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BEST APPROXIMATION OF HARTLEY-BESSEL MULTIPLIER OPERATORS ON WEIGHTED SOBOLEV SPACES

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Abstract. The main goal of this paper is to introduce the Hartley-Bessel L^2_α-multiplier operators and to give for them some new results as Plancherel's, Calderon's reproducing formulas and Heisenberg's, Donoho-Stark's uncertainty principles. Next, using the theory of reproducing kernels we give best approximation and an integral representation of the extremal functions related to these operators on weighted Sobolev spaces.

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1. INTRODUCTION

In their seminal papers, Hörmander's and Mikhlin's [10], [15] initiated the study of boundedness of the translation invariant operators on $\mathbb{R}^d$. The translation invariant operators on $\mathbb{R}^d$ were characterized using the classical Euclidean Fourier transform $\mathcal{F}(f)$ and therefore they are also known as Fourier multipliers. Given a measurable function

$$m : \mathbb{R}^d \longrightarrow \mathbb{C}$$

its Fourier multiplier is the linear map $\mathcal{T}_m$ given for all $\lambda \in \mathbb{R}^d$ by the relation

$$\mathcal{F}(\mathcal{T}_m(f))(\lambda) = m(\lambda)\mathcal{F}(f)(\lambda). \tag{1}$$

The Hörmander-Mikhlin fundamental condition gives a criterion for L^p-boundedness for all $1 < p < \infty$ of Fourier multiplier $\mathcal{T}_m$ in terms of derivatives of the symbol m, more precisely if

$$|\partial^\gamma_\lambda m(\lambda)| \lesssim |\lambda|^{-|\gamma|} \quad for \quad 0 \le |\gamma| \le \left[\tfrac{d}{2}\right] + 1, \tag{2}$$

then, $\mathcal{T}_m$ can be extended to a bounded linear operator from $L^p(\mathbb{R}^d)$ into itself.

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The condition (2) imposes m to be a bounded function, smooth over $\mathbf{R}^d\backslash\{0\}$ satisfying certain local and asymptotic behavior. Locally, m admits a singularity at 0 with a mild control of derivatives around it up to order $\left[\frac{d}{2}\right]+1$. This singularity links to deep concepts in harmonic analysis and justifies the key role of Hörmander-Mikhlin theorem in Fourier multiplier L_p-theory, this condition defines a large class of Fourier multipliers including Riesz transforms and Littelwood-Paley partitions of unity which are crucial in Fourier summability or Pseudo-differential operator. The boundedness of Fourier multipliers is useful to solve problems in the area of mathematical analysis as probability theory (see [13]) and stochastic processes (see [2]). For its importance, many researchers extended the theory of Fourier multiplier to different settings, for example in the Dunkl-Weinstein setting [20], in the Laguerre-Bessel setting [7] and in the Dunkl's setting [19].

The general theory of reproducing kernels started with Aronszajn's in [1] in 1950, next the authors in [12], [17], [18] applied this theory to study Tikhonov regularization problem and they obtained approximate solutions for bounded linear operator equations on Hilbert spaces with the viewpoint of numerical solutions by computers. This theory has gained considerable interest in various fields of mathematical sciences, especially in Engineering and numerical experiments by using computers [12], [18].

The Hartley transform is an integral transform attributed to Hartley see [5], [6], this transform shares several essential properties with the classical Fourier transform, including linearity, invertibility and Parseval's identity. These transforms find extensive applications across various fields of mathematics, physics and engineering, such as signal processing, data analysis and number theory see [5], [6], [11], [21].

The Hartley transform is a linear operator defined for a suitable function $\psi(x)$ as follows:

$$\mathscr{H}(\psi)(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\psi(x)\,\mathrm{cas}(\lambda x)dx, \tag{3}$$

where $\mathrm{cas}(x)$ is the cas function, defined as

$$\mathrm{cas}(x)=\sum_{n=0}^{\infty}\frac{(-1)^{\binom{n+1}{2}}}{n!}x^n, \tag{4}$$

with $\binom{n}{2}=\frac{n(n-1)}{2}$ being the binomial coefficient. The $\mathrm{cas}(x)$ function (4) can be seen as a generalization of the exponential function exp.

A simple computation shows that the cas function is the unique C^∞ solution of the following differential-reflection problem (see [5])

$$\begin{cases} R\partial_x u(x)=\lambda u(x),\\ u(0)=0.\end{cases}$$

Here, ∂_x represents the first-order derivative, and R is the reflection operator acting on functions $f(x)$ as:

$$(Rf)(x) = f(-x). \tag{5}$$

Furthermore, the function $\mathrm{cas}(x)$ is multiplicative on $\mathbb{R}$ in the sense

$$\mathrm{cas}(x)\,\mathrm{cas}(y) = \tfrac{1}{2}(\mathrm{cas}(x+y) - \mathrm{cas}(-x-y) + \mathrm{cas}(x-y) + \mathrm{cas}(y-x)). \tag{6}$$

Inspired by relation (6), the author in [3] generalized the relation (6) for the Hartley-Bessel function and introduced a generalized convolution product. This paper focuses on the generalized Hartley transform introduced in [3], [4] called the Hartley-Bessel transform, more precisely we consider the following dfferential-reflection operator Δ_α defined by

$$\Delta_\alpha = R\left(\partial_x + \tfrac{\alpha}{x}\right) + \tfrac{\alpha}{x}, \quad \alpha \geq 0, \tag{7}$$

where R is the reflection operator given by the relation (4).

The operator Δ_α is closely connected with the Dunkl's theory [9]. Furthermore, the eigenfunctions of this operator are related to Bessel functions and they satisfy a product formula which permits to develop a new harmonic analysis associated with this operator (see [3] for more information).

The Hartley-Bessel transform $\mathscr{H}_\alpha$ generalizes the classical Hartley transform (3) and it is defined on $L^1_\alpha(\mathbb{R})$ by

$$\mathscr{H}_\alpha(f)(\lambda) = \int_{\mathbb{R}} B_\alpha(\lambda x) f(x) d\mu_\alpha(x), \qquad \text{for } \lambda \in \mathbb{R},$$

where μ_α is the measure on $\mathbb{R}$ and $B_\alpha(\lambda\cdot)$ is the Hartley-Bessel kernel given later.

Let σ be a function in $L^2_\alpha(\mathbb{R})$ and $\beta > 0$ be a positive real number. The Hartley-Bessel L^2_α-multiplier operators are defined for a smooth function on $\mathbb{R}$ as

$$\mathcal{M}_{\sigma,\beta}(f)(x) := \mathscr{H}_\alpha^{-1}\left(\sigma_\beta \mathscr{H}_\alpha\right)(x). \tag{8}$$

These operators are a generalization of the classical multiplier operators given by the relation (1). The remainder of this paper is organized as follows. In Section 2 we recall the main results concerning the harmonic analysis associated with the Hartley-Bessel transform. In Section 3 we introduce the Hartley-Bessel L^2_α-multiplier operators $\mathcal{M}_{\sigma,\beta}$ and we give for them a Plancherel's point-wise reproducing formula and Heisenberg's, Donoho-Stark's uncertainty principles. Section 4 is devoted to give an application of the general theory of reproducing kernels to Fourier multiplier theory and to give best estimates and an integral representation of the extremal functions related to the Hartley-Bessel L^2_α-multiplier operators on weighted Sobolev spaces.

2. HARMONIC ANALYSIS ASSOCIATED WITH THE HARTLEY-BESSEL TRANSFORM

In this section we recall some results in harmonic analysis related to the Hartley-Bessel transform. For more details we refer the reader to [3].

- For $\alpha \geq 0$, μ_α is the weighted Lebesgue measure defined on $\mathbb{R}$ by

$$d\mu_\alpha(x) := \frac{|x|^{2\alpha}}{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+\frac{1}{2})}dx,$$

  where Γ is the Gamma function.

- $L^p_\alpha(\mathbb{R}), 1 \leq p \leq \infty$, the space of measurable functions on $\mathbb{R}$, satisfying

$$\|f\|_{p,\mu_\alpha} =: \begin{cases} \left(\int_{\mathbb{R}} |f(x)|^p d\mu_\alpha(x)\right)^{1/p} < \infty, & 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{x\in\mathbb{R}} |f(x)| < \infty, & p = \infty. \end{cases}$$

In particular, for $p = 2$, $L^2_\alpha(\mathbb{R})$ is a Hilbert space with inner product given by

$$\langle f, g\rangle_\alpha = \int_{\mathbb{R}} f(x)\overline{g(x)}d\mu_\alpha(x).$$

2.1. The Eigenfunctions of the differential-reflection operator Δ_α.

For $\lambda \in \mathbb{C}$ we consider the following Cauchy problem

$$(S) : \begin{cases} \Delta_\alpha(u)(x) = \lambda u(x), \\ u(0) = 1. \end{cases}$$

From [3], [4], the Cauchy problem (S) admits a unique solution $B_\alpha(\lambda.)$ given by

$$B_\alpha(\lambda x) = j_{\alpha-\frac{1}{2}}(\lambda x) + \tfrac{\lambda x}{2\alpha+1} j_{\alpha+\frac{1}{2}}(\lambda x), \tag{9}$$

where j_α denotes the normalized Bessel function of order α (see [16]).

The function $B_\alpha(\lambda.)$ is infinitely differentiable on $\mathbb{R}$ and we have the following important result

$$\forall \lambda, x \in \mathbb{R}, \quad |B_\alpha(\lambda x)| \leq \sqrt{2}. \tag{10}$$

Furthermore from [3], the Hartley-Bessel kernel (9) is multiplicative on $\mathbb{R}$ in the sense

$$\forall \lambda \in \mathbb{R}, x, y \in \mathbb{R}^* \quad B_\alpha(\lambda x)B_\alpha(\lambda y) = \int_{\mathbb{R}} B_\alpha(\lambda z)K_\alpha(x, y, z)d\mu_\alpha(z), \tag{11}$$

where K_α is the Bessel kernel given explicitly in [3].

The product formula (11) generalizes the relation (6) and permits to define a translation operator, a convolution product and to develop a new harmonic analysis associated to the Differential-reflection operator Δ_α.

2.2. The Hartley-Bessel transform.

Definition 1 ([3]). *The Hartley-Bessel transform* $\mathscr{H}_\alpha$ *defined on* $L^1_\alpha(\mathbb{R})$ *is given by*

$$\mathscr{H}_\alpha(f)(\lambda) = \int_{\mathbb{R}} B_\alpha(\lambda x) f(x) d\mu_\alpha(x), \qquad \text{for } \lambda \in \mathbb{R}.$$

Some basic properties of this transform are the following. For the proofs, we refer the reader to [3], [4], [5].

Proposition 2. *(1) For every* $f \in L^1_\alpha(\mathbb{R})$ *we have*

$$\|\mathscr{H}_\alpha(f)\|_{\infty,\mu_\alpha} \leq \sqrt{2}\|f\|_{1,\mu_\alpha}. \tag{12}$$

(2) (Inversion formula) For $f \in \left(L^1_\alpha \cap L^2_\alpha\right)(\mathbb{R})$ *such that* $\mathcal{F}_\alpha(f) \in L^1_\alpha(\mathbb{R})$ *we have*

$$f(x) = \int_{\mathbb{R}} B_\alpha(\lambda x) \mathscr{H}_\alpha(f)(\lambda) d\mu_\alpha(\lambda), \qquad a.e. \quad x \in \mathbb{R}. \tag{13}$$

(3) (Parseval formula) For all $f, g \in L^2_\alpha(\mathbb{R})$ *we have*

$$\langle f, g\rangle_\alpha = \langle \mathscr{H}_\alpha(f), \mathscr{H}_\alpha(g)\rangle_\alpha, \tag{14}$$

In particular we have

$$\|f\|_{2,\mu\alpha} = \|\mathscr{H}_\alpha(f)\|_{2,\mu\alpha}. \tag{15}$$

(4) (Plancherel theorem) The Hartley-Bessel transform $\mathscr{H}_\alpha$ *can be extended to an isometric isomorphism from* $L^2_\alpha(\mathbb{R})$ *into* $L^2_\alpha(\mathbb{R})$.

2.3. The translation operator associated with the Hartley-Bessel transform. The product formula (11) permits to define the translation operator as follows.

Definition 3. *Let* $x, y \in \mathbb{R}$ *and* f *be a measurable function on* $\mathbb{R}$. *The translation operator is defined by*

$$\tau^x_\alpha f(y) = \int_{\mathbb{R}} f(z) K_\alpha(x, y, z) d\mu_\alpha(z).$$

The following proposition summarizes some properties of the Hartley-Bessel translation operator (see [3]).

Proposition 4. *For all* $x, y \in \mathbb{R}$, *we have:*

(1)

$$\tau^x_\alpha f(y) = \tau^y_\alpha f(x). \tag{16}$$

(2)

$$\int_{\mathbb{K}} \tau^x_\alpha f(y) d\mu_\alpha(y) = \int_{\mathbb{R}} f(y) d\mu_\alpha(y). \tag{17}$$

(3) For $f \in L^p_\alpha(\mathbb{R})$ *with* $p \in [1; +\infty]$, $\quad \tau^x_\alpha f \in L^p_\alpha(\mathbb{R})$ *and we have*

$$\|\tau^x_\alpha f\|_{p,\mu_\alpha} \leq 4\|f\|_{p,\mu_\alpha}. \tag{18}$$

(4) For $f \in L^1_\alpha(\mathbb{R})$, $\tau^x_\alpha f \in L^1_\alpha(\mathbb{R})$ *and we have*

$$\mathscr{H}_\alpha\left(\tau^x_\alpha f\right)(\lambda) = B_\alpha(\lambda x)\mathcal{F}_\alpha(f)(\lambda), \quad \forall \lambda \in \mathbb{R}. \tag{19}$$

Relation (19) shows that the translation operator τ^x_α is a particular case of the Hartley-Bessel multiplier operator (8).

By using the translation, we define the generalized convolution product of f, g as

$$(f *_\alpha g)(x) = \int_{\mathbb{K}} \tau^x_\alpha(f)(y) g(y) d\mu_\alpha(y).$$

This convolution is commutative, associative and it satisfies the following properties (see [3]).

Proposition 5. *(1) (Young's inequality) for all* $p, q, r \in [1; +\infty]$ *such that:* $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ *and for all* $f \in L^p_\alpha(\mathbb{R}), g \in L^q_\alpha(\mathbb{R})$ *the function* $f *_\alpha g$ *belongs to the space* $L^r_\alpha(\mathbb{R})$ *and we have*

$$\|f *_\alpha g\|_{r,\mu_\alpha} \leq 4\|f\|_{p,\mu_\alpha}\|g\|_{q,\mu_\alpha} \tag{20}$$

(2) For $f, g \in L^2_\alpha(\mathbb{R})$ *the function* $f *_\alpha g$ *belongs to* $L^2_\alpha(\mathbb{R})$ *if and only if the function* $\mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g)$ *belongs to* $L^2_\alpha(\mathbb{R})$ *and in this case we have*

$$\mathscr{H}_\alpha\left(f *_\alpha g\right) = \mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g). \tag{21}$$

(3) For all $f, g \in L^2_\alpha(\mathbb{R})$ *we have*

$$\int_{\mathbb{R}} |f *_\alpha g(x,t)|^2 d\mu_\alpha(x) = \int_{\mathbb{R}} |\mathscr{H}_\alpha(f)(\lambda)|^2 |\mathscr{H}_\alpha(g)(\lambda)|^2 d\mu_\alpha(\lambda), \tag{22}$$

where both integrals are simultaneously finite or infinite.

3. THE HARTLEY-BESSEL L^2_α-MULTIPLIER OPERATORS

The main purpose of this section is to introduce the Hartley-Bessel L^2_α-multiplier operators on $\mathbb{R}$ and to establish for them some uncertainty principles and Calderon's reproducing formulas.

3.1. Calderon's reproducing formulas for the Hartley-Bessel L^2_α-multiplier operators.

Definition 6. *Let* $\sigma \in L^2_\alpha(\mathbb{R})$ *and* $\beta > 0$*. The Hartley-Bessel* L^2_α*-multiplier operators are defined for smooth functions on* $\mathbb{R}$ *as*

$$\mathcal{M}_{\sigma,\beta}(f)(x) := \mathscr{H}^{-1}_\alpha\left(\sigma_\beta \mathscr{H}_\alpha(f)\right)(x), \tag{23}$$

where the function σ_β *is given for all* $\lambda \in \mathbb{R}$ *by*

$$\sigma_\beta(\lambda) := \sigma(\beta\lambda),$$

By a simple change of variable we find that for all $\beta > 0, \sigma_\beta \in L^2_\alpha(\mathbb{R})$ and

$$\|\sigma_\beta\|_{2,\mu_\alpha} = \frac{1}{\beta^{\frac{2\alpha+1}{2}}}\|\sigma\|_{2,\mu_\alpha}. \tag{24}$$

REMARK 7. *According to relation (21) we find that*

$$\mathcal{M}_{\sigma,\beta}(f)(x) = \left(\mathscr{H}_\alpha^{-1}(\sigma_\beta) *_\alpha f\right)(x), \tag{25}$$

where

$$\mathscr{H}_\alpha^{-1}(\sigma_\beta)(x) = \tfrac{1}{\beta^{2\alpha+1}}\mathscr{H}_\alpha^{-1}(\sigma)\left(\tfrac{x}{\beta}\right). \tag{26}$$

We give some properties of the Hartley-Bessel L^2_α-multiplier operators.

PROPOSITION 8. *(i) For every $\sigma \in L^2_\alpha(\mathbb{R})$, and $f \in L^1_\alpha(\mathbb{R})$, the function $\mathcal{M}_{\sigma,\beta}(f)$ belongs to $L^2_\alpha(\mathbb{R})$, and we have*

$$\|\mathcal{M}_{\sigma,\beta}(f)\|_{2,\mu_\alpha} \leq \frac{4}{\beta^{\frac{2\alpha+1}{2}}}\|\sigma\|_{2,\mu_\alpha}\|f\|_{1,\mu_\alpha}.$$

(ii) For every $\sigma \in L^\infty_\alpha(\mathbb{R})$, and for every $f \in L^2_\alpha(\mathbb{R})$, the function $\mathcal{M}_{\sigma,\beta}(f)$ belongs to $L^2_\alpha(\mathbb{R})$, and we have

$$\|\mathcal{M}_{\sigma,\beta}(f)\|_{2,\mu_\alpha} \leq \|\sigma\|_{\infty,\mu_\alpha}\|f\|_{2,\mu_\alpha} \tag{27}$$

(iii) For every $\sigma \in L^2_\alpha(\mathbb{R})$, and for every $f \in L^2_\alpha(\mathbb{R})$, then $\mathcal{M}_{\sigma,\beta}(f) \in L^\infty_\alpha(\mathbb{R})$, and we have

$$\mathcal{M}_{\sigma,\beta}(f)(x) = \int_{\mathbb{R}} \sigma(\beta\lambda)B_\alpha(\lambda x)\mathscr{H}_\alpha(f)(\lambda)d\mu_\alpha(\lambda), \qquad a.e. \quad x \in \mathbb{R} \tag{28}$$

and

$$\|\mathcal{M}_{\sigma,\beta}(f)\|_{\infty,\mu_\alpha} \leq \frac{4}{\beta^{\frac{2\alpha+1}{2}}}\|\sigma\|_{2,\mu_\alpha}\|f\|_{2,\mu_\alpha}.$$

Proof. (i) By using the relations (20), (25) we find that

$$\|\mathcal{M}_{\sigma,\beta}(f)\|^2_{2,\mu_\alpha} = \left\|\mathscr{H}_\alpha^{-1}(\sigma_\beta) *_\alpha f\right\|^2_{2,\mu_\alpha} \leq 16\|f\|^2_{1,\mu_\alpha}\left\|\mathscr{H}_\alpha^{-1}(\sigma_\beta)\right\|^2_{1,\mu_\alpha}$$

Plancherel's formula (15) and relation (24) give the desired result.

(ii) It is a consequence of Plancherel's formula (15).

(iii) By relations (15), (20), (24) and (25) we find the result. On the other hand the relation (28) follows from inversion formula (13).

□

In the following result, we give Plancherel's and pointwise reproducing inversion formula for the Hartley-Bessel L^2_α-multiplier operators.

THEOREM 9. *Let $\sigma \in L^2_\alpha(\mathbb{R})$ satisfy the admissibility condition:*

$$\int_0^\infty |\sigma_\beta(\lambda)|^2\,\tfrac{d\beta}{\beta} = 1, \quad \lambda \in \mathbb{R}. \tag{29}$$

(i) (Plancherel formula) For all f in $L^2_\alpha(\mathbb{R})$, we have

$$\int_{\mathbb{R}} |f(x)|^2 d\mu_\alpha(x) = \int_0^\infty \|\mathcal{M}_{\sigma,\beta}(f)\|^2_{2,\mu_\alpha}\,\tfrac{d\beta}{\beta}. \tag{30}$$

(ii) (First Calderón's formula) Let $f \in L^1_\alpha(\mathbb{R})$ *such that* $\mathscr{H}_\alpha(f) \in L^1_\alpha(\mathbb{R})$. *Then we have*

$$f(x) = \int_0^\infty \left(\mathcal{M}_{\sigma,\beta}(f) *_\alpha \mathscr{H}_\alpha^{-1}(\overline{\sigma_\beta})\right)(x)\frac{d\beta}{\beta}, \qquad a.e.\ x \in \mathbb{R}.$$

Proof. (i) By using Fubini's theorem and the relations (22) and (25) we get

$$\begin{aligned}\int_0^\infty \|\mathcal{M}_{\sigma,\beta}(f)\|^2_{2,\mu_\alpha}\frac{d\beta}{\beta} &= \int_0^\infty \left[\int_{\mathbb{R}} \left|\mathscr{H}_\alpha^{-1}(\sigma_\beta) *_\alpha f(x)\right|^2 d\mu_\alpha(x)\right]\frac{d\beta}{\beta}\\ &= \int_0^\infty \left[\int_{\mathbb{R}} |\mathscr{H}_\alpha(f)(\lambda)|^2 d\mu_\alpha(\lambda)\right]|\sigma_\beta(\lambda)|^2\frac{d\beta}{\beta}.\end{aligned}$$

The admissibility condition (29) and Plancherel's formula (15) give the desired result.

(ii) Let $f \in L^1_\alpha(\mathbb{R})$ such that $\mathscr{H}_\alpha(f) \in L^1_\alpha(\mathbb{R})$. By using Fubini's theorem and relations (14), (19) we find that

$$\begin{aligned}&\int_0^\infty \left(\mathcal{M}_{\sigma,\beta}(f) *_\alpha \mathscr{H}_\alpha^{-1}(\overline{\sigma_\beta})\right)(x)\frac{d\beta}{\beta}\\ &= \int_0^\infty \left[\int_{\mathbb{R}} \mathcal{M}_{\sigma,\beta}(f)(y)\overline{\tau_\alpha^x\left(\mathscr{H}_\alpha^{-1}(\sigma_\beta)\right)(y)}d\mu_\alpha(y)\right]\frac{d\beta}{\beta}\\ &= \int_0^\infty \left[\int_{\mathbb{R}} \mathscr{H}_\alpha(f)(\lambda)B_\alpha(\lambda x)(x,t)d\mu_\alpha(\lambda)\right]|\sigma_\beta(\lambda)|^2\frac{d\beta}{\beta}.\end{aligned}$$

The admissibility condition (29) and inversion formula (15) give the desired result.

□

To establish the second Calderon's reproducing formula for the Hartley-Bessel L^2_α-multiplier operators, we need the following technical result.

PROPOSITION 10. *Let* $\sigma \in L^2_\alpha(\mathbb{R}) \cap L^\infty_\alpha(\mathbb{R})$ *satisfy the admissibility condition* (29). *Then the function defined by*

$$\Phi_{\gamma,\delta}(\lambda) = \int_\gamma^\delta |\sigma_\beta(\lambda)|^2\frac{d\beta}{\beta}$$

belongs to $L^2_\alpha(\mathbb{R}) \cap L^\infty_\alpha(\mathbb{R})$ *for all* $0 < \gamma < \delta < \infty$.

Proof. Using Hölder's inequality for the measure $\frac{d\beta}{\beta}$ and relation (24) we find that

$$\|\Phi_{\gamma,\delta}\|^2_{2,\mu_\alpha} \leq \log(\delta/\gamma)\|\sigma\|^2_{\infty,\mu_\alpha}\|\sigma\|^2_{2,\gamma_\alpha}\int_\gamma^\delta \frac{d\beta}{\beta^{\frac{2\alpha+3}{2}}} < \infty.$$

So, $\Phi_{\gamma,\delta}$ belongs to $L^2_\alpha(\mathbb{R})$. Furthermore, by using relation (29) we get $\|\Phi_{\gamma,\delta}\|_{\infty,\mu_\alpha} \leq 1$ and therefore $\Phi_{\gamma,\delta}$ belongs to $L^2_\alpha(\mathbb{R}) \cap L^\infty_\alpha(\mathbb{R})$. □

Theorem 11. *(Second Calderón's formula). Let* $f \in L^2_\alpha(\mathbb{R})$ *and* $\sigma \in L^2_\alpha(\mathbb{R}) \cap L^\infty_\alpha(\mathbb{R})$ *satisfy the admissibility condition* (29) *and* $0 < \gamma < \delta < \infty$. *Then the function*

$$f_{\gamma,\delta}(x) = \int_\gamma^\delta \left(\mathcal{M}_{\sigma,\beta}(f) *_\alpha \mathscr{H}_\alpha^{-1}\left(\overline{\sigma_\beta}\right)\right)(x)\frac{d\beta}{\beta}, \quad x \in \mathbb{R}$$

belongs to $L^2_\alpha(\mathbb{R})$ *and satisfies*

$$\lim_{(\gamma,\delta)\to(0,\infty)} \|f_{\gamma,\delta} - f\|_{2,\mu_\alpha} = 0. \tag{31}$$

Proof. By a simple computation we find that

$$f_{\gamma,\delta}(x) = \int_{\mathbb{R}} \Phi_{\gamma,\delta}(\lambda)B_\alpha(\lambda x)\mathscr{H}_\alpha(f)(\lambda)d\mu_\alpha(\lambda) = \mathscr{H}_\alpha^{-1}\left(\Phi_{\gamma,\delta}\mathscr{H}_\alpha(f)\right)(x).$$

Using Proposition 10 we find that $\Phi_{\gamma,\delta} \in L^\infty_\alpha(\mathbb{R})$. Then we have $f_{\gamma,\delta} \in L^2_\alpha(\mathbb{R})$ and

$$\mathscr{H}_\alpha\left(f_{\gamma,\delta}\right)(\lambda) = \Phi_{\gamma,\delta}(\lambda, m)\mathscr{H}_\alpha(f)(\lambda).$$

On the other hand, by using Plancherel's formula (15) we find that

$$\lim_{(\gamma,\delta)\to(0,\infty)} \|f_{\gamma,\delta} - f\|^2_{2,\mu_\alpha} = \lim_{(\gamma,\delta)\to(0,\infty)} \int_{\mathbb{R}} |\mathscr{H}_\alpha(f)(\lambda)|^2 \left(1 - \Phi_{\gamma,\delta}(\lambda)\right)^2 d\mu_\alpha(\lambda).$$

By using the admissibility condition (29), the relation (31) follows from the dominated convergence theorem. □

3.2. Uncertainty principles for the Hartley-Bessel L^2_α-multiplier operators.

The main purpose of this subsection is to establish Heisenberg's and Donoho-Stark's uncertainty principles for the Hartley-Bessel L^2_α-multiplier operators $\mathcal{M}_{\sigma,\beta}$.

3.2.1. *Heisenberg's uncertainty principle for* $\mathcal{M}_{\sigma,\beta}$.

In [14] the authors proved the following Heisenberg's inequality for $\mathscr{H}_\alpha$, there exist a positive constant c such that for all $f \in L^2_\alpha(\mathbb{R})$ we have

$$\|f\|^2_{2,\mu_\alpha} \le c \left\| |x|^2 f \right\|_{2,\mu_\alpha} \left\| |\lambda|^2 \mathscr{H}_\alpha(f) \right\|_{2,\mu_\alpha}. \tag{32}$$

We will generalize this inequality for $\mathcal{M}_{\sigma,\beta}$.

Theorem 12. *There exists a positive constant* c *such that for all* $f \in L^2_\alpha(\mathbb{R})$ *we have*

$$\|f\|^2_{2,\mu_\alpha} \le c \left\| |\lambda|^2 \mathscr{H}_\alpha(f) \right\|_{2,\mu_\alpha} \left[\int_0^\infty \left\| |x|^2 \mathcal{M}_{\sigma,\beta}(f) \right\|^2_{2,\mu_\alpha} \frac{d\beta}{\beta}\right]^{\frac{1}{2}}.$$

Proof. By using relation (32) we find that

$$\int_{\mathbb{R}} |\mathcal{M}_{\sigma,\beta}(f)(x)|^2 d\mu_\alpha(x) \le c \left\| |x|^2 \mathcal{M}_{\sigma,\beta}(f) \right\|_{2,\mu_\alpha} \left\| |\lambda|^2 \sigma_\beta \mathscr{H}_\alpha(f) \right\|_{2,\mu_\alpha}.$$

Integrating over $]0,+\infty[$ with respect to the measure $\frac{d\beta}{\beta}$ and by using Plancherel's formula (30) and Schwartz's inequality we get

$$\|f\|^2_{2,\mu_\alpha} \leq c\left[\int_0^\infty \left\||x|^2\mathcal{M}_{\sigma,\beta}(f)\right\|^2_{2,\mu_\alpha}\frac{d\beta}{\beta}\right]^{\frac{1}{2}}$$

$$\cdot\left[\int_0^\infty\left[\int_{\mathbb{R}}\left||\lambda|^4\sigma_\beta(\lambda)\right|^2|\mathscr{H}_\alpha(f)(\lambda)|^2|(\lambda)|d\mu_\alpha(\lambda)\right]\frac{d\beta}{\beta}\right]^{\frac{1}{2}}.$$

Fubini's theorem and the admissibility condition (29) give the desired result. □

3.2.2. *Donoho-Stark's uncertainty principle for $\mathcal{M}_{\sigma,\beta}$.* Building on the ideas of Donoho and Stark in [8], the main purpose of this subsection is to give an uncertainty inequality of concentration type in $L^2_\theta(\mathbb{R})$ where $L^2_\theta(\mathbb{R})$ is the space of measurables functions on $]0,+\infty[\times\mathbb{R}$ such that

$$\|f\|_{2,\theta_\alpha} = \left[\int_0^\infty \|f(\beta,.)\|^2_{2,\mu_\alpha}\frac{d\beta}{\beta}\right]^{\frac{1}{2}}.$$

We denote by θ_α the measure defined on $]0,+\infty[\times\mathbb{R}$ by

$$d\theta_\alpha(\beta,x) = d\mu_\alpha(x)\otimes\frac{d\beta}{\beta},$$

Definition 13 ([8]). *(i) Let E be a measurable subset of $\mathbb{R}$. We say that the function $f\in L^2_\alpha(\mathbb{R})$ is ϵ-concentrated on E if*

$$\|f-\mathbb{1}_E f\|_{2,\mu_\alpha} \leq \epsilon\|f\|_{2,\mu_\alpha}, \tag{33}$$

where $\mathbb{1}_E$ is the indicator function of the set E.

(ii) Let F be a measurable subset of $]0,+\infty[\times\mathbb{R}$. We say that the function $\mathcal{M}_{\sigma,\beta}(f)$ is ρ-concentrated on F if

$$\|\mathcal{M}_{\sigma,\beta}(f)-\mathbb{1}_F\mathcal{M}_{\sigma,\beta}(f)\|_{2,\theta_\alpha} \leq \rho\|\mathcal{M}_{\sigma,\beta}(f)\|_{2,\theta_\alpha}. \tag{34}$$

We have the following result.

Theorem 14. *Let $f\in L^2_\alpha(\mathbb{R})$ and $\sigma\in L^2_\alpha(\mathbb{R})\cap L^1_\alpha(\mathbb{R})$ satisfy the admissibility condition (29). If f is ϵ-concentrated on E and $\mathcal{T}_{\sigma,\beta}(f)$ is ρ-concentrated on F then we have*

$$\|\sigma\|_{1,\mu_\alpha}(\mu_\alpha(E))^{\frac{1}{2}}\left[\int_F\frac{d\theta_\alpha(\beta,x)}{\beta^{2\alpha+1}}\right]^{\frac{1}{2}} \geq 1-(\epsilon+\rho).$$

Proof. Let $f\in L^2_\alpha(\mathbb{R})$ and $\sigma\in L^2_\alpha(\mathbb{R})\cap L^\infty_\alpha(\mathbb{R})$ satisfying (29) and assume that $\mu_\alpha(E)<\infty$ and $\left[\int_F\frac{d\theta_\alpha(\beta,x)}{\beta^{2\alpha+1}}\right]^{\frac{1}{2}}<\infty$.

According to relations (33), (34) and Plancherel's relation (30) we find that

$$\begin{aligned}\|\mathcal{M}_{\sigma,\beta}(f)\|_{2,\theta_\alpha} &\leq \|\mathcal{M}_{\sigma,\beta}(f)-\mathbb{1}_F\mathcal{M}_{\sigma,\beta}(\mathbb{1}_E f)\|_{2,\theta_\alpha}+\|\mathbb{1}_F\mathcal{M}_{\sigma,\beta}(\mathbb{1}_E f)\|_{2,\theta_\alpha}\\ &\leq(\epsilon+\rho)\|f\|_{2,\mu_\alpha}+\|\mathbb{1}_F\mathcal{M}_{\sigma,\beta}(\mathbb{1}_E f)\|_{2,\theta_\alpha}.\end{aligned}\tag{35}$$

On the other hand by the relations (13), (28) and Hölder's inequality we find that

$$\|\mathbb{1}_F \mathcal{M}_{\sigma,\beta}(\mathbb{1}_E f)\|_{2,\theta_\alpha} \leq \|f\|_{2,\mu_\alpha} \|\sigma\|_{1,\mu_\alpha} (\mu(E))^{\frac{1}{2}} \left[\int_F \frac{d\theta_\alpha(\beta,x)}{\beta^{2\alpha+1}} \right]^{\frac{1}{2}}. \tag{36}$$

By using relations (35), (36) we deduce that

$$\|\mathcal{M}_{\sigma,\beta}(f)\|_{2,\theta_\alpha} \leq \|f\|_{2,\mu_\alpha} \left[(\epsilon+\rho) + \|\sigma\|_{1,\gamma_\alpha} (\mu_\alpha(E))^{\frac{1}{2}} \left[\int_F \frac{d\theta_\alpha(\beta,x}{\beta^{2\alpha+1}} \right]^{\frac{1}{2}} \right].$$

Plancherel's formula (30) for $\mathcal{M}_{\sigma,\beta}$ gives the desired result. □

4. EXTREMAL FUNCTIONS ASSOCIATED WITH THE HARTLEY-BESSEL L^2_α-MULTIPLIER OPERATORS

In this section we study the extremal functions associated with the Hartley-Bessel L^2_α-multiplier operators.

Definition 15. *Let ψ be a positive function on $\mathbb{R}$ satisfying the following conditions*

$$\frac{1}{\psi} \in L^1_\alpha(\mathbb{R}) \tag{37}$$

and

$$\psi(\lambda) \geq 1, \quad (\lambda) \in \mathbb{R}. \tag{38}$$

We define the Sobolev-type space $\mathcal{H}_\psi(\mathbb{R})$ by

$$\mathcal{H}_\psi(\mathbb{R}) = \left\{ f \in L^2_\alpha(\mathbb{R}) : \sqrt{\psi}\mathscr{H}_\alpha(f) \in L^2_\alpha(\mathbb{R}) \right\},$$

provided with inner product

$$\langle f,g\rangle_\psi = \int_{\mathbb{R}} \psi(\lambda,m)\mathscr{H}_\alpha(f)(\lambda)\overline{\mathscr{H}_\alpha(g)(\lambda)}d\mu_\alpha(\lambda),$$

and the norm

$$\|f\|_\psi = \sqrt{\langle f,f\rangle_\psi}.$$

Proposition 16. *Let σ be a function in $L^\infty_\alpha(\mathbb{R})$. Then the Hartley-Bessel L^2_α multiplier operators $\mathcal{M}_{\sigma,\beta}$ are bounded and linear from $\mathcal{H}_\psi(\mathbb{R})$ into $L^2_\alpha(\mathbb{R})$ and we have for all $f \in \mathcal{H}_\psi(\mathbb{R})$*

$$\|\mathcal{M}_{\sigma,\beta}(f)\|_{2,\mu_\alpha} \leq \|\sigma\|_{\infty,\gamma_\alpha}\|f\|_\psi. \tag{39}$$

Proof. By using relations (15), (27), (38) we get the result. □

Definition 17. *Let $\eta > 0$ and let σ be a function in $L^\infty_\alpha(\mathbb{R})$. We denote by $\langle f,g\rangle_{\psi,\eta}$ the inner product defined on the space $\mathcal{H}_\psi(\mathbb{R})$ by*

$$\langle f,g\rangle_{\psi,\eta} = \int_{\mathbb{R}} \left(\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2\right) \mathscr{H}_\alpha(f)(\lambda)\overline{\mathscr{H}_\alpha(g)(\lambda)}d\mu_\alpha(\lambda),$$

and the norm

$$\|f\|_{\psi,\eta} = \sqrt{\langle f, f\rangle_{\psi,\eta}}.$$

THEOREM 18. *Let $\sigma \in L^{\infty}_{\alpha}(\mathbb{R})$. The Sobolev-type space $(\mathcal{H}_{\psi}(\mathbb{R})), \langle\cdot,\cdot\rangle_{\psi,\eta})$ is a reproducing kernel Hilbert space with kernel*

$$\mathcal{K}_{\psi,\eta}(x,y) = \int_{\mathbb{R}} \frac{B_\alpha(\lambda x)B_\alpha(\lambda y)}{\eta\psi(\lambda)+\left|\sigma_\beta(\lambda)\right|^2} d\mu_\alpha(\lambda),$$

that is

(i) For all $y \in \mathbb{R}$, the function $x \mapsto \mathcal{K}_{\psi,\eta}(x,y)$ belongs to $\mathcal{H}_{\psi}(\mathbb{R})$.

(ii) For all $f \in \mathcal{H}_{\psi}(\mathbb{R})$ and $y \in \mathbb{R}$, we have the reproducing property

$$f(y) = \langle f, \mathcal{K}_{\psi,\eta}(\cdot,(y))\rangle_{\psi,\eta}.$$

Proof. (i) Let $y \in \mathbb{R}$, from the relations (10), (37) we have the function

$$g_y : \lambda \longrightarrow \frac{B_\alpha(\lambda y)}{\eta\psi(\lambda)+\left|\sigma_\beta(\lambda)\right|^2}$$

belongs to $L^1_\alpha(\mathbb{R}) \cap L^2_\alpha((\mathbb{R})$. Hence the function $\mathcal{K}_{\psi,\eta}$ is well defined and by the inversion formula (13), we get

$$\mathcal{K}_{\psi,\eta}(x,y) = \mathscr{H}_\alpha^{-1}(g_y)(x).$$

By using Plancherel's theorem for $\mathscr{H}_\alpha$ we find that $\mathcal{K}_{\psi,\eta}(\cdot,y)$ belongs to $L^2_\alpha(\mathbb{R})$ and we have

(40) $$\mathscr{H}_\alpha(\mathcal{K}_{\psi,\eta}(\cdot,y))(\lambda) = \frac{B_\alpha(\lambda y)}{\eta\psi(\lambda)+\left|\sigma_\beta(\lambda)\right|^2}.$$

By using relations (10), (37) and (40) we find that

$$\|\sqrt{\psi}\mathscr{H}_\alpha(\mathcal{K}_{\psi,\eta}(\cdot,y))\|_{2,\mu_\alpha} \leq \frac{1}{\eta^2}\left\|\frac{1}{\psi}\right\|_{1,\mu_\alpha} < \infty,$$

This proves that for every $y \in \mathbb{R}$ the function $x \mapsto \mathcal{K}_{\psi,\eta}(x,y)$ belongs to $\mathcal{H}_{\psi}(\mathbb{R})$.

(ii) By using the relation (4.4) we find that for all $f \in \mathcal{H}_{\psi}(\mathbb{R})$,

$$\begin{aligned}\langle f, \mathcal{K}_{\psi,\eta}(\cdot,y)\rangle_{\psi,\eta} &= \int_{\mathbb{R}} \left(\eta\psi(\lambda) + |\sigma_\beta(\lambda)|^2\right) \mathscr{H}_\alpha(f)(\lambda)\overline{\mathscr{H}_\alpha(\mathcal{K}_{\psi,\eta}(\cdot,y)(\lambda)} d\mu_\alpha(\lambda) \\ &= \int_{\mathbb{R}} B_\alpha(\lambda y)\mathscr{H}_\alpha(f)(\lambda) d\mu_\alpha(\lambda).\end{aligned}$$

Inversion formula (13) gives the desired result.

□

By taking σ a null function and $\eta = 1$ we find the following result.

Corollary 19. *The Sobolev-type space* $(\mathcal{H}_\psi(\mathbb{R}))\,,\langle\cdot,\cdot\rangle_\psi)$ *is a reproducing kernel Hilbert space with kernel*

$$\mathcal{K}_\psi(x,y)=\int_{\mathbb{R}}\frac{B_\alpha(\lambda x)B_\alpha(\lambda y)}{\eta\psi(\lambda)}d\mu_\alpha(\lambda).$$

The main result of this section can be stated as follows.

Theorem 20. *Let* $\sigma\in L^\infty_\alpha(\mathbb{R})$ *and* $\beta>0$*, for any* $h\in L^2_\alpha(\mathbb{R})$ *and for any* $\eta>0$*, there exist a unique function* $f^*_{\eta,\beta,h}$ *where the infimum*

$$\inf_{f\in\mathcal{H}_\psi(\mathbb{R})}\left\{\eta\|f\|^2_\psi+\|h-\mathcal{M}_{\sigma,\beta}(f)\|^2_{2,\mu_\alpha}\right\}\tag{41}$$

is attained. Moreover the extremal function $f^*_{\eta,\beta,h}$ *is given by*

$$f^*_{\eta,\beta,h}(y)=\int_{\mathbb{R}}h(x)\overline{\Theta_{\eta,\beta}(x,y)}d\mu_\alpha(x),$$

where $\Theta_{\eta,\beta}$ *is given by*

$$\Theta_{\eta,\beta}(x,y)=\int_{\mathbb{R}}\frac{\sigma_\beta(\lambda)B_\alpha(\lambda x)B_\alpha(\lambda y)}{\eta\psi(\lambda)+|\sigma_\beta(\lambda)|^2}d\mu_\alpha(\lambda)$$

Proof. The existence and the unicity of the extremal function $f^*_{\eta,\beta,h}$ satisfying (41) is given in [17], [18]. Furthermore, $f^*_{\eta,\beta,h}$ is given by

$$f^*_{\eta,\beta,h}(y)=\langle h,\mathcal{M}_{\sigma,\beta}(\mathcal{K}_{\psi,\eta}(\cdot,y))\rangle_{\mu_\alpha}.$$

By using the inversion formula (13) and relation (40) we get

$$\begin{aligned}\mathcal{M}_{\sigma,\beta}(\mathcal{K}_{\psi,\eta}(\cdot,y)(x)&=\int_{\mathbb{R}}\frac{\sigma_\beta(\lambda)B_\alpha(\lambda x)B_\alpha(\lambda y)}{\eta\psi(\lambda)+|\sigma_\beta(\lambda)|^2}d\mu_\alpha(\lambda)\\&=\Theta_{\eta,\beta}(x,y)\end{aligned}$$

and the proof is complete. □

Theorem 21. *If* $\sigma\in L^\infty_\alpha(\mathbb{R})$ *and* $h\in L^2_\alpha(\mathbb{R})$*, then the function* $f^*_{\eta,\beta,h}$ *satisfies the following properties*

$$\mathscr{H}_\alpha(f^*_{\eta,\beta,h})(\lambda)=\frac{\overline{\sigma_\beta(\lambda)}}{\eta\psi(\lambda)+|\sigma_\beta(\lambda)|^2}\mathscr{H}(\lambda)\tag{42}$$

and

$$\|f^*_{\eta,\beta,h}\|_\psi\le\frac{1}{\sqrt{2\eta}}\|h\|_{2,\mu_\alpha}.$$

Proof. Let $y\in\mathbb{R}$. Then the function

$$k_y:(\lambda)\longrightarrow\frac{\sigma_\beta(\lambda)B_\alpha(\lambda y)}{\eta\psi(\lambda)+\left|\sigma_\beta(\lambda)\right|^2}$$

belongs to $L^2_\alpha(\mathbb{R})\cap L^1_\alpha(\mathbb{R})$ and by using inversion formula (13) we get

$$\Theta_{\eta,\beta}(x,y)=\mathscr{H}_\alpha^{-1}(k_y)(x).$$

Using Plancherel's theorem and Parseval's relation (14) we find that $\Theta_{\eta,\beta}(\cdot,y)\in L^2_\alpha(\mathbb{R})$ and

$$f^*_{\eta,\beta,h}(y)=\int_{\mathbb{R}}\mathscr{H}_\alpha(f)(\lambda)\overline{k_y(\lambda)}d\mu_\alpha(\lambda)=\int_{\mathbb{R}}\frac{\overline{\sigma_\beta(\lambda)}}{\eta\psi(\lambda)+|\sigma_\beta(\lambda)|^2}\mathscr{H}_\alpha(h)(\lambda)B_\alpha(\lambda y)d\mu_\alpha(\lambda).$$

On the other hand the function

$$F:\lambda\longrightarrow\frac{\overline{\sigma_\beta(\lambda)}\mathscr{H}_\alpha(h)(\lambda)}{\eta\psi(\lambda)+\left|\sigma_\beta(\lambda)\right|^2}$$

belongs to $L^1_\alpha(\mathbb{R})\cap L^\infty_\alpha(\mathbb{R})$. By using the inversion formula (13) and Plancherel's theorem we find that $f^*_{\eta,\beta,h}$ belongs to $L^2_\alpha(\mathbb{R})$ and

$$\mathscr{H}_\alpha(f^*_{\eta,\beta,h})(\lambda)=F(\lambda).$$

On the other hand we have

$$|\mathscr{H}_\alpha(f^*_{\eta,\beta,h})(\lambda)|^2=\frac{\left|\sigma_\beta(\lambda)\right|^2}{\left(\eta\psi(\lambda)+\left|\sigma_\beta(\lambda)\right|^2\right)^2}|\mathscr{H}_\alpha(h)(\lambda)|^2\le\frac{1}{2\eta\psi(\lambda)}|\mathscr{H}_\alpha(h)(\lambda)|^2.$$

By Plancherel's formula (15) we find that

$$\|f^*_{\eta,\beta,h}\|_\psi\le\frac{1}{\sqrt{2\eta}}\|h\|_{2,\mu_\alpha}.$$

□

THEOREM 22. *(Third Calderón's formula) Let $\sigma\in L^\infty_\alpha(\mathbb{R})$ and $f\in\mathcal{H}_\psi(\mathbb{R})$. Then the extremal function given by*

$$f^*_{\eta,\beta,h}(y)=\int_{\mathbb{R}}\mathcal{M}_{\sigma,\beta}(f)(x)\overline{\Theta_{\eta,\beta}(x,y)}d\mu_\alpha(x),$$

satisfies

$$\lim_{\eta\to0^+}\left\|f^*_{\eta,\beta}-f\right\|_{2,\mu_\alpha}=0. \tag{43}$$

Moreover we have $f^*_{\eta,\beta}\longrightarrow f$ uniformly when $\eta\longrightarrow0^+$.

Proof. $f\in\mathcal{H}_\psi(\mathbb{R})$, we put $h=\mathcal{M}_{\sigma,\beta}(f)$ and $f^*_{\eta,\beta,h}=f^*_{\eta,\beta}$ in the relation (42) and we find that

$$\mathscr{H}_\alpha(f^*_{\eta,\beta,h}-f)(\lambda)=\frac{-\eta\psi(\lambda)\mathscr{H}_\alpha(f)(\lambda)}{\eta\psi(\lambda)+\left|\sigma_\beta(\lambda)\right|^2}. \tag{44}$$

Therefore

$$\left\|f^*_{\eta,\beta}-f\right\|^2_\psi=\int_{\mathbb{R}}\frac{\eta^2(\psi(\lambda))^3}{\eta\psi(\lambda)+|\sigma_\beta(\lambda)|^2}\left|\mathscr{H}_\alpha(f)(\lambda)\right|^2d\mu_\alpha(\lambda).$$

On the other hand we have

$$\frac{\eta^2(\psi(\lambda))^3}{\eta\psi(\lambda)+|\sigma_\beta(\lambda)|^2}\left|\mathscr{H}_\alpha(f)(\lambda)\right|^2\le\psi(\lambda)\left|\mathscr{H}_\alpha(f)(\lambda)\right|^2. \tag{45}$$

The result (43) follows from (45) and the dominated convergence theorem. Now, for all $f \in \mathcal{H}_\psi(\mathbb{R})$ we have $\mathscr{H}_\alpha(f) \in L^2_\alpha(\mathbb{R}) \cap L^1_\alpha(\mathbb{R})$ and by using the relations (13), (44) we find that

$$f^*_{\eta,\beta(y,s)} - f(y) = \int_{\mathbb{R}} \frac{-\eta\psi(\lambda)\mathscr{H}_\alpha(f)(\lambda)}{\eta\psi(\lambda)+\left|\sigma_\beta(\lambda)\right|^2} B_\alpha(\lambda y)\,d\mu_\alpha(\lambda)$$

and

$$\left|\frac{-\eta\psi(\lambda)\mathscr{H}_\alpha(f)(\lambda)}{\eta\psi(\lambda)+\left|\sigma_\beta(\lambda)\right|^2} B_\alpha(\lambda y)\right| \leq \left|\mathscr{H}_\alpha(f)(\lambda, m)\right|. \tag{46}$$

Using relation (46) and the dominated convergence theorem we deduce that

$$\lim_{\eta\to 0^+} \left|f^*_{\eta,\beta}(y) - f(y)\right| = 0,$$

which completes the proof of the theorem. □

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