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ON GENERATION AND PROPERTIES OF TRIPLE SEQUENCE-INDUCED FRAMES IN HILBERT SPACES

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Abstract. In this paper, we present the innovative idea of "t-frames", frames produced by triple sequences within Hilbert spaces. The paper explores various properties of these t-frames, delving into topics like frame operators, alternative dual frames, and the stability inherent in t-frames.

MSC. 42C15, 46C50.

Keywords. frame, triple sequence, t-frame, alternate dual t-frame, frame operator.

1. INTRODUCTION AND PRELIMINARIES

In functional analysis and related fields, the concept of frames provides a generalized notion of basis, which allows for redundant and stable representations of elements in a Hilbert space. In a Hilbert space, a frame comprises vectors that enable the representation of any space vector in a stable and surplus fashion. In contrast to a basis, a frame permits multiple ways to represent a vector, offering redundancy that proves beneficial in fields like signal processing and data compression. Frames find utility across mathematics and engineering, impacting signal processing, image compression, and quantum mechanics. They present a versatile and resilient method for analyzing and representing signals or functions within a non-orthogonal basis.

Frames were introduced by Duffin and Schaeffer [10] with a focus on nonharmonic Fourier series, serving as an alternative to orthonormal or Riesz bases within Hilbert spaces. Their paper elegantly presents a substantial portion of the abstract framework for frames. Subsequently, Daubechies *et al.* [8] extended frames to $L^2(R)$ using time-frequency or time-scale translated functions, a development crucial in Gabor and wavelet analysis. The connections between these developments are explored in explanatory discussions found in

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[7] and [12]. Gröchenig's work [11] extended frames to Banach spaces, presenting a significant nontrivial advancement. For more recent research about the frame theory, refer [1, 2, 3, 4, 5, 9, 18].

This paper presents a novel concept termed "t-frames," denoting frames generated by triple sequences within Hilbert spaces. Section 2 will present the concept of t-frames along with their features. Section 3 and Section 4 will then delve into the examination of alternate dual and stability of t-frames, respectively.

Throughout this paper, the symbols $\mathcal{H}$ and $\mathbb{F}$ represent an infinite dimensional Hilbert space and a scalar field of real and complex numbers, respectively. The sets $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of natural, real, and complex numbers, respectively.

Next, we will offer explanations and context related to the concept of frames and triple sequences.

Definition 1 ([6]). *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a Hilbert space $\mathcal{H}$ is said to be a frame for $\mathcal{H}$ if there exist positive constants λ_1 and λ_2 such that*

$$\lambda_1\|x\|^2 \leq \sum_{n\in\mathbb{N}} |\langle x, x_n\rangle|^2 \leq \lambda_2\|x\|^2, \quad x\in\mathcal{H} \tag{1}$$

The positive constants λ_1 and λ_2 are called the lower and upper frame bounds respectively. If $\lambda_1 = \lambda_2$, then $\{x_n\}_{n\in\mathbb{N}}$ is said to be a tight frame and if $\lambda_1 = \lambda_2 = 1$, then $\{x_n\}_{n\in\mathbb{N}}$ is called Parseval frame.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ satisfying the upper frame condition, i.e.,

$$\sum_{n\in\mathbb{N}} |\langle x, x_n\rangle|^2 \leq \lambda_2\|x\|^2$$

is called a Bessel sequence with Bessel bound λ_2.

In this setting, it is crucial to note that not every Bessel sequence within a Hilbert space inherently meets the criteria for being a frame. Nevertheless, it is feasible to convert these sequences into frames by introducing additional elements or by selectively omitting elements from the sequence. In light of this observation, Sharma *et al.* [21] have recently attempted to generate frames for Hilbert spaces using Bessel sequences that do not originally serve as frames for those specific spaces. In essence, they introduced the following definition.

Definition 2 ([21]). *Let $\mathcal{H}$ be a Hilbert space and $\{x_{n,i}\}_{i=1,2,\ldots,m_n}^{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$, where $\{m_n\}$ be an increasing sequence of positive integers. Then, $\{x_{n,i}\}_{\substack{i=1,2,\ldots,m_n\\ n\in\mathbb{N}}}$ is called an approximative frame for $\mathcal{H}$ if there exist positive constants λ_1 and λ_2 such that*

$$\lambda_1\|x\|^2 \leq \lim_{n\to\infty}\sum_{i=1}^{m_n} |\langle x, x_{n,i}\rangle|^2 \leq \lambda_2\|x\|^2, \quad x\in\mathcal{H}. \tag{2}$$

The positive constants λ_1 and λ_2 are called the lower and upper approximative frame bounds, respectively. If $\lambda_1 = \lambda_2$, then $\{x_{n,i}\}_{i=1,2,\ldots,m_n}$ is a tight *approximative frame and if* $\lambda_1 = \lambda_2 = 1$*, then it is called a Parseval approximative frame. A sequence* $\{x_{n,i}\}_{\substack{i=1,2,\ldots,m_n \\ n\in\mathbb{N}}}$ *is said to be an approximative Bessel sequence if right-hand side of inequality* (2) *is satisfied.*

Now, we define a new generalization of frame with the help of triple sequences and named it as t-frame.

The concept of triple sequence and triple series builds upon the foundation of single, double, or regular sequences and series. The function $X : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \mathbb{R}(C)$ serves as a means to characterize a triple sequence, whether it be real or complex. The triple series, represented by the infinite sum $\sum\limits_{i,j,k\in\mathbb{N}} x_{ijk}$, is an integral component of this extension. To define t-frames and establish results regarding the properties of t-frames and frame operators, we employ specific definitions and concepts.

At first, Sahiner *et al.* [20] introduced and explored different ideas associated with triple sequences and their statistical convergence.

Definition 3 ([20]). *A triple sequence* $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ *is said to be convergent to* l *in the Pringsheim's sense if for every* $\epsilon > 0$ *there exists* $N_\epsilon \in \mathbb{N}$ *such that* $|x_{ijk} - l| < \epsilon$ *whenever* $i, j, k \geq N_\epsilon$*, where* l *is called the Pringsheim limit of* $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$.

A triple sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$ *is said to be Cauchy sequence if for every* $\epsilon > 0$ *there exists* $N_\epsilon \in \mathbb{N}$ *such that* $|x_{pqr} - x_{ijk}| < \epsilon$ *for all* $p \geq i \geq N_\epsilon, q \geq j \geq N_\epsilon, r \geq k \geq N_\epsilon$.

The sequence of partial sums of triple sequence $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ *is defined by*

$$S = \{S_{lmn}\}_{l,m,n\in\mathbb{N}}\,, \text{ where } \quad S_{mn} = \sum_{i,j,k=1}^{l,m,n} x_{ijk}, \quad \text{for all } l, m, n \in \mathbb{N}.$$

If $\lim\limits_{l,m,n\to\infty} S_{lmn} = l$*, then the triple series* $\sum_{i,j,k\in\mathbb{N}} x_{ijk}$ *is said to be convergent and vice versa. Also,*

$$\lim_{l,m,n\to\infty} S_{lmn} = \lim_{l,m,n\to\infty} \sum_{i,j,k=1}^{l,m,n} x_{ijk} = \sum_{i,j,k\in\mathbb{N}} x_{ijk}.$$

If no such limit exists then the triple series is divergent.

If every x_{ijk} is non-negative then $\sum_{i,j,k\in\mathbb{N}} x_{ijk}$ is convergent if and only if $\{S_{lmn}\}_{l,m,n\in\mathbb{N}}$ is bounded above. For the further information on triple sequence, refer to [13, 14, 15, 16, 17, 19].

2. t-FRAMES

In this section, we will present the idea of t-frames along with their characteristics.

Definition 4. *The triple sequence* $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ *in* $\mathcal{H}$ *is said to be a* t*-frame for* $\mathcal{H}$ *if there exist positive constants* λ_1 *and* λ_2 *such that*

$$\lambda_1\|x\|^2 \leq \lim_{l,m,n\to\infty}\sum_{i,j,k=1}^{l,m,n} |\langle x, x_{ijk}\rangle|^2 \leq \lambda_2\|x\|^2, \quad \text{for all } x\in\mathcal{H}. \tag{3}$$

The constants λ_1 *and* λ_2 *are called lower and upper* t*-frame bounds respectively. If* $\lambda_1 = \lambda_2$*, then* $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ *is called tight* t*-frame, and if* $\lambda_1 = \lambda_2 = 1$*, then it is called Parseval* t*-frame.*

A triple sequence frame, or t-frame, in a Hilbert space $\mathcal{H}$ can be particularly useful in contexts where data or functions are naturally indexed by three parameters. For example, this can occur in the study of functions of three variables, in multi-dimensional signal processing, or in quantum mechanics where states can be parameterized in three dimensions. Extending the concept of frames to triple sequences, can provide more flexibility and finer granularity in analyzing multi-dimensional data or functions. The redundancy and stability provided by t-frames ensure that even when data is incomplete or corrupted by noise, meaningful reconstructions can still be achieved. The lower bound λ_1 ensures that no information is lost, meaning the frame elements provide a complete and stable representation of any vector in the Hilbert space. The upper bound λ_2 prevents excessive redundancy, which could otherwise lead to inefficiencies or numerical instability.

In physical research, t-frames can be particularly useful in contexts involving multi-dimensional data sets, such as [3, 4, 5]:

(1) **Quantum Mechanics:** In the study of quantum states, where the state of a system might be described by a wave function depending on three parameters (*e.g.*, three spatial dimensions), t-frames can provide a way to decompose and analyze these states.

(2) **Signal Processing:** In applications involving three-dimensional signals (such as video signals where each frame is a 2D image evolving over time), t-frames offer a means to analyze and reconstruct signals in a stable manner, even in the presence of noise or incomplete data.

(3) **Medical Imaging:** Techniques such as MRI or CT scans produce data that can be naturally represented as triple sequences, where i and j could index the pixel coordinates in a slice and k could index the slice number. Using t-frames in this context ensures stable reconstruction and analysis of the medical images, leading to more accurate diagnostics.

Overall, the introduction of t-frames allows for the handling of more complex and higher-dimensional data, ensuring stable representations and facilitating advanced analysis techniques in various scientific and engineering fields.

Remark 5. *A triple sequence* $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ *in Hilbert space* $\mathcal{H}$ *is called t Bessel sequence if it satisfies upper t-frame inequality i.e.,*

$$\lim_{l,m,n\to\infty}\sum_{i,j,k=1}^{l,m,n}|\langle x, x_{ijk}\rangle|^2 \leq \lambda_2\|x\|^2, \quad \text{for all } x\in\mathcal{H}.$$

Remark 6. *Let* $\{y_k\}_{k\in\mathbb{N}}$ *is a frame for Hilbert space* $\mathcal{H}$ *with lower and upper frame bounds* λ_1 *and* λ_2*, respectively. Then, we define a triple sequence* $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ *as*

$$x_{ijk} = \begin{cases} y_k, & i=j \\ 0, & \text{otherwise} \end{cases}$$

which is a t-frame for $\mathcal{H}$ *with the same bounds* λ_1 *and* λ_2*.*

Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$. Following examples vindicate the Definition 4.

Example 7. *Define a sequence* $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ *by*

$$x_{ijk} = \begin{cases} e_k, & i=j \\ 0, & \text{otherwise.} \end{cases}$$

Then $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ *is a Parseval t-frame for* $\mathcal{H}$*.*

We know that every Bessel sequence is not a frame always. One can construct a triple sequence from a given Bessel sequence, which becomes a *t*-frame.

Example 8. *Given a sequence* $\{x_n\}$ *such that* $x_n = \frac{e_n}{\sqrt{n}}$*, for all* $n\in\mathbb{N}$*. Then,* $\{x_n\}$ *is a Bessel sequence but not a frame for* $\mathcal{H}$ *because it does not satisfy the lower condition of frame. Define a sequence* $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ *in* $\mathcal{H}$ *by*

$$\begin{aligned}
&x_{111} = e_1, \\
&x_{211} = x_{212} = x_{221} = x_{222} = \tfrac{e_2}{\sqrt{2}}, \\
&x_{311} = x_{312} = x_{313} = \tfrac{e_3}{\sqrt{3}}, \\
&\quad\vdots \\
&x_{n11} = x_{n12} = x_{n13} = \cdots = x_{nnn} = \tfrac{e_n}{\sqrt{n}}, \\
&\quad\vdots \\
&x_{ijk} = 0, \quad \text{for all } i<j \text{ and } i<k.
\end{aligned}$$

Then, $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ *is a Parseval t-frame.*

Example 9. *Consider the Hilbert space* $\mathcal{H} = L^2([0,1]^3)$*, the space of square-integrable functions on the unit cube* $[0,1]^3$*. Define a triple sequence*

$\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ in $\mathcal{H}$ by:

$$x_{ijk}(u,v,w) = \sqrt{2}\sin(\pi iu)\sin(\pi jv)\sin(\pi kw),$$

for $(u,v,w) \in [0,1]^3$ and $i,j,k \in \mathbb{N}$.

To show that $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ forms a t-frame for $\mathcal{H}$, we need to verify the existence of positive constants λ_1 and λ_2 such that for all $x \in \mathcal{H}$,

$$\lambda_1\|x\|^2 \leq \lim_{l,m,n\to\infty} \sum_{i,j,k=1}^{l,m,n} |\langle x, x_{ijk}\rangle|^2 \leq \lambda_2\|x\|^2.$$

For a function $x \in L^2([0,1]^3)$, the inner product $\langle x, x_{ijk}\rangle$ is given by:

$$\langle x, x_{ijk}\rangle = \int_0^1\int_0^1\int_0^1 x(u,v,w)\sqrt{2}\sin(\pi iu)\sin(\pi jv)\sin(\pi kw)\,du\,dv\,dw.$$

The sequence $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ can be seen as an extension of the trigonometric system to three dimensions, analogous to the Fourier basis. In this case, the Parseval's identity for the trigonometric system ensures that:

$$\sum_{i,j,k=1}^{\infty} |\langle x, x_{ijk}\rangle|^2 = \|x\|^2.$$

Thus, for this triple sequence, we can choose $\lambda_1 = \lambda_2 = 1$, showing that $\{x_{ijk}\}_{i,j,k\in\infty}$ is a Parseval t-frame for $L^2([0,1]^3)$. This example demonstrates the practical application of t-frames in representing and analyzing functions in a three-dimensional domain.

Example 10. *Consider a Hilbert space $\mathcal{H}$ and a triple sequence $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ in $\mathcal{H}$ that satisfies the t-frame condition. An example might be found in image processing, where x_{ijk} represents pixel values in a three-dimensional structure (e.g., a sequence of color images over time). Each pixel's value could depend on its position in the 2D image grid (indexed by i and j) and the time or sequence number k.*

If $\lambda_1 = 0.5$ and $\lambda_2 = 1.5$, the triple sequence $\{x_{ijk}\}$ forms a t-frame if for any image x in $\mathcal{H}$, the inequality

$$0.5\|x\|^2 \leq \lim_{l,m,n\to\infty} \sum_{i,j,k=1}^{l,m,n} |\langle x, x_{ijk}\rangle|^2 \leq 1.5\|x\|^2$$

holds. This ensures that the sequence $\{x_{ijk}\}$ provides a stable and reliable representation of any image x in the space.

Example 11. *The sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n = e_n + e_{n+1} + e_{n+2}$, for all $n \in \mathbb{N}$ is a Bessel sequence for $\mathcal{H}$, but not a frame for $\mathcal{H}$. Define a sequence*

$\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ *in* $\mathcal{H}$ *by*

$$x_{ijk} = \begin{cases} e_i + e_j + e_k, & i = j = k \ or \ i = j+1, k = j+2 \\ & or \ j = i+1, k = i+2 \\ & or \ i = k+1, j = k+2 \\ 0, & otherwise \end{cases}$$

which is a t-frame for $\mathcal{H}$ *with lower and upper t-frame bounds* $\lambda_1 = 9$ *and* $\lambda_2 = 27$ *respectively.*

For the rest part of this paper, we define the space as

$$l^2(\mathbb{N}\times\mathbb{N}\times\mathbb{N}) = \left\{ \{\alpha_{ijk}\}_{i,j,k\in\mathbb{N}} : \alpha_{ijk} \in \mathbb{F}, \lim_{l,m,n\to\infty} \sum_{i,j,k=1}^{l,m,n} |\alpha_{ijk}|^2 < \infty \right\}.$$

Then $l^2(\mathbb{N}\times\mathbb{N}\times\mathbb{N})$ is a Hilbert space with the norm induced by the inner product which is given by

$$\left\langle \{\alpha_{ijk}\}_{i,j,k\in\mathbb{N}}, \{\beta_{ijk}\}_{i,j,k\in\mathbb{N}} \right\rangle = \lim_{l,m,n\to\infty} \sum_{i,j,k=1}^{l,m,n} \alpha_{ijk}\overline{\beta_{ijk}}$$

for all $\{\alpha_{ijk}\}_{i,j,k\in\mathbb{N}}, \{\beta_{ijk}\}_{i,j,k\in\mathbb{N}} \in l^2(\mathbb{N}\times\mathbb{N}\times\mathbb{N})$.

Let $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ be a t-Bessel sequence. Define operator $\mathcal{K} : l^2(\mathbb{N}\times\mathbb{N}\times\mathbb{N}) \to \mathcal{H}$ as

$$\mathcal{K}\left(\{\alpha_{ijk}\}_{i,j,k\in\mathbb{N}}\right) = \lim_{l,m,n\to\infty} \sum_{i,j,k=1}^{l,m,n} \alpha_{ijk}x_{ijk}, \text{ for all } \{\alpha_{ijk}\}_{i,j,k\in\mathbb{N}} \in \iota^2(\mathbb{N}\times\mathbb{N}\times\mathbb{N}).$$

If $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ is a t-frame then operator $\mathcal{K}$ is called pre t-frame (synthesis) operator and the adjoint operator $\mathcal{K}^*$ of $\mathcal{K}$ is called analysis operator for t-frame.

THEOREM 12. *A triple sequence* $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ *in* $\mathcal{H}$ *is a t-Bessel sequence with t-Bessel bound* λ_2 *if and only if the operator* $\mathcal{K}$ *is linear, well defined and bounded with* $\|\mathcal{K}\| \le \sqrt{\lambda_2}$.

Proof. From the definition of $\mathcal{K}$, it is obvious that $\mathcal{K}$ is linear.
Let $\{\alpha_{ijk}\}_{i,j,k\in\mathbb{N}} \in l^2(\mathbb{N}\times\mathbb{N}\times\mathbb{N})$. For any $l,m,n,l',m',n' \in \mathbb{N}$ with $l > l', m > m', n > n'$, we have

$$\begin{aligned} &\left\| \sum_{i,j,k=1}^{l,m,n} \alpha_{ijk}x_{ijk} - \sum_{i,j,k=1}^{l',m',n'} \alpha_{ijk}x_{ijk} \right\| = \\ &= \sup_{\|y\|=1} \left(\left| \left\langle \sum_{i,j,k=1}^{l,m,n} \alpha_{ijk}x_{ijk} - \sum_{i,j,k=1}^{l',m',n'} \alpha_{ijk}x_{ijk}, y \right\rangle \right| \right) \\ &\le \sup_{\|y\|=1} \left(\sum_{i,j,k=1}^{l,m,n} \left| \alpha_{ijk} \left\langle x_{ijk}, y \right\rangle \right| + \sum_{i,j,k=1}^{l',m',n'} \left| \alpha_{ijk} \left\langle x_{ijk}, y \right\rangle \right| \right) \end{aligned}$$

$$\leq \sup_{\|y\|=1}\Bigg(\Big(\sum_{i,j,k=1}^{l,m,n}\Big|\alpha_{ijk}\Big|^2\Big)^{1/2}\Big(\sum_{i,j,k=1}^{l,m,n}\Big|\Big\langle x_{ijk},y\Big\rangle\Big|^2\Big)^{1/2}$$

$$+\Big(\sum_{i,j,k=1}^{l',m',n'}\Big|\alpha_{ijk}\Big|^2\Big)^{1/2}\Big(\sum_{i,j=1}^{l',m',n'}\Big|\Big\langle x_{ijk},y\Big\rangle\Big|^2\Big)^{1/2}\Bigg)$$

$$\leq sqrt\lambda_2\Bigg(\Big(\sum_{i,j,k=1}^{m,n}\Big|\alpha_{ijk}\Big|^2\Big)^{1/2}+\Big(\sum_{i,j,k=1}^{l',m',n'}\Big|\alpha_{ijk}\Big|^2\Big)^{1/2}\Bigg),$$

implies that $\lim\limits_{l,m,n\to\infty}\sum\limits_{i,j,k=1}^{l,m,n}\alpha_{ijk}x_{ijk}$ exists. Hence, $\mathcal{K}$ is well defined.

Further,

$$\begin{aligned}\left\|\mathcal{K}\left(\{\alpha_{ijk}\}_{i,j,k\in\mathbb{N}}\right)\right\| &= \sup_{\|x\|=1}\left|\left\langle\mathcal{K}\left(\{\alpha_{ijk}\}_{i,j,k\in\mathbb{N}}\right),x\right\rangle\right|\\ &= \sup_{\|x\|=1}\lim_{l,m,n\to\infty}\left|\left\langle\sum_{i,j,k=1}^{l,m,n}\alpha_{ijk}x_{ijk},x\right\rangle\right|\\ &\leq\sqrt{\lambda_2}\lim_{l,m,n\to\infty}\left(\sum_{i,j,k=1}^{l,m,n}|\alpha_{ijk}|^2\right)^{1/2}.\end{aligned}$$

This implies that $\left\|\mathcal{K}\left(\{\alpha_{ijk}\}_{i,j,k\in\mathbb{N}}\right)\right\|\leq\sqrt{\lambda_2}\left\|\{\alpha_{ijk}\}_{i,j,k\in\mathbb{N}}\right)\right\|$, hence $\mathcal{K}$ is bounded operator with $\|\mathcal{K}\|\leq\sqrt{\lambda_2}$.

Conversely, for any $x\in\mathcal{H}$, we have

$$\begin{aligned}\left\langle x,\mathcal{K}\left(\{\alpha_{ijk}\}_{i,j,k\in\mathbb{N}}\right)\right\rangle &= \left\langle x,\lim_{l,m,n\to\infty}\sum_{i,j,k=1}^{l,m,n}\alpha_{ijk}x_{ijk}\right\rangle\\ &= \lim_{l,m,n\to\infty}\sum_{i,j,k=1}^{l,m,n}\bar{\alpha}_{ijk}\langle x,x_{ijk}\rangle\\ &= \left\langle\{\langle x,x_{ijk}\rangle\}_{i,j,k\in\mathbb{N}},\{\alpha_{ijk}\}_{i,j,k\in\mathbb{N}}\right\rangle.\end{aligned}$$

Hence,

$$\mathcal{K}^*(x)=\{\langle x,x_{ijk}\rangle\}_{i,j,k\in\mathbb{N}},\quad\text{for all } x\in\mathcal{H}. \tag{4}$$

Thus,

$$\|\mathcal{K}^*(x)\|^2=\lim_{l,m,n\to\infty}\sum_{i,j,k=1}^{l,m,n}|\langle x,x_{ijk}\rangle|^2\leq\sqrt{\lambda_2}\|x\|^2.$$

Hence, $\{x_{ij}\}_{i,j,k\in\mathbb{N}}$ is a t-Bessel sequence with bound λ_2.

Now, define t-frame operator $\mathcal{S}:\mathcal{H}\to\mathcal{H}$ for t-frame $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ by

$$\begin{aligned}\mathcal{S}(x) &= \mathcal{K}\mathcal{K}^*(x)\\ &= \mathcal{K}\left(\{\langle x, x_{ijk}\rangle\}_{i,j,k\in\mathbb{N}}\right)\\ &= \lim_{l,m,n\to\infty}\sum_{i,j,k=1}^{l,m,n}\langle x, x_{ijk}\rangle\, x_{ijk}, \qquad \text{for all } x\in\mathcal{H}.\end{aligned}$$

Since $\mathcal{K}$ and $\mathcal{K}^*$ both are linear, so $\mathcal{S}$ is also linear. □

THEOREM 13. *$\mathcal{S}$ is bounded, self adjoint, positive and invertible operator.*

Proof. $\|\mathcal{S}\| = \|\mathcal{K}\mathcal{K}^*\| \le \|\mathcal{K}\|^2 \le \lambda_2$ and $\mathcal{S}^* = (\mathcal{K}\mathcal{K}^*)^* = \mathcal{K}\mathcal{K}^* = \mathcal{S}$. Hence, $\mathcal{S}$ is bounded and self adjoint operator.

For $x\in\mathcal{H}$,

$$\begin{aligned}\langle\mathcal{S}(x), x\rangle &= \left\langle \lim_{l,m,n\to\infty}\sum_{i,j,k=1}^{l,m,n}\langle x, x_{ijk}\rangle\, x_{ijk}, x\right\rangle\\ &= \lim_{l,m,n\to\infty}\sum_{i,j,k=1}^{l,m,n}\left|\langle x, x_{ijk}\rangle\right|^2.\end{aligned}$$

Using definition of t-frame, we have

$$\lambda_1\langle\mathcal{I}(x), x\rangle \le \langle\mathcal{S}(x), x\rangle \le \lambda_2\langle\mathcal{I}(x), x\rangle, \qquad \text{for all } x\in\mathcal{H}.$$

Hence,

$$\lambda_1\cdot\mathcal{I} \le \mathcal{S} \le \lambda_2\cdot\mathcal{I}. \tag{5}$$

Thus, $\mathcal{S}$ is a positive operator. Moreover,

$$\mathcal{I} - \lambda_2^{-1}\mathcal{S} \le \tfrac{\lambda_2-\lambda_1}{\lambda_2}\mathcal{I},$$

which implies that $\left\|\mathcal{I} - \lambda_2^{-1}\mathcal{S}\right\| < 1$, *i.e.*, $\mathcal{S}$ is invertible. □

THEOREM 14. *A triple sequence $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ in $\mathcal{H}$ is a t-frame for $\mathcal{H}$ if and only if the operator $\mathcal{T}$ is well defined, bounded, linear and surjective.*

Proof. It is clear from Theorem 12 that, the operator $\mathcal{K}$ is well defined, bounded and linear. Since $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ is a t-frame hence the t-frame operator $\mathcal{S} = \mathcal{K}\mathcal{K}^*$ is invertible (bijective) which implies $\mathcal{K}$ is also surjective.

Conversely, let $\mathcal{K}$ is well defined, bounded, linear and surjective.

From Theorem 12, it is already clear that $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ is a Bessel sequence. Now we prove the lower d-frame inequality.

Since $\mathcal{K}$ is surjective and $\mathcal{K}^*$ is one-one operator, then the operator $S = \mathcal{K}\mathcal{K}^*$ is invertible and positive.

For any $a, b\in\mathcal{H}$, $|\langle a, b\rangle| \le \|a\|\|b\|$ (Cauchy Schwarz inequality).

Consider $a = \mathcal{S}^{-\frac{1}{2}}(x)$ and $b = \mathcal{S}^{\frac{1}{2}}(x)$, then

$$\left|\left\langle \mathcal{S}^{-\frac{1}{2}}(x), \mathcal{S}^{\frac{1}{2}}(x)\right\rangle\right| \leq \left\|\mathcal{S}^{-\frac{1}{2}}(x)\right\| \left\|\mathcal{S}^{\frac{1}{2}}(x)\right\|$$

$\implies \left|\left\langle \mathcal{S}^{\frac{1}{2}}\mathcal{S}^{-\frac{1}{2}}(x), x\right\rangle\right| \leq \left\langle \mathcal{S}^{-\frac{1}{2}}(x), \mathcal{S}^{-\frac{1}{2}}(x)\right\rangle^{\frac{1}{2}} \left\langle \mathcal{S}^{\frac{1}{2}}(x), \mathcal{S}^{\frac{1}{2}}(x)\right\rangle^{\frac{1}{2}}$, from which we get

$$\|x\|^2 \leq \left\langle \mathcal{S}^{-1}(x), x\right\rangle^{\frac{1}{2}} \langle \mathcal{S}(x), x\rangle^{\frac{1}{2}}.$$

Squaring both side and using Cauchy Schwarz inequality for $\langle \mathcal{S}^{-1}x, x\rangle$ we have

$$\|x\|^4 \leq \left\|\mathcal{S}^{-1}(x)\right\| \|x\| \langle \mathcal{S}(x), x\rangle.$$

Hence, since $\mathcal{S}$ is bounded,

$$\|x\|^4 \leq \left\|\mathcal{S}^{-1}\right\| \|x\|^2 \langle \mathcal{S}(x), x\rangle.$$

Finally,

$$\frac{1}{\|\mathcal{S}^{-1}\|}\|x\|^2 \leq \langle \mathcal{S}(x), x\rangle = \langle \mathcal{K}\mathcal{K}^*(x), x\rangle = \left\langle \lim_{l,m,n\to\infty} \sum_{i,j,k=1}^{m,n} \langle x, x_{ijk}\rangle\, x_{ij}, x\right\rangle$$
$$= \lim_{l,m,n\to\infty} \sum_{i,j,k=1}^{l,m,n} |\langle x, x_{ijk}\rangle|^2 .$$

Hence $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ is a t-frame for $\mathcal{H}$. □

Now, we establish following result to characterize t-frames in terms of bounded linear operators.

Theorem 15. *The image of a t-frame $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ under a linear bounded operator $\mathbb{K}$ on $\mathcal{H}$ is again a t-frame for $\mathcal{K}$ if and only if there exist a positive constant λ such that the adjoint operator $\mathbb{K}^*$ satisfies*

$$\|\mathbb{K}^*(x)\|^2 \geq \lambda\|x\|^2, \quad \textit{for all } x \in \mathcal{H}.$$

Proof. Since $\mathbb{K}$ is a linear bounded operator hence $\mathbb{K}^*$ is also linear bounded. Taking $\mathbb{K}^*(x) \in \mathcal{H}$ and using the definition of t-frame $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$,

$$\lambda_1 \|\mathbb{K}^*(x)\|^2 \leq \lim_{l,m,n\to\infty} \sum_{i,j,k=1}^{l,m,n} |\langle \mathbb{K}^*(x), x_{ijk}\rangle|^2 \leq \lambda_2 \|\mathbb{K}^*(x)\|^2 .$$

By the given condition, we get

$$\lambda\lambda_1\|x\|^2 \leq \lambda_1 \|\mathbb{T}^*(x)\|^2 \leq \lim_{l,m,n\to\infty} \sum_{i,j,k=1}^{m,n} |\langle x, \mathbb{K}(x_{ijk})\rangle|^2$$
$$\leq \lambda_2 \|\mathbb{K}^*(x)\|^2 \leq \lambda_2 \|\mathbb{K}^*\|^2 \|x\|^2.$$

Thus $\{\mathbb{T}(x_{ijk})\}_{i,j,k\in\mathbb{N}}$ is a t-frame for $\mathcal{H}$. Converse is obvious by the definition of t-frame. □

REMARK 16. *From Theorem 15, it is clear that image of a t-frame under a linear bounded operator is always a t-Bessel sequence.*

In the following theorem, we construct a t-frame with the help of t-frame operator.

THEOREM 17. *For a t-frame $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ with bounds λ_1 and λ_2 respectively and t-frame operator $\mathcal{S}$, the triple sequence $\{\mathcal{S}^{-1}(x_{ijk})\}_{i,j,k\in\mathbb{N}}$ is again a t-frame.*

Proof. From equation (5) of Theorem 13, we have

$$\lambda_1 \cdot \mathcal{I} \leq \mathcal{S} \leq \lambda_2 \cdot \mathcal{I}$$

which implies

$$\lambda_2^{-1} \cdot \mathcal{I} \leq \mathcal{S}^{-1} \leq \lambda_1^{-1}\mathcal{I}$$

Taking inner product with x, we get

$$\lambda_2^{-1}\|x\|^2 \leq \left\langle \mathcal{S}^{-1}x, x \right\rangle \leq \lambda_1^{-1}\|x\|^2. \tag{6}$$

Now,

$$\begin{aligned}\left\langle \mathcal{S}^{-1}(x), x \right\rangle &= \left\langle \mathcal{S}^{-1}\mathcal{S}\mathcal{S}^{-1}(x), x \right\rangle \\ &= \left\langle \mathcal{S}^{-1}\left(\lim_{l,m,n\to\infty} \sum_{i,j,k=1}^{l,m,n} \left\langle \mathcal{S}^{-1}x, x_{ijk} \right\rangle x_{ijk} \right), x \right\rangle \\ &= \left\langle \lim_{l,m,n\to\infty} \sum_{i,j,k=1}^{l,m,n} \left\langle x, \mathcal{S}^{-1}(x_{ijk}) \right\rangle \mathcal{S}^{-1}(x_{ijk}), x \right\rangle \\ &= \lim_{l,m,n\to\infty} \sum_{i,j,k=1}^{l,m,n} \left| \left\langle x, \mathcal{S}^{-1}(x_{ijk}) \right\rangle \right|^2.\end{aligned}$$

Hence, by equation (6), $\{\mathcal{S}^{-1}(x_{ijk})\}_{i,j,k\in\mathbb{N}}$ is a t-frame for $\mathcal{H}$ with lower and upper bound λ_2^{-1} and λ_1^{-1} respectively *i.e.*,

$$\lambda_2^{-1}\|x\|^2 \leq \lim_{l,m,n\to\infty} \sum_{i,j,k=1}^{l,m,n} \left| \left\langle x, \mathcal{S}^{-1}(x_{ijk}) \right\rangle \right|^2 \leq \lambda_1^{-1}\|x\|^2. \tag{7}$$

□

REMARK 18. *In above theorem, equations (6) and (7) show that $\mathcal{S}^{-1}$ is a t-frame operator for the t-frame $\{\mathcal{S}^{-1}(x_{ijk})\}, j \in \mathbb{N}$. And for any $x \in \mathcal{H}$,*

$$x = \lim_{l,m,n\to\infty} \sum_{i,j,k=1}^{m,n} \left\langle x, \mathcal{S}^{-1}(x_{ij}) \right\rangle x_{ijk} = \lim_{l,m,n\to\infty} \sum_{i,j,k=1}^{l,m,n} \langle x, x_{ijk} \rangle \mathcal{S}^{-1}(x_{ijk}).$$

COROLLARY 19. *For a t-frame $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ and operator $\mathcal{S}$, $\left\{\mathcal{S}^{-1/2}(x_{ijk})\right\}_{i,j,k\in\mathbb{N}}$ is Parseval t-frame, where $\mathcal{S}^{-1/2}$ is square root of $\mathcal{S}^{-1}$.*

3. ALTERNATE DUAL t-FRAMES

In this section, we examine the alternate or canonical dual of a t-frame along with its associated characteristics.

DEFINITION 20. *Let $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ be a t-frame for Hilbert space $\mathcal{H}$. A t-frame $\{\tilde{x}_{ijk}\}_{i,j,k\in\mathbb{N}}$ is called alternate dual t-frame for $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$, if for all $x \in \mathcal{H}$*

$$x = \lim_{l,m,n\to\infty} \sum_{i,j,k=1}^{m,n} \langle x, \tilde{x}_{ijk}\rangle x_{ijk}$$

or

$$x = \sum_{i,j,k\in\mathbb{N}} \langle x, \tilde{x}_{ijk}\rangle x_{ijk}.$$

REMARK 21. $\{\mathcal{S}^{-1}(x_{ij})\}_{i,j,k\in\mathbb{N}}$ *is a special type of dual t-frame for* $\{x_{ij}\}_{i,j,k\in\mathbb{N}}$, *called canonical dual t-frame.*

THEOREM 22. *Let $\{\tilde{x}_{ijk}\}_{i,j,k\in\mathbb{N}}$ be an alternate dual t-frame for a t-frame $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$. Then, for every $P \times Q \times R \subseteq \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ and $x \in \mathcal{H}$,*

$$\sum_{(i,j,k)\in P\times Q\times R} \langle x, \tilde{x}_{ijk}\rangle \overline{\langle x, x_{ijk}\rangle} - \left\| \sum_{(i,j,k)\in P\times Q\times R} \langle x, \tilde{x}_{ijk}\rangle x_{ijk} \right\|^2 = \tag{8}$$

$$= \overline{\sum_{(i,j,k)\in P^c\times Q^c\times R^c} \langle x, \tilde{x}_{ijk}\rangle \overline{\langle x, x_{ijk}\rangle}} - \left\| \sum_{(i,j,k)\in P^c\times Q^c\times R^c} \langle x, \tilde{x}_{ijk}\rangle x_{ijk} \right\|^2.$$

Proof. For $x \in \mathcal{H}$ and $P \times Q \times R \subseteq \mathbb{N}\times\mathbb{N}\times\mathbb{N}$, define the operator $\mathcal{K}_{P\times Q\times R}$ as

$$\mathcal{K}_{P\times Q\times R}(x) = \sum_{(i,j,k)\in P\times Q\times R} \langle x, \tilde{x}_{ijk}\rangle x_{ijk}.$$

It is obvious that the operator $\mathcal{K}_{P\times Q\times R}(x)$ is well defined, linear and bounded. And by the definition of dual t-frame, we have

$$\mathcal{K}_{P\times Q\times R} + \mathcal{K}_{P^c\times Q^c\times R^c} = I.$$

Therefore,

$$\begin{aligned}\mathcal{K}_{P\times Q\times R} - \mathcal{K}^*_{P\times Q\times R}\mathcal{K}_{P\times Q\times R} &= \left(I - \mathcal{K}^*_{P\times Q\times R}\right)\mathcal{K}_{P\times Q\times R}\\ &= \mathcal{K}^*_{P^c\times Q^c\times R^c}\left(I - \mathcal{K}_{P^c\times Q^c\times R^c}\right)\\ &= \mathcal{K}^*_{P^c\times Q^c\times R^c} - \mathcal{K}^*_{P^c\times Q^c\times R^c}\mathcal{K}_{P^c\times Q^c\times R^c}.\end{aligned}$$

Hence,

$$\sum_{(i,j,k)\in P\times Q\times R} \langle x, \tilde{x}_{ijk}\rangle \overline{\langle x, x_{ijk}\rangle} - \left\| \sum_{(i,j,k)\in P\times Q\times R} \langle x, \tilde{x}_{ijk}\rangle x_{ijk} \right\|^2 =$$

$$= \langle \mathcal{K}_{P\times Q\times R}(x), x\rangle - \left\langle \mathcal{K}^*_{P\times Q\times R}\mathcal{K}_{P\times Q\times R}(x), x\right\rangle$$

$$= \left\langle \mathcal{K}^*_{P^c\times Q^c\times R^c}(x), x \right\rangle - \left\langle \mathcal{K}^*_{P^c\times Q^c\times R^c}\mathcal{K}_{P^c\times Q^c\times R^c}(x), x \right\rangle$$
$$= \langle x, \mathcal{K}_{P^c\times Q^c\times R^c}(x)\rangle - \langle \mathcal{K}_{P^c\times Q^c\times R^c}(x), \mathcal{K}_{P^c\times Q^c\times R^c}(x)\rangle .$$

□

REMARK 23. *Every Parseval t-frame is dual t-frame of itself. Hence identity* (8) *becomes*

$$\sum_{(i,j)\in N\times M} |\langle x, x_{ijk}\rangle|^2 - \left\| \sum_{(i,j,k)\in N\times M} \langle x, x_{ijk}\rangle x_{ij} \right\|^2 =$$
$$= \sum_{(i,j)\in P^c\times Q^c\times R^c} |\langle x, x_{ijk}\rangle|^2 - \left\| \sum_{(i,j,k)\in P^c\times Q^c\times R^c} \langle x, x_{ijk}\rangle x_{ij} \right\|^2 ,$$

which is called Parseval t-frame identity.

4. STABILITY OF t-FRAMES

In this section, we investigate the stability of t-frames and establish similar results regarding the stability of the corresponding canonical dual t-frame.

THEOREM 24. *Let* $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ *be a t-frame with lower and upper t-frame bounds* λ_1, λ_2 *respectively, and* $\{y_{ijk}\}_{i,j,k\in\mathbb{N}}$ *be a triple sequence in* $\mathcal{H}$ *such that* $\exists\ \lambda, \mu \geq 0$ *with* $\left(\lambda + \frac{\mu}{\sqrt{\lambda_1}}\right) < 1$ *and*

(9)
$$\lim_{m,n\to\infty} \left\| \sum_{i,j,k=1}^{l,m,n} \alpha_{ij}\left(x_{ijk} - y_{ijk}\right) \right\| \leq \lambda \lim_{l,m,n\to\infty} \left\| \sum_{i,j=1}^{m,n} \alpha_{ijk} x_{ijk} \right\| + \mu \left\| \{\alpha_{ijk}\}_{i,j,k\in\mathbb{N}} \right\| ,$$

for all $\{\alpha_{ijk}\}_{i,j\in\mathbb{N}} \in l^2(\mathbb{N}\times\mathbb{N}\times\mathbb{N})$.

Then, $\{y_{ijk}\}_{i,j,k\in\mathbb{N}}$ *is also a t-frame for* $\mathcal{H}$ *with lower and upper t-frame bounds* $\lambda_1\left(1 - \left(\lambda + \frac{\mu}{\sqrt{\lambda_1}}\right)\right)^2$ *and* $\lambda_2\left(1 + \lambda + \frac{\mu}{\sqrt{\lambda_2}}\right)^2$ *respectively.*

Proof. Given $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ is a t-frame for $\mathcal{H}$. Let $\mathcal{K}$ be the pre t-frame operator. From Theorem 12, we have

$$\left\| \mathcal{K}\left(\{\alpha_{ijk}\}_{i,j,k\in\mathbb{N}}\right) \right\| = \lim_{l,m,n\to\infty} \left\| \sum_{i,j,k=1}^{l,m,n} \alpha_{ijk} x_{ijk} \right\| \leq \sqrt{\lambda_2} \left\| \{\alpha_{ijk}\}_{i,j,k\in\mathbb{N}} \right\| , \tag{10}$$

for $\{\alpha_{ijk}\}_{i,j,k\in\mathbb{N}} \in l^2(\mathbb{N}\times\mathbb{N}\times\mathbb{N})$.

For the given triple sequence $\{y_{ijk}\}_{i,j,k\in\mathbb{N}}$, we have

$$\lim_{l,m,n\to\infty} \left\| \sum_{i,j,k=1}^{l,m,n} \alpha_{ijk} y_{ijk} \right\| \leq$$

$$\leq \lim_{l,m,n\to\infty}\left\|\sum_{i,j,k=1}^{l,m,n}\alpha_{ij}\left(x_{ijk}-y_{ijk}\right)\right\|+\lim_{l,m,n\to\infty}\left\|\sum_{i,j,k=1}^{l,m,n}\alpha_{ijk}x_{ijk}\right\|$$
$$\leq (1+\lambda)\lim_{l,m,n\to\infty}\left\|\sum_{i,j,k=1}^{l,m,n}\alpha_{ijk}x_{ijk}\right\|+\mu\left\|\{\alpha_{ijk}\}_{i,j,k\in\mathbb{N}}\right\|.$$

By equation (10), we get

$$\lim_{l,m,n\to\infty}\left\|\sum_{i,j,k=1}^{l,m,n}\alpha_{ijk}y_{ijk}\right\|\leq\left((1+\lambda)\sqrt{\lambda_2}+\mu\right)\left\|\{\alpha_{ijk}\}_{i,j,k\in\mathbb{N}}\right\|. \tag{11}$$

Now, define an another operator $\mathcal{U}: l^2(\mathbb{N}\times\mathbb{N}\times\mathbb{N})\to\mathcal{H}$ as

$$\mathcal{U}\left(\{\alpha_{ijk}\}_{i,j,k\in\mathbb{N}}\right)=\lim_{l,m,n\to\infty}\sum_{i,j,k=1}^{l,m,n}\alpha_{ijk}y_{ijk}.$$

So, for $l>l', m>m'$ and $n>n'$, where $l,m,n,,l',m',n'\in\mathbb{N}$, we have

$$\left\|\sum_{i,j,k=1}^{l,m,n}\alpha_{ijk}y_{ijk}-\sum_{i,j,k=1}^{m',n'}\alpha_{ijk}y_{ijk}\right\|\leq\left\|\sum_{i,j,k=1}^{l,m,n}\alpha_{ijk}y_{ijk}\right\|+\left\|\sum_{i,j,k=1}^{l',m',n'}\alpha_{ijk}y_{ijk}\right\|.$$

Using (10), we get

$$\left\|\sum_{i,jk,=1}^{l,m,n}\alpha_{ijk}y_{ijk}-\sum_{i,j,k=1}^{l',m',n'}\alpha_{ijk}y_{ijk}\right\|\leq 2\left((1+\lambda)\sqrt{\lambda_2}+\mu\right)\left\|\{\alpha_{ijk}\}_{i,j,k\in\mathbb{N}}\right\|.$$

Since $\{\alpha_{ijk}\}_{i,j,k\in\mathbb{N}}\in l^2(\mathbb{N}\times\mathbb{N})$, hence sequence of partial sums of $\lim_{l,m,n\to\infty}\sum_{i,j,k=1}^{l,m,n}\alpha_{ijk}y_{ijk}$ is Cauchy, *i.e.*, $\lim_{l,m,n\to\infty}\sum_{i,j,k=1}^{l,m,n}\alpha_{ijk}y_{ijk}$ exists. Which implies

$$\left\|\mathcal{U}\left(\{\alpha_{ij}\}_{i,j,k\in\mathbb{N}}\right)\right\|\leq\left((1+\lambda)\sqrt{\lambda_2}+\mu\right)\left\|\{\alpha_{ijk}\}_{i,j,k\in\mathbb{N}}\right\|. \tag{12}$$

Therefore, operator $\mathcal{U}$ is linear, well defined and bounded. Thus, by Theorem 12, $\{y_{ijk}\}_{i,j,k\in\mathbb{N}}$ is a Bessel sequence for $\mathcal{H}$ with bound $\lambda_2\left(1+\lambda+\frac{\mu}{\sqrt{\lambda_2}}\right)^2$ *i.e.*,

$$\lim_{l,m,n\to\infty}\sum_{i,j,k=1}^{l,m,n}|\langle y_{ijk},x\rangle|^2\leq\lambda_2\left(1+\lambda+\frac{\mu}{\sqrt{\lambda_2}}\right)^2. \tag{13}$$

Now, using $\mathcal{K}$ and $\mathcal{U}$ in equation (9)

$$\left\|\mathcal{K}\left(\{\alpha_{ijk}\}_{i,j,k\in\mathbb{N}}\right)-\mathcal{U}\left(\{\alpha_{ij}\}_{i,j,k\in\mathbb{N}}\right)\right\|\leq\lambda\left\|\mathcal{K}\left(\{\alpha_{ijk}\}_{i,j,k\in\mathbb{N}}\right)\right\|+\mu\left\|\{\alpha_{ijk}\}_{i,j,k\in\mathbb{N}}\right\|. \tag{14}$$

By Theorem 17, we know that $S=\mathcal{K}^*$ is a d-frame operator for $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ with upper bound λ_1^{-1}.

Again, consider $\mathcal{K}^{\dagger}:\mathcal{H}\to l^2(\mathbb{N}\times\mathbb{N})$ as

$$\begin{aligned}\mathcal{K}^{\dagger}(x)&=\mathcal{K}^{*}\left(\mathcal{K}\mathcal{K}^{*}\right)^{-1}(x)\\&=\left\{\left\langle x,\left(\mathcal{K}\mathcal{K}^{*}\right)^{-1}(x_{ijk})\right\rangle\right\}_{i,j,k\in\mathbb{N}},\quad\text{for all } x\in\mathcal{H}.\end{aligned}$$

So,

$$\begin{aligned}\left\|\mathcal{K}^{\dagger}(x)\right\|^2&=\lim_{l,m,n\to\infty}\sum_{i,j,k=1}^{l,m,n}\left|\left\langle x,\left(\mathcal{K}\mathcal{K}^{*}\right)^{-1}(x_{ijk})\right\rangle\right|^2\\&\leq\lambda_1^{-1}\|x\|^2,\quad\text{for all } x\in\mathcal{H}.\end{aligned}$$

Put $\{\alpha_{ijk}\}_{i,j,k\in\mathbb{N}}=\mathcal{K}^{\dagger}(x)$ in equation (14), we get

$$\left\|x-\mathcal{U}\mathcal{K}^{\dagger}(x)\right\|\leq\left(\lambda+\tfrac{\mu}{\sqrt{\lambda_1}}\right)\|x\|,\quad\text{for all } x\in\mathcal{H}.$$

Given that, $\left(\lambda+\frac{\mu}{\sqrt{\lambda_1}}\right)<1$. Therefore, $\mathcal{U}\mathcal{T}^{\dagger}$ is invertible and

$$\left\|\mathcal{U}\mathcal{K}^{\dagger}\right\|\leq1+\lambda+\tfrac{\mu}{\sqrt{\lambda_1}},\quad\left\|\left(\mathcal{U}\mathcal{K}^{\dagger}\right)^{-1}\right\|\leq\frac{1}{1-\left(\lambda+\frac{\mu}{\sqrt{\lambda_1}}\right)}.$$

For $x\in\mathcal{H}$,

$$\begin{aligned}x&=\left(\mathcal{U}\mathcal{K}^{\dagger}\right)\left(\mathcal{U}\mathcal{K}^{\dagger}\right)^{-1}(x)\\&=\lim_{m,n\to\infty}\sum_{i,j,k=1}^{m,n}\left\langle\left(\mathcal{U}\mathcal{K}^{\dagger}\right)^{-1}(x),\left(\mathcal{K}\mathcal{K}^{*}\right)^{-1}(x_{ijk})\right\rangle y_{ijk}.\end{aligned}$$

This implies

$$\|x\|^2=\langle x,x\rangle=\left|\lim_{m,n\to\infty}\sum_{i,j,k=1}^{l,m,n}\left\langle(\mathcal{U}\mathcal{K}^{\dagger})^{-1}(x),(\mathcal{K}^{*})^{-1}(x_{ijk})\right\rangle\left\langle y_{ijk},x\right\rangle\right|.$$

Squaring both sides, we get

$$\begin{aligned}\|x\|^4&=\left|\lim_{l,m,n\to\infty}\sum_{i,j,k=1}^{l,m,n}\left\langle\left(\mathcal{U}\mathcal{K}^{\dagger}\right)^{-1}(x),\left(\mathcal{K}\mathcal{K}^{*}\right)^{-1}(x_{ijk})\right\rangle\langle y_{ijk},x\rangle\right|^2\\&\leq\lim_{l,m,n\to\infty}\frac{1}{\lambda_1}\left\|\left(\mathcal{U}\mathcal{K}^{\dagger}\right)^{-1}(x)\right\|^2\sum_{i,j,k=1}^{m,n}|\langle y_{ijk},x\rangle|^2\\&\leq\tfrac{1}{\lambda_1}\frac{1}{\left(1-\left(\lambda+\frac{\mu}{\sqrt{\lambda_1}}\right)\right)^2}\|x\|^2\lim_{l,m,n\to\infty}\sum_{i,j,k=1}^{l,m,n}|\langle y_{ijk},x\rangle|^2,\quad\text{for all } x\in\mathcal{H}.\end{aligned}$$

Therefore,

$$\lambda_1\left(1-\left(\lambda+\tfrac{\mu}{\sqrt{\lambda_1}}\right)\right)^2\|x\|^2\leq\lim_{l,m,n\to\infty}\sum_{i,j,k=1}^{l,m,n}|\langle y_{ijk},x\rangle|^2.\tag{15}$$

From (13) and (15), we have $\{y_{ijk}\}_{i,j,k\in\mathbb{N}}$ a t-frame for $\mathcal{H}$. □

The lemma below is employed to examine the stability theorem for the canonical dual t-frame.

LEMMA 25. *Let $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ be a t-Bessel sequence in $\mathcal{H}$ with t-Bessel bound β. Then for any $\{c_{ijk}\}_{i,j,k\in\mathbb{N}} \in \mathfrak{l}^2(\mathbb{N}\times\mathbb{N}\times\mathbb{N})$,*

$$\left\| \sum_{i,j,k\in\mathbb{N}} c_{ijk}x_{ijk} \right\|^2 \leq \beta \sum_{i,j,k\in\mathbb{N}} |c_{ijk}|^2 .$$

Proof. For any $\{c_{ijk}\}_{i,j,k\in\mathbb{N}} \in l^2(\mathbb{N}\times\mathbb{N}\times\mathbb{N})$, we have

$$\begin{aligned}\left\| \sum_{i,j,k\in\mathbb{N}} c_{ijk}x_{ijk} \right\|^2 &= \sup_{\|x\|=1} \left| \left\langle \sum_{i,j,k\in\mathbb{N}} c_{ijk}x_{ijk}, x \right\rangle \right|^2 \\ &= \sup_{\|x\|=1} \left| \sum_{i,j,k\in\mathbb{N}} c_{ijk} \langle x_{ijk}, x\rangle \right|^2 \\ &\leq \sup_{\|x\|=1} \sum_{i,j,k\in\mathbb{N}} |c_{ijk}|^2 \sum_{i,j,k\in\mathbb{N}} |\langle x_{ijk}, x\rangle|^2 \\ &\leq \beta \sum_{i,j,k\in\mathbb{N}} |c_{ijk}|^2 .\end{aligned}$$

□

THEOREM 26. *Let $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ and $\{y_{ijk}\}_{i,j,k\in\mathbb{N}}$ be t-frames for $\mathcal{H}$ with (λ_1, μ_1) and (λ_2, μ_2) as respective lower and upper bounds. And let $\{\tilde{x}_{ijk}\}_{i,j,k\in\mathbb{N}}$ and $\{\tilde{y}_{ijk}\}_{i,j,k\in\mathbb{N}}$ are their respective canonical dual t-frames.*

(i) *If $\{x_{ijk} - y_{ijk} : i,j,k\in\mathbb{N}\}$ is a t-Bessel sequence with bound β, then $\{\tilde{x}_{ijk} - \tilde{y}_{ijk} : i,j,k\in\mathbb{N}\}$ is also t-Bessel sequence with bound*

$$\beta \left(\frac{\lambda_1+\lambda_2+\lambda_2^{\frac{1}{2}}\mu_2^{\frac{1}{2}}}{\lambda_1\mu_1} \right)^2 .$$

(ii) If

$$\left| \sum_{i,j,k\in\mathbb{N}} \left| \langle x_{ijk}, x\rangle \right|^2 - \sum_{i,j,k\in\mathbb{N}} |\langle y_{ijk}, x\rangle|^2 \right| \leq \gamma\|x\|^2, \qquad \text{for all } x\in\mathcal{H}$$

then

$$\left| \sum_{i,j,k\in\mathbb{N}} \left| \langle \tilde{x}_{ijk}, x\rangle \right|^2 - \sum_{i,j,k\in\mathbb{N}} \left| \langle \tilde{y}_{ijk}, x\rangle \right|^2 \right| \leq \frac{\gamma}{\lambda_1\mu_1}\|x\|^2, \qquad \text{for all } x\in\mathcal{H}$$

where γ is a real positive number.

Proof. Let $\mathcal{S}$ and $\mathcal{K}$ are t-frame operator for $\{x_{ijk}\}_{i,j,k\in\mathbb{N}}$ and $\{y_{ijk}\}_{i,j,k\in\mathbb{N}}$ respectively. Then, for $x \in \mathcal{H}$

$$\mathcal{S}(x) = \sum_{i,j,k\in\mathbb{N}} \langle x, x_{ijk}\rangle x_{ijk} \quad \text{and} \quad \mathcal{K}(x) = \sum_{i,j,k\in\mathbb{N}} \langle x, y_{ijk}\rangle y_{ijk}.$$

Since $\{\tilde{x}_{ijk}\}_{i,j,k\in\mathbb{N}}$ and $\{\tilde{y}_{ijk}\}_{i,j,k\in\mathbb{N}}$ are canonical dual t-frames, so $\{\tilde{x}_{ijk}\}_{i,j,k\in\mathbb{N}} = \{S^{-1}(x_{ijk})\}_{i,j,k\in\mathbb{N}}$ and $\{\tilde{y}_{ijk}\}_{i,j\in\mathbb{N}} = \{K^{-1}(y_{ijk})\}_{i,j,k\in\mathbb{N}}$.

(i) Given that $\{x_{ijk} - y_{ijk} : i, j, k \in \mathbb{N}\}$ is a t-Bessel sequence with bound β. So,

$$\begin{aligned}\|\mathcal{S}(x) - \mathcal{K}(x)\| &= \left\| \sum_{i,j,k\in\mathbb{N}} \left(\langle x, x_{ijk}\rangle x_{ijk} - \langle x, y_{ijk}\rangle y_{ijk}\right)\right\| \\ &\le \left\| \sum_{i,j,k\in\mathbb{N}} \langle x, x_{ijk}\rangle (x_{ijk} - y_{ijk})\right\| + \left\| \sum_{i,j,k\in\mathbb{N}} \langle x, x_{ijk} - y_{ijk}\rangle y_{ijk}\right\|.\end{aligned}$$

Using Lemma 25, we get

$$\begin{aligned}&\left\|\mathcal{S}(x) - \mathcal{K}(x)\right\| \le \\ &\le \beta^{1/2}\Bigg(\sum_{i,j,k\in\mathbb{N}} \left|\left\langle x, x_{ijk}\right\rangle\right|^2\Bigg)^{1/2} + \mu_2^{1/2}\Bigg(\sum_{i,j,k\in\mathbb{N}} \left|\left\langle x, x_{ijk} - y_{ijk}\right\rangle\right|^2\Bigg)^{1/2} \\ &\le \beta^{1/2}(\lambda_2^{1/2} + \mu_2^{1/2})\|x\|.\end{aligned}$$

Hence,

$$\|\mathcal{S} - \mathcal{K}\| \le \beta^{1/2}\left(\lambda_2^{1/2} + \mu_2^{1/2}\right)$$

and

$$\begin{aligned}\left\|\mathcal{S}^{-1} - \mathcal{K}^{-1}\right\| &= \left\|\mathcal{K}^{-1}(\mathcal{K} - \mathcal{S})\mathcal{S}^{-1}\right\| \\ &\le \left\|\mathcal{K}^{-1}\right\| \|\mathcal{K} - \mathcal{S}\| \left\|\mathcal{S}^{-1}\right\| \\ &\le \tfrac{1}{\lambda_1\mu_1}\beta^{1/2}\left(\lambda_2^{1/2} + \mu_2^{1/2}\right).\end{aligned}$$

Now, we prove that $\{\tilde{x}_{ijk} - \tilde{y}_{ijk} : i, j, k \in \mathbb{N}\}$ is a t-Bessel sequence.

$$\begin{aligned}\sum_{i,j,k\in\mathbb{N}} |\left\langle \tilde{x}_{ijk} - \tilde{y}_{ijk}, x\right\rangle^2 &= \sum_{i,j,k\in\mathbb{N}} \left|\left\langle \mathcal{S}^{-1}(x_{ij}) - \mathcal{K}^{-1}(y_{ijk}), x\right\rangle\right|^2 \\ &= \sum_{i,j,k\in\mathbb{N}} |\left\langle x_{ijk}, \mathcal{S}^{-1}(x)\right\rangle - \left\langle x_{ijk}, \mathcal{K}^{-1}(x)\right\rangle \\ &\quad + \left\langle x_{ijk}, \mathcal{K}^{-1}(x)\right\rangle - \left\langle y_{ijk}, \mathcal{K}^{-1}(x)\right\rangle\Big|^2 \\ &= \sum_{i,j\in\mathbb{N}} \left|\left\langle x_{ij}, \left(\mathcal{S}^{-1} - \mathcal{K}^{-1}\right)(x)\right\rangle + \left\langle x_{ijk} - y_{ijk}, \mathcal{K}^{-1}(x)\right\rangle\right|^2.\end{aligned} \tag{16}$$

For the computation on the right-hand side, we possess;

$$\sum_{i,j,k\in\mathbb{N}} \left|\left\langle x_{ijk}, \left(\mathcal{S}^{-1}-\mathcal{K}^{-1}\right)(x)\right\rangle\right|^2 \leq \lambda_2 \left\|\left(\mathcal{S}^{-1}-\mathcal{K}^{-1}\right)(x)\right\|^2 \leq \frac{\lambda_2}{\lambda_1^2\mu_1^2}\beta\left(\lambda_2^{1/2}+\mu_2^{1/2}\right)^2\|x\|^2 \tag{17}$$

and

$$\sum_{i,j,k\in\mathbb{N}} \left|\left\langle x_{ijk}-y_{ijk}, \mathcal{K}^{-1}(x)\right\rangle\right|^2 \leq \beta\left\|\mathcal{K}^{-1}(x)\right\|^2 \leq \frac{\beta}{\mu_1^2}\|x\|^2. \tag{18}$$

Using equation (17) and (18) in equation (16), we get

$$\sum_{i,j,k\in\mathbb{N}} |\langle \tilde{x}_{ijk}-\tilde{y}_{ijk}, x\rangle|^2 \leq \beta\left(\frac{\lambda_1+\lambda_2+\lambda_2^{\frac{1}{2}}\mu_2^{\frac{1}{2}}}{\lambda_1\mu_1}\right)^2\|x\|^2.$$

(ii) Since both $\mathcal{S}$ and $\mathcal{K}$ are self adjoint, we have

$$\begin{aligned}\|\mathcal{S}-\mathcal{K}\| &= \sup_{\|x\|=1}\left|\left\langle(\mathcal{S}-\mathcal{K})(x),x\right\rangle\right| \\ &= \sup_{\|x\|=1}\left|\left\langle\mathcal{S}(x),x\right\rangle-\left\langle\mathcal{K}(x),x\right\rangle\right| \\ &= \sup_{\|x\|=1}\left|\sum_{i,j,k\in\mathbb{N}}\left|\left\langle x_{ijk},x\right\rangle\right|^2-\sum_{i,j,k\in\mathbb{N}}\left|\left\langle y_{ijk},x\right\rangle\right|^2\right| \leq \gamma,\end{aligned}$$

and

$$\left\|\mathcal{S}^{-1}-\mathcal{K}^{-1}\right\| \leq \frac{1}{\lambda_1\mu_1}\gamma.$$

We have $\tilde{x}_{ijk}=\mathcal{S}^{-1}(x_{ijk})$, so

$$\begin{aligned}\sum_{i,j,k\in\mathbb{N}}|\langle\tilde{x}_{ijk},x\rangle|^2 &= \sum_{i,j,k\in\mathbb{N}}\left|\left\langle\mathcal{S}^{-1}(x_{ijk}),x\right\rangle\right|^2 = \sum_{i,j\in\mathbb{N}}\left|\left\langle x_{ijk},\mathcal{S}^{-1}x\right\rangle\right|^2 \\ &= \left\langle\mathcal{S}\mathcal{S}^{-1}(x),\mathcal{S}^{-1}(x)\right\rangle = \left\langle x,\mathcal{S}^{-1}(x)\right\rangle.\end{aligned}$$

Similarly,

$$\sum_{i,j,k\in\mathbb{N}}|\langle\tilde{y}_{ijk},x\rangle|^2 = \left\langle x,\mathcal{K}^{-1}(x)\right\rangle.$$

Hence,

$$\begin{aligned}\left|\sum_{i,j,k\in\mathbb{N}}\left|\langle\tilde{x}_{ijk},x\rangle\right|^2-\sum_{i,j,k\in\mathbb{N}}|\langle\tilde{y}_{ijk},x\rangle|^2\right| &= \left|\left\langle x,\mathcal{S}^{-1}(x)\right\rangle-\left\langle x,\mathcal{K}^{-1}(x)\right\rangle\right| \\ &\leq \left\|\mathcal{S}^{-1}-\mathcal{K}^{-1}\right\|\|x\|^2 \leq \frac{\gamma}{\lambda_1\mu_1}\|x\|^2.\end{aligned}$$

□

5. CONCLUSION

The paper presents the concept of frames produced by triple sequences within Hilbert spaces, referred to as t-frames. It thoroughly explores various characteristics of t-frames, including frame operators, alternate dual t-frames, and stability for t-frames. The paper also delves into potential applications of t-frames in diverse fields of study, with a particular focus on signal processing, indicating future avenues for research. Furthermore, By defining and studying t-frames, researchers can develop new mathematical tools and techniques for analyzing and processing multi-dimensional data, leading to advancements in both theoretical and applied disciplines.

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