF THE CHENEY-SHARMA CHLODOVSKY OPERATORS

In the second section of our paper we combine the ideas presented by Bostanci and Başcanbaz-Tunca in [3], respectively by Cătinaş and Buda in [6], with the results presented by Söylemez and Taşdelen in [15]. Hence, we construct a Stancu type extension of the Cheney-Sharma Chlodovsky operators and study its properties.

Definition 4. *Let $n, k, r \in \mathbb{N}$ be such that $n > 2r$ and let $\beta > 0$ be a real number. Also, let (λ_n) be a positive sequence such that $\lim_{n\to\infty} \lambda_n = \infty$ and $\lim_{n\to\infty} (\lambda_n/n) = 0$. For every $x \in [0, \lambda_n]$ we define*

- *the Stancu type extension of the first Cheney-Sharma Chlodovsky operator, given by the formula*

$$\mathcal{P}^{*}_{n,r}(f;x) = \sum_{k=0}^{n-r} p^{*}_{n-r,k}(x)\Big[\Big(1-\tfrac{x}{\lambda_n}\Big)f\Big(\tfrac{k}{n}\lambda_n\Big) + \tfrac{x}{\lambda_n}f\Big(\tfrac{k+r}{n}\lambda_n\Big)\Big], \tag{8}$$

where

$$p^*_{n,k}(x) = \binom{n}{k} \frac{\frac{x}{\lambda_n}\left(\frac{x}{\lambda_n}+k\beta\right)^{k-1}\left(1-\frac{x}{\lambda_n}+(n-k)\beta\right)^{n-k}}{(1+n\beta)^{n-1}}$$

respectively,

- *the Stancu type extension of the second Cheney-Sharma Chlodovsky operator, given by the formula*

$$\mathcal{Q}^*_{n,r}(f;x) = \sum_{k=0}^{n-r} q^*_{n-r,k}(x)\left[\left(1-\frac{x}{\lambda_n}\right)f\left(\frac{k}{n}\lambda_n\right) + \frac{x}{\lambda_n} f\left(\frac{k+r}{n}\lambda_n\right)\right], \tag{9}$$

where

$$q^*_{n,k}(x) = \binom{n}{k} \frac{\frac{x}{\lambda_n}\left(\frac{x}{\lambda_n}+k\beta\right)^{k-1}\left(1-\frac{x}{\lambda_n}\right)\left(1-\frac{x}{\lambda_n}+(n-k)\beta\right)^{n-k-1}}{(1+n\beta)^{n-1}}.$$

Remark 5. *It is not difficult to observe that*

- *if $r=0$, then the operator $\mathcal{Q}^*_{n,r}$ given by formula* (9) *reduces to the Cheney-Sharma Chlodovsky operator G^*_n defined by Söylemez and Taşdelen in* [15]*, as follows*

$$G^*_n(f;x) = \sum_{k=0}^{n} q^*_{n,k}(x) f\left(\frac{k}{n}\lambda_n\right), \tag{10}$$

for $0 \le x \le \lambda_n$ and $f(x)$ for $x > \lambda_n$ (see formula (1.5) in [15]*), where (λ_n) is a positive sequence such that $\lim_{n\to\infty} \lambda_n = \infty$ and $\lim_{n\to\infty} (\lambda_n/n) = 0$;*

- *on the other hand, for the particular case $\lambda_n = 1$, we obtain the operators $L^{\beta}_{P_{n,r}}$, respectively $L^{\beta}_{Q_{n,r}}$, considered by Cătinaş and Buda in* [6]*, respectively by Bostanci and Başcanbaz-Tunca in* [3]*;*
- *finally, if $r=0$ and $\lambda_n = 1$, then the operators $\mathcal{P}^*_{n,r}$ and $\mathcal{Q}^*_{n,r}$ reduce to the Cheney-Sharma operators P^{β}_n and Q^{β}_n defined by Cheney and Sharma in* [9] *(see also formulas* (4) *and* (5) *in the first section).*

Notice that all the operators presented above are generalizations of the Bernstein operator B_n from Definition 2. Moreover, some of them are Stancu type extensions of the operator B_n (see [17]*).*

Related to the operator G^*_n presented in formula (10) we have the following expressions of the moments of G^*_n (cf. [15, Lemma 2.3]):

Lemma 6 ([15, Lemma 2.3]). *For every $x \in [0, \lambda_n]$ and for the operator given by relation* (10)*, we have that*

$$G^*_n(e_0;x) = 1,$$
$$G^*_n(e_1;x) = x$$

and

$$G^*_n(e_2;x) \le x(x+2\lambda_n\beta)(1+n\beta) + x\frac{\lambda_n}{n}(n\beta)^2(1+n\beta) + x(x+2\lambda_n\beta)n\beta + x\frac{\lambda_n}{n}(n\beta)^3 + x\frac{\lambda_n}{n}.$$

It is important to observe that following the ideas presented in the proof of [15, Lemma 2.3], we can prove a similar result for the operator $\mathcal{Q}^*_{n,r}$.

2.1. Properties of the operator $\mathcal{Q}^*_{n,r}$. Taking into account the previous result (proved by Söylemez and Taşdelen in [15]) we can obtain similar properties for the Stancu type extension of the Cheney-Sharma Chlodovsky operator $\mathcal{Q}^*_{n,r}$ as can be seen in the following two results. Notice that in the proofs of our results we follow arguments similar to those for Lemma 2.3. in [15].

PROPOSITION 7. *For every $x \in [0, \lambda_n]$ and $n, r \in \mathbb{N}$ with $n > 2r$ we have*

i) $\mathcal{Q}^*_{n,r}(e_0; x) = 1$
ii) $\mathcal{Q}^*_{n,r}(e_1; x) = x$.

Proof. Let $x \in [0, \lambda_n]$ and $n, r \in \mathbb{N}$ be such that $n > 2r$. Next, we give a complete proof of this result (for an arbitrary $r \in \mathbb{N}$) following the ideas presented by Söylemez and Taşdelen in [15].

i) Taking into account that $e_0(x) = 1$ for all $x \geq 0$, we have that

$$\begin{aligned}\mathcal{Q}^*_{n,r}(e_0; x) &= \sum_{k=0}^{n-r} q^*_{n-r,k}(x)\left(1 - \tfrac{x}{\lambda_n} + \tfrac{x}{\lambda_n}\right)\\ &= \sum_{k=0}^{n-r} \binom{n-r}{k} \frac{\frac{x}{\lambda_n}\left(\frac{x}{\lambda_n}+k\beta\right)^{k-1}\left(1-\frac{x}{\lambda_n}\right)\left(1-\frac{x}{\lambda_n}+(n-r-k)\beta\right)^{n-r-k-1}}{[1+(n-r)\beta]^{n-r-1}}.\end{aligned}$$

If we denote by $u = \frac{x}{\lambda_n}$, $v = 1 - \frac{x}{\lambda_n}$ and $m = n - r$, then

$$\begin{aligned}\mathcal{Q}^*_{n,r}(e_0; x) &= [1 + m\beta]^{1-m} \sum_{k=0}^{m} \binom{m}{k} u(u + k\beta)^{k-1} v[v + (m - k)\beta]^{m-k-1}\\ &= [1 + m\beta]^{1-m}[1 + m\beta]^{m-1} = 1,\end{aligned}$$

in view of equality (3). Hence,

$$\mathcal{Q}^*_{n,r}(e_0; x) = 1. \tag{11}$$

ii) Next, let us consider $e_1(x) = x$ for all $x \geq 0$. Then

$$\begin{aligned}\mathcal{Q}^*_{n,r}(e_1; x) &= \sum_{k=0}^{n-r} q^*_{n-r,k}(x)\left[\left(1 - \tfrac{x}{\lambda_n}\right)\tfrac{k}{n}\lambda_n + \tfrac{x}{\lambda_n}\tfrac{k+r}{n}\lambda_n\right]\\ &= \sum_{k=0}^{n-r} q^*_{n-r,k}(x)\left(\tfrac{k}{n}\lambda_n + \tfrac{xr}{n}\right)\\ &= \tfrac{n-r}{n}\sum_{k=0}^{n-r} \tfrac{k}{n-r}\lambda_n q^*_{n-r,k}(x) + x\tfrac{r}{n}\sum_{k=0}^{n-r} q^*_{n-r,k}(x)\\ &= \tfrac{n-r}{n}\sum_{k=0}^{n-r} \tfrac{k}{n-r}\lambda_n q^*_{n-r,k}(x) + x\tfrac{r}{n}\mathcal{Q}^*_{n,r}(e_0; x)\end{aligned}$$

$$= \tfrac{n-r}{n} \sum_{k=0}^{n-r} \tfrac{k}{n-r} \lambda_n q^*_{n-r,k}(x) + x\tfrac{r}{n},$$

in view of relation (11). Let us denote by

$$A(x) = \tfrac{n-r}{n} \sum_{k=0}^{n-r} \tfrac{k}{n-r} \lambda_n q^*_{n-r,k}(x).$$

Then

$$\mathcal{Q}^*_{n,r}(e_1;x) = A(x) + x\tfrac{r}{n},$$

where

$$\begin{aligned} A(x) &= \tfrac{n-r}{n} \sum_{k=0}^{n-r} \tfrac{k}{n-r} \lambda_n q^*_{n-r,k}(x) \\ &= [1+(n-r)\beta]^{1-n+r} \tfrac{n-r}{n} \sum_{k=1}^{n-r-1} \tbinom{n-r-1}{k-1} \lambda_n \tfrac{x}{\lambda_n} \Big(\tfrac{x}{\lambda_n} + k\beta\Big)^{k-1} \\ &\quad \times \Big(1 - \tfrac{x}{\lambda_n}\Big)\Big(1 - \tfrac{x}{\lambda_n} + (n-r-k)\beta\Big)^{n-r-k-1}. \end{aligned}$$

If we replace k by $k+1$, then we can write

$$\begin{aligned} A(x) = [1+(n-r)\beta]^{1-n+r} \tfrac{(n-r)\lambda_n}{n} \sum_{k=0}^{n-r-1} \tbinom{n-r-1}{k} \tfrac{x}{\lambda_n} \\ \times \Big(\tfrac{x}{\lambda_n} + (k+1)\beta\Big)^{k} \Big(1 - \tfrac{x}{\lambda_n}\Big)\Big(1 - \tfrac{x}{\lambda_n} + (n-r-k-1)\beta\Big)^{n-r-k-2} \end{aligned}$$

and then

$$\begin{aligned} A(x) = [1+(n-r)\beta]^{1-n+r} \lambda_n \sum_{k=0}^{n-r-1} \tfrac{x}{\lambda_n} \tbinom{n-r-1}{k} \Big(\tfrac{x}{\lambda_n} + \beta + k\beta\Big)^{k-1} \\ \times [1-(n-r)\beta]\Big(1 - \tfrac{x}{\lambda_n}\Big)\Big(1 - \tfrac{x}{\lambda_n} + (n-r-k-1)\beta\Big)^{n-k-r-2} \\ - \tfrac{n-r}{n}[1+(n-r)\beta]^{1-n+r} \lambda_n \sum_{k=0}^{n-r-1} \tfrac{x}{\lambda_n} \tbinom{n-r-1}{k} \\ \times \Big(\tfrac{x}{\lambda_n} + \beta + k\beta\Big)^{k-1} \Big(1 - \tfrac{x}{\lambda_n}\Big)\Big(1 - \tfrac{x}{\lambda_n} + (n-r-k-1)\beta\Big)^{n-k-r-1}. \end{aligned}$$

If, in relations (2) and (3), we consider $u = \tfrac{x}{\lambda_n} + \beta$, $v = 1 - \tfrac{x}{\lambda_n}$ and $m = n-r-1$, then

$$\begin{aligned} (1+\beta)[1+(n-r)\beta]^{n-r-2} = \tfrac{x+\beta\lambda_n}{\lambda_n} \sum_{k=0}^{n-r-1} \tbinom{n-r-1}{k} \Big(\tfrac{x}{\lambda_n} + \beta + k\beta\Big)^{k-1} \\ \times \Big(1 - \tfrac{x}{\lambda_n}\Big)\Big(1 - \tfrac{x}{\lambda_n} + (n-r-k-1)\beta\Big)^{n-r-k-2} \end{aligned}$$

and

$$[1+(n-r)\beta]^{n-r-1} = \tfrac{x+\beta\lambda_n}{\lambda_n}\sum_{k=0}^{n-r-1}\binom{n-r-1}{k}\left(\tfrac{x}{\lambda_n}+\beta+k\beta\right)^{k-1} \times\left(1-\tfrac{x}{\lambda_n}+(n-r-k-1)\beta\right)^{n-r-k-1}.$$

Finally, $A(x)$ reduces to

$$A(x) = \tfrac{n-r}{n}\left[(1+\beta)\tfrac{x\lambda_n}{x+\lambda_n\beta} - \left(1-\tfrac{x}{\lambda_n}\right)\tfrac{x\lambda_n}{x+\lambda_n\beta}\right] = \tfrac{n-r}{n}x.$$

and we obtain that

$$\begin{aligned}\mathcal{Q}^*_{n,r}(e_1;x) &= A(x)+\tfrac{r}{n}x\\ &= \tfrac{n-r}{n}x+\tfrac{r}{n}x\\ &= x,\end{aligned} \tag{12}$$

as desired.

□

REMARK 8. *It is clear that if we consider $r=0$ in the previous result, then Proposition 7 reduces to Lemma 2.3. from* [15].

LEMMA 9. *For every $x\in[0,\lambda_n]$ and $n,r\in\mathbb{N}$ with $n>2r$ we have*

$$\begin{aligned}&\mathcal{Q}^*_{n,r}(e_2;x)\le\\ &\le \tfrac{(n-r)^2}{n^2}\left\{\left[x+2\lambda_n\beta+\lambda_n\beta^2(n-r)\right][1+2(n-r)\beta]+\tfrac{(n-r)(\lambda_n+2rx)+r^2\lambda_n}{(n-r)^2}\right\}x.\end{aligned} \tag{13}$$

Proof. From Definition 4 and Lemma 6 we know that

$$\begin{aligned}&\mathcal{Q}^*_{n,r}(e_2;x) =\\ &= \sum_{k=0}^{n-r} q^*_{n-r,k}(x)\left[\left(1-\tfrac{x}{\lambda_n}\right)\tfrac{k^2}{n^2}\lambda_n^2+\tfrac{x}{\lambda_n}\left(\tfrac{k+r}{n}\right)^2\lambda_n^2\right]\\ &= \sum_{k=0}^{n-r} q^*_{n-r,k}(x)\left[\tfrac{k^2}{n^2}\lambda_n^2+\tfrac{2xkr}{n^2}\lambda_n+\tfrac{xr^2}{n^2}\lambda_n\right]\\ &= \tfrac{(n-r)^2}{n^2}\sum_{k=0}^{n-r} q^*_{n-r,k}(x)\tfrac{k^2}{(n-r)^2}\lambda_n^2+\tfrac{2rx(n-r)}{n^2}\sum_{k=0}^{n-r} q^*_{n-r,k}(x)\tfrac{k}{n-r}\lambda_n\\ &\quad+\tfrac{xr^2\lambda_n}{n^2}\sum_{k=0}^{n-r} q^*_{n-r,k}(x)\\ &= \tfrac{(n-r)^2}{n^2}G^*_{n-r}(e_2;x)+\tfrac{2r(n-r)}{n^2}xG^*_{n-r}(e_1;x)+\tfrac{r^2}{n^2}x\lambda_n G^*_{n-r}(e_0;x)\\ &= \tfrac{(n-r)^2}{n^2}G^*_{n-r}(e_2;x)+\tfrac{2r(n-r)}{n^2}x^2+\tfrac{r^2}{n^2}x.\end{aligned}$$

Simple computations lead to

$$\mathcal{Q}^*_{n,r}(e_2;x) \leq$$
$$\leq \tfrac{(n-r)^2}{n^2}\Big\{\big[x+2\lambda_n\beta+\lambda_n\beta^2(n-r)\big]\big[1+2(n-r)\beta\big]+\tfrac{(n-r)(\lambda_n+2rx)+r^2\lambda_n}{(n-r)^2}\Big\}x$$

and this completes the proof. □

Proposition 10. *For every* $f\in C[0,\lambda_n]$*, we have that*

$$\|\mathcal{Q}^*_{n,r}f\| \leq \|f\|, \tag{14}$$

where $\|\cdot\|$ *is the uniform norm on the space* $C[0,\lambda_n]$*.*

Proof. Taking into account the expression of the operator $\mathcal{Q}^*_{n,r}$ given by formula (9) and Proposition 7, we obtain that

$$\begin{aligned}
|\mathcal{Q}^*_{n,r}(f;x)| &= \Big|\sum_{k=0}^{n-r} q^*_{n-r,k}(x)\Big[\Big(1-\tfrac{x}{\lambda_n}\Big)f\Big(\tfrac{k}{n}\lambda_n\Big)+\tfrac{x}{\lambda_n}f\Big(\tfrac{k+r}{n}\lambda_n\Big)\Big]\Big|\\
&\leq \sum_{k=0}^{n-r} q^*_{n-r,k}(x)\Big|\Big(1-\tfrac{x}{\lambda_n}\Big)f\Big(\tfrac{k}{n}\lambda_n\Big)+\tfrac{x}{\lambda_n}f\Big(\tfrac{k+r}{n}\lambda_n\Big)\Big|\\
&\leq \sum_{k=0}^{n-r} q^*_{n-r,k}(x)\Big[\Big(1-\tfrac{x}{\lambda_n}\Big)\Big|f\Big(\tfrac{k}{n}\lambda_n\Big)\Big|+\tfrac{x}{\lambda_n}\Big|f\Big(\tfrac{k+r}{n}\lambda_n\Big)\Big|\Big]\\
&\leq \|f\|\sum_{k=0}^{n-r} q^*_{n-r,k}(x)\Big(1-\tfrac{x}{\lambda_n}+\tfrac{x}{\lambda_n}\Big)\\
&= \|f\|\mathcal{Q}^*_{n,r}(e_0;x)\\
&= \|f\|
\end{aligned}$$

and this completes the proof. □

Based on the proofs of the results presented in [3, Theorem 3.2] and [14, Theorem 2], we can prove the following result that provides the property of the preservation of Lipschitz constant and order of a Lipschitz continuous function by each operator $\mathcal{Q}^*_{n,r}$. Hence, let us denote by

$$\mathrm{Lip}_M(\alpha,A)=\Big\{f\in C(A): |f(x)-f(y)|\leq M|x-y|^\alpha, \forall x,y\in A, \alpha\in(0,1]\Big\},$$

where $A\subseteq[0,\infty)$, $M>0$ is a positive constant and $0<\alpha\leq 1$.

Theorem 11. *If* $f\in \mathrm{Lip}_M(\alpha,[0,\lambda_n])$*, then* $\mathcal{Q}^*_{n,r}(f;x)\in \mathrm{Lip}_M(\alpha,[0,\lambda_n])$*.*

Proof. Without loss of generality, let us consider $x,y\in[0,\lambda_n]$ be such that $y\geq x$. Following similar ideas to those presented in the proofs of [3, Theorem 3.2] and [14, Theorem 2], we deduce that

$$\mathcal{Q}^*_{n,r}(f;y)=\sum_{k=0}^{n-r} q^*_{n-r,k}(y)\Big[\Big(1-\tfrac{y}{\lambda_n}\Big)f\Big(\tfrac{k}{n}\lambda_n\Big)+\tfrac{y}{\lambda_n}f\Big(\tfrac{k+r}{n}\lambda_n\Big)\Big]$$

$$= \tfrac{1}{[1+(n-r)\beta]^{n-r-1}} \sum_{m=0}^{n-r} \sum_{k=0}^{m} \tbinom{n-r}{m} \tbinom{m}{k} \tfrac{x}{\lambda_n} \left(\tfrac{x}{\lambda_n} + k\beta\right)^{k-1}$$
$$\times \tfrac{y-x}{\lambda_n} \left[\tfrac{y-x}{\lambda_n} + (m-k)\beta\right]^{m-k-1} \left(1 - \tfrac{y}{\lambda_n}\right)$$
$$\times \left[1 - \tfrac{y}{\lambda_n} + (n-r-m)\beta\right]^{n-r-m-1}$$
$$\times \left[\left(1 - \tfrac{y}{\lambda_n} f\left(\tfrac{m}{n}\lambda_n\right) + \tfrac{y}{\lambda_n} f\left(\tfrac{m+r}{n}\lambda_n\right)\right],$$

according to relations (3) and (9). Next, let us change the order of the summation and letting $m - k = j$ in the previous relation. Then

$$\mathcal{Q}^*_{n,r}(f;y) = \tfrac{1}{[1+(n-r)\beta]^{n-r-1}} \sum_{k=0}^{n-r} \sum_{j=0}^{n-r-k} \tfrac{(n-r)!}{(n-r-k-j)!k!j!} \tfrac{x}{\lambda_n}$$
$$\times \left(\tfrac{x}{\lambda_n} + k\beta\right)^{k-1} \tfrac{y-x}{\lambda_n} \left(\tfrac{y-x}{\lambda_n} + j\beta\right)^{j-1} \left(1 - \tfrac{y}{\lambda_n}\right)$$
$$\times \left[(1 - \tfrac{y}{\lambda_n} + (n-r-k-j)\beta\right]^{n-r-k-j-1}$$
$$\times \left[\left(1 - \tfrac{y}{\lambda_n} f\left(\tfrac{k+j}{n}\lambda_n\right) + \tfrac{y}{\lambda_n} f\left(\tfrac{k+j+r}{n}\lambda_n\right)\right].$$

If we consider $u = y - x$, $v = 1 - y$ and $m = n - r - k$ in formula (3), then we can obtain the expression of $\mathcal{Q}^*_{n,r}(f;x)$ as follows

$$\mathcal{Q}^*_{n,r}(f;x) =$$
$$= \tfrac{1}{[1+(n-r)\beta]^{n-r-1}} \sum_{k=0}^{n-r} \sum_{j=0}^{n-r-k} \tfrac{(n-r)!}{(n-r-k-j)!k!j!} \tfrac{x}{\lambda_n} \left(\tfrac{x}{\lambda_n}+k\beta\right)^{k-1} \tfrac{y-x}{\lambda_n} \left(\tfrac{y-x}{\lambda_n}+j\beta\right)^{j-1} \left(1-\tfrac{y}{\lambda_n}\right)$$
$$\times \left[(1 - \tfrac{y}{\lambda_n} + (n-r-k-j)\beta\right]^{n-r-k-j-1} \left[\left(1 - \tfrac{x}{\lambda_n} f\left(\tfrac{k}{n}\lambda_n\right) + \tfrac{x}{\lambda_n} f\left(\tfrac{k+r}{n}\lambda_n\right)\right].$$

Finally, we deduce that

$$\mathcal{Q}^*_{n,r}(f;y) - \mathcal{Q}^*_{n,r}(f;x) =$$
$$= \tfrac{1}{[1+(n-r)\beta]^{n-r-1}} \sum_{k=0}^{n-r} \sum_{j=0}^{n-r-k} \tfrac{(n-r)!}{(n-r-k-j)!k!j!} \tfrac{x}{\lambda_n} \left(\tfrac{x}{\lambda_n} + k\beta\right)^{k-1} \tfrac{y-x}{\lambda_n} \left(\tfrac{y-x}{\lambda_n} + j\beta\right)^{j-1}$$
$$\times \left(1-\tfrac{y}{\lambda_n}\right) \left[(1-\tfrac{y}{\lambda_n}+(n-r-k-j)\beta\right]^{n-r-k-j-1} \left\{ \left(1-\tfrac{y}{\lambda_n}\right) \left[f\left(\tfrac{k+j}{n}\lambda_n\right) - f\left(\tfrac{k}{n}\lambda_n\right)\right]\right.$$
$$\left. + \tfrac{x}{\lambda_n} \left[f\left(\tfrac{k+j+r}{n}\lambda_n\right) - f\left(\tfrac{k+r}{n}\lambda_n\right)\right] + \tfrac{y-x}{\lambda_n} \left[f\left(\tfrac{k+j+r}{n}\lambda_n\right) - f\left(\tfrac{k}{n}\lambda_n\right)\right] \right\}.$$

Taking into account the hypothesis that $f \in \mathrm{Lip}_M(\alpha)$ and the assumption that $y \geq x$, we obtain the inequality

$$\left|\mathcal{Q}^*_{n,r}(f;y) - \mathcal{Q}^*_{n,r}(f;x)\right| \leq$$

$$
\begin{aligned}
&\le \tfrac{1}{[1+(n-r)\beta]^{n-r-1}} \sum_{k=0}^{n-r} \sum_{j=0}^{n-r-k} \tfrac{(n-r)!}{(n-r-k-j)!k!j!} \tfrac{x}{\lambda_n}\Big(\tfrac{x}{\lambda_n}+k\beta\Big)^{k-1} \tfrac{y-x}{\lambda_n}\Big(\tfrac{y-x}{\lambda_n}+j\beta\Big)^{j-1} \\
&\quad \times \Big(1-\tfrac{y}{\lambda_n}\Big)\Big[(1-\tfrac{y}{\lambda_n}+(n-r-k-j)\beta\Big]^{n-r-k-j-1} \\
&\quad \times \Big\{\Big(1-\tfrac{y}{\lambda_n}\Big)\Big|f\Big(\tfrac{k+j}{n}\lambda_n\Big)-f\Big(\tfrac{k}{n}\lambda_n\Big)\Big| + \tfrac{x}{\lambda_n}\Big|f\Big(\tfrac{k+j+r}{n}\lambda_n\Big)-f\Big(\tfrac{k+r}{n}\lambda_n\Big)\Big| \\
&\quad + \tfrac{y-x}{\lambda_n}\Big|f\Big(\tfrac{k+j+r}{n}\lambda_n\Big)-f\Big(\tfrac{k}{n}\lambda_n\Big)\Big|\Big\} \\
&\le \tfrac{M}{[1+(n-r)\beta]^{n-r-1}} \sum_{k=0}^{n-r} \sum_{j=0}^{n-r-k} \tfrac{(n-r)!}{(n-r-k-j)!k!j!} \\
&\quad \times \tfrac{x}{\lambda_n}\Big(\tfrac{x}{\lambda_n}+k\beta\Big)^{k-1} \tfrac{y-x}{\lambda_n}\Big(\tfrac{y-x}{\lambda_n}+j\beta\Big)^{j-1} \\
&\quad \times \Big(1-\tfrac{y}{\lambda_n}\Big)\Big[(1-\tfrac{y}{\lambda_n}+(n-r-k-j)\beta\Big]^{n-r-k-j-1} \\
&\quad \times \Big[\Big(1-\tfrac{y-x}{\lambda_n}\Big)\Big(\tfrac{j}{n}\lambda_n\Big)^{\alpha} + \tfrac{y-x}{\lambda_n}\Big(\tfrac{j+r}{n}\lambda_n\Big)^{\alpha}\Big]
\end{aligned}
$$

that leads to

$$
\begin{aligned}
&\big|\mathcal{Q}^*_{n,r}(f;y)-\mathcal{Q}^*_{n,r}(f;x)\big| \le \\
&\le \tfrac{M}{[1+(n-r)\beta]^{n-r-1}} \sum_{k=0}^{n-r} (n-rk)\, \tfrac{y-x}{\lambda_n}\Big(\tfrac{y-x}{\lambda_n}+k\beta\Big)^{k-1}\Big(1-\tfrac{y-x}{\lambda_n}\Big) \\
&\quad \times \Big[1-\tfrac{y-x}{\lambda_n}+(n-r-k)\beta\Big]^{n-r-k-1}\Big[\Big(1-\tfrac{y-x}{\lambda_n}\Big)\Big(\tfrac{k}{n}\lambda_n\Big)^{\alpha} + \tfrac{y-x}{\lambda_n}\Big(\tfrac{k+r}{n}\Big)^{\alpha}\Big].
\end{aligned}
$$

Using relation (3) after a suitable changing of the order of the summation and considering the Hölder inequality with $p=\frac{1}{\alpha}$ and $q=\frac{1}{1-\alpha}$, we obtain the estimates

$$
\begin{aligned}
&\big|\mathcal{Q}^*_{n,r}(f;y)-\mathcal{Q}^*_{n,r}(f;x)\big| \le \\
&\le M\Big\{\tfrac{1}{[1+(n-r)\beta]^{n-r-1}} \sum_{k=0}^{n-r} \Big(\tfrac{k}{n}\lambda_n+\tfrac{xr}{n}\Big)\tbinom{n-r}{k} \tfrac{y-x}{\lambda_n}\Big(\tfrac{y-x}{\lambda_n}+k\beta\Big)^{k-1}\Big(1-\tfrac{y-x}{\lambda_n}\Big) \\
&\quad \times \Big[1-\tfrac{y-x}{\lambda_n}+(n-r-k)\beta\Big]^{n-r-k-1}\Big\}^{\alpha}\Big\{\tfrac{1}{[1+(n-r)\beta]^{n-r-1}} \sum_{k=0}^{n-r}\tbinom{n-r}{k} \\
&\quad \times \tfrac{y-x}{\lambda_n}\Big(\tfrac{y-x}{\lambda_n}+k\beta\Big)^{k-1}\Big(1-\tfrac{y-x}{\lambda_n}\Big)\Big[1-\tfrac{y-x}{\lambda_n}+(n-r-k)\beta\Big]^{n-r-k-1}\Big\}^{\alpha} \\
&= M\big[\mathcal{Q}^*_{n,r}(e_1;y-x)\big]^{\alpha}\big[\mathcal{Q}^*_{n,r}(e_0;x)\big]^{\alpha} \\
&= M(y-x)^{\alpha},
\end{aligned}
$$

according also to Proposition 7. Hence,

$$\left|\mathcal{Q}^*_{n,r}(f;y) - \mathcal{Q}^*_{n,r}(f;x)\right| \le M|y-x|^\alpha$$

and this completes the proof. □

Another important property that can be studied for the operator $\mathcal{Q}^*_{n,r}$ is related to convexity. Based on the ideas presented in [2, Theorem 3] and [14, Theorem 3], we obtain the following result:

PROPOSITION 12. *If f is convex on $[0,\lambda_n]$, then $\mathcal{Q}^*_{n,r}(f;x) \ge f(x)$, for all $x \in [0,\lambda_n]$.*

Proof. Based on relation (9) we have that

$$\mathcal{Q}^*_{n,r}(f;x) = \sum_{k=0}^{n-r} q^*_{n-r,k}(x)\Big[\Big(1-\tfrac{x}{\lambda_n}\Big) f\Big(\tfrac{k}{n}\lambda_n\Big) + \tfrac{x}{\lambda_n} f\Big(\tfrac{k+r}{n}\lambda_n\Big)\Big],$$

where the coefficients $q^*_{n-r,k}$ have the property that

$$\sum_{k=0}^{n-r} q^*_{n-r,k}(x) = \mathcal{Q}^*_{n,r}(e_0;x) = 1,$$

by Proposition 7. If we denote by $t = \frac{x}{\lambda_n}$ and $u_k = \frac{k}{n}\lambda_n$, then

$$\begin{aligned}
\mathcal{Q}^*_{n,r}(f;x) &= \sum_{k=0}^{n-r} q^*_{n-r,k}(x)\big[(1-t)f(u_k) + tf(u_{k+r})\big] \\
&= (1-t)\sum_{k=0}^{n-r} q^*_{n-r,k}(x) f(u_k) + t\sum_{k=0}^{n-r} q^*_{n-r,k}(x) f(u_{k+r}) \\
&\ge (1-t) f\Big(\sum_{k=0}^{n-r} q^*_{n-r,k}(x) u_k\Big) + t f\Big(\sum_{k=0}^{n-r} q^*_{n-r,k}(x) u_{k+r}\Big) \\
&\ge f\Big((1-t)\sum_{k=0}^{n-r} q^*_{n-r,k}(x) u_k + t\sum_{k=0}^{n-r} q^*_{n-r,k}(x) u_{k+r}\Big) \\
&= f\Big(\sum_{k=0}^{n-r} q^*_{n-r,k}(x)\big[(1-t)u_k + tu_{k+r}\big]\Big).
\end{aligned}$$

According to Proposition 7 we know that

$$\sum_{k=0}^{n-r} q^*_{n-r,k}(x)\big[(1-t)u_k + tu_{k+r}\big] = \mathcal{Q}^*_{n,r}(e_1;x)$$

and then

$$\mathcal{Q}^*_{n,r}(f;x) \ge f(\mathcal{Q}^*_{n,r}(e_1;x)) = f(x), \quad x \in [0,\lambda_n]$$

and this completes the proof. □

REMARK 13. *In particular, if $r = 0$, then Proposition 12 reduces to* [14, Theorem 3] *proved by Söylemez and Taşdelen.*

2.2. Korovkin type theorems for the operator $\mathcal{Q}^*_{n,r}$. Following the ideas presented by Söylemez and Taşdelen in [15], we consider the following families of functions:

$$B_\rho[0,\infty) = \Big\{ f : [0,\infty) \to \mathbb{R} : |f(x)| \le M(f)\rho(x) \Big\},$$

where $M(f) > 0$ is a constant that depends on f and $\rho(x) = 1 + x^2$,

$$C_\rho[0,\infty) = \Big\{ f \in B_\rho[0,\infty) : f \text{ is continuous on } [0,\infty) \Big\}$$

and

$$C^k_\rho[0,\infty) = \Big\{ f \in C_\rho[0,\infty) : \lim_{x\to\infty} \tfrac{f(x)}{\rho(x)} = k(f) \Big\},$$

where $k(f)$ is a constant that depends on f. Also from [15] we know that $B_\rho[0,\infty)$ is a normed linear space with the norm $\|f\|_\rho = \sup\big\{ \frac{|f(x)|}{\rho(x)} : x \ge 0 \big\}$.

In order to prove an approximation result for the operator $\mathcal{Q}^*_{n,r}$, we recall two important results related to the weighted Korovkin type theorem (see [15, Lemma 2.1] and [15, Theorem 2.2]; cf. [11] and [12]).

Lemma 14 ([15, Lemma 2.1]). *Let $n \ge 1$. Then a family of positive linear operators (F_n) act from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$ if and only if*

$$|F_n(\rho; x)| \le \rho(x) K_n,$$

holds for $x \in [0,\infty)$ and $K_n > 0$ a positive constant.

Theorem 15 ([15, Theorem 2.2]). *Let $(F_n)_{n\ge1}$ be a family of linear operators acting from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$ such that*

$$\lim_{n\to\infty} \|F_n(e^j; x) - x^j\|_\rho = 0,$$

for every $j = 0, 1, 2$. Then for any function $f \in C^k_\rho[0,\infty)$ we have that

$$\lim_{n\to\infty} \|F_n f - f\|_\rho = 0,$$

where $\rho(x) = 1 + x^2$, for all $x \in [0,\infty)$.

Hence, according to the Korovkin type theorems presented above, we can prove the following approximation results for the operator $\mathcal{Q}^*_{n,r}$:

Theorem 16. *Let $n, r \in \mathbb{N}$ be such that $n > 2r$ and (λ_n), (β_n) be two sequences of positive numbers with the property that $\lim_{n\to\infty} \lambda_n = \infty$, $\lim_{n\to\infty} (\lambda_n/n) = 0$ and $\lim_{n\to\infty} (n\beta_n) = 0$. Then for each $f \in C^k_\rho[0,\infty)$ we have that*

$$\lim_{n\to\infty} \|\mathcal{Q}^*_{n,r} f - f\|_\rho = 0, \tag{15}$$

where $\rho(x) = 1 + x^2$ for all $x \in [0,\infty)$.

Proof. In view of relations (9), (11) and (13) we obtain that

$$\begin{aligned}
&\mathcal{Q}^*_{n,r}(\rho;x) = \\
&= \sum_{k=0}^{n-r} q^*_{n-r,k}(x)\Big[\Big(1-\tfrac{x}{\lambda_n}\Big)\rho\Big(\tfrac{k}{n}\lambda_n\Big)+\tfrac{x}{\lambda_n}\rho\Big(\tfrac{k+r}{n}\lambda_n\Big)\Big]\\
&= \sum_{k=0}^{n-r} q^*_{n-r,k}(x)\Big[\Big(1-\tfrac{x}{\lambda_n}\Big)\Big(1+\tfrac{k^2}{n^2}\lambda_n^2\Big)+\tfrac{x}{\lambda_n}\Big(1+\tfrac{(k+r)^2}{n^2}\lambda_n^2\Big)\Big]\\
&= \sum_{k=0}^{n-r} q^*_{n-r,k}(x)\Big[1+\Big(1-\tfrac{x}{\lambda_n}\Big)\tfrac{k^2}{n^2}\lambda_n^2+\tfrac{x}{\lambda_n}\tfrac{(k+r)^2}{n^2}\lambda_n^2\Big]\\
&= \mathcal{Q}^*_{n,r}(e_0;x)+\mathcal{Q}^*_{n,r}(e_2;x)\\
&\le 1+\tfrac{(n-r)^2}{n^2}\Big\{\big[x+2\lambda_n\beta+\lambda_n\beta^2(n-r)\big]\big[1+2(n-r)\beta\big]+\tfrac{(n-r)(\lambda_n+2rx)+r^2\lambda_n}{(n-r)^2}\Big\}x
\end{aligned}$$

and then

$$\big|\mathcal{Q}^*_{n,r}(\rho;x)\big| \le (1+x^2)K_{n,r},$$

where

$$K_{n,r}=1+\Big(\tfrac{n-r}{n}\Big)^2\Big[1+\lambda\beta+\tfrac{\lambda\beta^2(n-r)[5+2\beta(n-r)]}{2}+2\beta(n-r)+\tfrac{(n-r)(\lambda+4r)+r^2\lambda}{2(n-r)^2}\Big]$$

is a positive constant. Then, according to Lemma 14 we deduce that $(\mathcal{Q}^*_{n,r})$ is a family of linear operators between the spaces $C_\rho[0,\infty)$ and $B_\rho[0,\infty)$. Moreover, following the arguments presented in Proposition 7 and Lemma 9 we have that

$$\lim_{n\to\infty}\|\mathcal{Q}^*_{n,r}(e_j;x)-x^j\|_\rho=0, \quad \text{for} \quad j=0,1,2.$$

Hence, based on Theorem 15 we obtain that

$$\lim_{n\to\infty}\|\mathcal{Q}^*_{n,r}f-f\|_\rho=0,$$

for all $f\in C^k_\rho[0,\infty)$ and this completes the proof. □

THEOREM 17. *For any function $f\in C[0,\infty)\cap E$, we have that*

$$\lim_{n\to\infty}\mathcal{Q}^*_{n,r}(f;x)=f(x)$$

uniformly on each compact subset of $[0,\infty)$, where

$$E=\Big\{f:[0,\infty)\to\mathbb{R}:\tfrac{f(x)}{1+x^2}\ \textit{is convergent as}\ x\to\infty\Big\}$$

and $\mathcal{Q}^*_{n,r}$ is given by (9).

Proof. Using Korovkin-type theorem (see, for example [1], [11] and [12]), it is sufficient to prove that the operators $\mathcal{Q}^*_{n,r}$ verify the conditions

$$\lim_{n\to\infty}\mathcal{Q}^*_{n,r}(e_j;x)=x^j,$$

for $j = 0, 1, 2$ uniformly on each compact subset of $[0, \infty)$. According to Proposition 7 and Lemma 9 we deduce that the above conditions are satisfied and this completes the proof. □

REMARK 18. *Finally, it is important to mention that similar results can be obtained also for the operator* $\mathcal{P}^*_{n,r}$ *defined by* (8). *Such extensions will generalize the results proved by Cătinaş and Buda in* [6].

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