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BEST APPROXIMATION OF HARTLEY-BESSEL MULTIPLIER OPERATORS ON WEIGHTED SOBOLEV SPACES

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Abstract. The main goal of this paper is to introduce the Hartley-Bessel L^2_α-multiplier operators and to give for them some new results as Plancherel's, Calderon's reproducing formulas and Heisenberg's, Donoho-Stark's uncertainty principles. Next, using the theory of reproducing kernels we give best approximation and an integral representation of the extremal functions related to these operators on weighted Sobolev spaces.

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1. INTRODUCTION

In their seminal papers, Hörmander's and Mikhlin's [10, 15] initiated the study of boundedness of the translation invariant operators on $\mathbb{R}^d$. The translation invariant operators on $\mathbb{R}^d$ characterized using the classical Euclidean Fourier transform $\mathcal{F}(f)$ therefore they also known as Fourier multipliers. Given a measurable function

$$m : \mathbb{R}^d \longrightarrow \mathbb{C}$$

its Fourier multiplier is the linear map $\mathcal{T}_m$ given for all $\lambda \in \mathbb{R}^d$ by the relation

$$\mathcal{F}(\mathcal{T}_m(f))(\lambda) = m(\lambda)\mathcal{F}(f)(\lambda) \tag{1}$$

The Hörmander-Mikhlin fundamental condition gives a criterion for L^p-boundedness for all $1 < p < \infty$ of Fourier multiplier $\mathcal{T}_m$ in terms of derivatives of the symbol m, more precisely if

$$|\partial^\gamma_\lambda m(\lambda)| \lesssim |\lambda|^{-|\gamma|} \quad for \quad 0 \leq |\gamma| \leq \left[\frac{d}{2}\right] + 1. \tag{2}$$

Then, $\mathcal{T}_m$ can be extended to a bounded linear operator from $L^p(\mathbb{R}^d)$ into itself.

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The condition (2) imposes m to be a bounded function, smooth over $\mathbf{R}^d\setminus\{0\}$ satisfying certain local and asymptotic behavior. Locally, m admits a singularity at 0 with a mild control of derivatives around it up to order $\left[\frac{d}{2}\right]+1$. This singularity links to deep concepts in harmonic analysis and justifies the key role of Hörmander-Mikhlin theorem in Fourier multiplier L_p-theory, this condition defines a large class of Fourier multipliers including Riesz transforms and Littelwood-Paley partitions of unity which are crucial in Fourier summability or Pseudo-differential operator.The boundedness of Fourier multipliers is useful to solve problems in the area of mathematical analysis as Probability theory see [13], Stochastic processus see [2]. For its importance many researcher extend the theory of Fourier multiplier to different setting for example in the Dunkl-Weinstein setting [20], in the Laguerre-Bessel setting [7], in the Dunkl's setting [19]. The general theory of reproducing kernels is stared with Aronszajn's in [1] in 1950, next the authors in [12, 17, 18] applied this theory to study Tikhonov regularization problem and they obtained approximate solutions for bounded linear operator equations on Hilbert spaces with the viewpoint of numerical solutions by computers. This theory has gained considerable interest in various field of mathematical sciences especially in Engineering and numerical experiments by using computers [12, 18].
The Hartley transform is an integral transform attributed to Hartley see [5, 6], this transform shares several essential properties with the classical Fourier transform, including linearity, invertibility and Parseval's identity. These transforms find extensive applications across various field of mathematics, physics and engineering, such as signal processing, data analysis and number theory see [5, 6, 11, 21].
The Hartley transform is a linear operator defined for a suitable function $\psi(x)$ as follows:

$$H(\psi)(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\psi(x)\operatorname{cas}(\lambda x)dx, \tag{3}$$

where $\operatorname{cas}(x)$ is the cas function, defined as:

$$\operatorname{cas}(x)=\sum_{n=0}^{\infty}\frac{(-1)^{\binom{n+1}{2}}}{n!}x^n, \tag{4}$$

with $\binom{n}{2}=\frac{n(n-1)}{2}$ being the binomial coefficient. The cas(x) function (1.4) can be seen as a generalization of the exponential function exp.
A simple computation shows that the cas function is the unique C^∞ solution of the following differential-reflection problem see [5]

$$\begin{cases} R\partial_x u(x)=\lambda u(x),\\ u(0)=0.\end{cases}$$

Here, ∂_x represents the first-order derivative, and R is the reflection operator acting on functions $f(x)$ as:

$$(Rf)(x) = f(-x). \tag{5}$$

Furthermore, the function $cas(x)$ is multiplicative on $\mathbb{R}$ in the sens

$$\operatorname{cas}(x)\operatorname{cas}(y) = \frac{1}{2}(\operatorname{cas}(x+y) - \operatorname{cas}(-x-y) + \operatorname{cas}(x-y) + \operatorname{cas}(y-x)). \tag{6}$$

Inspired by the relation (1.6), the author in [3] generalized the relation (1.6) for the Hartley-Bessel function and introduce a generalized convolution product. This paper focuses on the generalized Hartley transform introduced in [3, 4] called the Hartley-Bessel transform, more precisely we consider the following dfferential-reflection operator Δ_α defined by

$$\Delta_\alpha = R\left(\partial_x + \frac{\alpha}{x}\right) + \frac{\alpha}{x}, \quad \alpha \geq 0. \tag{7}$$

Where R is the reflection operator given by the relation (1.4).
The operator Δ_α is closely connected with the Dunkl's theory [9], furthermore the eigenfunctions of this operator are related to Bessel functions and they satisfies a product formula which permits to develop a new harmonic analysis associated with this operator see [3] for more information.
The Hartly-Bessel transform H_α generalizing the classical Hartley transform (1.3) and it is defined on $L^1_\alpha(\mathbb{R})$ by

$$H_\alpha(f)(\lambda) = \int_{\mathbb{R}} B_\alpha(\lambda x) f(x) d\mu_\alpha(x), \qquad \text{for } \lambda \in \mathbb{R}.$$

Where μ_α is the measure on $\mathbb{R}$ and $B_\alpha(\lambda\cdot)$ is the Hartley-Bessel kernel given later. Let σ be a function in $L^2_\alpha(\mathbb{R})$ and $\beta > 0$ be a positive real number, the Hartley-Bessel L^2_α-multiplier operators are defined for smooth function on $\mathbb{R}$ as

$$\mathcal{M}_{\sigma,\beta}(f)(x) := H_\alpha^{-1}\left(\sigma_\beta H_\alpha\right)(x) \tag{8}$$

These operators are a generalization of the classical multiplier operators given by the relation (1.1). The remainder of this paper is arranged as follows, in section 2 we recall the main results concerning the harmonic analysis associated with the Hartley-Bessel transform, in section 3, we introduce the Hartley-Bessel L^2_α-multiplier operators $\mathcal{M}_{\sigma,\beta}$ and we give for them a Plancherel's, point- wise reproducing formulas and Heisenberg's, Donoho-Stark's uncertainty principles. The last section of this paper is devoted to give an application of the general theory of reproducing kernels to Fourier multiplier theory and to give best estimates and an integral representation of the extremal functions related to the Hartley-Bessel L^2_α-multiplier operators on weighted Sobolev spaces.

2. HARMONIC ANALYSIS ASSOCIATED WITH THE HARTLEY-BESSEL TRANSFORM

In this section we recall some results in harmonic analysis related to the Hartley-Bessel transform, for more details we refer the reader to [3].

- For $\alpha \geq 0$, μ_α is the weighted Lebesgue measure defined on $\mathbb{R}$ by

$$d\mu_\alpha(x) := \frac{|x|^{2\alpha}}{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+\frac{1}{2})}dx.,$$

where Γ is the Gamma function.

- $L^p_\alpha(\mathbb{R}), 1 \leq p \leq \infty$, the space of measurable functions on $\mathbb{R}$, satisfying

$$\|f\|_{p,\mu_\alpha} =: \begin{cases} (\int_{\mathbb{R}} |f(x)|^p d\mu_\alpha(x))^{1/p} < \infty, & 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{x\in\mathbb{R}} |f(x)| < \infty, & p = \infty. \end{cases}$$

In particular, for $p = 2$, $L^2_\alpha(\mathbb{R})$ is a Hilbert space with inner product given by

$$\langle f, g\rangle_\alpha = \int_{\mathbb{R}} f(x)\overline{g(x)}d\mu_\alpha(x)$$

2.1. The Eigenfunctions of the differential-reflection operator Δ_α.

For $\lambda \in \mathbb{C}$ we consider the following Cauchy problem

$$(S) : \begin{cases} \Delta_\alpha(u)(x) = \lambda u(x), \\ u(0) = 1. \end{cases}$$

From [3, 4], the Cauchy problem (S) admits a unique solution $B_\alpha(\lambda.)$ given by

$$B_\alpha(\lambda x) = j_{\alpha-\frac{1}{2}}(\lambda x) + \frac{\lambda x}{2\alpha+1} j_{\alpha+\frac{1}{2}}(\lambda x), \tag{9}$$

where j_α denotes the normalized Bessel function of order α see [16].
The function $B_\alpha(\lambda.)$ is infinitely differentiable on $\mathbb{R}$ and we have the following important result

$$\forall \lambda, x \in \mathbb{R}, \quad |B_\alpha(\lambda x)| \leq \sqrt{2}. \tag{10}$$

Furthermore from [3], the Hartley-Bessel kernel (2.1) is multiplicative on $\mathbb{R}$ in the sens

$$\forall \lambda \in \mathbb{R}, x, y \in \mathbb{R}^* \quad B_\alpha(\lambda x)B_\alpha(\lambda y) = \int_{\mathbb{R}} B_\alpha(\lambda z)K_\alpha(x, y, z)d\mu_\alpha(z) \tag{11}$$

where K_α is the Bessel kernel given explicitly in [3].
The product formula (2.3) generalize the relation (1.6) and permits to define a translation operator, convolution product and to develop a new harmonic analysis associated to the Differential-reflection operator Δ_α.

2.2. The Hartley-Bessel transform. ([3]) The Hartley-Bessel transform H_α defined on $L^1_\alpha(\mathbb{R})$ by

$$H_\alpha(f)(\lambda)=\int_{\mathbb{R}} B_\alpha(\lambda x)f(x)d\mu_\alpha(x), \qquad \text{for } \lambda\in\mathbb{R}$$

Some basic properties of this transform are as follows, for the proofs, we refer the reader to [3, 4, 5].

(1) For every $f\in L^1_\alpha(\mathbb{R})$ we have

$$\|H_\alpha(f)\|_{\infty,\mu_\alpha}\leq\sqrt{2}\|f\|_{1,\mu_\alpha}. \tag{12}$$

(2)(Inversion formula) For $f\in\left(L^1_\alpha\cap L^2_\alpha\right)(\mathbb{R})$ such that $\mathcal{F}_\alpha(f)\in L^1_\alpha(\mathbb{R})$ we have

$$f(x)=\int_{\mathbb{R}} B_\alpha(\lambda x)H_\alpha(f)(\lambda)d\mu_\alpha(\lambda), \qquad \text{a.e} \qquad x\in\mathbb{R}. \tag{13}$$

(3) (Parseval formula) For all $f,g\in L^2_\alpha(\mathbb{R})$ we have

$$\langle f,g\rangle_\alpha=\langle H_\alpha(f),H_\alpha(g)\rangle_\alpha\,, \tag{14}$$

In particular we have

$$\|f\|_{2,\mu\alpha}=\|H_\alpha(f)\|_{2,\mu\alpha}\,. \tag{15}$$

(4) (Plancherel theorem) The Hartley-Bessel transform H_α can be extended to an isometric isomorphism from $L^2_\alpha(\mathbb{R})$ into $L^2_\alpha(\mathbb{R})$.

2.3. The translation operator associated with the Hartley-Bessel transform. The product formula (2.3) permits to define the translation operator as follows Let $x,y\in\mathbb{R}$ and f is a measurable function on $\mathbb{R}$ the translation operator is defined by

$$\tau^x_\alpha f(y)=\int_{\mathbb{R}} f(z)K_\alpha(x,y,z)d\mu_\alpha(z),$$

The following proposition summarizes some properties of the Hartley-Bessel translation operator see [3]. For all $x,y\in\mathbb{R}$,we have:

(1)

$$\tau^x_\alpha f(y)=\tau^y_\alpha f(x). \tag{16}$$

(2)

$$\int_{\mathbb{K}}\tau^x_\alpha f(y)d\mu_\alpha(y)=\int_{\mathbb{R}} f(y)d\mu_\alpha(y). \tag{17}$$

(3) for $f\in L^p_\alpha(\mathbb{R})$ with $p\in[1;+\infty]$ $\quad\tau^x_\alpha f\in L^p_\alpha(\mathbb{R})$ and we have

$$\|\tau^x_\alpha f\|_{p,\mu_\alpha}\leq 4\|f\|_{p,\mu_\alpha}, \tag{18}$$

(4) For $f\in L^1_\alpha(\mathbb{R})$, $\quad\tau^x_\alpha f\in L^1_\alpha(\mathbb{R})$ and we have

$$H_\alpha\left(\tau^x_\alpha f\right)(\lambda)=B_\alpha(\lambda x)\mathcal{F}_\alpha(f)(\lambda), \quad \forall\lambda\in\mathbb{R}. \tag{19}$$

The relation (2.11) shows that the translation operator τ^x_α is a particular case of the Hartley-Bessel multiplier operator (1.8).

By using the translation, we define the generalized convolution product of f, g by

$$(f *_\alpha g)(x) = \int_{\mathbb{K}} \tau_\alpha^x(f)(y) g(y) d\mu_\alpha(y).$$

This convolution is commutative, associative and its satisfies the following properties see [3].

(1)(Young's inequality) for all $p, q, r \in [1; +\infty]$ such that: $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ and for all $f \in L_\alpha^p(\mathbb{R}), g \in L_\alpha^q(\mathbb{R})$ the function $f *_\alpha g$ belongs to the space $L_\alpha^r(\mathbb{R})$ and we have

$$\|f *_\alpha g\|_{r,\mu_\alpha} \leq 4\|f\|_{p,\mu_\alpha}\|g\|_{q,\mu_\alpha} \tag{20}$$

(2) For $f, g \in L_\alpha^2(\mathbb{R})$ the function $f *_\alpha g$ belongs to $L_\alpha^2(\mathbb{R})$ if and only if the function $H_\alpha(f)H_\alpha(g)$ belongs to $L_\alpha^2(\mathbb{R})$ and in this case we have

$$H_\alpha(f *_\alpha g) = H_\alpha(f)H_\alpha(g). \tag{21}$$

(3) For all $f, g \in L_\alpha^2(\mathbb{R})$ then we have

$$\int_{\mathbb{R}} |f *_\alpha g(x,t)|^2 d\mu_\alpha(x) = \int_{\mathbb{R}} |H_\alpha(f)(\lambda)|^2 |H_\alpha(g)(\lambda)|^2 d\mu_\alpha(\lambda), \tag{22}$$

where both integrals are simultaneously finite or infinite.

3. THE HARTLEY-BESSEL L_α^2-MULTIPLIER OPERATORS

The main purpose of this section is to introduce the Hartley-Bessel L_α^2-multiplier operators on $\mathbb{R}$ and to establish for them some uncertainty principles and Calderon's reproducing formulas.

3.1. Calderon's reproducing formulas for the Hartley-Bessel L_α^2-multiplier operators. Let $\sigma \in L_\alpha^2(\mathbb{R})$ and $\beta > 0$, the Hartley-Bessel L_α^2-multiplier operators are defined for smooth functions on $\mathbb{R}$ as

$$\mathcal{M}_{\sigma,\beta}(f)(x) := H_\alpha^{-1}(\sigma_\beta H_\alpha(f))(x), \tag{23}$$

where the function σ_β is given for all $\lambda \in \mathbb{R}$ by

$$\sigma_\beta(\lambda) := \sigma(\beta\lambda),$$

By a simple change of variable we find that for all $\beta > 0, \sigma_\beta \in L_\alpha^2(\mathbb{R})$ and

$$\|\sigma_\beta\|_{2,\mu_\alpha} = \frac{1}{\beta^{\frac{2\alpha+1}{2}}}\|\sigma\|_{2,\mu_\alpha}. \tag{24}$$

According to the relation (2.13) we find that

$$\mathcal{M}_{\sigma,\beta}(f)(x) = \left(H_\alpha^{-1}(\sigma_\beta) *_\alpha f\right)(x), \tag{25}$$

where

$$H_\alpha^{-1}(\sigma_\beta)(x) = \frac{1}{\beta^{2\alpha+1}} H_\alpha^{-1}(\sigma)\left(\frac{x}{\beta}\right). \tag{26}$$

We give some properties of the Hartley-Bessel L^2_α-multiplier operators. (i) For every $\sigma \in L^2_\alpha(\mathbb{R})$, and $f \in L^1_\alpha(\mathbb{R})$, the function $\mathcal{M}_{\sigma,\beta}(f)$ belongs to $L^2_\alpha(\mathbb{R})$, and we have

$$\|\mathcal{M}_{\sigma,\beta}(f)\|_{2,\mu_\alpha} \leq \frac{4}{\beta^{\frac{2\alpha+1}{2}}} \|\sigma\|_{2,\mu_\alpha} \|f\|_{1,\mu_\alpha}.$$

(ii) For every $\sigma \in L^\infty_\alpha(\mathbb{R})$, and for every $f \in L^2_\alpha(\mathbb{R})$, the function $\mathcal{M}_{\sigma,\beta}(f)$ belongs to $L^2_\alpha(\mathbb{R})$, and we have

$$\|\mathcal{M}_{\sigma,\beta}(f)\|_{2,\mu_\alpha} \leq \|\sigma\|_{\infty,\mu_\alpha} \|f\|_{2,\mu_\alpha} \tag{27}$$

(iii) For every $\sigma \in L^2_\alpha(\mathbb{R})$, and for every $f \in L^2_\alpha(\mathbb{R})$, then $\mathcal{M}_{\sigma,\beta}(f) \in L^\infty_\alpha(\mathbb{R})$, and we have

$$\mathcal{M}_{\sigma,\beta}(f)(x) = \int_{\mathbb{R}} \sigma(\beta\lambda) B_\alpha(\lambda x) H_\alpha(f)(\lambda) d\mu_\alpha(\lambda), \quad \text{a.e} \quad x \in \mathbb{R} \tag{28}$$

and

$$\|\mathcal{M}_{\sigma,\beta}(f)\|_{\infty,\mu_\alpha} \leq \frac{4}{\beta^{\frac{2\alpha+1}{2}}} \|\sigma\|_{2,\mu_\alpha} \|f\|_{2,\mu_\alpha}.$$

Proof. (i) By using the relations (2.12),(3.3) we find that

$$\|\mathcal{M}_{\sigma,\beta}(f)\|^2_{2,\mu_\alpha} = \left\| H^{-1}_\alpha(\sigma_\beta) *_\alpha f \right\|^2_{2,\mu_\alpha} \leq 16 \|f\|^2_{1,\mu_\alpha} \left\| H^{-1}_\alpha(\sigma_\beta) \right\|^2_{1,\mu_\alpha}$$

Plancherel's formula (2.7) and the relation (3.2) gives the desired result.
(ii) Is a consequence of Plancherel's formula (2.7).
(iii) By the relations (2.7),(2.12),(3.2) and (3.3) we find the result, on the other hand the relation (3.6) follows from inversion formula (2.5). □

In the following result, we give Plancherel's and pointwise reproducing inversion formula for the Hartley-Bessel L^2_α-multiplier operators.

Theorem 1. *Let $\sigma \in L^2_\alpha(\mathbb{R})$ satisfying the admissibility condition:*

$$\int_0^\infty |\sigma_\beta(\lambda)|^2 \frac{d\beta}{\beta} = 1, \quad \lambda \in \mathbb{R}. \tag{29}$$

(i) (Plancherel formula) For all f in $L^2_\alpha(\mathbb{R})$, we have

$$\int_{\mathbb{R}} |f(x)|^2 d\mu_\alpha(x) = \int_0^\infty \|\mathcal{M}_{\sigma,\beta}(f)\|^2_{2,\mu_\alpha} \frac{d\beta}{\beta}. \tag{30}$$

(ii) (First calderón's formula) Let $f \in L^1_\alpha(\mathbb{R})$ such that $H_\alpha(f) \in L^1_\alpha(\mathbb{R})$ then we have

$$f(x) = \int_0^\infty \left(\mathcal{M}_{\sigma,\beta}(f) *_\alpha H^{-1}_\alpha(\overline{\sigma_\beta}) \right)(x) \frac{d\beta}{\beta}, \quad \textit{a.e. } x \in \mathbb{R}.$$

Proof. (i) By using Fubini's theorem and the relations (2.15) and (3.3) we get

$$\int_0^\infty \|\mathcal{M}_{\sigma,\beta}(f)\|_{2,\mu_\alpha}^2 \frac{d\beta}{\beta} = \int_0^\infty \left[\int_{\mathbb{R}} \left|H_\alpha^{-1}(\sigma_\beta) *_\alpha f(x)\right|^2 d\mu_\alpha(x)\right] \frac{d\beta}{\beta}$$
$$= \int_0^\infty \left[\int_{\mathbb{R}} |H_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda)\right] |\sigma_\beta(\lambda)|^2 \frac{d\beta}{\beta}$$

the admissibility condition (3.7) and Plancherel's formula (2.7) gives the desired result.
(ii) Let $f \in L^1_\alpha(\mathbb{R})$ such that $H_\alpha(f) \in L^1_\alpha(\mathbb{R})$, by using Fubini's theorem and the relations (2.6),(2.11) we find that

$$\int_0^\infty \left(\mathcal{M}_{\sigma,\beta}(f) *_\alpha H_\alpha^{-1}(\overline{\sigma_\beta})\right)(x)\frac{d\beta}{\beta} = \int_0^\infty \left[\int_{\mathbb{R}} \mathcal{M}_{\sigma,\beta}(f)(y)\overline{\tau_\alpha^x\left(H_\alpha^{-1}(\sigma_\beta)\right)(y)}d\mu_\alpha(y)\right] \frac{d\beta}{\beta}$$
$$= \int_0^\infty \left[\int_{\mathbb{R}} H_\alpha(f)(\lambda)B_\alpha(\lambda x)(x,t)d\mu_\alpha(\lambda)\right] |\sigma_\beta(\lambda)|^2 \frac{d\beta}{\beta}$$

the admissibility condition (3.7),inversion formula (2.7) gives the desired result. □

To establish the second Calderon's reproducing formula for the Hartley-Bessel L^2_α-multiplier operators, we need the following technical result. Let $\sigma \in L^2_\alpha(\mathbb{R}) \cap L^\infty_\alpha(\mathbb{R})$ satisfy the admissibility condition (3.7) then the function defined by

$$\Phi_{\gamma,\delta}(\lambda) = \int_\gamma^\delta |\sigma_\beta(\lambda)|^2 \frac{d\beta}{\beta}$$

belongs to $L^2_\alpha(\mathbb{R}) \cap L^\infty_\alpha(\mathbb{R})$ for all $0 < \gamma < \delta < \infty$.

Proof. Using Hölder's inequality for the measure $\frac{d\beta}{\beta}$ and the relation (3.2) we find that

$$\|\Phi_{\gamma,\delta}\|_{2,\mu_\alpha}^2 \leq \log(\delta/\gamma)\|\sigma\|_{\infty,\mu_\alpha}^2 \|\sigma\|_{2,\gamma_\alpha}^2 \int_\gamma^\delta \frac{d\beta}{\beta^{\frac{2\alpha+3}{2}}} < \infty.$$

So $\Phi_{\gamma,\delta}$ belongs to $L^2_\alpha(\mathbb{R})$, furthermore by using the relation (3.7) we get $\|\Phi_{\gamma,\delta}\|_{\infty,\mu_\alpha} \leq 1$ therefore $\Phi_{\gamma,\delta}$ belongs to $L^2_\alpha(\mathbb{R}) \cap L^\infty_\alpha(\mathbb{R})$. □

Theorem 2. *(Second Calderón's formula). Let $f \in L^2_\alpha(\mathbb{R})$ and $\sigma \in L^2_\alpha(\mathbb{R}) \cap L^\infty_\alpha(\mathbb{R})$ satisfy the admissibility condition (3.7) and $0 < \gamma < \delta < \infty$. Then the function*

$$f_{\gamma,\delta}(x) = \int_\gamma^\delta \left(\mathcal{M}_{\sigma,\beta}(f) *_\alpha H_\alpha^{-1}(\overline{\sigma_\beta})\right)(x)\frac{d\beta}{\beta}, \quad x \in \mathbb{R}$$

belongs to $L^2_\alpha(\mathbb{R})$ and satisfies

$$\lim_{(\gamma,\delta)\to(0,\infty)} \|f_{\gamma,\delta} - f\|_{2,\mu_\alpha} = 0 \tag{31}$$

Proof. By a simple computation we find that

$$f_{\gamma,\delta}(x) = \int_{\mathbb{R}} \Phi_{\gamma,\delta}(\lambda) B_\alpha(\lambda x) H_\alpha(f)(\lambda) d\mu_\alpha(\lambda) = H_\alpha^{-1}\left(\Phi_{\gamma,\delta} H_\alpha(f)\right)(x),$$

by using proposition 3.2 we find that $\Phi_{\gamma,\delta} \in L^\infty_\alpha(\mathbb{R})$ then we have $f_{\gamma,\delta} \in L^2_\alpha(\mathbb{R})$ and

$$H_\alpha\left(f_{\gamma,\delta}\right)(\lambda) = \Phi_{\gamma,\delta}(\lambda, m) H_\alpha(f)(\lambda)$$

on the other hand by using Plancherel's formula (2.7) we find that

$$\lim_{(\gamma,\delta)\to(0,\infty)} \|f_{\gamma,\delta} - f\|^2_{2,\mu_\alpha} = \lim_{(\gamma,\delta)\to(0,\infty)} \int_{\mathbb{R}} |H_\alpha(f)(\lambda)|^2 \left(1 - \Phi_{\gamma,\delta}(\lambda)\right)^2 d\mu_\alpha(\lambda)$$

by using the admissibility condition (3.7), the relation (3.9) follows from the dominated convergence theorem. □

3.2. Uncertainty principles for the Hartley-Bessel L^2_α-multiplier operators. The main purpose of this subsection is to establish Heisenberg's and Donoho-Stark's uncertainty principles for the Hartley-Bessel L^2_α-multiplier operators $\mathcal{M}_{\sigma,\beta}$.

3.2.1. *Heisenberg's uncertainty principle for $\mathcal{M}_{\sigma,\beta}$.* In [14] the authors proved the following Heisenberg's inequality for H_α, there exist a positive constant c such that for all $f \in L^2_\alpha(\mathbb{R})$ we have

$$\|f\|^2_{2,\mu_\alpha} \leq c \left\| |x|^2 f \right\|_{2,\mu_\alpha} \left\| |\lambda|^2 H_\alpha(f) \right\|_{2,\mu_\alpha}. \tag{32}$$

We will generalize this inequality for $\mathcal{M}_{\sigma,\beta}$.

Theorem 3. *There exist a positive constant c such that for all $f \in L^2_\alpha(\mathbb{R})$ we have*

$$\|f\|^2_{2,\mu_\alpha} \leq c \left\| |\lambda|^2 H_\alpha(f) \right\|_{2,\mu_\alpha} \left[\int_0^\infty \left\| |x|^2 \mathcal{M}_{\sigma,\beta}(f) \right\|^2_{2,\mu_\alpha} \frac{d\beta}{\beta} \right]^{\frac{1}{2}}$$

Proof. By using the relation (3.10) we find that

$$\int_{\mathbb{R}} |\mathcal{M}_{\sigma,\beta}(f)(x)|^2 d\mu_\alpha(x) \leq c \left\| |x|^2 \mathcal{M}_{\sigma,\beta}(f) \right\|_{2,\mu_\alpha} \left\| |\lambda|^2 \sigma_\beta H_\alpha(f) \right\|_{2,\mu_\alpha},$$

integrating over $]0, +\infty[$ with respect to measure $\frac{d\beta}{\beta}$ and by using Plancherel's formula (3.8) and Schwartz's inequality we get

$$\|f\|^2_{2,\mu_\alpha} \leq c \left[\int_0^\infty \left\| |x|^2 \mathcal{M}_{\sigma,\beta}(f) \right\|^2_{2,\mu_\alpha} \frac{d\beta}{\beta} \right]^{\frac{1}{2}} \left[\int_0^\infty \left[\int_{\mathbb{R}} \left| |\lambda|^4 \sigma_\beta(\lambda) \right|^2 |H_\alpha(f)(\lambda)|^2 |(\lambda)| d\mu_\alpha(\lambda) \right] \frac{d\beta}{\beta} \right]^{\frac{1}{2}}$$

Fubini's theorem and the admissibility condition (3.7) gives the desired result. □

3.2.2. *Donoho-Stark's uncertainty principle for* $\mathcal{M}_{\boldsymbol{\sigma},\boldsymbol{\beta}}$. Building on the ideas of Donoho and Stark In [8], the main purpose of this subsection is to give an uncertainty inequality of concentration type in $L^2_\theta(\mathbb{R})$ where $L^2_\theta(\mathbb{R})$ the space of measurables functions on $]0,+\infty[\times\mathbb{R}$ such that

$$\|f\|_{2,\theta_\alpha} = \left[\int_0^\infty \|f(\beta,.)\|^2_{2,\mu_\alpha}\frac{d\beta}{\beta}\right]^{\frac{1}{2}}$$

We denote by θ_α the measure defined on $]0,+\infty[\times\mathbb{R}$ by

$$d\theta_\alpha(\beta,x) = d\mu_\alpha(x)\otimes\frac{d\beta}{\beta},$$

[8]

(i) Let E be a measurable subset of $\mathbb{R}$, we say that the function $f \in L^2_\alpha(\mathbb{R})$ is ϵ-concentrated on E if

$$\|f - \mathbb{1}_E f\|_{2,\mu_\alpha} \leq \epsilon\|f\|_{2,\mu_\alpha}, \tag{33}$$

where $\mathbb{1}_E$ is the indicator function of the set E.
(ii) Let F be a measurable subset of $]0,+\infty[\times\mathbb{R}$, we say that the function $\mathcal{M}_{\sigma,\beta}(f)$ is ρ-concentrated on F if

$$\|\mathcal{M}_{\sigma,\beta}(f) - \mathbb{1}_F\mathcal{M}_{\sigma,\beta}(f)\|_{2,\theta_\alpha} \leq \rho\|\mathcal{M}_{\sigma,\beta}(f)\|_{2,\theta_\alpha}. \tag{34}$$

We have the following result

Theorem 4. *Let $f \in L^2_\alpha(\mathbb{R})$ and $\sigma \in L^2_\alpha(\mathbb{R})\cap L^1_\alpha(\mathbb{R})$ satisfying the admissibility condition (3.7), if f is ϵ-concentrated on E and $\mathcal{T}_{\sigma,\beta}(f)$ is ρ-concentrated on F then we have*

$$\|\sigma\|_{1,\mu_\alpha}(\mu_\alpha(E))^{\frac{1}{2}}\left[\int_F \frac{d\theta_\alpha(\beta,x)}{\beta^{2\alpha+1}}\right]^{\frac{1}{2}} \geq 1-(\epsilon+\rho).$$

Proof. Let $f \in L^2_\alpha(\mathbb{R})$ and $\sigma \in L^2_\alpha(\mathbb{R})\cap L^\infty_\alpha(\mathbb{R})$ satisfying (3.7) and assume that $\mu_\alpha(E) < \infty$ and $\left[\int_F \frac{d\theta_\alpha(\beta,x)}{\beta^{2\alpha+1}}\right]^{\frac{1}{2}} < \infty$.
According to the relations (3.11),(3.12) and Plancherel's relation (3.8) we find that

$$\begin{aligned}\|\mathcal{M}_{\sigma,\beta}(f)\|_{2,\theta_\alpha} &\leq \|\mathcal{M}_{\sigma,\beta}(f) - \mathbb{1}_F\mathcal{M}_{\sigma,\beta}(\mathbb{1}_E f)\|_{2,\theta_\alpha} + \|\mathbb{1}_F\mathcal{M}_{\sigma,\beta}(\mathbb{1}_E f)\|_{2,\theta_\alpha}\\ &\leq (\epsilon+\rho)\|f\|_{2,\mu_\alpha} + \|\mathbb{1}_F\mathcal{M}_{\sigma,\beta}(\mathbb{1}_E f)\|_{2,\theta_\alpha},\end{aligned} \tag{35}$$

on the other hand by the relations (2.5), (3.6) and Hölder's inequality we find that

$$\|\mathbb{1}_F\mathcal{M}_{\sigma,\beta}(\mathbb{1}_E f)\|_{2,\theta_\alpha} \leq \|f\|_{2,\mu_\alpha}\|\sigma\|_{1,\mu_\alpha}(\mu(E))^{\frac{1}{2}}\left[\int_F \frac{d\theta_\alpha(\beta,x)}{\beta^{2\alpha+1}}\right]^{\frac{1}{2}}, \tag{36}$$

by using the relations (3.13),(3.14) we deduce that

$$\|\mathcal{M}_{\sigma,\beta}(f)\|_{2,\theta_\alpha} \leq \|f\|_{2,\mu_\alpha}\left[(\epsilon+\rho) + \|\sigma\|_{1,\gamma_\alpha}(\mu_\alpha(E))^{\frac{1}{2}}\left[\int_F \frac{d\theta_\alpha(\beta,x}{\beta^{2\alpha+1}}\right]^{\frac{1}{2}}\right]$$

Plancherel's formula (3.8) for $\mathcal{M}_{\sigma,\beta}$ gives the desired result. $\square$

4. EXTREMAL FUNCTIONS ASSOCIATED WITH THE HARTLEY-BESSEL L^2_α-MULTIPLIER OPERATORS

In the following, we study the extremal functions associated with the Hartley-Bessel L^2_α-multiplier operators. . Let ψ be a positive function on $\mathbb{R}$ satisfying the following conditions

$$\frac{1}{\psi} \in L^1_\alpha(\mathbb{R}) \tag{37}$$

and

$$\psi(\lambda) \geq 1, \quad (\lambda) \in \mathbb{R}. \tag{38}$$

We define the Sobolev-type space $\mathcal{H}_\psi(\mathbb{R})$ by

$$\mathcal{H}_\psi(\mathbb{R}) = \left\{ f \in L^2_\alpha(\mathbb{R}) : \sqrt{\psi} H_\alpha(f) \in L^2_\alpha(\mathbb{R}) \right\}$$

provided with inner product

$$\langle f, g \rangle_\psi = \int_{\mathbb{R}} \psi(\lambda, m) H_\alpha(f)(\lambda) \overline{H_\alpha(g)(\lambda)} d\mu_\alpha(\lambda),$$

and the norm

$$\|f\|_\psi = \sqrt{\langle f, f \rangle_\psi}.$$

Let σ be a function in $L^\infty_\alpha(\mathbb{R})$. Then the Hartley-Bessel L^2_α multiplier operators $\mathcal{M}_{\sigma,\beta}$ are bounded and linear from $\mathcal{H}_\psi(\mathbb{R})$ into $L^2_\alpha(\mathbb{R})$ and we have for all $f \in \mathcal{H}_\psi(\mathbb{R})$

$$\|\mathcal{M}_{\sigma,\beta}(f)\|_{2,\mu_\alpha} \leq \|\sigma\|_{\infty,\gamma_\alpha} \|f\|_\psi. \tag{39}$$

Proof. By using the relations (2.7),(3.5),(4.2) we get the result $\square$

Let $\eta > 0$ and let σ be a function in $L^\infty_\alpha(\mathbb{R})$. We denote by $\langle f, g \rangle_{\psi,\eta}$ the inner product defined on the space $\mathcal{H}_\psi(\mathbb{R})$ by

$$\langle f, g \rangle_{\psi,\eta} = \int_{\mathbb{R}} \left(\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2 \right) H_\alpha(f)(\lambda) \overline{H_\alpha(g)(\lambda)} d\mu_\alpha(\lambda),$$

and the norm

$$\|f\|_{\psi,\eta} = \sqrt{\langle f, f \rangle_{\psi,\eta}}$$

THEOREM 5. *Let $\sigma \in L^\infty_\alpha(\mathbb{R})$ the Sobolev-type space $(\mathcal{H}_\psi(\mathbb{R})), \langle \cdot, \cdot \rangle_{\psi,\eta})$ is a reproducing kernel Hilbert space with kernel*

$$\mathcal{K}_{\psi,\eta}(x, y) = \int_{\mathbb{R}} \frac{B_\alpha(\lambda x) B_\alpha(\lambda y)}{\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2} d\mu_\alpha(\lambda),$$

that is

(i) For all $y \in \mathbb{R}$, the function $x \mapsto \mathcal{K}_{\psi,\eta}(x, y)$ belongs to $\mathcal{H}_\psi(\mathbb{R})$.

(ii) For all $f \in \mathcal{H}_\psi(\mathbb{R})$ and $y \in \mathbb{R}$, we have the reproducing property

$$f(y) = \langle f, \mathcal{K}_{\psi,\eta}(\cdot, (y)) \rangle_{\psi,\eta}.$$

Proof. (i) Let $y \in \mathbb{R}$, from the relations (2.2),(4.1) we have the function

$$g_y : \lambda \longrightarrow \frac{B_\alpha(\lambda y)}{\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2}$$

belongs to $L^1_\alpha(\mathbb{R}) \cap L^2_\alpha((\mathbb{R})$. Hence the function $\mathcal{K}_{\psi,\eta}$ is well defined and by the inversion formula (2.5), we get

$$\mathcal{K}_{\psi,\eta}(x, y) = H^{-1}_\alpha(g_y)(x)$$

by using Plancherel's theorem for H_α we find that $\mathcal{K}_{\psi,\eta}(\cdot, y)$ belongs to $L^2_\alpha(\mathbb{R})$ and we have

$$H_\alpha(\mathcal{K}_{\psi,\eta}(\cdot, y))(\lambda) = \frac{B_\alpha(\lambda y)}{\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2} \tag{40}$$

by using the relations (2.2),(4.1) and (4.4) we find that

$$\|\sqrt{\psi} H_\alpha(\mathcal{K}_{\psi,\eta}(\cdot, y))\|_{2,\mu_\alpha} \leq \frac{1}{\eta^2} \left\| \frac{1}{\psi} \right\|_{1,\mu_\alpha} < \infty,$$

this prove that for every $y \in \mathbb{R}$ the function $x \mapsto \mathcal{K}_{\psi,\eta}(x, y)$ belongs to $\mathcal{H}_\psi(\mathbb{R})$.
(ii) By using the relation (4.4) we find that for all $f \in \mathcal{H}_\psi(\mathbb{R})$,

$$\begin{aligned}\langle f, \mathcal{K}_{\psi,\eta}(\cdot, y)\rangle_{\psi,\eta} &= \int_{\mathbb{R}} \left(\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2\right) H_\alpha(f)(\lambda)\overline{H_\alpha(\mathcal{K}_{\psi,\eta}(\cdot, y)(\lambda)} d\mu_\alpha(\lambda) \\ &= \int_{\mathbb{R}} B_\alpha(\lambda y) H_\alpha(f)(\lambda) d\mu_\alpha(\lambda),\end{aligned}$$

inversion formula (2.5) gives the desired result. □

By taking σ a null function and $\eta = 1$ we find the following result The Sobolev-type space $(\mathcal{H}_\psi(\mathbb{R}), \langle\cdot,\cdot\rangle_\psi)$ is a reproducing kernel Hilbert space with kernel

$$\mathcal{K}_\psi(x, y) = \int_{\mathbb{R}} \frac{B_\alpha(\lambda x) B_\alpha(\lambda y)}{\eta\psi(\lambda)} d\mu_\alpha(\lambda).$$

The main result of this section can be stated as follows

Theorem 6. *Let $\sigma \in L^\infty_\alpha(\mathbb{R})$ and $\beta > 0$, for any $h \in L^2_\alpha(\mathbb{R})$ and for any $\eta > 0$, there exist a unique function $f^*_{\eta,\beta,h}$ where the infimum*

$$\inf_{f \in \mathcal{H}_\psi(\mathbb{R})} \left\{ \eta \|f\|^2_\psi + \|h - \mathcal{M}_{\sigma,\beta}(f)\|^2_{2,\mu_\alpha} \right\} \tag{41}$$

is attained. Moreover the extremal function $f^*_{\eta,\beta,h}$ is given by

$$f^*_{\eta,\beta,h}(y) = \int_{\mathbb{R}} h(x)\overline{\Theta_{\eta,\beta}(x, y)} d\mu_\alpha(x),$$

where $\Theta_{\eta,\beta}$ is given by

$$\Theta_{\eta,\beta}(x, y) = \int_{\mathbb{R}} \frac{\sigma_\beta(\lambda) B_\alpha(\lambda x) B_\alpha(\lambda y)}{\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2} d\mu_\alpha(\lambda)$$

Proof. The existence and the unicity of the extremal function $f^*_{\eta,\beta,h}$ satisfying (4.5) is given in [17, 18], furthermore $f^*_{\eta,\beta,h}$ is given by

$$f^*_{\eta,\beta,h}(y) = \langle h, \mathcal{M}_{\sigma,\beta}(\mathcal{K}_{\psi,\eta}(\cdot, y))\rangle_{\mu_\alpha},$$

by using inversion formula (2.5) and the relation (4.4) we get

$$\begin{aligned}\mathcal{M}_{\sigma,\beta}(\mathcal{K}_{\psi,\eta}(\cdot, y)(x) &= \int_{\mathbb{R}} \frac{\sigma_\beta(\lambda)B_\alpha(\lambda x)B_\alpha(\lambda y)}{\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2} d\mu_\alpha(\lambda) \\ &= \Theta_{\eta,\beta}(x, y)\end{aligned}$$

and the proof is complete. □

THEOREM 7. *$\sigma \in L^\infty_\alpha(\mathbb{R})$ and $h \in L^2_\alpha(\mathbb{R})$ then the function $f^*_{\eta,\beta,h}$ satisfies the following properties*

$$H_\alpha(f^*_{\eta,\beta,h})(\lambda) = \frac{\overline{\sigma_\beta(\lambda)}}{\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2} H(\lambda) \tag{42}$$

and

$$\|f^*_{\eta,\beta,h}\|_\psi \le \frac{1}{\sqrt{2\eta}} \|h\|_{2,\mu_\alpha}.$$

Proof. Let $y \in \mathbb{R}$ then the function

$$k_y : (\lambda) \longrightarrow \frac{\sigma_\beta(\lambda)B_\alpha(\lambda y)}{\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2}$$

belongs to $L^2_\alpha(\mathbb{R}) \cap L^1_\alpha(\mathbb{R})$ and by using inversion formula (2.5) we get

$$\Theta_{\eta,\beta}(x, y) = H^{-1}_\alpha(k_y)(x)$$

using Plancherel's theorem and Parseval's relation (2.6) we find that $\Theta_{\eta,\beta}(\cdot, y) \in L^2_\alpha(\mathbb{R})$ and

$$f^*_{\eta,\beta,h}(y) = \int_{\mathbb{R}} H_\alpha(f)(\lambda)\overline{k_y(\lambda)} d\mu_\alpha(\lambda) = \int_{\mathbb{R}} \frac{\overline{\sigma_\beta(\lambda)}}{\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2} H_\alpha(h)(\lambda)B_\alpha(\lambda y) d\mu_\alpha(\lambda)$$

on the other hand the function

$$F : \lambda \longrightarrow \frac{\overline{\sigma_\beta(\lambda)} H_\alpha(h)(\lambda)}{\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2}$$

belongs to $L^1_\alpha(\mathbb{R}) \cap L^\infty_\alpha(\mathbb{R})$, by using inversion formula (2.5), Plancherel's theorem we find that $f^*_{\eta,\beta,h}$ belongs to $L^2_\alpha(\mathbb{R})$ and

$$H_\alpha(f^*_{\eta,\beta,h})(\lambda) = F(\lambda)$$

on the other hand we have

$$|H_\alpha(f^*_{\eta,\beta,h})(\lambda)|^2 = \frac{|\sigma_\beta(\lambda)|^2}{\left(\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2\right)^2} |H_\alpha(h)(\lambda)|^2 \le \frac{1}{2\eta\psi(\lambda)} |H_\alpha(h)(\lambda)|^2$$

by Plancherel's formula (2.7) we find that

$$\|f^*_{\eta,\beta,h}\|_\psi \leq \frac{1}{\sqrt{2\eta}}\|h\|_{2,\mu_\alpha}.$$

□

THEOREM 8. *(Third Calderón's formula) Let $\sigma \in L^\infty_\alpha(\mathbb{R})$ and $f \in \mathcal{H}_\psi(\mathbb{R})$ then the extremal function given by*

$$f^*_{\eta,\beta,h}(y) = \int_{\mathbb{R}} \mathcal{M}_{\sigma,\beta}(f)(x)\overline{\Theta_{\eta,\beta}(x,y)}d\mu_\alpha(x),$$

satisfies

$$\lim_{\eta\to 0^+} \left\| f^*_{\eta,\beta} - f\right\|_{2,\mu_\alpha} = 0 \tag{43}$$

moreover we have $f^*_{\eta,\beta} \longrightarrow f$ uniformly when $\eta \longrightarrow 0^+$.

Proof. $f \in \mathcal{H}_\psi(\mathbb{R})$, we put $h = \mathcal{M}_{\sigma,\beta}(f)$ and $f^*_{\eta,\beta,h} = f^*_{\eta,\beta}$ in the relation (4.6) we find that

$$H_\alpha(f^*_{\eta,\beta,h} - f)(\lambda) = \frac{-\eta\psi(\lambda)H_\alpha(f)(\lambda)}{\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2} \tag{44}$$

therefore

$$\left\| f^*_{\eta,\beta} - f\right\|^2_\psi = \int_{\mathbb{R}} \frac{\eta^2\,(\psi(\lambda))^3}{\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2}\,|H_\alpha(f)(\lambda)|^2\,d\mu_\alpha(\lambda)$$

On the other hand we have

$$\frac{\eta^2\,(\psi(\lambda))^3}{\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2}\,|H_\alpha(f)(\lambda)|^2 \leq \psi(\lambda)\,|H_\alpha(f)(\lambda)|^2 \tag{45}$$

the result (4.7) follows from (4.9) and the dominated convergence theorem. Now, for all $f \in \mathcal{H}_\psi(\mathbb{R})$ we have $H_\alpha(f) \in L^2_\alpha(\mathbb{R}) \cap L^1_\alpha(\mathbb{R})$ and by using the relations (2.5), (4.8) we find that

$$f^*_{\eta,\beta(y,s)} - f(y) = \int_{\mathbb{R}} \frac{-\eta\psi(\lambda)H_\alpha(f)(\lambda)}{\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2} B_\alpha(\lambda y)d\mu_\alpha(\lambda)$$

and

$$\left|\frac{-\eta\psi(\lambda)H_\alpha(f)(\lambda)}{\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2} B_\alpha(\lambda y)\right| \leq |H_\alpha(f)(\lambda, m)| \tag{46}$$

By using the relation (4.10) and the dominated convergence theorem we deduce that

$$\lim_{\eta\to 0^+} \left| f^*_{\eta,\beta}(y) - f(y)\right| = 0$$

which complete the proof of the theorem. □

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