TWO STEP STEFFENSEN-TYPE METHODS WITH OR WITHOUT MEMORY FOR NONDIFFERENTIABLE EQUATIONS

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Abstract. In the current study, two step Steffensen-Type methods with and without memory free from derivatives are considered for the nondifferentiable equations and the local as well as semilocal convergence analysis is proved under more generalized conditions. Numerical applications are provided which demonstrate the theoretical results. Better results in terms of radii of convergence balls and number of iterations are obtained using the proposed approach as compared to the existing ones.

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1. INTRODUCTION

A significant and interesting challenge in numerical analysis is the problem of solving a nonlinear equation or system of nonlinear equations of the form

$$F(x) = 0, \tag{1}$$

where F is a Fréchet differentiable operator mapping from a Banach space X into X, and D be an open convex subset of X. Formulation of problems as an equation like (1) using Mathematical Modeling [2, 5, 11, 14, 23] arises in multiple disciplines of science and engineering. Obtaining a solution $x^* \in D$ of (1) is in analytic form is another important issue. The non-analytic and complex functions are thereby handled using a useful computational tool called iterative methods which approximate the solution x^* of (1). To further overcome the issues like slow or no convergence, divergence and inefficiency, an extensive literature can be found on convergence of iterative methods based on algebraic or geometrical considerations [14, 23]. As a result, researchers all around the world are persistently endeavoring to create higher order iterative methods [1, 3, 6, 7, 8, 9, 13, 15, 16, 17, 18, 19, 20]. Let $L(X)$ denote the space of bounded linear operators from X into itself and $\xi \in \mathbb{R} - \{0\}$ be a

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parameter.

We examine the convergence of a general Steffensen-type method free from derivatives developed by Chicharro et. al [6] which is defined for all $m = 0, 1, 2, \ldots$ by

$$x_0 \in D, \quad u_m = x_m + \xi F(x_m),$$
$$y_m = x_m - [u_m, x_m; F]^{-1} F(x_m),$$
(2) $$\qquad x_{m+1} = y_m - [u_m, y_m; F]^{-1} F(y_m),$$

where $[\cdot, \cdot; F] : D \times D \to L(X)$ is divided difference of order one [2, 14]. The beauty of the method (2) lies in the fact that if ξ is a nonzero constant then the method is without memory and if is taken to be Kurchatov operator [6] then it becomes method with memory. The fifth convergence order of the method (2) is proved utilizing Taylor series expansions in [6] provided that $X = \mathbb{R}^k$ and by assuming the existence of at least the fourth derivative which is not on the method. Hence, the application is limited to solving nonlinear equation (1) where the operator is at least that many times differentiable. But the method may converge even if $F^{(4)}$ does not exist.

For an instance, consider D to be an interval $\left[\frac{-5}{2}, 2\right]$ and the function f defined on D as

$$f(t) = \begin{cases} t^3 \log(\pi^2 t^2) + t^5 \sin\left(\frac{1}{t}\right), & t \neq 0 \\ 0, & t = 0. \end{cases}$$

Clearly, $f^{(3)}(t)$ is not continuous at $t = 0$. As a result the convergence of method (2) to the solution $t^* = \frac{1}{\pi}$ cannot be assured using results in [6] although the method converges.

Moreover, notice that the method (2) does not have any derivatives. Also, the iterates which contain $x_m + F(x_m)$ may lead to loss of convergence when F has large variations near the solution [24].

Some other limitations with local convergence analysis provided in [6] are:

(C_1) The initial guess x_0 is "a shot in dark" and no information is available on the uniqueness of the solution.

(C_2) A priori upper bounds on $\|x_m - x^*\|$ are not given, $x^* \in D$ being a solution of the equation (1). The number of iterations to be performed to reach a predecided error tolerance are not known.

(C_3) The convergence of the method is not assured (although it may converge to x^*) if at least $f^{(3)}$ does not exist.

(C_4) The results are limited to the case only when $X = \mathbb{R}^k$.

(C_5) The more interesting semi-local convergence is not given in [6].

The same concerns exist for numerous other methods with no derivatives [6, 20]. So, the technique of this study can also be used to extend the applicability of such methods along the same lines.

The main feature of current study is that it takes up all the aforementioned limitations positively. In particular, the local convergence is based on the

more general ω-continuity condition [1, 2, 16] and uses only information from the operators appearing on the method. Moreover, the semi-local convergence utilizes majorizing sequences [1, 2] is also provided.

The novelty of the article lies in the fact that the process leading to the aforementioned benefits does not rely on the particular method (2). But it can be utilized on other methods involving inverses of linear operators in a similar manner. The numerical study includes the results considering both cases of the method (2) i.e with memory and without memory. The results show that lesser number of iterations are required to obtain the solution and larger convergence radii are obtained using the presented approach as compared to existing ones. The development of efficiency and computational benefits have been discussed in [6], hence are not repeated in present article.

The paper is structured as follows: The local convergence of the method (2) is followed by the semi-local convergence. The numerical applications and concluding remarks, respectively complete the paper.

ANALYSIS I: LOCAL

In this section, local convergence of (2) for solving (1) is established. Suppose $U(x, r)$ and $U[x, r]$ denote the open and closed balls, respectively with center x and radius r. Let $B = [0, +\infty)$. The hypotheses for the local convergence analysis are: Assume there exist

(T_1) Functions $\Phi_0 : B \times B \to B$, $w_1 : B \to B$, which are continuous, symmetric and nondecreasing (CSNF) such that $\Phi_0(w_1(t), t) - 1 = 0$ admits a minimal positive solution (MPS) denoted as k_0. Let $B_0 = [0, k_0)$.

(T_2) A CSNF $\Phi : B_0 \times B_0 \to B$ such that for $J_1 : B_0 \to B$, where

$$J_1(t) = \frac{\Phi(w_1(t), t)}{1 - \Phi_0(w_1(t), t)},$$

$J_1(t) - 1 = 0$ admits MPS in B_0 denoted as k_1. Let $B_1 = [0, k_1)$.

(T_3) $\Phi_0(w_1(t), J_1(t), t) - 1 = 0$ admits a MPS in B_1 denoted by k_2. Let $B_2 = [0, k_2)$.

(T_4) For $J_2 : B_2 \to B$, where

$$J_2(t) = \frac{\Phi(w_1(t), J_1(t)t)}{1 - \Phi_0(w_1(t), J_1(t)t)},$$

$J_2(t) - 1 = 0$ has a MPS in B_2 denoted by k_3.

(3) $\qquad\qquad$ Let $\quad k^* = \min\{k_1, k_3\} \quad$ and $\quad B^* = [0, k^*).$

It follows by this definition that for all $t \in B^*$

(4) $\qquad\qquad\qquad 0 \leq \Phi_0(w_1(t), t) < 1,$

(5) $\qquad\qquad\qquad 0 \leq \Phi_0(w_1(t), J_1(t)t) < 1,$

and

(6) $$0 \le J_j(t) < 1 \quad \text{for} \quad j = 1, 2.$$

(T_5) There exist $x^* \in D$ solving equation $F(x) = 0$ and an invertible operator L so that for all $x \in D$, $u = x + \xi F(x)$

$$\|u - x^*\| \le w_1(\|x - x^*\|)$$

and

$$\|L^{-1}(F'(x) - L)\| \le \Phi_0(\|u - x^*\|, \|x - x^*\|).$$

Define the set $D_0 = D \cap U(x^*, k_0)$.

(T_6) For all $x \in D_0$, $u = x + \xi F(x)$

$$\|L^{-1}([u, x; F] - [x, x^*; F])\| \le \Phi(\|u - x^*\|, \|x - x^*\|).$$

and

(T_7) $U[x^*, \overline{k}] \subset D$, where
$\overline{k} = \max\{k^*, w_1(k^*)\}$.

The analysis follows for the method (2) via means of hypotheses (T_1)-(T_7).

THEOREM 1. *Assume hypotheses (T_1)-(T_7) are validated. Then, the assertions are validated*

(7) $$\{x_m\} \subset U(x^*, k^*),$$

(8) $$\|y_m - x_m\| \le J_1(\|x_m - x^*\|)\|x_m - x^*\| \le \|x_m - x^*\| < k^*,$$

(9) $$\|x_{m+1} - x^*\| \le J_2(\|x_m - x^*\|)\|x_m - x^*\| \le \|x_m - x^*\|$$

and the sequence $\{x_m\}$ is convergent to x^*, provided that the initial point $x_0 \in U(x^*, k^*) - \{x^*\}$.

Proof. The application of hypotheses (T_1), (T_5) and (4), (7) give

$$
\begin{aligned}
\|L^{-1}(F'(x_0) - L)\| &\le \Phi_0(\|u - x^*\|, \|x_0 - x^*\|) \\
&\le \Phi_0(w_1(\|x_0 - x^*\|), \|x_0 - x^*\|) \\
(10) \qquad &\le \Phi_0(w_1(k^*), k^*) < 1.
\end{aligned}
$$

The estimate (10) and the celebrated lemma attributed to Banach on linear operators imply the invertibility of $[u_0, x_0; F]$ and the estimate

(11) $$\|[u_0, x_0; F]^{-1}L\| \le \frac{1}{1 - \Phi_0(w_1(\|x_0 - x^*\|), \|x_0 - x^*\|)}.$$

Thus, the iterate y_0 exists and

(12) $$y_0 - x^* = x_0 - x^* - [u_0, x_0; F]^{-1}F(x_0)$$

by the first substep of (2) if $m = 0$. It follows by the estimate (3), (6) (if $j = 1$), (11), (12) and (T_6) that

$$y_0 - x^* = [u_0, x_0; F]^{-1}([u_0, x_0; F] - [x_0, x^*; F])(x_0 - x^*),$$

so

$$\|y_0 - x^*\| \leq \frac{\Phi(\|u_0 - x^*\|, \|x_0 - x^*\|)\|x_0 - x^*\|}{1 - \Phi_0(\|u_0 - x^*\|, \|x_0 - x^*\|)}$$

(13) $$\leq J_1(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < k^*.$$

Hence, the iterate $y_0 \in U(x^*, k^*)$ and the assertion (8) is validated if $m = 0$. As in (10) but using (3), (5) and (T_5) we obtain

(14) $$\|[u_0, y_0; F]^{-1}L\| \leq \frac{1}{1 - \Phi_0(\|u_0 - x^*\|, \|y_0 - x^*\|)},$$

so the iterate x_1 exists.

Moreover,we can have if $m = 0$ in the method (2)

$$x_1 - x^* = [u_0, y_0; F]^{-1}([u_0, y_0; F] - [y_0, x^*; F])(y_0 - x^*)$$

leading as in (13)

$$\|x_1 - x^*\| \leq \frac{\Phi(\|u_0 - x^*\|, \|y_0 - x^*\|)\|y_0 - x^*\|}{1 - \Phi_0(\|u_0 - x^*\|, \|y_0 - x^*\|)}$$

(15) $$\leq J_2(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\|.$$

So, the iterate $x_1 \in U(x^*, k^*)$ and the assertion (7), (9) are validated if $m = 1$ and $m = 0$, respectively. The induction for assertions (7)-(9) is terminated if x_0, u_0, y_0, x_1 are exchanged by x_m, u_m, y_m, x_{m+1} in previous calculations. Furthermore, the estimate

(16) $$\|x_{m+1} - x^*\| \leq b(\|x_m - x^*\|) \leq k^*,$$

for $b = J_2(\|x_0 - x^*\|) \in [0, 1)$ give $\lim_{m \to +\infty} x_m = x^*$ and complete the induction for assertions (7)-(9). $\qquad \square$

REMARK 2. *The real function w_1 can be selected using the assumption (18) and the following calculations:*

$$u - x^* = x - x^* + \xi F(x) = x - x^* + \xi[x, x^*; F](x - x^*)$$
$$= (I + \xi LL^{-1}([x, x^*; F] - L + L))(x - x^*),$$
$$= ((I + \xi L) + \xi LL^{-1}([x, x^*; F] - L))(x - x^*).$$

Thus, a possible choice for the function w_1 is

(17) $$w_1(t) = (\|1 + \xi L\| + \|\xi\|\Phi_1(t))t.$$

The function can be further specified if the linear operator L is precised.
A popular choice is $L = F'(x^*)$. But in this case although there are no derivatives on the method (2) it cannot be used to solve non differentiable equations under previous assumptions, since we assume x^* is a simple solution (i.e $F'(x^*)$ is invertible). Thus L should be chosen so that functions "Φ" are as tight as possible but not $L = F'(x^*)$ in the case of nondifferentiable equations.

The isolation of the solution domain is specified in the next result.

PROPOSITION 3. *Assume:*
The conditions (T_2) and (T_5) are validated on the ball $U(x^*, p)$ for some $p > 0$ and there exists $p_1 \geq p$ such that such that

$$\|L^{-1}([y, x^*; F] - L)\| \leq \Phi_1(\|y - x^*\|),$$
$$\Phi_1(p_1) < 1,$$

where $\Phi_1 : B_0 \times B \to \mathbb{R}$ is CSNF.
Then, the equation $F(x) = 0$ is uniquely solvable by x^* in the domain $D_1 = D \cap U[x^*, p]$.

Proof. Suppose there exists a solution $\lambda \in D_1$ such that $F(\lambda) = 0$ and $\lambda \neq x^*$. Define the divided difference $[\lambda, x^*; F]$. Then, we get

(18)
$$\|L^{-1}([\lambda, x^*; F] - L)\| \leq \Phi_1(\|\lambda - x^*\|)$$
$$\leq \Phi_1(p_1) < 1.$$

Thus, from identity

$$\lambda - x^* = [\lambda, x^*; F]^{-1}(F(\lambda) - F(x^*))$$
$$= [\lambda, x^*; F]^{-1}(0) = 0,$$

we conclude $\lambda = x^*$. $\hspace{2cm} \square$

REMARK 4. *Notice that a possible choice for $p = k^*$.*

ANALYSIS II: SEMI-LOCAL

This time the role of $x^* \in D$ and the functions "Φ"are switched by $x_0 \in D$ and the functions "Ψ"precised below.
Assume there exist:

(M_1) Functions CSNF $\Psi_0 : B \times B \to B$, $w_2 : B \to B$ so that $\Psi_0(w_2(t), t) - 1 = 0$ has MPS in B denoted as q. Let $B_3 = [0, q)$.

(M_2) Functions CSNF $\Psi : B_3 \times B_3 \times B_3 \to B$, $\Psi_1 : B_3 \times B_3 \times B_3 \times B_3 \to B$. Let $D_1 = D \cap U(x_0, q)$

(M_3) There exist $x_0 \in D$ and an invertible linear operator L so that for $x \in D$, $u = x + \xi F(x)$

$$\|u - x_0\| \leq w_2(\|x - x_0\|)$$

and

$$\|L^{-1}([u, x; F] - L)\| \leq \Psi_0(\|u - x_0\|, \|x - x_0\|).$$

It follows by (M_1) and (M_3) that

$$\Psi_0(\|u_0 - x_0\|, \|x_0 - x_0\|) \leq \Psi(w_2(0), 0) < 1,$$

so, $[u_0, x_0; F]$ is invertible, thus y_0 exists and consequently we can let $\|[u_0, x_0; F_0]^{-1} F(x_0)\| \leq \gamma_0$.

Consider the sequence $\{\delta_m\}$ generated for $\delta_0 = 0$ and each $m = 0, 1, 2 \ldots$ as

$$
\begin{aligned}
\delta_{m+1} =& \gamma_m + \frac{\Psi(w_2(\delta_m), \delta_m, \gamma_m)(\gamma_m - \delta_m)}{1 - \Psi_0(w_2(\delta_m), \gamma_m)}, \\
\zeta_{m+1} =& \Psi_1(w_2(\delta_m), \delta_m, \gamma_m, \delta_{m+1})
\end{aligned}
$$

(19)

and

$$
\gamma_{m+1} = \delta_{m+1} + \frac{\zeta_{m+1}}{1 - \Psi_0(w_2(\delta_{m+1}), \delta_{m+1})}.
$$

(M_4)

$$
\Psi_0(w_2(\delta_m), \gamma_m) < 1 \quad \Psi_0(w_2(\delta_m), \delta_m) < 1
$$

$m = 0, 1, 2, \ldots$ and $\delta_m \leq q_0$ for some $q_0 \in B_3 - \{0\}$. It follows from the definition (19) and this hypothesis that $0 \leq \delta_m \leq \gamma_m \leq \delta_{m+1} \leq q_0$ and there exists $\delta^* \in [0, q_0]$ so that $\lim_{m \to +\infty} \delta_m = \delta^*$. Notice that δ^* is the least upper bound of the scalar sequence $\{\delta_m\}$ which is unique.

(M_5) For each $x, y, u \in D_1$, $u = x + \xi F(x)$

$$
\|L^{-1}([y, x, ; F] - [u, x; F])\| \leq \Psi(\|x - x_0\|, \|u - x_0\|, \|y - x_0\|)
$$

and

$$
\|L^{-1}([x, y, ; F] - [u, v; F])\| \leq \Psi_1(\|x - x_0\|, \|y - x_0\|, \|u - x_0\|, \|v - x_0\|).
$$

and

(M_6) $U(x_0, q^*) \subset D$, where

$$
q^* = \max\{\delta^*, w_2(\delta^*)\}.
$$

Then as in local case we present the semi-local result.

THEOREM 5. *Under the Assumptions* $(M_1) - (M_4)$ *there exists* $x^* \in U[x_0, q^*]$ *solving the equation* $F(x) = 0$ *so that*

(20) $$\|x_m - x^*\| \leq \delta^* - \delta_m \quad \text{for all} \quad m = 0, 1, 2, \ldots.$$

Proof. The assertions

(21) $$\|y_i - x_i\| \leq \gamma_i - \delta_i$$

and

(22) $$\|x_{i+1} - y_i\| \leq \delta_{i+1} - \gamma_i,$$

must be shown by induction. Notice that (21) holds if $m = 0$ by the choice of δ_0, γ_0 and the first substep of the method (2) if $m = 0$. Then, as in local

proof we have in turn the estimates

$$\|[u_i, y_i; F]^{-1}L\| \leq \frac{1}{1 - \Psi_0(\|u_i - x_0\|, \|x_i - x_0\|)}$$

$$\leq \frac{1}{1 - \Psi_0(w_2(\delta_i), \gamma_i)},$$

$$F(y_i) = F(y_i) - F(x_i) - [u_i, x_i; F](y_i - x_i),$$

$$= ([y_i, x_i; F] - [u_i, x_i; F])(y_i - x_i),$$

$$\|L^{-1}F(y_i)\| \leq \Psi(\|u_i - x_0\|, \|x_i - x_0\|, \|y_i - x_0\|)$$

$$\leq \Psi(w_2(\delta_i), \delta_i, \gamma_i) = P_i,$$

$$x_{i+1} - y_i = -[u_i, y_i; F]^{-1}F(y_i),$$

$$\|x_{i+1} - y_i\| \leq \|[u_i, y_i; F]^{-1}L\|\|L^{-1}F(y_i)\|$$

$$\leq \frac{P_i}{1 - \Psi_0(w_2(\delta_i), \gamma_i)} = \delta_{i+1} - \gamma_i,$$

$$\|x_{i+1} - x_0\| \leq \|x_{i+1} - y_i\| + \|y_i - x_0\|$$

$$\leq \delta_{i+1} - \gamma_i + \gamma_i - \delta_0 = \delta_{i+1} < \delta^*,$$

$$F(x_{i+1}) = F(x_{i+1}) - F(y_i) - [u_i, x_i; F](x_{i+1} - y_i),$$

$$= ([x_{i+1}, y_i; F] - [u_i, x_i; F])(x_{i+1} - y_i),$$

$$\|L^{-1}F(x_{i+1})\| \leq \Psi_1(\|u_i - x_0\|, \|x_i - x_0\|, \|y_i - x_0\|, \|x_{i+1} - x_0\|),$$

$$(23) \qquad \leq \Psi(w_2(\delta_i), \delta_i, \gamma_i, \delta_{i+1}) = \zeta_{i+1},$$

$$\|[u_{i+1}, x_{i+1}; F]^{-1}L\| \leq \frac{1}{1 - \Psi(w_2(\delta_{i+1}), \delta_{i+1})},$$

$$\|y_{i+1} - x_{i+1}\| \leq \|[u_{i+1}, x_{i+1}; F]^{-1}L\|\|L^{-1}F(x_{i+1})\|$$

$$\leq \frac{\zeta_{i+1}}{1 - \Psi_0(w_2(\delta_{i+1}), \delta_{i+1})} = \gamma_{i+1} - \delta_{i+1},$$

$$\|y_{i+1} - x_0\| \leq \|y_{i+1} - x_{i+1}\| + \|x_{i+1} - x_0\|$$

$$\leq \gamma_{i+1} - \delta_{i+1} + \delta_{i+1} - \delta_0 = \gamma_{i+1} < \delta^*.$$

Thus, $\{x_i\}$, $\{y_i\} \in U(x^*, \delta^*)$, (21), (22) are validated and the sequences are Cauchy in Banach space X. Hence, there exists $x^* \in U[x^*, \delta^*]$ so that $\lim_{i \to +\infty} x_i = x^*$. By letting $i \to +\infty$ in (23) we get $F(x^*) = 0$. Finally, for $i \geq 0$ the estimate

$$\|x_{i+m} - x_i\| \leq \delta_{i+m} - \delta_i$$

implies (20) if $i \to +\infty$.   □

REMARK 6. *The function w_2 can be determined analogously to the function w_1 as follows:*

$$u - x_0 = x - x_0 + \xi F(x)$$
$$= (I + \xi[x, x_0; F])(x - x_0) + \xi F(x_0),$$
$$= [(I + \xi L) + \xi L L^{-1}([x, x_0; F] - L)](x - x_0) + \xi F(x_0).$$

Assume: There exist FCN $\Psi_2 : B_3 \to B$ such that

$$\|L^{-1}([x, x_0; F] - L))\| \leq \Psi_2(\|x - x_0\|)$$

for all $x \in B_3$. Then, we can choose

$$w_2(t) = (\|I + \xi L\| + \|\xi\|\|L\|\Psi_2(t))t + \|\xi\|\|F(x_0)\|.$$

The next result determines the isolation of a solution region.

PROPOSITION 7. *Assume :*
There exists a solution $\tau \in B(x_0, \upsilon)$ of solution $F(x) = 0$ for some $\upsilon > 0$, The first condition in (M_3) is validated on the ball $B(x_0, \upsilon)$ and there exists $\varpi \geq \upsilon$ such that

$$\|L^{-1}([\tau, x; F] - L)\| \leq \Psi_0(\|\tau - x_0\|, \|x - x_0\|),$$
$$\Psi_0(\upsilon, \varpi) < 1,$$

where $\Psi_0 : B \times B \to \mathbb{R}$ is FCN.
Let $D_3 = D \cap B[x_0, \varpi]$. Then, the only solution of the equation $F(x) = 0$ is τ in the region D_3.

Proof. Let $\tau^* \in D_3$ satisfy $F(\tau^*) = 0$. Define the divided difference $[\tau, \tau^*; F]$. This is possible if $\tau^* \neq \tau$. Then, we obtain in turn by the conditions

$$\|L^{-1}([\tau, \tau^*; F] - L)\| \leq \Psi_0(\|\tau - x_0\|, \|\tau^* - x_0\|),$$
$$\leq \Psi_0(\upsilon, \varpi) < 1,$$

thus $[\tau, \tau^*; F]^{-1} \in L(B)$. But the identity

$$0 = F(\tau) - F(\tau^*) = [\tau, \tau^*; F](\tau - \tau^*)$$

leads to a contradiction and the divided difference $[\tau, \tau^*; F]$ can not be defined. Therefore, we conclude that $\tau^* = \tau$. $\square$

REMARK 8.   (1) *The point q can also be replaced by δ^* in condition (M_4).*
    (2) *If conditions (M_1)-(M_4) are all validated, let $x^* = \tau$ and $\delta^* = \upsilon$.*

APPLICATIONS

Method (2) can turn from without memory to with memory if the constant ξ turns into a suitable linear operator. As in [6] choose ξ to be the Kurchatov operator [2, 14]

$$(24) \qquad \xi \approx -[2x_m - x_{m-1}, x_{m-1}; F]^{-1}.$$

That is we obtain the method with memory derived by (2) for ξ replaced by (24). Then, the local as well as the semi-local results hold say if w_1, w_2 are adjusted. In view of the calculation

$$\|[2x_m - x_{m-1}, x_{m-1}; F]^{-1}LL^{-1}\| \leq \frac{\|L^{-1}\|}{1 - \overline{\overline{\Phi}}_0(2x_m - x_{m-1} - x^*\|, \|x_m - x^*\|)}$$

$$\leq \frac{\|L^{-1}\|}{1 - \overline{\overline{\Phi}}_0(2\|x_m - x^*\| + \|x_{m-1} - x^*\|, \|x_m - x^*\|)}$$

provided that

$$\|L^{-1}([z, x; F] - L)\| \leq \overline{\overline{\Phi}}_0(\|z - x^*\|, \|x - x^*\|),$$

where $\overline{\overline{\Phi}}_0 : \overline{X} \to \overline{X} \to X$ is CSNF for all $x, z \in D$.
Then, ξ can be replaced in Remark 2 by $\frac{\|L^{-1}\|}{1 - \overline{\overline{\Phi}}_0(3t, t)} = a^*$ provided that
$\overline{\overline{\Phi}}_0(3t, t) - 1$ has a MPS denoted as $\overline{\overline{s}}$.
In this case we also require (T_7) to be replaced by
$(T_7)'$ $U[x^*, \varsigma]$ where $\varsigma = \max\{3k^*, w_1(k^*)\}$.

EXAMPLE 9. *Let $X = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and $D = B[x^*, 1]$. Consider the mapping on the ball D be given for $\rho = (\rho_1, \rho_2, \rho_3)^T$ as*

$$F(\rho) = \left(\frac{e-1}{2}\rho_1^2 + \rho_1, \rho_2, e^{\rho_3} - 1\right)^T.$$

The Jacobian is given by

$$F'(\rho) = \begin{pmatrix} (e-1)\rho_1 + 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{\rho_3} \end{pmatrix}.$$

It follows that the solution $x^* = (0, 0, 0)^T$ and $F'(x^*) = I$ the identity mapping with choice $L = F'(x^*)$. The divided difference is defined by $[x, y; F] = \int_0^1 F'(x + \theta(y - x))d\theta$:. Then, the conditions (T_3)-(T_4) are validated provided that

$$\Phi_0(u_1, u_2) = \frac{1}{2}(e-1)(u_1 + u_2)$$

$$\Phi(u_1, u_2) = \frac{1}{2}(e-1)u_2$$

$$\Phi_1(u_1) = \frac{e-1}{2}u_1.$$

Table 1. Comparison of radius of convergence Example 9

Method	SM_1	SM_2	$PM1_4$	M_4	MWM	HLM_5	$PM2_5$
Convergence Radius	0.274772	0.330032	0.111497	0.131305	0.184350	0.02763	0.118532

Then, taking $\xi = 1$ i.e considering the case of method (2) (denoted by SM_1) without memory, the radius of convergence k^* from (4) is given as

$$k^* = 0.2747721282852604\ldots$$

If we consider the method (2) (denoted by SM_2) with memory, then from (24) the radius of convergence k^* is given by

$$k^* = 0.33003245785816543\ldots$$

We compare the radius of convergence obtained by fourth order Kung-Traub method (M_4) given by Sharma et al. [21], fifth-order Weerakoon method (MWM) in Sharma and Parhi [22], fourth and fifth order methods ($PM1_4$), (HLM_5), ($PM2_5$) in Maroju et al. [12].

Thus, from Table 1 it is clear that enlarged convergence radius is obtained by the our approach for the methods SM_1 and SM_2 in comparison to existing methods.

REMARK 10. *Notice that using the iterates of the type $u_m = x_m + x_i F(x_m)$ may lead to very small radius of attraction balls for the method, unless a small x_i is used.*

EXAMPLE 11. *Let $Q : X \to X$ be a mapping. Recall that the standard divided difference of order one when $B = \mathbb{R}^k$ is defined for $\overline{x} = (x_1, x_2, \ldots, x_k)$, $\overline{y} = (y_1, y_2, \ldots, y_k)$, $i = 0, 1, 2, \ldots, k$, $j = 0, 1, 2, \ldots, k$ by*

$$[\overline{y}, \overline{x}; Q]_{ji} = \frac{Q_j(y_1, \ldots, y_{i-1}, y_i, x_{i+1}, \ldots, x_k) - Q_j(y_1, \ldots, y_{i-1}, x_i, x_{i+1}, \ldots, x_k)}{y_i - x_i}$$

provided that $y_i \neq x_i$. It is known that for certain pairs of distinct vectors x, y, the formula is not applicable when some of the components are equal.
The solution is sought for the nonlinear system

$$t_1^2 - t_2 + 1 + \frac{1}{9}|t_1 - 1| = 0$$

$$t_1 + t_2^2 - 7 + \frac{1}{9}|t_2| = 0$$

Let $Q = (Q_1, Q_2)^T$ for $(t_1, t_2) \in \mathbb{R} \times \mathbb{R}$, where

$$Q_1 = t_1^2 - t_2 + 1 + \frac{1}{9}|t_1 - 1| \quad and$$

$$Q_2 = t_1 + t_2^2 - 7 + \frac{1}{9}|t_2|.$$

Then, the system becomes

$$Q(t) = 0 \quad for \quad t = (t_1, t_2)^T.$$

The divided difference $L = [\cdot, \cdot; Q]$ belongs in the space $L_{2 \times 2}(\mathbb{R})$ and is the standard 2×2 matrix in $\mathbb{R}^2$ [10]. Let us choose $x_0 = (1.15, 2.36)^T$ and $\xi = 1$. Then, the application of the method (2) gives the solution x^* after three iterations. The solution

Table 2. Comparison of the performances of methods for Example 12

Methods	k	$\|x_{k+1} - x_k\|$	$\|F(x_k)\|$
SM_1	1	$0.283 - 001$	$1.29e - 001$
	2	$1.31e - 002$	$1.61e - 007$
	3	$1.62e - 008$	$3.13e - 025$
SM_2	1	$0.284 - 001$	$6.92e - 003$
	2	$6.98e - 004$	$1.80e - 013$
	3	$1.82e - 014$	$3.11e - 048$

$x^* = (x_1^*, x_2^*)^T = (1.159360850193\ldots, 2.361824342093\ldots)^T$.

Taking into account (24) and $x_{-1} = (1.22, 2.37)^T$, if we consider the method (2) with memory, the solution x^* is obtained after two iterations. The number of iterations required to obtain solution x^* by two Kurchatov methods presented in [4] are four and five, respectively. Thus, less number of iterations are required to obtain the solution by the methods SM_1 and SM_2 in comparison to Kurchatov methods [4].

EXAMPLE 12. *Consider the following system of ten equations:*

$$\sum_{j=1, j\neq i}^{10} x_j - e^{-x_i} = 0, \quad 1 \leq i \leq 10.$$

The initial approximations chosen are $x_0 = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)^T$ and $x_{-1} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T$ to obtain the solution

$$x^* = (0.10048840033731707\ldots, 0.10048840033731707\ldots, \ldots, 0.10048840033731707\ldots)^T$$

The comparison of error and norm of function for methods SM_1 and SM_2 taking three iterations are presented in Table 2. The stopping criterion used is $\|x_{k+1} - x_k\| + \|F(x_k)\| < 10^{-100}$. The number of iterations required to obtain solution x^* by methods SM_1 and SM_2 are three and two, respectively. Thus, the method (2) with memory requires lesser number of iterations than the method without memory taking $\xi = 0.5$.

REMARK 13. *Summing up what we did in this study is that the local and semilocal convergence of the method (2) are presented without Taylor series expansions. Moreover, the limitations $(C_1) - (C_5)$ have been addressed as follows:*

$(C_1)'$ *The radius of convergence k^* is provided in (3). So, the initial point is selected from a certain ball centered at x^* and of radius k^*. Moreover, the uniqueness of the solution is established in the Propositions 3 and 7. Furthermore, the convergence conditions use only the first derivative (see Theorem 1 and 5).*

$(C_2)'$ *Computable error bounds on $\|x_m - x^*\|$ or $\|x_{m+1} - x_m\|$ become available in Theorem 1 and Theorem 5, respectively. Thus, the number of iterations to be performed to reach a certain error tolerance is known in advance.*

$(C_3)'$ *The method ([2](#)) converges $t^* = \frac{1}{\pi}$ in the motivational example of the introduction if t_0 is chosen close enough to t^* and inside D for $\xi = 1$.*

$(C_4)'$ *The results are established in more general setting of a Banach space.*

$(C_5)'$ *The more interesting and challenging semi-local convergence analysis is provided using majorizing sequences [2, 14].*

CONCLUDING REMARKS

The present study is based on the local and semi-local analysis of two step Steffensen-type methods with and without memory. It is applicable to the case when the problems formulated from varied areas of science and engineering are nondifferentiable. However, this methodology can also be applied to solve the differentiable equations and on other methods utilizing the inverses of linear operators. Further, numerical experiments are performed on various examples for both the cases i.e with and without memory that demonstrates the theoretical results. The enlarged convergence radii have been obtained by the presented approach as compared to the existing ones. In our future research the methodology shall be extended to extend the applicability of multipoint and multi-step methods [2, 14, 20, 23].

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