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A NUMERICAL STUDY OF AN INFEASIBLE INTERIOR-POINT ALGORITHM FOR CONVEX QUADRATIC SEMI-DEFINITE OPTIMIZATION

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Abstract. The focus of this research is to apply primal-dual interior-point path-following methods, specifically those derived from Newton's method for solving convex quadratic semidefinite optimization (CQSDO) problems. In this paper, we present a numerical study of an infeasible primal-dual interior-point method for tackling this class of optimization problems. Unlike the feasible interior-point algorithms, the proposed algorithm can be start with any initial positive definite matrix and does not require the strictly feasible initial points. Under certain conditions, the Newton system is well defined and its Jacobian is nonsingular at the solution. For computing an iteration throughout the algorithm, a Newton direction and a step-size are determined. Here, our search direction is based on Alizadeh-Haeberly-Overton (AHO) symmetrization. However, for the step size along this direction two alternatives are suggested. Preliminary numerical results demonstrate the efficiency of our algorithm.

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1. INTRODUCTION

Let $\mathcal{S}^n$ and $\mathcal{S}^n_+$ denote the cones of $n \times n$ real symmetric and symmetric positive semi-definite matrices, respectively. The convex quadratic semidefinite optimization (CQSDO) problem is defined as follows

$$(\mathcal{P}) \quad p^\star = \min_{X \in \mathcal{S}^n} f(X) \quad \text{s.t. } A_i \bullet X = b_i, i = 1, \ldots, m, X \succeq 0,$$

and its dual

$$(\mathcal{D}) \quad d^\star = \max_{y \in \mathcal{R}^m, Z \in \mathcal{S}^n} g(y, Z) \text{ s.t. } \sum_{i=1}^{m} y_i A_i + Z = C + \mathcal{Q}(X),\ Z \succeq 0,$$

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where

$$f(X) = \tfrac{1}{2} X \bullet \mathcal{Q}(X) + C \bullet X,$$

and

$$g(y, Z) = -\tfrac{1}{2} X \bullet \mathcal{Q}(X) + b^\top y,$$

$b, y \in \mathcal{R}^m$, $X, Z \in \mathcal{S}^n_+$, C, $A_i \in \mathcal{S}^n, i = 1, \cdots, m$, $\mathcal{Q}(X)$ is a given linear transformation on $\mathcal{S}^n$ and the inequality $M \succeq 0$ means that $M \in \mathcal{S}^n_+$. Recall that the notation $A \bullet B = \mathrm{tr}(AB)$, $A, B \in \mathcal{S}^n$ denotes the trace inner product.

The CQSDO problem has wide applications such as the computation of nearest correlation matrix problem (NCMP) also it arises in nearest Euclidean distance matrix problems and other matrix least square problems (SDLS) (see [17]). Many problems in metric embeddings, covariance estimations, eigenvalues and max-cut problems and molecular conformations can also be restated as CQSDO. It also includes the semidefinite optimization (SDO) as a special case when $\mathcal{Q}(X) = 0$.

Over the last decades, many feasible primal-dual interior-point algorithms (FIPA) for solving CQSDO have been proposed (see, *e.g.*, [20, 22, 6, 9, 15]). These algorithms are of Newton type methods and which enjoy certain property such as the locally quadratic convergence and have polynomial complexity. Further they are highly efficient in practice. It is well-known that FIPAs are iterative methods and start with a strictly feasible starting point and follow the central-path with keeping feasibility and centrality during the solution process. However, experimental issue of these algorithms show that getting a strictly feasible centered point is a hard task and even impossible. Therefore, to overcome this drawback many remedies are suggested. Among them, infeasible interior-point (IFIPA) algorithms which require that the starting initial point is any arbitrary symmetric positive definite matrix where feasibility is reached as optimality is approached [24, 27].

The main aim of this work is to deal with the numerical implementation of an IFIPAs for solving the CQSDO problems. Here, the determination of Newton search directions is based on the symmetrization of Alizadeh-Haeberly-Overton (AHO) [8]. Further, efficient step-size are suggested to keep iterations positive definite during Newton's process. Our tested examples of CQSDO are reformulated from some known optimization problems such as SDO, SDLS, NCMP, the computation of the smallest eigenvalue of a symmetric positive definite matrix and the Max-cut problem.

Throughout the paper, the following notations are used. $\mathcal{R}^{n\times n}$ denotes the set of all $n \times n$ real matrices. The trace inner-product and the Frobenius norm in $\mathcal{S}^n$ are denoted, respectively, by: $A \bullet B = \mathrm{tr}(AB) = \sum_{i,j} a_{ij} b_{ij}$ and $\|A\|_F = (\mathrm{tr}(A^2))^{\frac{1}{2}}$, $A, B \in \mathcal{S}^n$. For a matrix A, $\lambda_i(A)$ denote its eigenvalues with $\lambda_{\min(A)}$ ($\lambda_{\max(A)}$) as the smallest (largest) one, respectively, and $\det A$ stand for its determinant. The square root matrix of any $X \in \mathcal{S}^n_+$ is denoted by $X^{1/2}$ and the identity matrix is denoted by I, and e is the vector of all l's.

The paper is organized as follows. In Section 2, preliminaries, the concept of the central-path and the AHO symmetrization search directions for CQSDO are discussed. Further, the numerical computation of the step-size and the search directions are stated. We described in Section 3 the generic primal-dual IFIPA. In Section 4, some numerical results are reported. In Section 5, a conclusion and future remarks end the paper.

2. A PRIMAL-DUAL IFIPA FOR CQSDO

2.1. Preliminaries. In this subsection, we first provide some mathematical useful tools which are crucial for the development of our proposed algorithm.

LEMMA 1 (Lemma 2.1 in [22]). *If $X, Z \in \mathcal{S}^n_+$. Then the following statements*

1) $X \bullet Z = 0$,
2) $XZ = 0$,
3) $\frac{XZ+ZX}{2} = 0$

are equivalent.

Let A and B two matrices, their Kronecker product is denoted by $A \otimes B$. For a matrix $A \in \mathcal{R}^{n\times n}$, $\mathrm{vec}(A)$ is the $n^2 \times 1$ vector given by

$$\mathrm{vec}(A) = (a_{11}, a_{21}, \cdots, a_{nn})^\top.$$

We make use of the functions svec and smat, formally defined as follows

DEFINITION 2. *For $A \in \mathcal{S}^n$, $\mathrm{svec} \in \mathcal{R}^{\frac{1}{2}n(n+1)}$ is given by*

$$\mathrm{svec}(A) = (a_{11}, \sqrt{2}a_{21}, \cdots, \sqrt{2}a_{n1}, a_{22}, \sqrt{2}a_{32}, \cdots, \sqrt{2}a_{n2}, \cdots, a_{nn})^\top.$$

Further, the operator smat *is the inverse operator of* svec. *That is,*

$$\mathrm{smat}(\mathrm{svec}(A)) = A.$$

In all the paper, the feasible sets and the strict feasible sets of $\mathcal{P}$ and $\mathcal{D}$ are the subsets of $\mathcal{S}^n$ and $\mathcal{R}^m \times \mathcal{S}^n$, respectively,

$$\begin{aligned}
\mathcal{F}_\mathcal{P} &= \{X \in \mathcal{S}^n_+ \colon A_i \bullet X = b_i, i = 1, \ldots, m\},\\
\mathcal{F}^0_\mathcal{P} &= \{X \in \mathcal{F}_\mathcal{P} \colon X \in \mathcal{S}^n_{++}\},\\
\mathcal{F}_\mathcal{D} &= \{(y, Z) \in \mathcal{R}^m \times \mathcal{S}^n_+ \colon C + \mathcal{Q}(X) - \sum_{i=1}^m y_i A_i = Z\},\\
\mathcal{F}^0_\mathcal{D} &= \{(y, Z) \in \mathcal{F}_\mathcal{D} \colon Z \in \mathcal{S}^n_{++}\}.
\end{aligned}$$

The optimal sets of $\mathcal{P}$ and $\mathcal{D}$ are the sets

$$S^\mathcal{P}_{opt} = \{X \in \mathcal{S}^n : X \succeq 0, A_i \bullet X = b_i, i = 1, \ldots, m,\ f(X) = p^\star\},$$

and

$$S^\mathcal{D}_{opt} = \{(y, Z) \in \mathcal{R}^m \times \mathcal{S}^n_+ \colon C + \mathcal{Q}(X) - \sum_{i=1}^m y_i A_i = Z,\ g(y, Z) = d^\star\}.$$

It is assumed that the two problems satisfy the following conditions:

- $\mathcal{F}_{\mathcal{P}}^0 \times \mathcal{F}_{\mathcal{D}}^0$ is nonempty.
- The $A_i, i = 1, \ldots, m$, are linearly independent.
- The linear transformation $\mathcal{Q}(X)$ is monotone and self-adjoint, *i.e.*, $X \bullet \mathcal{Q}(X) \geq 0$ and $X \bullet \mathcal{Q}(Y) = \mathcal{Q}(X) \bullet Y,\ \forall X, Y \in \mathcal{S}^n$.

The first assumption implies

- $-\infty < p^\star = d^\star < \infty$.
- $S_{opt}^{\mathcal{P}}$ and $S_{opt}^{\mathcal{D}}$ are two non empty bounded convex sets.

Meanwhile, the second assumption is only to ensure that the dual variables Z and y are in one to one correspondence. By assumption three, $\mathcal{P}$ and $\mathcal{D}$ are two convex optimization problems, therefore the set of primal and dual optimal solutions consists of all solutions $(X, y, Z) \in \mathcal{S}_+^n \times \mathcal{R}^m \times \mathcal{S}_+^n$ of the following Karush-Khun-Tucker optimality system:

$$\begin{cases} A_i \bullet X - b_i = 0, \forall i = 1, \ldots, m, X \succeq 0, \\ C + \mathcal{Q}(X) - \sum_{i=1}^{m} y_i A_i - Z = 0_{\mathcal{S}^n}, Z \succeq 0, \\ X \bullet Z = 0. \end{cases} \tag{1}$$

Therefore, by Lemma 1 finding an optimal solution for $\mathcal{P}$ and $\mathcal{D}$ is equivalent to solving the system

$$\begin{cases} A_i \bullet X = b_i,\ i = 1, \ldots, m,\ X \succeq 0, \\ C + \mathcal{Q}(X) - Z - \sum_{i=1}^{m} y_i A_i = 0_{\mathcal{S}^n},\ Z \succ 0, \\ \frac{XZ+ZX}{2} = 0. \end{cases} \tag{2}$$

2.2. The central-path of CQSDO. The basic idea behind primal-dual IPAs is to replace the third equation in (2) by the parameterized equation

$$\frac{XZ+ZX}{2} = \sigma\mu I,$$

with $\mu > 0$. Thus we consider the system

$$\begin{cases} A_i \bullet X - b_i = 0, \forall i = 1, \ldots, m,\ X \succ 0, \\ C + \mathcal{Q}(X) - Z - \sum_{i=1}^{m} y_i A_i = 0_{\mathcal{S}^n},\ Z \succ 0, \\ \frac{XZ+ZX}{2} = \sigma\mu I,\ \mu > 0, \end{cases} \tag{3}$$

where $\sigma \in (0, 1)$ is the centrality parameter. Under our assumptions, system (3) has a unique solution $(X(\mu), y(\mu), Z(\mu))$, for each $\mu > 0$. The set

$$\mathcal{C} = \{(X(\mu), y(\mu), Z(\mu)) : \mu > 0\}$$

of μ-centers is called the central-path of $\mathcal{P}$ and $\mathcal{D}$. If μ tends to zero then the limit of $\mathcal{C}$ exists and since the limit point satisfies the complementarity condition, the limit yields an approximated optimal solution of $\mathcal{P}$ and $\mathcal{D}$ (*e.g.*, [8, 10]).

2.3. The Newton search directions for CQSDO. Next, we want to define search directions $(\Delta X, \Delta y, \Delta Z)$ that move in the direction of the central-path $\mathcal{C}$. Applying *Newton's method* for (3), for a given $\mu > 0$ and an infeasible point $(X \succ 0, y, Z \succ 0)$. Then the Newton direction $(\Delta X, \Delta y, \Delta Z)$ at this point is the unique symmetric solution of the linear system:

$$(4) \qquad \begin{cases} A_i \bullet \Delta X = \mathcal{R}_{\mathcal{P}}, \ i = 1, \ldots, m, \\ \sum_{i=1}^{m} (\Delta y)_i A_i + \Delta Z - \mathcal{Q}(\Delta X) = \mathcal{R}_{\mathcal{D}}, \\ \Delta X Z + X \Delta Z + \Delta Z X + Z \Delta X = \mathcal{R}_{\mathcal{C}}, \ X \succ 0, \ Z \succ 0, \end{cases}$$

with

$$\mathcal{R}_{\mathcal{P}} = b_i - A_i \bullet X, i = 1, \ldots, m,$$
$$\mathcal{R}_{\mathcal{D}} = C - (Z - \mathcal{Q}(X) + \sum_{i=1}^{m} y_i A_i),$$
$$\mathcal{R}_{\mathcal{C}} = 2\sigma\mu I - (XZ + ZX).$$

The search direction obtained via system (4) is named as the AHO-direction. It is worth mentioning that beside the AHO direction, there are other important search directions such as Kojima *et al.* [16], Helmberg *et al.* [12], Monteiro and Nesterov and Todd (NT) [19].

We will refer to the assignment:

$$X^+ = X + \alpha \Delta X, \ y^+ = y + \alpha \Delta y, \ Z^+ = Z + \alpha \Delta Z$$

as a damped Newton step with $\alpha > 0$ is the step-size along this direction.

2.4. Computation of AHO directions. We have seen that the direction of **AHO** is given by system (4). So to compute this direction, we apply the "svec" operator to the three equations of (4). For the first equation

$$\begin{aligned} A_i \bullet \Delta X &= \mathcal{R}_{\mathcal{P}}, i = 1, \ldots, m, \\ \operatorname{svec}^{\top}(A_i) \operatorname{svec}(\Delta X) &= \mathcal{R}_{\mathcal{P}}, i = 1, \ldots, m, \\ \mathcal{A} \operatorname{svec}(\Delta X) &= \mathcal{R}_{\mathcal{P}}, \end{aligned}$$

such that

$$\mathcal{A} = \begin{bmatrix} \operatorname{svec}^{\top}(A_1) \\ \vdots \\ \operatorname{svec}^{\top}(A_m) \end{bmatrix} \in \mathcal{R}^{m \times \bar{n}} \text{ and } \operatorname{svec}(\Delta X) \in \mathcal{R}^{\bar{n}}, \text{ with } \bar{n} = \tfrac{n(n+1)}{2}.$$

The second equation

$$\begin{aligned} \operatorname{svec}(\sum_{i=1}^{m} (\Delta y)_i A_i + \Delta Z) - \operatorname{svec}\left[\tfrac{1}{2}\left(\mathcal{Q}(X)\Delta X + \Delta X \mathcal{Q}(X)\right)\right] &= \mathcal{R}_{\mathcal{D}}, \\ \sum_{i=1}^{m} \operatorname{svec}(A_i)(\Delta y)_i + \operatorname{svec}(\Delta Z) - (\mathcal{Q}(X) \otimes_s I) \operatorname{svec}(\Delta X), &= \mathcal{R}_{\mathcal{D}}, \end{aligned}$$

$$\mathcal{A}^\top \Delta y + \operatorname{svec}(\Delta Z) - \mathcal{Q}(X) svec(\Delta X) \quad = \quad \mathcal{R}_\mathcal{D},$$

such that

$$\mathcal{A}^\top = \left[\operatorname{svec}(A_1), \quad \ldots, \quad \operatorname{svec}(A_m) \right] \in \mathcal{R}^{\bar{n} \times m}.$$

The third equation becomes

$$\begin{aligned} (Z\Delta X + \Delta X Z) + (X\Delta Z + \Delta Z X) &= \mathcal{R}_\mathcal{C}, \\ E \operatorname{svec}(\Delta X) + F \operatorname{svec}(\Delta Z) &= \mathcal{R}_\mathcal{C}, \end{aligned}$$

such that

$$E = (Z \otimes_s I), \; F = (X \otimes_s I), \; \mathcal{R}_\mathcal{C} = \operatorname{svec}(\mu\sigma I - \tfrac{1}{2}(XZ + ZX)),$$

where $\otimes_s$ denotes the symmetric Kronecker product.

Finally, we obtain the following system

$$\begin{cases} \mathcal{A} \operatorname{svec}(\Delta X) = \mathcal{R}_\mathcal{P}, \\ \mathcal{A}^\top \Delta y + \operatorname{svec}(\Delta Z) - \mathcal{Q} \operatorname{svec}(\Delta X) = \mathcal{R}_\mathcal{D}, \\ E \operatorname{svec}(\Delta X) + F \operatorname{svec}(\Delta Z) = \mathcal{R}_\mathcal{C}, \end{cases}$$

which can be written as the following matrix form:

$$\begin{bmatrix} 0 & \mathcal{A} & 0 \\ \mathcal{A}^\top & -\mathcal{Q} & I_{\bar{n}} \\ 0 & E & F \end{bmatrix} \begin{bmatrix} \Delta y \\ \operatorname{svec}(\Delta X) \\ \operatorname{svec}(\Delta Z) \end{bmatrix} = \begin{bmatrix} \mathcal{R}_\mathcal{P} \\ \mathcal{R}_\mathcal{D} \\ \mathcal{R}_\mathcal{C} \end{bmatrix}. \tag{5}$$

We note that the system (5) is a linear system whose blocked matrix is a square matrix of order $(m + 2\bar{n})$, by using any linear system solver we get

$$\left(\operatorname{svec}(\Delta X), \; \Delta y, \; \operatorname{svec}(\Delta X)\right).$$

Then, we introduce smat the reciprocal application of svec to this last result.

2.5. Computation of a step-size α. For computing the step size $\alpha > 0$ in each iteration such that

$$X + \alpha\Delta X \succ 0 \, \text{and} \, Z + \alpha\Delta Z \succ 0,$$

we need to determine the maximum step-size $\alpha_{\max}$ so that if $0 < \alpha \leq \alpha_{\max}$ then $X + \alpha\Delta X \succ 0$ and $Z + \alpha\Delta Z \succ 0$. Let α_X and α_Z be the maximum possible step-size in the direction ΔX and ΔZ, respectively. The natural step length for a Newton direction is $\alpha = 1$, but a step length of 1 may cause us to exceed the positive definite region to which we are constrained. Therefore, we need to determine the maximum possible step length $\alpha_{\max}$ such that if $0 < \alpha < \alpha_{\max}$, then $Z + \alpha\Delta Z \succ 0$. and $X + \alpha\Delta X \succ 0$. The maximum step length in the direction ΔX can easily be seen as being the smallest positive number α_X such that

$$\det(X + \alpha\Delta X) = 0,$$

or $\alpha_X = \infty$ if no positive solutions exist. Similarly, the maximum step length in the direction ΔZ is defined as the smallest positive number α_Z such that

$$\det(Z + \alpha\Delta Z) = 0,$$

or $\alpha_Z = \infty$ if no positive solutions exist. Our maximum possible step length is then the minimum of these two numbers:

$$\alpha_{\max} = \min(\alpha_X, \alpha_Z).$$

We can compute α_X and α_Z as the minimum positive eigenvalues of the respective generalized eigenvalue problems

$$X\nu = \lambda(-\Delta X)\nu \text{ and } Z\nu = \lambda(-\Delta Z)\nu.$$

Since algorithms for solving generalized eigenvalue problems prefer to have the matrix on the right-hand-side be positive definite [17], it is better computationally to find the maximum eigenvalues $\lambda_{\max}$ and $\lambda'_{\max}$ of the respective problems

$$-\Delta X\nu = \lambda X\nu \text{ and } -\Delta Z\nu = \lambda Z\nu.$$

Thus

$$\alpha_X = \begin{cases} \frac{1}{\lambda_{\max}} & \text{if} \quad \lambda_{\max} > 0 \\ \infty & \text{otherwise,} \end{cases}$$

and

$$\alpha_Z = \begin{cases} \frac{1}{\lambda'_{\max}} & \text{if} \quad \lambda'_{\max} > 0 \\ \infty & \text{otherwise.} \end{cases}$$

Once these two allowed maximum step-sizes are determined, then the step size α is taken as follows

$$\alpha = \min(1,\ \rho \min(\alpha_X, \alpha_Z)) : \rho \in (0, 1).$$

For specifying the barrier parameter $\mu > 0$, it is easily seen from the last equation in (3) that $\mu = \frac{X \bullet Z}{\sigma n}$.

3. AN INFEASIBLE PATH-FOLLOWING ALGORITHM FOR CQSDO

We present an infeasible path-following interior-point algorithm for computing an optimal solution of CQSDO that uses the primal-dual interior-point framework proposed by many authors. In each iteration the algorithm starts with guesses (matrices) $X^0, Z^0 \succ 0, y^0 \in \mathcal{R}^m$ not necessarily feasible but only symmetric positive definite. We would like to update these matrices until we are within our desired tolerance of satisfying equations. We will stop our algorithm when the

$$\max\left(\|\mathcal{R}_{\mathcal{D}}\|_F, \|\mathcal{R}_{\mathcal{P}}\|_2, n\mu\right)$$

is small enough.

Algorithm 1 The generic IFIPA for CQSDO.

Input:
An accuracy parameter $\epsilon > 0$;
initial guesses $(X^0, Z^0 \succ 0, y^0 \in \mathcal{R}^m)$ and $\mu > 0$;
begin
$k := 0$;
While $\max\left(\left\|\mathcal{R}_{\mathcal{D}}{}^k\right\|_F, \left\|\mathcal{R}_{\mathcal{P}}{}^k\right\|_2, n\mu^k\right) > \epsilon$ **do**
• Compute $(\Delta X^k, \Delta y^k, \Delta Z^k)$ by solving system (5);
• Determine a step-size $\alpha > 0$ s.t. $X^k + \alpha\Delta X^k \succ 0$ and $Z^k + \alpha\Delta Z^k \succ 0$;
• Update $X^{k+1} := X^k + \alpha\Delta X^k$, $y^{k+1} := y^k + \alpha\Delta y^k$, $Z^{k+1} := Z^k + \alpha\Delta Z^k$;
$k := k + 1$;
endWhile
end.

4. NUMERICAL RESULTS

In this section, we implemented Algorithm 1 in the software **Matlab** environment and run it on a personal computer. In the implementation, we take $\epsilon \in [10^{-4},\ 10^{-6}]$ as our tolerance and $(X^0 \succ 0, y^0, Z^0 \succ 0)$ stand for the initial guessing point of the algorithm. In below tables of the obtained numerical results, the number of iterations and the time obtained by the algorithm are denoted by "**Iter**" and "**CPU**", respectively. Our testing examples are inspired from different well known optimization problems. For each example, $(X^\star, y^\star, Z^\star)$ denotes an approximated primal-dual optimal solution for CQSDO problems.

EXAMPLE 1. *Consider the CQSDO problem whose primal-dual pair $\mathcal{P}$ and $\mathcal{D}$ have the following data:*

$$\mathcal{Q}(X) = 0_{\mathcal{S}^4},\ A_1 = \begin{bmatrix} 0.9 & 0 & 0 & -1.5 \\ 0 & 3 & 0.75 & 0 \\ 0 & 0.75 & 1.098 & 0 \\ -1.5 & 0 & 0 & -0.5 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1.414 & 1.386 & 0 & 0 \\ 1.386 & 1.732 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}, C = \begin{bmatrix} 1.7071 & 0.6931 & -0.1 & 0 \\ 0.6931 & 1.366 & -0.5 & 0.02 \\ -0.1 & -0.5 & 2 & 0 \\ 0 & 0.02 & 0 & 0 \end{bmatrix}$$

$$A_3 = \mathrm{diag}(1,\ 0.5,\ 1.333,\ -0.333),\ b = [7,\ 4,\ 2]^\top.$$

The initial point is defined as follows

$$X^0 = \begin{bmatrix} 1 & 0 & -0.1 & 0 \\ 0 & 0.5 & 0 & 0.02 \\ -0.1 & 0 & 1 & 0 \\ 0 & 0.02 & 0 & 1 \end{bmatrix}, y^0 = \begin{bmatrix} 0 \\ 0.5 \\ 0 \end{bmatrix}, Z^0 = I.$$

The obtained primal-dual optimal solution is

$$X^\star = \begin{bmatrix} 0.4419 & -1.0091 & 0.3849 & -0.2336 \\ -1.0091 & 2.3044 & -0.8790 & 0.5335 \\ 0.3849 & -0.8790 & 0.3353 & -0.2035 \\ -0.2336 & 0.5335 & -0.2035 & 0.1235 \end{bmatrix}, \; y^\star = \begin{bmatrix} 0.2015 \\ 0.1656 \\ 0.1603 \end{bmatrix},$$

$$Z^\star = \begin{bmatrix} 1.1314 & 0.4636 & 0.1000 & 0.3023 \\ 0.4636 & 0.3946 & 0.5144 & 0.0200 \\ 0.1000 & 0.5144 & 1.2338 & 0 \\ 0.3023 & 0.0200 & 0 & 0.4853 \end{bmatrix}.$$

EXAMPLE 2 ([4]). *The semidefinite least squares optimization, (SDLS) is a convex optimization problem which is defined as follows:*

$$\begin{cases} \min_X f(X) = \frac{1}{2} \|BX - N\|_F^2 \\ s.t. \\ A_i \bullet X = b_i, i = 1, \ldots, m, \; X \succeq 0, \end{cases}$$

where $b \in \mathcal{R}^m$ *and* $N, \; B \in \mathcal{S}^n$. *Because*

$$\tfrac{1}{2} \|BX - N\|_F^2 = \tfrac{1}{2} B^2 X \bullet X - (NB \bullet X + BN \bullet X)/2 + \tfrac{1}{2} N \bullet N,$$

then, the SDLS can be restated as a CQSDO problem with

$$f(X) = C \bullet X + \tfrac{1}{2} X \bullet \mathcal{Q}(X) + \tfrac{1}{2} N \bullet N, \; \mathcal{Q}(X) = B^2 X \; and C = -(NB + BN)/2.$$

For the size $(m = 5, n = 7)$, *the data of the SDLS is stated as follows:*
$A_1 = \mathrm{diag}(6.4,\ 1.03,\ 1.03,\ 8.21,\ 1.69,\ 0,\ 4.2)$, $b = \left[-4,\ 4,\ 9,\ 2.16,\ 3\right]^\top$,
$A_2 = \mathrm{diag}(1.22,\ 3,\ 3.33,\ 0,\ 0,\ 0,2,\ 0)$, $A_3 = \mathrm{diag}(0,\ 0,\ 6.67,\ 4,\ 1,\ 0,8,\ 0)$,

$$A_4 = \begin{bmatrix} 0.66 & 4 & 5.6 & 0 & 0 & 0 & 5.9 \\ 4 & 0 & 0 & 0 & -1.6 & 0 & 0 \\ 5.6 & 0 & 8.88 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.93 & 0 & 0 & 0 \\ 0 & -1.6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.03 & 0 \\ 5.9 & 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 0.1 & 10.2 & 0 & 0 & 0 & 2.828 & -1.4 \\ 10.2 & 2 & 6.7 & 0 & -1.5 & 0 & 0 \\ 0 & 6.7 & 7.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.5 & 0 & 0 & 0 & 0 & 0 \\ 2.828 & 0 & 0 & 0 & 0 & 7.6 & 0 \\ -1.4 & 0 & 0 & 0 & 0 & 0 & 8 \end{bmatrix},$$

$$N = \begin{bmatrix} -1.376 & -19.758 & -2.711 & -0.102 & 0 & -5.42 & -1.485 \\ -19.758 & -3.501 & -11.287 & -0.032 & 3.487 & -0.0043 & 0 \\ -2.711 & -11.287 & -27.252 & -0.067 & 0 & 0.287 & -0.175 \\ -0.102 & -0.0329 & -0.067 & -4.7048 & 0.015 & 0 & 0 \\ 0 & 3.4875 & 0 & 0.015 & -0.5 & 0.0008 & 0 \\ -5.4205 & -0.0043 & 0.2877 & 0 & 0.0008 & -16.0725 & 0 \\ -1.485 & 0 & -0.175 & 0 & 0 & 0 & -23.678 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.9947 & -10.2 & 0.3705 & 0 & 0 & 2.3284 & 1.4 \\ -10.2 & -0.3335 & -6.7 & -0.0209 & 1.5 & 0.0008 & 0 \\ 0.3705 & -6.7 & -8.1762 & 0 & 0 & 0 & 0 \\ 0 & -0.0209 & 0 & -0.4452 & 0 & 0 & 0 \\ 0 & 1.5 & 0 & 0 & 0.5 & 0 & 0 \\ -2.3284 & -0.0008 & 0 & 0 & 0 & -7.1 & 0 \\ 1.4 & 0 & 0 & 0 & 0 & 0 & -6.945 \end{bmatrix}.$$

The initial point is defined as follows

$$X^0 = \begin{bmatrix} 1.4 & 0 & -0.25 & 0 & 0 & 0 & 0 \\ 0 & 0.55 & 0 & 0.02 & 0 & 0.001 & 0 \\ -0.25 & 0 & 1.7 & 0 & 0 & 0 & 0 \\ 0 & 0.02 & 0 & 1.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0.001 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \; y^0 = \begin{bmatrix} 0. \\ 0.75 \\ 0 \\ 0.75 \\ 0 \end{bmatrix},$$

$$Z^0 = \begin{bmatrix} 2.106 & 0 & -0.017 & 0 & 0 & 0.5 & 0 \\ 0 & 1.1818 & 0 & 0.0016 & 0 & 0 & 0 \\ -0.017 & 0 & 0.9 & 0 & 0 & 0 & 0 \\ 0 & 0.0016 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.9 \end{bmatrix}.$$

The approximated primal-dual optimal solution is

$$X^\star = \begin{bmatrix} 0.938 & 0.1464 & -0.368 & -0.2052 & -0.0855 & -0.0359 & -0.002 \\ 0.1464 & 0.2592 & 0.041 & -0.048 & -0.1211 & 0.0926 & 0.072 \\ -0.368 & 0.0419 & 1.494 & 0.044 & -0.0857 & -0.1702 & -0.043 \\ -0.2052 & -0.0480 & 0.044 & 2.2875 & 0.2740 & 0.0616 & -0.052 \\ -0.0855 & -0.1211 & -0.085 & 0.274 & 0.1010 & 0.0668 & 0.002 \\ -0.0359 & 0.0926 & -0.170 & 0.0616 & 0.0668 & 0.7594 & 0.041 \\ -0.0021 & 0.0729 & -0.043 & -0.0523 & 0.0018 & 0.0413 & 1.337 \end{bmatrix}$$

$$Z^\star = \begin{bmatrix} 0 & -0.0007 & 0 & 0.0002 & -0.001 & 0.0002 & 0 \\ -0.0007 & 0.0983 & 0.0058 & -0.0211 & 0.196 & -0.026 & -0.0054 \\ 0 & 0.0058 & 0.0004 & -0.0013 & 0.011 & -0.0015 & -0.0003 \\ 0.0002 & -0.0211 & -0.0013 & 0.0045 & -0.042 & 0.0056 & 0.0012 \\ -0.001 & 0.196 & 0.0117 & -0.0422 & 0.393 & -0.052 & -0.011 \\ 0.0002 & -0.026 & -0.0015 & 0.0056 & -0.05 & 0.0069 & 0.0014 \\ 0 & -0.0054 & -0.0003 & 0.0012 & -0.011 & 0.0014 & 0.0003 \end{bmatrix}$$

$$y^\star = \left[0.0849,\ 0.7007,\ 0.1591,\ 1.0552,\ 0.0357\right]^\top.$$

Example 3. *This example is reformulated from the following Nearest Correlation Matrix Problem (NCMP):*

$$(\mathcal{P}) \quad \begin{cases} \min_X f(X) = \frac{1}{2}\left\|X - N\right\|_F^2 \\ s.t. \\ A_i \bullet X = b_i, i = 1, \ldots, m,\ X \succeq 0. \end{cases}$$

Here, $f(X) = -N \bullet X + \frac{1}{2} X \bullet \mathcal{Q}(X) + \frac{1}{2} N \bullet N$, $\mathcal{Q}(X) = X$ *and* $C = -N$. *For the size* $m = 4$, $n = 8$, *the data of NCMP is given by*

$$A_1 = \begin{bmatrix} -1.2 & 0 & 0.3 & 0.22 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 5.3 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0 & 0 & 1.87 & 0 & 0 & 0 \\ 0.22 & 5.3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.87 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.5 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -3 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 6 & 2.55 & 0 & 0 & 6.3 & 0 & 0 & 0 \\ 2.55 & -5 & 0 & 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 1.1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4.4 & 0 & 0 & 6.03 & 0 \\ 6.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 6.03 & 0 & 0 & -0.22 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0.1 & 10.2 & 0 & 0 & 0 & 2.828 & -1.4 & -3 \\ 10.2 & 2 & 6.7 & 0 & -1.5 & 0 & 0 & 0 \\ 0 & 0 & 7.6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.7321 & 0 & 0 & 0 & 0 & 7.6 & 0 & 0 \\ -1.4 & 0 & 0 & 0 & 0 & 0 & 1.05 & 0 \\ -3 & 0 & 0 & 0 & 0 & 0 & 0 & -3 \end{bmatrix},$$

$$b = \begin{bmatrix} -0.4 \\ 4 \\ 11 \\ 14.5 \end{bmatrix}, A_2 = \begin{bmatrix} 0.66 & 4 & 5.6 & 0 & 0 & 0 & 9.5 & 0 \\ 4 & 0 & 0 & 0 & -10.6 & 0 & 0 & 0 \\ 5.6 & 0 & 8.8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.3 & 0 & 0 & 0 & 0 \\ 0 & -10.6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.03 & 0 & 0 \\ 9.5 & 0 & 0 & 0 & 0 & 0 & -6.9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix},$$

$$N = \begin{bmatrix} -0.772 & -10.65 & -4.201 & 0 & 0 & -1.304 & -6.075 & 2.25 \\ -10.65 & -2.768 & 0 & 0.0005 & 9.075 & 0.001 & 0 & 0 \\ -4.201 & 0 & -12.286 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0005 & 0 & -2.461 & 0 & 0 & 0 & 0 \\ 0 & 9.075 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.304 & 0.001 & 0 & 0 & 0 & -7.972 & 0 & 0 \\ -6.075 & 0 & 0 & 0 & 0 & 0 & 4.446 & 0 \\ 2.25 & 0 & 0 & 0 & 0 & 0 & 0 & 5.25 \end{bmatrix}.$$

The initial point is defined as follows:

$$X^0 = \begin{bmatrix} 0.904 & 0 & -0.0025 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.55 & 0 & 0.0019 & 0 & 0.001 & 0 & 0 \\ -0.0025 & 0 & 1.007 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0019 & 0 & 1.007 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0.001 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.03 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$y^0 = [0,\ 0.25,\ 0,\ 0.3,\ 0]^\top,$$

$$Z^0 = \begin{bmatrix} 1.106 & 0 & -0.0013 & 0 & 0 & 0.005 & 0 & 0 \\ 0 & 1.818 & 0 & 0.0014 & 0 & 0 & 0 & 0 \\ -0.0013 & 0 & 0.993 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0014 & 0 & 0.993 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.9708 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

An approximated optimal solution is:

$$X^\star = \begin{bmatrix} 0.4253 & 0.1903 & -0.0251 & -0.0633 & 0.1488 & 0.1709 & -0.0891 & -0.1536 \\ 0.1903 & 0.0932 & -0.0044 & -0.0419 & 0.056 & 0.1339 & -0.0371 & -0.0617 \\ -0.0251 & -0.0044 & 0.6 & -0.0006 & -0.0543 & 0.0008 & -0.0001 & -0.0014 \\ -0.0633 & -0.0419 & -0.0006 & 0.2219 & 0.0107 & -0.0016 & 0.1956 & -0.0009 \\ 0.1488 & 0.056 & -0.0543 & 0.0107 & 0.0741 & -0.0006 & -0.018 & -0.0600 \\ 0.1709 & 0.1339 & 0.0008 & -0.0016 & -0.0006 & 0.5576 & 0.1037 & -0.0169 \\ -0.0891 & -0.0371 & -0.0001 & 0.1956 & -0.018 & 0.1037 & 0.2044 & 0.0237 \\ -0.1536 & -0.0617 & -0.0014 & -0.0009 & -0.0600 & -0.0169 & 0.0237 & 0.0644 \end{bmatrix}$$

$$Z^\star = \begin{bmatrix} 0.3486 & -0.6608 & 0.0008 & -0.0569 & -0.1009 & 0.0466 & 0.0423 & 0.0997 \\ -0.6608 & 1.4689 & -0.0044 & 0.1134 & 0.1379 & -0.1446 & -0.0371 & -0.0617 \\ 0.0008 & -0.0044 & 0.0001 & -0.0006 & 0.0007 & 0.0008 & -0.0001 & -0.0014 \\ -0.0569 & 0.1134 & -0.0006 & 0.0429 & 0.0107 & -0.0016 & -0.0434 & -0.0009 \\ -0.1009 & 0.1379 & 0.0007 & 0.0107 & 0.0447 & -0.0006 & -0.018 & -0.06 \\ 0.0466 & -0.1446 & 0.0008 & -0.0016 & -0.0006 & 0.0225 & -0.0139 & -0.0169 \\ 0.0423 & -0.0371 & -0.0001 & -0.0434 & -0.018 & -0.0139 & 0.0559 & 0.0237 \\ 0.0997 & -0.0617 & -0.0014 & -0.0009 & -0.06 & -0.0169 & 0.0237 & 0.1096 \end{bmatrix}$$

$$y^\star = \left[-0.0294,\ 0.2472,\ 0.0396,\ 0.3746\right]^\top.$$

EXAMPLE 4. *This example of CQSDO is reformulated from the computation of the smallest eigenvalue of a symmetric positive definite matrix denoted by (EVP):*

$$\lambda_{\min}(B) = \begin{cases} \min_x f(x) = x^\top Bx \\ s.t.\ \|x\|_2 = 1. \end{cases}$$

The EVP becomes a CQSDO as follows:

$$\lambda_{\min}(B) = \begin{cases} \min_X f(X) = B \bullet X \\ s.t. \\ I \bullet X = 1,\ X \succeq 0, \end{cases}$$

since

$$x^\top Bx = \operatorname{tr}(x^\top Bx) = \operatorname{tr}(Bxx^\top) = \operatorname{tr}(BX) = B \bullet X,$$

and

$$\|x\|_2^2 = x^\top x = \operatorname{tr}(x^\top x) = \operatorname{tr}(xx^\top) = \operatorname{tr}(XI) = I \bullet X$$

with $X = xx^\top$. For the size $n = 8$, we consider the following EVP where its data is given by

$$B = \begin{bmatrix} 7.29 & 0 & -1 & 0 & 4.301 & 0 & 0 & -4 & 0 \\ 0 & 13.29 & 0 & 7.2 & 0 & -2 & 0 & 0 & 2 \\ -1 & 0 & 15.3869 & 0 & 0 & 0 & 0 & 3.3 & 0 \\ 0 & 7.2 & 0 & 5.69 & 0 & 0 & 0 & 0 & 0 \\ 4.301 & 0 & 0 & 0 & 4.4969 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 13.25 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6.76 & 0 & 0 \\ -4 & 0 & 3.3 & 0 & 0 & 0 & 0 & 10 & 0 \\ 0 & 2 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}.$$

The initial point is defined as follows: $Z^0 = 2I$, $y^0 = 0.5$, *and*

$$X^0 = \begin{bmatrix} 9.21 & 1 & -6.75 & 1 & 1 & 1 & 1 & 0 & 4.4 \\ 1 & 11.1125 & 1 & 1.4845 & 1 & 2.9 & 1 & 0 & 3 \\ -6.75 & 1 & 10.14 & 1 & 1 & 1 & 1 & 0 & 0.5 \\ 1 & 1.4845 & 1 & 3.9261 & 1 & 1 & 1 & 0 & 3 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 & 0 & 3 \\ 1 & 2.9 & 1 & 1 & 1 & 2 & 1 & 0 & 3 \\ 1 & 1 & 1 & 1 & 1 & 1 & 5 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 4.4 & 3 & 0.5 & 3 & 3 & 3 & 3 & 0 & 10 \end{bmatrix}.$$

An approximated optimal primal-dual solution is:

$$X^\star = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1458 & 0 & -0.1854 & 0 & -0.0007 & 0 & 0 & -0.3003 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.1854 & 0 & 0.2357 & 0 & 0.0008 & 0 & 0 & 0.3818 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.0007 & 0 & 0.0008 & 0 & 0 & 0 & 0 & 0.0013 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.3003 & 0 & 0.3818 & 0 & 0.0013 & 0 & 0 & 0.6185 \end{bmatrix}$$

$$Z^\star = \begin{bmatrix} 7.2634 & 0 & -1 & 0 & 4.301 & 0 & 0 & -4 & 0 \\ 0 & 13.2634 & 0 & 7.2 & 0 & -2 & 0 & 0 & 2 \\ -1 & 0 & 15.3603 & 0 & 0 & 0 & 0 & 3.3 & 0 \\ 0 & 7.2 & 0 & 5.6634 & 0 & 0 & 0 & 0 & 0 \\ 4.301 & 0 & 0 & 0 & 4.4703 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 13.2234 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6.7334 & 0 & 0 \\ -4 & 0 & 3.3 & 0 & 0 & 0 & 0 & 9.9734 & 0 \\ 0 & 2 & 0 & 0 & 0 & -1 & 0 & 0 & 0.9734 \end{bmatrix}$$

$$y^\star = 0.0266,$$

finally we get $p^\star = d^\star = \lambda_{\min}(B) = 0.0266$. *With the help of Matlab, the exact spectrum of* B, *denoted by Sp* B *is given by*

$$\mathrm{Sp}(B) = \{0.0266, 0.693, 1.991, 6.76, 6.9907, 11.6838, 12.7608, 17.805, 18.4509\}.$$

So it is clear that our algorithm gives the exact smallest eigenvalue of B.

Example 5 ([12]). *This example is reformulated from the following Max-cut problem (MCP):*

$$(\mathcal{MCP}) \quad \begin{cases} \max_x f(x) = \frac{1}{4} x^\top C x \\ s.t.\ \ x_i \in \{-1, 1\}^n, \end{cases}$$

where its reformulation as a CQSDO is as follows:

$$\begin{cases} \max\limits_{X} f(X) = C \bullet X \\ s.t. \quad \mathrm{diag}(X) = \frac{1}{4}e, \\ \qquad X \succeq 0, \end{cases}$$

with $X = \frac{1}{4}xx^\top$*, the matrix* C *is called the Laplacian matrix associated with the graph. For the size* $n = m = 6$*, we consider the following MCP where*

$$C = \begin{bmatrix} 0.5225 & 0.1 & 0.075 & 0.3325 & 0 & 0 \\ 0.1 & 0.5925 & 0 & 0.1 & 0.275 & 0 \\ 0.075 & 0 & 0.25 & 0.275 & 0 & 0 \\ 0.3325 & 0.1 & 0.275 & 0.975 & 0.3348 & 0.3250 \\ 0 & 0.275 & 0 & 0.3348 & 0.5152 & 0.2575 \\ 0 & 0 & 0 & 0.325 & 0.2575 & 0.25 \end{bmatrix}.$$

The initial point is defined as follows

$$Z^0 = 2I,\ y^0 = 0.2e,$$

and

$$X^0 = \begin{bmatrix} 0.5 & 0.725 & 0.125 & 0 & 0.275 & 0 \\ 0.725 & 1.715 & 0.575 & 0 & 0.5225 & 0 \\ 0.125 & 0.575 & 0.4225 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.25 & 0 & 0 \\ 0.275 & 0.5225 & 0 & 0 & 0.55257 & 0.25 \\ 0 & 0 & 0 & 0 & 0.25 & 0.5 \end{bmatrix}.$$

An approximated optimal primal-dual solution is:

$$X^\star = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$Z^\star = \begin{bmatrix} 0.5077 & -0.1 & -0.075 & -0.3325 & 0 & 0 \\ -0.1 & 0.4768 & 0 & -0.1 & -0.275 & 0 \\ -0.075 & 0 & 0.3496 & -0.275 & 0 & 0 \\ -0.3325 & -0.1 & -0.275 & 1.3681 & -0.3347 & -0.325 \\ 0 & -0.275 & 0 & -0.3347 & 0.8671 & -0.2575 \\ 0 & 0 & 0 & -0.325 & -0.2575 & 0.5806 \end{bmatrix},$$

$$y^\star = [1.0302,\ 1.0693,\ 0.5996,\ 2.3431,\ 1.3823,\ 0.8305]^\top.$$

In Table 1, the number of iterations and the elapsed time for each problem are summarized.

	Algorithm 1		
	$(n, m)\downarrow$	ITER	CPU
Example 1	(4,3)	11	0.0528
Example 2	(7,5)	15	0.3887
Example 3	(8,4)	12	0.5901
Example 4	(9,1)	13	0.0506
Example 5	(6,6)	40	0.0279

Table 1. Number of iterations and CPU time for the previous examples.

4.1. Comparison study. In this subsection, in order to compare our obtained numerical results, we suggest beside the first alternative described in subsection 2.5 where we denote it by *Alternative 1*, a second alternative.

4.1.1. *Alternative 2.* The latter was suggested by Touil *et al.* [23] for solving the SDO problems using a feasible starting point. Here we adapted it for our IFIPA.

This alternative is described as follows, let

$$\alpha_X = \begin{cases} \frac{-1}{\bar{\lambda}_X - \delta_X\sqrt{n-1}} - \epsilon & \text{if} \quad \frac{-1}{\bar{\lambda}_X - \delta_X\sqrt{n-1}} > 0 \text{ and } \min_{i=1}^{n} \lambda_i(\Delta X) < 0 \\ \omega & \text{if} \quad \frac{-1}{\bar{\lambda}_X - \delta_X\sqrt{n-1}} < 0 \text{ and } \min_{i=1}^{n} \lambda_i(\Delta X) < 0 \\ 1 & \text{if} \quad \min_{i=1}^{n} \lambda_i(\Delta X) > 0, \end{cases}$$

such that

$$\bar{\lambda}_X = \frac{1}{n}\sum_{i=1}^{n}(L_X^{-1}\Delta X L_X^{-\top})_{ii},$$

$$\delta_X^2 = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}(L_X^{-1}\Delta X L_X^{-\top})_{ii}^2 - \bar{\lambda}_X^2,$$

$\lambda_i(X),\ i = 1, \ldots, n$, are the eigenvalues of $L_X^{-1}\Delta X L_X^{-\top}$, ω is a small positive real and $X = L_X L_X^{\top}$ is the Cholesky factorization.

We mention that the same formula α_Z with respect to the dual variable Z. The numerical results obtained through these enhancements are presented in Table 2. For consistency, all experiments were conducted under the same parameters, μ, ϵ, and ρ, to ensure a fair comparison.

4.2. Analysis. Results indicate that **Alter 1** often demonstrates improved convergence rates across multiple problem types, especially in cases involving complex data matrices or stricter tolerance levels. While **Alter 2** is competitive, particularly in problems with simpler structure or fewer constraints,

			Alter 1	Alter 2
Example 1	(4, 3)	Iter	11	23
		CPU	0.0528	0.0877
Example 2	(7, 5)	Iter	15	24
		CPU	0.3887	0.6201
Example 3	(8, 4)	Iter	12	21
		CPU	0.5901	1.1124
Example 4	(9, 1)	Iter	13	32
		CPU	0.0506	0.1536
Example 5	(6, 6)	Iter	40	67
		CPU	0.0279	0.0651

Table 2. Number of iterations and CPU time for the ameliorated Algorithm.

where it shows fewer iterations yet longer CPU times. Overall, these findings emphasize that **Alter 1** (Algorithm 1) offers significant advantages for CQSDO problems, balancing computational efficiency with robustness. These comparatives confirm that this algorithm is effective and show that it could be useful for a wide range of complex tasks in semidefinite programming.

5. CONCLUSION AND FUTURE WORK

In this paper, we implemented an infeasible primal-dual IPA for solving CQSDO based on AHO search directions and on efficient step-sizes for computing an approximated optimal solution of it. The advantage of this algorithm is that it can be started with any initial arbitrary symmetric positive definite matrix. The obtained numerical results on a set of problems which are taken from different benchmarked examples like NCMP, SDLS, the Max-Cut problem and eigenvalue problems are very encouraging. Finally, using other symmetrization scheme for computing the search directions remains a good topic of research in the future.

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