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PRESERVING PROPERTIES OF SOME SZÁSZ-MIRAKYAN TYPE OPERATORS

JORGE BUSTAMANTE[1]

Abstract. For a family of Szász-Mirakyan type operators we prove that they preserve convex-type functions and that a monotonicity property verified by Cheney and Sharma in the case Szász-Mirakyan operators holds for the variation study here. We also verify that several modulus of continuity are preserved.

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1. INTRODUCTION

Throughout the work $\mathbb{N}$ is the set of all positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{P}_n$ is the family of all algebraic polynomials of degree non greater than n. Moreover, for each $j \in \mathbb{N}_0$, we use the notations

$$e_j(x) = x^j, \qquad x \in \mathbb{R},$$

and $I = [0, \infty)$. Let $C(I)$ the family of all continuous functions $f : I \to \mathbb{R}$.

The Szász-Mirakyan operators are defined by (see [5] and the references therein)

$$S_n(f, x) = e^{-nx} \sum_{k=0}^{\infty} \tfrac{n^k}{k!} f\left(\tfrac{k}{n}\right) x^k, \quad x \in I.$$

It is known that $S_n(e_0, x) = 1$ and $S_n(e_1, x) = x$ (see [5]).

For a fixed real $p \geq 0$ and $n \in \mathbb{N}$, Schurer defined ([26] and [27])

$$S_{n,p}(f, x) = e^{-(n+p)x} \sum_{k=0}^{\infty} \tfrac{(n+p)^k}{k!} f\left(\tfrac{k}{n}\right) x^k, \quad x \in I. \tag{1}$$

Some studies concerning these operators were given by Sikkema in [28] and [29] (see also [25]).

It is known that (see [25, p. 82]), for each $x \geq 0$ and $n \in \mathbb{N}$, $S_{n,p}(e_0, x) = 1$ and

$$S_{n,p}(e_1, x) = x + \tfrac{p\,x}{n}.$$

[1]Benemérita Universidad Autónoma de Puebla, Facultad de Ciencias Físico-Matemáticas, Avenida San Claudio y 18 Sur, Colonia San Manuel, Edificio FM1-101B, Ciudad Universitaria, C.P. 72570, Puebla, México. e-mail: jbusta@fcfm.buap.mx.

Hence one has $S_{n,p}(e_1,x)=x$ only when $p=0$.

In this work we study properties of a modification $M_{n,p}$ of Schurer operators satisfying $M_{n,p}(e_0,x)=1$ and $M_{n,p}(e_1,x)=x$.

Let $\{\beta(n)\}$ be an strictly increasing sequence of positive real numbers such that $\lim_{n\to\infty}\beta(n)=\infty$. For $p\geq 0$, $n\in\mathbb{N}$, $x\geq 0$, and a function $f\in C(I)$ consider the operator

$$M_{n,p}(f,x)=e^{-(\beta(n)+p)x}\sum_{k=0}^{\infty}\frac{(\beta(n)+p)^k}{k!}f\Big(\frac{k}{\beta(n)+p}\Big)x^k, \tag{2}$$

whenever the series converges absolutely. Let $\mathcal{L}(I)$ be the family of all functions $f\in C(I)$ such that, for each $n\in\mathbb{N}$, the series $M_{n,p}(f)$ converges absolutely.

Notice that $M_{n,p}$ can be considered a more natural extension of Szász-Mirakyan operators. This modification appeared in [7] and [8]. In [7] they were studied in spaces defined by the weight $\varrho_m(x)=1/(1+x)^m$, with $m\in\mathbb{N}$ and in [8] some weighted space of bounded functions were considered.

There is a long list of papers devoted to study properties of Szász-Mirakyan operators. Here we recall some of them: [1], [3], [4], [5], [10], [17], [20], [21], [22], [32], [33], [34], [35], and [36]. It is worth asking when the results presented in the cited articles can be extended to the case $M_{n,p}$ operators.

For a fixed $p\geq 0$, $n\in\mathbb{N}$, and $x\geq 0$ we use the notations

$$g_{n,p}(x)=e^{-(\beta(n)+p)x}\qquad\text{and}\qquad a_{n,p}=\beta(n)+p. \tag{3}$$

For $r\in\mathbb{N}_0$, $C_r(I)$ is the family of all $f\in C(I)$ such that

$$\|f\|_r=\sup_{x\in I}\frac{|f(x)|}{(1+x)^r}<\infty. \tag{4}$$

For $r\in\mathbb{N}_0$, let $C_{r,\infty}(I)$ be the class of all functions $f\in C_r(I)$ such that $f(x)/(1+x)^r$ has a finite limit as $x\to\infty$.

In Section 2 we present some general properties of operators $M_{n,p}$. In Section 3 we show that some known properties related with monotone and convex functions and Szász-Mirakyan operators also holds for the operators $M_{n,p}$. In Section 4 we prove that several modulus of continuity are preserved (up to a constant) by the operators $M_{n,p}$.

2. SOME BASIC PROPERTIES

Since the series

$$\sum_{k=0}^{\infty}\frac{a_{n,p}^k}{k!}x^k=e^{(\beta(n)+p)x}=g_{n,p}(x), \tag{5}$$

converges uniformly on each interval $[0,a]$, $a>0$, it can be differentiated term by term. For $i\in\mathbb{N}$, we will use several times the equations

$$g_{n,p}^{(i)}(x)=\sum_{k=i}^{\infty}\frac{a_{n,p}^k}{(k-i)!}x^{k-i}=\sum_{k=0}^{\infty}\frac{a_{n,p}^{k+i}}{k!}x^k=a_{n,p}^i g_{n,p}(x). \tag{6}$$

THEOREM 1. *If $i \in \mathbb{N}_0$ and*

$$P_{i+1}(x) = x\left(x - \tfrac{1}{a_{n,p}}\right)\cdots\left(x - \tfrac{i}{a_{n,p}}\right), \quad x \ge 0, \tag{7}$$

then

$$M_{n,p}(P_{i+1}, x) = x^{i+1}. \tag{8}$$

In particular, for each $n \in \mathbb{N}$ and $i \in \mathbb{N}_0$, $e_i \in \mathcal{L}[0,\infty)$ and $M_{n,p}(e_i, x) \in \mathbb{P}_i$.

Proof. Notice that $P_{i+1}(x) \in \mathbb{P}_{i+1}$ and, for $k \in \mathbb{N}_0$,

$$a_{n,p}^{i+1} P_{i+1}\left(\tfrac{k}{a_{n,p}}\right) = k(k-1)\cdots(k-i).$$

In particular $P_{i+1}(k/a_{n,p})) = 0$ for $0 \le k \le i$. Therefore, for each fixed $x > 0$,

$$\begin{aligned} a_{n,p}^{i+1} g_{n,p}(x) M_{n,p}(P_{i+1}, x) &= \sum_{k=i+1}^{\infty} \tfrac{a_{n,p}^k x^k}{(k-i-1)!} = x^{i+1} \sum_{k=i+1}^{\infty} \tfrac{a_{n,p}^k x^{k-i-1}}{(k-i-1)!} \\ &= x^{i+1} \sum_{k=0}^{\infty} \tfrac{a_{n,p}^{k+i+1}}{k!} x^k = x^{i+1} g_n^{(i+1)}(x), \end{aligned}$$

where we use (6). Therefore $M_{n,p}(P_{i+1}, x) = x^{i+1} \in \mathbb{P}_{i+1}$, for each $i \ge 0$.

Since, for $i \ge 0$, x^i can be written as a linear combination of the polynomials $P_1, \dots, P_i$, we know that $e_i \in \mathcal{L}[0,\infty)$ and $M_{n,p}(e_i, x) \in \mathbb{P}_i$. For $i = 0$ it is a simple assertion because $M_{n,p}(e_0, x) = 1$. □

For the case of Szász-Mirakyan operators the last assertion in Theorem 1 was verified by Becker in [5, Lemma 3].

PROPOSITION 2. *If $r \in \mathbb{N}$, there exists a constant $C(r) \ge 1$ such that, for every real $a > 0$,*

$$M_{n,p}((a + e_1)^r, x) \le C(r)(1 + a + x)^r.$$

Proof. From Theorem 1 we know that, for each $i \in \mathbb{N}$, there is an algebraic polynomial $P_i \in \mathbb{P}_n$, say $P_i(x) = \sum_{k=0}^{i} b_{i,k} x^k$, such that

$$M_{n,p}(e_i, x) = \sum_{k=0}^{i} b_{i,k} x^k.$$

If $0 \le x \le 1$, then

$$\left|\sum_{k=0}^{i} b_{i,k} x^k\right| \le \sum_{k=0}^{i} | b_{i,k} | \le (1+x)^i \sum_{k=0}^{i} | b_{i,k} | .$$

If $1 \le x$, then

$$\left|\sum_{k=0}^{i} b_{i,k} x^k\right| \le x^i \sum_{k=0}^{i} | b_{i,k} | \le (1+x)^i \sum_{k=0}^{i} | b_{i,k} | .$$

Therefore $0 \le M_{n,p}(e_i, x) \le C_i(1+x)^i$, where the constant C_i depends only on i.

If $a > 0$,

$$M_{n,p}\Big((a+e_1)^r, s\Big) = \sum_{j=0}^{r} \tbinom{r}{j} a^{r-j} M_{n,p}(e_j, s)$$
$$\leq C\Big(a^r + \sum_{j=1}^{r} \tbinom{r}{j} a^{r-j}(1+x)^j = C(1+a+x)^r.\square$$

Theorem 3 was proved in [8] when $\beta(n) = n$, but it can be easily extended to the case of a general $\beta(n)$.

THEOREM 3. *The operators $M_{n,p}$ has the following properties:*
(i) $M_{n,p} : \mathcal{L}(I) \to C^1(I)$.
(ii) $M_{n,p}(e_0, x) = 1$
(iii) *For every $n, m \in \mathbb{N}$, $f \in \mathcal{L}(I)$ and $x > 0$,*

$$\frac{1}{a_{n,p}^m} M_{n,p}^{(m)}(f, x) = M_{n,p}\Big(\Delta_{1/a_{n,p}}^m f(t), x\Big), \tag{9}$$

where $\Delta_h^k g(u)$ stands the usual k-th forward difference of the function g at u with step h.

The following result can be proved as Theorem 1 in [30] (it is a consequence of the Korovkin theorem).

THEOREM 4. *If $f \in \mathcal{L}(I)$ and $a > 0$ then $M_{n,p}(f, x)$ converges uniformly to $f(x)$ on $[0, a]$.*

3. MONOTONICITY AND CONVEX FUNCTIONS

For $k \in \mathbb{N}$, a function $g : I \to \mathbb{R}$ is said to be k-convex, if $\Delta_h^k g(u) \geq 0$ for each $h > 0$. In particular, 2-convexity agrees with the usual notion of convex functions.

For each $k \in \mathbb{N}$, Szász-Mirakyan operators preserve k-convexity [24]. That is, if $\Delta_h^k g(u) \geq 0$ and $S_n(g, x)$ is well defined, then $\Delta_h^k S_n(g, u) \geq 0$. If follows from (9) that the operators $M_{n,p}$ share this property Szász-Mirakyan operators. But the assertion must be presented in a more convenient form. Let us explain why we need that. In [38, Th. 1], Zhen proved that, if $f'(x) > 0$, then $S_n'(f, x) > 0$, and if $f''(x) > 0$, then $S_n''(f, x) > 0$. Theorem 5 shows that these types of results are trivial.

THEOREM 5. (i) *If $f \in \mathcal{L}(I)$ increases, then $M_{n,p}'(f, x) \geq 0$.*
(ii) *If $f \in \mathcal{L}(I)$ is convex, then $M_{n,p}''(f, x) \geq 0$.*

Proof. It follows directly from (9). □

Cheney and Sharma proved in [9] that, if f is convex, for each x and every $n \in \mathbb{N}$, $S_{n+1}(f, x) \leq S_n(f, x)$. Horová [14] obtained a converse theorem. In Theorem 6 we verify that a similar result holds for the operators $M_{n,p}$. A converse result can also be proved (see [14] and [18]). But we do not want to consider that problem here.

THEOREM 6. (i) *If* $f \in \mathcal{L}(I)$ *is convex then, for each* $x \geq 0$ *and* $n \in \mathbb{N}$,

$$f(x) \leq M_{n+1,p}(f,x) \leq M_{n,p}(f,x).$$

(ii) *If* $f \in \mathcal{L}(I)$ *is concave then, for each* $x \geq 0$ *and* $n \in \mathbb{N}$,

$$M_{n,p}(f,x) \leq M_{n+1,p}(f,x) \leq f(x).$$

Proof. Assume f is convex. If we set

$$c_{n,k} = (\beta(n+1) - \beta(n))^k \quad \text{and} \quad b_{n,p} = \tfrac{a_{n,p}}{\beta(n+1)-\beta(n)},$$

then

$$c_{n,k} \sum_{r=0}^{k} \tbinom{k}{r} b_{n,p}^r = (\beta(n+1) - \beta(n))^k \Big(\tfrac{\beta(n)+p}{(\beta(n+1)-\beta(n))} + 1 \Big)^k = a_{n+1,p}^k.$$

That is

$$\tfrac{c_{n,k}}{a_{n+1,p}^k}\, k! \sum_{r=0}^{k} \tfrac{1}{r!} \tfrac{b_{n,p}^r}{(k-r)!} = 1.$$

Therefore

$$\begin{aligned} \tfrac{c_{n,k}}{a_{n+1,p}^k} k! \sum_{r=0}^{k} \tfrac{r}{a_{n,p}} \tfrac{1}{r!} \tfrac{b_{n,p}^r}{(k-r)!} &= \tfrac{k}{a_{n+1,p}^k} \tfrac{c_{n,k}}{(\beta(n+1)-\beta(n))} \sum_{r=1}^{k} \tbinom{k-1}{r} b_{n,p}^{r-1} \\ &= \tfrac{k}{a_{n+1,p}^k} \tfrac{c_{n,k}}{(\beta(n+1)-\beta(n))} \Big(\tfrac{\beta(n)+p}{(\beta(n+1)-\beta(n))} + 1 \Big)^{k-1} \\ &= \tfrac{k}{a_{n+1,p}^k} \Big(\beta(n+1) + p \Big)^{k-1} = \tfrac{k}{a_{n+1,p}}. \end{aligned}$$

This proves that $k/a_{n+1,p}$ is a convex combination of the points $\{r/a_{n,p} : 0 \leq r \leq k\}$.

If f is convex, then

$$f\Big(\tfrac{k}{a_{n+1,p}}\Big) \leq \tfrac{c_{n,k}}{a_{n+1,p}^k} k! \sum_{r=0}^{k} f\Big(\tfrac{r}{a_{n,p}}\Big) \tfrac{1}{r!} \tfrac{b_{n,p}^r}{(k-r)!} = \tfrac{k!}{a_{n+1,p}^k} \sum_{r=0}^{k} c_{n,k}^{k-r} f\Big(\tfrac{r}{a_{n,p}}\Big) \tfrac{1}{r!} \tfrac{a_{n,p}^r}{(k-r)!}.$$

By the Cauchy multiplication rule for product of series,

$$\begin{aligned} e^{(\beta(n+1)-\beta(n))x} \sum_{k=0}^{\infty} \tfrac{a_{n,p}^k}{k!} f\Big(\tfrac{k}{a_{n,p}}\Big) x^k &= \sum_{k=0}^{\infty} \Big\{ \sum_{m+r=k} \tfrac{((\beta(n+1)-\beta(n))x)^m}{m!} \tfrac{a_{n,p}^r}{r!} f\Big(\tfrac{r}{a_{n,p}}\Big) x^r \Big\} \\ &= \sum_{k=0}^{\infty} \Big\{ \sum_{r=0}^{k} \tfrac{(\beta(n+1)-\beta(n))^{k-r}}{(k-r)!} \tfrac{a_{n,p}^r}{r!} f\Big(\tfrac{r}{a_{n,p}}\Big) \Big\} x^k. \end{aligned}$$

Therefore

$$\begin{aligned} & e^{a_{n+1,p}x} \Big(M_{n,p}(f,x) - M_{n+1,p}(f,x) \Big) = \\ & = e^{(\beta(n+1)-\beta(n))x} \sum_{k=0}^{\infty} \tfrac{a_{n,p}^k}{k!} f\Big(\tfrac{k}{a_{n,p}}\Big) x^k - \sum_{k=0}^{\infty} \tfrac{a_{n+1,p}^k}{k!} f\Big(\tfrac{k}{a_{n+1,p}}\Big) x^k \end{aligned}$$

$$= \sum_{k=0}^{\infty} \Big\{ \sum_{r=0}^{k} \frac{(\beta(n+1)-\beta(n))^{k-r}}{(k-r)!} \frac{a_{n,p}^r}{r!} f\Big(\frac{r}{a_{n,p}}\Big) - \frac{a_{n+1,p}^k}{k!} f\Big(\frac{k}{a_{n+1,p}}\Big) \Big\} x^k \geq 0.$$

This proves that $M_{n,p}(f,x) \geq M_{n+1,p}(f,x)$. From Theorem 4 we know that $M_{n,p}(f,x) \to f(x)$ as $n \to \infty$ (pointwise convergence). Thus $M_{n+1,p}(f,x) \geq f(x)$.

The concave functions follows by changing f by $-f$. □

Fix $n \in \mathbb{N}$ and let $f \in C_r(I)$ be a non-negative function (see (4)).

For a non-negative function $f \in C_r(I)$, in [37], Zhao proved that if $f(x)/x$ is non-increasing on $(0,\infty)$, then for each $n \geq 1$, $S_n(f,x)/x$ is non-increasing. A similar result can be proved for the operators $M_{n,p}$ by modifying the arguments of Zhao. Since the work [37] is not well known, we include the complete proof. Notice that the condition $f \in C_r(I)$ (assumed by Zhao) will be replaced by the more general $f \in \mathcal{L}(I)$.

THEOREM 7. *Let $f \in \mathcal{L}(I)$ be a non-negative function. If $f(x)/x$ is non-increasing on $(0,\infty)$, then for each $n \in \mathbb{N}$, $M_{n,p}(f,x)/x$ is non-increasing.*

Proof. We will prove that $(d/dx)(M_{n,p}(f,x)/x) \leq 0$. We use the notations in (3).

Since

$$\frac{M_{n,p}(f,x)}{x} = f(0)\frac{g_{n,p}(x)}{x} + g_{n,p}(x) \sum_{k=1}^{\infty} \frac{a_{n,p}^k}{k!} f\Big(\frac{k}{a_n}\Big) x^{k-1}$$

and

$$\frac{d}{dx}\frac{g_{n,p}(x)}{x} = \frac{g_{n,p}(x)}{x^2}\Big(- a_{n,p}x - 1\Big) < 0,$$

we should consider the derivative of the previous series. Note that

$$\sum_{k=1}^{\infty} \frac{a_{n,p}^k}{k!} f\Big(\frac{k}{a_{n,p}}\Big) \frac{d}{dx}\Big(g_{n,p}(x)x^{k-1}\Big) =$$

$$= g_{n,p}(x) \sum_{k=2}^{\infty} \frac{a_{n,p}^k(k-1)}{k!} f\Big(\frac{k}{a_{n,p}}\Big) x^{k-2} - g_{n,p}(x) \sum_{k=1}^{\infty} \frac{a_{n,p}^{k+1}}{k!} f\Big(\frac{k}{a_n}\Big) x^{k-1}$$

$$= g_{n,p}(x) \sum_{k=1}^{\infty} \frac{a_{n,p}^{k+1}\, k}{(k+1)!} f\Big(\frac{k+1}{a_{n,p}}\Big) x^{k-1} - g_{n,p}(x) \sum_{k=1}^{\infty} \frac{a_{n,p}^{k+1}}{k!} f\Big(\frac{k}{a_{n,p}}\Big) x^{k-1}$$

$$= g_{n,p}(x) \sum_{k=1}^{\infty} \Big\{ \frac{a_{n,p}}{k+1} f\Big(\frac{k+1}{a_{n,p}}\Big) - \frac{a_n}{k} f\Big(\frac{k}{a_n}\Big) \Big\} \frac{a_{n,p}^k\, x^{k-1}}{(k-1)!} \leq 0.$$

The result is proved. □

4. PRESERVATION OF MODULUS OF CONTINUITY

DEFINITION 8. *A function $\omega : I \to \mathbb{R}^+$ is called a modulus of continuity if $\omega(0) = 0$, $\lim_{t\to 0}\omega(t) = 0$, ω is non-negative and non-decreasing in I and $\omega(t)$ is continuous in $\mathbb{R}^+$.*

Definition 9. *A function* $\omega : I \to \mathbb{R}^+$ *is called subadditive if for any* $s, t \geq 0$

$$\omega(s+t) \leq \omega(s) + \omega(t).$$

If a subadditive function $\omega : I \to \mathbb{R}^+$ is continuous at zero and $\omega(0) = 0$, then it is continuous. If ω is subadditive, then $\omega(2t) \leq 2\omega(t)$ and it follows from standard arguments that, if $t, \lambda > 0$, then

$$\omega(\lambda t) \leq (1+\lambda)\, \omega(f, t). \tag{10}$$

It is known that (see [11, p. 43]), for any modulus of continuity ω on I, there exists a concave modulus of continuity (the least concave majorant) $\widetilde{\omega}$ such that

$$\omega(t) \leq \widetilde{\omega}(t) \leq 2\omega(t). \tag{11}$$

For Szász-Mirakyan operators preservation of the usual modulus of continuity has been considered in [31], [15] and [3]. For instance, if $\omega(t)$ is a concave modulus of continuity and

$$\Lambda(\omega, A) = \Big\{ f \in C(I) : \ \omega(f,t) \leq A\omega(t) \Big\},$$

it is asserted in [15] that $f \in \Lambda(\omega, A)$ if and only if $S_n(f) \in \Lambda(\omega, A)$, for each each $n \in \mathbb{N}$. On the other hand, in [3] the authors considered functions f such that $0 < \omega(f, 1) < \infty$, where $\omega(f, t)$ is the usual modulus of continuity. Of course the condition $0 < \omega(f, 1)$ holds whenever f is not a constant function.

Of course, since the usual modulus of continuity is not well defined for all $f \in C(I)$, such a result must be handled with care. In fact in [13] Hermann presented a negative result. Let

$$C_0 = \Big\{ f \in C(I) : \sup_{x \in I} \mid f(x+\delta) - f(x) \mid < \infty \ \text{ for any } \delta > 0 \Big\}.$$

Notice that for any $f \in C_0$ the usual modulus of continuity is well defined, but the conditions $f \in C_0$ and $\delta \to 0$ does not necessarily imply $\omega(f, \delta) \to 0$.

Set $C_0^* = \{ f \in C_0 : \omega(f,t) > 0 \}$. In [13] Hermann proved that

$$\sup_{f \in C_0^*} \tfrac{\|S_n(f) - f\|_C}{\omega(f, 1/n)} = \infty.$$

In this section we prove some results related with preservation of some modulus of continuity by the operators $M_{n,p}$.

Although Theorem 3 is sufficient to prove the preservation of convexity of different order by the operators $M_{n,p}$, we need other kind of representations for studying modulus of continuity.

The ideas for the proof of Proposition 10 have been used for different authors in the case of Szász-Mirakyan operators (see [31] and [15]).

Proposition 10. *If* $f \in \mathcal{L}(I)$, $n \in \mathbb{N}$, $x \in I$ *and* $s > 0$, *then*

$$M_{n,p}(f, x+s) - M_{n,p}(f,x) = e^{-a_{n,p}(x+s)} \sum_{k=0}^{\infty} \tfrac{a_{n,p}^k x^k}{k!} \sum_{i=1}^{\infty} \tfrac{a_{n,p}^i}{i!} s^i \Big(f\Big(\tfrac{k+i}{a_{n,p}}\Big) - f\Big(\tfrac{k}{a_{n,p}}\Big)\Big).$$

Proof. Notice that

$$e^{a_{n,p}(x+s)}M_{n,p}(f,x+s)=$$
$$=\sum_{j=0}^{\infty} f\left(\frac{j}{a_{n,p}}\right)\frac{a_{n,p}^j}{j!}(x+s)^j=\sum_{j=0}^{\infty} f\left(\frac{j}{a_{n,p}}\right)\frac{a_{n,p}^j}{j!}\sum_{k=0}^{j}\binom{j}{k}x^k s^{j-k}=$$
$$=\sum_{k=0}^{\infty}\sum_{j=k}^{\infty}\binom{j}{k}f\left(\frac{j}{a_{n,p}}\right)\frac{a_{n,p}^j}{j!}x^k s^{j-k}=\sum_{k=0}^{\infty}\frac{x^k}{k!}\sum_{j=k}^{\infty}f\left(\frac{j}{a_{n,p}}\right)\frac{a_{n,p}^j}{(j-k)!}s^{j-k}$$
$$=\sum_{k=0}^{\infty}\frac{x^k}{k!}\sum_{i=0}^{\infty}f\left(\frac{i+k}{a_{n,p}}\right)\frac{a_{n,p}^{i+k}}{i!}s^i.$$

On the other hand,

$$e^{a_{n,p}(x+s)}M_{n,p}(f,x)=e^{a_{n,p}s}\sum_{k=0}^{\infty}f\left(\frac{k}{a_{n,p}}\right)\frac{a_{n,p}^k}{k!}x^k=\sum_{k=0}^{\infty}\frac{a_{n,p}^k x^k}{k!}\left(\sum_{i=0}^{\infty}\frac{a_{n,p}^i}{i!}s^i f\left(\frac{k}{a_{n,p}}\right)\right).$$

It follows from the equation given above the announced result. □

Let $UC_b(I)$ the class of all bounded uniformly continuous functions $f : I \to \mathbb{R}$. For $f \in UC_b(I)$ and $t \geq 0$, define

$$\omega(f,t)=\sup_{0\leq h\leq t}\sup_{x\geq 0}\mid f(x+h)-f(x)\mid . \tag{12}$$

It can be proved that $\omega(f,t)$ is subadditive modulus of continuity in the sense of Definition 8.

Theorem 11. *If $f \in UC_b(I)$, $n \in \mathbb{N}$, and $s > 0$, then $M_{n,p}(f,x)$ is uniformly continuous and*

$$\omega(M_{n,p}(f),s)\leq 2\,\omega(f,s).$$

Proof. Let $\widetilde{\omega}(f,t)$ be the least concave majorant of $\omega(f,t)$.
If $f \in UC_b(I)$, then $f \in \mathcal{L}(I)$. From Proposition 10 one has

$$\mid M_{n,p}(f,x+s)-M_{n,p}(f,x)\mid\leq$$
$$\leq e^{-a_{n,p}(x+s)}\sum_{k=0}^{\infty}\frac{a_{n,p}^k x^k}{k!}\sum_{i=0}^{\infty}\frac{a_{n,p}^i}{i!}s^i\mid f\left(\frac{k+i}{a_{n,p}}\right)-f\left(\frac{k}{a_{n,p}}\right)\mid$$
$$\leq e^{-a_{n,p}(x+s)}\sum_{k=0}^{\infty}\frac{a_{n,p}^k x^k}{k!}\sum_{i=0}^{\infty}\frac{a_{n,p}^i}{i!}s^i\omega\left(f,\frac{i}{a_{n,p}}\right)$$
$$=M_{n,p}(\omega(f,e_1),s)\leq M_{n,p}(\widetilde{\omega}(f,e_1),s).$$

Since $\widetilde{\omega}(f,t)$ is a concave function, it follows from Theorem 6 that

$$M_{n,p}(\widetilde{\omega}(f),s)\leq\widetilde{\omega}(f,s)\leq 2\,\omega(f,s).$$

In particular if $\varepsilon > 0$, $\omega(f,s) \leq \varepsilon/2$, $0 \leq y < x$, $x - y \leq s$ and we set $x = y + t$

$$\mid M_{n,p}(f,x)-M_{n,p}(f,y)\mid=\mid M_{n,p}(f,y+t)-M_{n,p}(f,y)\mid\leq\varepsilon.$$

This proves that $M_{n,p}(f)$ is uniformly continuous. $\square$

For $f \in UC_b(I)$, $0 < \alpha \leq 1$, and $t > 0$ define

$$\theta_\alpha(f,t) = \sup_{0<s\leq t} \sup_{x\in I,\, 0<h\leq s} \tfrac{|f(x+h)-f(x)|}{h^\alpha},$$

$\theta_\alpha(f,0) = 0$, and

$$K^\alpha(f) = \sup_{0\leq t} \theta_\alpha(f,t).$$

For $0 < \alpha \leq 1$, let us set $\mathrm{Lip}^\alpha(I)$ for the family of all $f \in UC_b(I)$ such that

$$K^\alpha(f) < \infty.$$

For $0 < \alpha < 1$, we also we consider the subspace

$$\mathrm{lip}^\alpha(I) = \left\{ f \in \mathrm{Lip}^\alpha(I) : \lim_{t\to 0} \theta_\alpha(f,t) = 0 \right\}. \tag{13}$$

This type of spaces appears when we study the approximation in Hölder type norms (see [6]).

We will analyze the problem of the preservation of the constants $K^\alpha(f)$ and the class $\mathrm{lip}^\alpha(I)$ by the operators $M_{n,p}$.

For an analogous of Theorem 12 for Szász-Mirakyan operators see [15] and [12].

Theorem 12. (i) *If* $0 < \alpha \leq 1$ *and* $f \in \mathrm{Lip}^\alpha(I)$, *then* $M_{n,p}(f) \in \mathrm{Lip}^\alpha(I)$, *and*

$$K^\alpha(M_{n,p}(f)) \leq K^\alpha(f), \tag{14}$$

for each $n \in \mathbb{N}$.

(ii) *If* $0 < \alpha \leq 1$, $f \in \mathcal{L}(I)$, $M_{n,p}(f) \in \mathrm{Lip}^\alpha(I)$, *for each* $n \in \mathbb{N}$, *and*

$$K := \sup_{n\in\mathbb{N}} K^\alpha(M_{n,p}(f)) < \infty,$$

then $f \in \mathrm{Lip}^\alpha(I)$.

Proof. (i) Set $g(x) = x^\alpha$. Since the function $g(x)$ is concave function and $M_{n,p}(g,x) \to g(x)$ (Theorem 4) and it follows Theorem 6 that $M_{n,p}(g,x) \leq g(x)$.

For any $k \in \mathbb{N}_0$ and $i \in \mathbb{N}$,

$$\mid f\left(\tfrac{k+i}{a_{n,p}}\right) - f\left(\tfrac{k}{a_{n,p}}\right) \mid\leq \omega\left(f, \tfrac{i}{a_{n,p}}\right) \leq \theta_\alpha\left(f, \tfrac{i}{a_{n,p}}\right)\left(\tfrac{i}{a_{n,p}}\right)^\alpha \leq K^\alpha(f) g\left(\tfrac{i}{a_{n,p}}\right).$$

From Proposition 10 we know that, for $x \in I$ and $h > 0$,

$$\mid M_{n,p}(f,x+h) - M_{n,p}(f,x) \mid\leq K^\alpha(f) M_{n,p}(g,h) \leq K^\alpha(f) h^\alpha.$$

(ii) From Theorem 4 we know that, for each fixed $x \in I$, $M_{n,p}(f,x) \to f(x)$, as $n \to \infty$.

For $x \geq 0$, $h > 0$ fixed, and each $n \in \mathbb{N}$, one has

$\mid f(x+h) - f(x) \mid\leq\mid f(x+h) - M_{n,p}(f,x+h) \mid$

$$+ \mid M_{n,p}(f, x+h) - M_{n,p}(f,x) \mid + \mid M_{n,p}(f,x) - f(x) \mid$$
$$\leq \mid f(x+h) - M_{n,p}(f, x+h) \mid + Kh^{\alpha} + \mid M_{n,p}(f,x) - f(x) \mid .$$

The result follows by taking $n \to \infty$. □

For the preservation of the class $\mathrm{lip}^{\alpha}(I)$ we need some previous results.

PROPOSITION 13. *For $0 < \alpha < 1$ and each $f \in \mathrm{lip}^{\alpha}(I)$,*

$$\theta_{\alpha}(f,t) = \sup_{0<s\leq t} \sup_{0<h\leq s} \frac{\omega(f,h)}{h^{\alpha}},$$

where $\omega(f,t)$ is defined by (12).

Proof. By definition, if $f \in UC_b(I)$, then $\omega(f,s)$ is well defined. It is clear that

$$\sup_{x\in I,\, 0<h\leq s} \frac{|f(x+h)-f(x)|}{h^{\alpha}} \leq \sup_{0<h\leq s} \frac{\omega(f,h)}{h^{\alpha}}.$$

On the other hand, given $\varepsilon > 0$, for any $0 < h \leq s$, there exists $x_h \in I$ such that

$$\omega(f,h) \leq \varepsilon h^{\alpha} + \mid f(x_h + h) - f(x_h) \mid .$$

Therefore

$$\frac{\omega(f,h)}{h^{\alpha}} \leq \varepsilon + \frac{|f(x_h+h)-f(x_h)|}{h^{\alpha}} \leq \varepsilon + \sup_{x\in I,\, 0<h\leq s} \frac{|f(x+h)-f(x)|}{h^{\alpha}}. \square$$

PROPOSITION 14. *If $0 < \alpha < 1$, for each $f \in \mathrm{lip}^{\alpha}(I)$, the functional $\theta_{\alpha}(f,t)$ is a subadditive modulus of continuity.*

Proof. (a) By definition $\theta_{\alpha}(f,0) = 0$ and $\theta_{\alpha}(f,t) \to 0$ as $t \to 0$. Moreover it is clear that $\theta_{\alpha}(f,t)$ is non-negative and non-decreasing in I

(b) Let us verify that $\theta_{\alpha}(f,t)$ is subadditive. Assume $0 < v \leq t$ and fix any s and h such that $0 < s \leq v + t$ and $0 < h \leq s$.

If $x \in I$ and $h \leq t$ it is clear that

$$\frac{|f(x+h)-f(x)|}{h^{\alpha}} \leq \sup_{0<u\leq t} \sup_{y\in I,\, 0<w\leq u} \frac{|f(y+w)-f(y)|}{w^{\alpha}} = \theta_{\alpha}(f,t).$$

We still have to consider the case $v \leq t < h$. Since $t < h$ and $0 < h - t < h$, one has

$$\frac{|f(x+h)-f(x)|}{h^{\alpha}} \leq \frac{|f(x+h-t+t)-f(x+h-t)|}{t^{\alpha}} + \frac{|f(x+h-t)-f(x)|}{(h-t)^{\alpha}}$$
$$\leq \theta_{\alpha}(f,t) + \theta_{\alpha}(f,h-t) \leq \theta_{\alpha}(f,t) + \theta_{\alpha}(f,v),$$

because $\theta_{\alpha}(f,t)$ increases and $h - t \leq s - t \leq v$. Therefore

$$\theta_{\alpha}(f, t+v) \leq \theta_{\alpha}(f,t) + \theta_{\alpha}(f,v).$$

(c) Taking into account that $\theta_{\alpha}(f,0) = 0$ and $\theta(f,t)$ is subadditive, it is a continuous function. □

THEOREM 15. *If $0 < \alpha < 1$, $f \in \mathrm{lip}^{\alpha}(I)$, $n \in \mathbb{N}$, and $t > 0$, then*

$$\theta_{\alpha}(M_{n,p}(f), t) \leq 2\theta_{\alpha}(f,t).$$

Proof. If $0 < s \le t$, taking into account Theorem 11, one has

$$|\, M_{n,p}(f,x+s) - M_{n,p}(f,x)\,| \le 2\,\omega(f,s) = 2\,\frac{\omega(f,s)}{s^\alpha} s^\alpha \le 2\,\theta_\alpha(f,s)s^\alpha \le 2\,\theta_\alpha(f,t)s^\alpha.$$

This is sufficient to prove the result. □

For each $r \ge 0$, $f \in C_r(I)$ (see (4)), and $t \ge 0$, define

$$\Omega_r(f,t) = \sup_{0\le s\le t}\ \sup_{x\ge 0} \frac{|f(x+s)-f(x)|}{(1+x+s)^r}.$$

We will use this modulus only in the case $f \in C_{r,\infty}(I)$.

Before proving some properties of this modulus, let us compare them with others that have been used previously.

The following functional was considered by Kratz and Stadtmüller in [19]. For $r \in \mathbb{N}$ and a function $f \in C_r(I)$ set

$$\widetilde{\Omega}_r(f,t) = \sup_{s,v\in I, |s-v|\le t} \frac{|f(s)-f(v)|}{(1+s+v)^r} = \sup_{x\ge 0}\ \sup_{0<s\le t} \frac{|f(x+s)-f(x)|}{(1+2x+s)^r}.$$

Taking into account that $1+x+s \le 1+2x+s \le 2(1+x+s)$, we know that

$$\frac{1}{2^r}\Omega_r(f,t) \le \widetilde{\Omega}_r(f,t) \le \Omega_r(f,t).$$

Kratz and Stadtmüller proved that, for Szász-Mirakyan operators, there exists a constant C such that, for all $f \in C_r(I)$, every $t \ge 0$ and each $n \in \mathbb{N}$,

$$\widetilde{\Omega}_r(S_n(f),t) \le C\widetilde{\Omega}_r(f,t).$$

They did not proved that $\lim_{t\to 0^+} \widetilde{\Omega}_r(f,t) = 0$. We will verify that, if $f \in C_{r,\infty}(I)$, then $\lim_{t\to 0^+} \Omega_r(f,t) = 0$.

For $f \in C_{2,\infty}(I)$, another modulus was considered in [2] by setting

$$\Omega(f,t) = \sup_{0\le s\le t}\ \sup_{x\in I} \frac{|f(x+s)-f(x)|}{(1+s)^2(1+x)^2}.$$

For $0 \le t \le 1$, $\Omega(f,t)$ and $\Omega_2(f,t)$ are equivalent. In fact, suppose that $s \le 1$. First one has

$$(1+s^2)(1+x^2) = 1+s^2+x^2+s^2x^2 \le 2(1+s^2+x^2) \le 2(1+x+s)^2.$$

On the other hand, if $x \le 1$,

$$(1+s+x)^2 = 1+2x+2s+x^2+2xs+s^2 \le 7(1+s^2+x^2) \le 7(1+s^2)(1+x^2).$$

and, if $x > 1$,

$$(1+s+x)^2 \le 3+5x^2+s^2 \le 5(1+s^2+x^2) \le 5(1+s^2)(1+x^2).$$

Therefore

$$\frac{1}{2}\Omega_2(f,t) \le \Omega(f,t) \le 7\Omega_2(f,t).$$

Proposition 16. *If r is a non negative real and $f \in C_{r,\infty}(I)$, then $\Omega_r(f,t)$ is a subadditive modulus of continuity in the sense of* Definition 8.

Proof. It is clear that $\Omega_r(f,0)=0$ and $\Omega_r(f,t)$ is non-negative and non-decreasing in I

(a) We consider first the case $r=0$. As in the case of the classical modulus of continuity, it is easy to prove that the functional $\Omega_r(f,t)$ is a subadditive. In order to prove continuity, it is sufficient to verify continuity a zero, but if follows from the condition $\lim_{x\to\infty} f(x)=0$.

(b) Assume $r>0$. Denote $A=\lim_{x\to\infty} f(x)/(1+x)^r$. Given $\varepsilon>0$, there exists x_0 such that

$$\left|\frac{f(x)}{(1+x)^r}-A\right|<\frac{\varepsilon}{2}, \qquad x\ge x_0.$$

If $t>0$ and $0<s\le t\le 1$, then

$$\begin{aligned}\sup_{x\ge 0}\frac{|f(x+s)-f(x)|}{(1+x+s)^r} &\le \sup_{0\le x\le x_0}\frac{|f(x+s)-f(x)|}{(1+x+s)^r}+\sup_{x\ge x_0}\frac{|f(x+s)-f(x)|}{(1+x+s)^r}\\ &\le \sup_{0\le x\le x_0}\mid f(x+s)-f(x)\mid+\sup_{x\ge x_0}\frac{|f(x+s)-A|}{(1+x+s)^r}+\sup_{x\ge x_0}\frac{|f(x)-A|}{(1+x)^r}\\ &\le \omega_1(f,t)_{[0,x_0+1]}+\varepsilon,\end{aligned}$$

where $\omega_1(f,t)_{[0,x_0+1]}$ is the usual modulus of continuity in the interval $[0,x_0+1]$.

This is sufficient to prove that $\lim_{t\to 0}\Omega_r(f,t)\to 0=0$.

(c) Let us verify that $\Omega_r(f,t)$ is subadditive: $\Omega_r(f,v+t)\le\Omega_r(f,v)+\Omega_r(f,t)$. Without losing generality we assume that $0<v\le t$.

Fijemos $x\ge 0$ and $0<s\le t+v$.

If $s\le t$, it is clear that

$$\frac{|f(x+s)-f(x)|}{1+(x+s)^r}\le\sup_{0<s\le t}\frac{|f(x+s)-f(x)|}{1+(x+s)^r}=\Omega_r(f,t).$$

Let us consider the case $v\le t<s$. Since $0<s-t$, one has $(1+x+s-t)^r<(1+x+s)^r$. Therefore

$$\begin{aligned}\frac{|f(x+s)-f(x)|}{(1+x+s)^r} &\le \frac{|f(x+s-t+t)-f(x+s-t)|}{(1+(x+s-t)+t)^r}+\frac{|f(x+s-t)-f(x)|}{(1+x+s-t)^r}\\ &\le \Omega_r(f,t)+\Omega_r(f,s-t)\le\Omega_r(f,t)+\Omega_r(f,v).\end{aligned}$$

It is sufficient to prove that $\Omega_r(f,t)$ is a modulus of continuity. □

THEOREM 17. *If r is a non negative real, there exists a constant C such that, for $f\in C_{r,\infty}(I)$, $n\in\mathbb{N}$, and $t>0$,*

$$\Omega_r(M_{n,p}(f),t)\le C\Omega_r(f,t).$$

Proof. Notice that, for $s>0$, taking into account (10), with $t=s$ and $\lambda=i/(sa_{n,p})$,

$$\begin{aligned}\left|f\left(\frac{k+i}{a_{n,p}}\right)-f\left(\frac{k}{a_{n,p}}\right)\right| &\le \Omega_r\left(\frac{i}{a_{n,p}}\right)\left(1+\frac{k}{a_{n,p}}+\frac{i}{a_{n,p}}\right)^r\\ &\le \Omega_r(f,s)\left(1+\frac{i}{a_{n,p}s}\right)\left(1+\frac{k}{a_{n,p}}+\frac{i}{a_{n,p}}\right)^r,\end{aligned}$$

because $\Omega_r(f,s)$ is a subadditive modulus.

Therefore (see Proposition 10)

$$
\begin{aligned}
&\mid M_{n,p}(f,x+s)-M_{n,p}(f,x)\mid=\\
&=\left|e^{-a_{n,p}(x+s)}\sum_{k=0}^{\infty}\frac{a_{n,p}^k x^k}{k!}\sum_{i=0}^{\infty}\frac{a_{n,p}^i}{i!}s^i\Big(f\Big(\tfrac{k+i}{a_{n,p}}\Big)-f\Big(\tfrac{k}{a_{n,p}}\Big)\Big)\right|\\
&\le\frac{\Omega_r(f,s)}{e^{a_{n,p}(x+s)}}\sum_{k=0}^{\infty}\frac{a_{n,p}^k x^k}{k!}\sum_{i=0}^{\infty}\frac{a_{n,p}^i}{i!}s^i\Big(1+\tfrac{i}{a_{n,p}s}\Big)\Big(1+\tfrac{k}{a_{n,p}}+\tfrac{i}{a_{n,p}}\Big)^r.
\end{aligned}
$$

Taking into account Proposition 2 (with $a=1+k/a_{n,p}$), we obtain

$$
\begin{aligned}
&e^{-a_{n,p}s}\sum_{i=0}^{\infty}\frac{a_{n,p}^i}{i!}s^i\Big(1+\tfrac{k}{a_{n,p}}+\tfrac{i}{a_{n,p}}\Big)^r=\\
&=M_{n,p}\Big(\Big(1+\tfrac{k}{a_{n,p}}+e_1\Big)^r,s\Big)\le C(r)\Big(2+\tfrac{k}{a_{n,p}}+s\Big)^r\le 2^rC(r)\Big(1+\tfrac{k}{a_{n,p}}+s\Big)^r.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
&\frac{e^{-a_{n,p}s}}{s}\sum_{i=1}^{\infty}\frac{a_{n,p}^i}{i!}s^i\frac{i}{a_{n,p}}\Big(1+\tfrac{k}{a_{n,p}}+\tfrac{i}{a_{n,p}}\Big)^r=\\
&=e^{-a_{n,p}s}\sum_{i=1}^{\infty}\frac{a_{n,p}^{i-1}}{(i-1)!}s^{i-1}\Big(1+\tfrac{k}{a_{n,p}}+\tfrac{i}{a_{n,p}}\Big)^r\\
&=M_{n,p}\Big(\Big(1+\tfrac{1}{a_{n,p}}+\tfrac{k}{a_{n,p}}+e_1\Big)^r,s\Big)\le M_{n,p}\Big(\Big(2+\tfrac{k}{a_{n,p}}+e_1\Big)^r,s\Big)\\
&\le C(r)\Big(3+\tfrac{k}{a_{n,p}}+s\Big)^r\le 3^rC(r)\Big(1+\tfrac{k}{a_{n,p}}+s\Big)^r.
\end{aligned}
$$

From the estimates given above one has

$$
\begin{aligned}
\mid M_{n,p}(f,x+s)-M_{n,p}(f,x)\mid &\le 3^rC(r)\frac{\Omega_r(f,s)}{e^{a_{n,p}x}}\sum_{k=0}^{\infty}\frac{a_{n,p}^k x^k}{k!}\Big(1+\tfrac{k}{a_{n,p}}+s\Big)^r\\
&\le 6^rC(r)\Omega_r(f,s)(1+x+s)^r,
\end{aligned}
$$

where we use again Proposition 2. □

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