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PRODUCTS OF PARAMETRIC EXTENSIONS: REFINED ESTIMATES[†]

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Abstract. We present pointwise estimates on approximation by bounded linear operators of real-valued continuous functions defined on the cartesian product of d compact intervals. The main purpose is to provide a unified theory to deal with pointwise estimates on approximation processes of the above type which are generated by the tensor product method. This will constitute an extension and a refinement of earlier work of Haussmann and Pottinger. As an example a new estimate for approximation by multivariate positive linear operators is given.

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1. INTRODUCTION

In the present paper we deal with pointwise estimates on approximation by bounded linear operators of real-valued continuous functions defined on the cartesian product of d compact intervals I_δ. This space will be denoted by $C(\times_{\delta=1}^{d} I_\delta)$. The main purpose is to provide a unified theory to deal with *pointwise* estimates on approximation processes of the above type which are generated by the tensor product method. Thus it constitutes an extension and a refinement of papers of W. Haussmann and P. Pottinger [3], [4], [5] who treated the case of uniform estimates. Since all function to be approximated are defined on a rectangular domain in d dimensions, it is possible to take full advantage of refined estimates for the univariate case, many of which were obtained only recently. This is exemplified for the case of positive operators.

While Section 2 will deal with the case of products of arbitrary bounded linear operators, several more instructive pointwise inequalities on tensor product of positive linear operators will be given in Section 3.

See for instance W. Haussmann and P. Pottinger [5] for references concerning among other things existence and uniqueness theorems or non-quantitative

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assertions on convergence. Throughout the paper we write $I_\delta = [a_\delta, b_\delta], 1 \leq \delta \leq d \in N$, where $[a_\delta, b_\delta]$ are compact intevals with non-empty interior. The space of continuous functions on such an interval will be $C(I_\delta)$. The definition of some further notation used in this paper may be found in Haussmann's and Pottinger's article.

2. ESTIMATES ON APPROXIMATION BY BOUNDED LINEAR OPERATORS

The following is a modification of a result due to W. Haussmann and P. Pottinger [5, Proposition 1].

THEOREM 1. *Let $d \in N$. Let I_δ be a non-trivial compact interval, $1 \leq \delta \leq d$, and $\delta_0 \in \{1, \dots d\}$ be fixed. If $\mu : C(I_{\delta_0}) \to R$ is a continous linear functional, then for each $h \in C\left(\times_{\delta=1}^{d} I_\delta\right)$ we have*

$$\left\|(id^1 \widehat{\otimes} \dots \widehat{\otimes}\ id^{\delta_0-1} \widehat{\otimes} \mu \widehat{\otimes} id^{\delta_0+1} \widehat{\otimes} \dots \widehat{\otimes} id^d)(h)\right\|_\varepsilon =$$

$$= \sup\left\{\left|\mu(h_{\delta_0}(x_{1,\dots,x_{\delta_0-1},x_{\delta_0+1},\dots,x_d}))\right| : x_\delta \in I_\delta, 1 \leq \delta \leq d, \delta \neq \delta_0\right\}.$$

Here for $1 \leq \delta \leq d$ the symbol id^δ denotes the identity of $C(I_\delta)$,

$$id^1 \widehat{\otimes} \dots \widehat{\otimes} id^{\delta_0-1} \widehat{\otimes} \mu \widehat{\otimes} id^{\delta_0+1} \widehat{\otimes} \dots \widehat{\otimes} id^d$$

is the extension of

$$id^1 \otimes \dots \otimes id^{\delta_0-1} \otimes \mu \otimes \times id^{\delta_0+1} \otimes \dots \otimes id^d : \otimes^\varepsilon_{1\leq\delta\leq d} C(I_\delta) \to \otimes^d_{\delta=1,\delta\neq\delta_0} C(I_\delta)$$

to the space $\widehat{\otimes}^\varepsilon_{1\leq\delta\leq d} C(I_\delta)$, and $h_{\delta_0}^{(x_1,\dots,x_{\delta_0-1},x_{\delta_0-1},\dots,x_d)}$ is the δ_0-th partial mapping of h belonging to the fixed points $x_1, \dots, x_{\delta_0-1}, x\delta_0 \dots, x_d$, which is defined by

$$I_{\delta_0} \ni x \mapsto h(x_1, \dots, x_{\delta_0-1}, x, x_{\delta_0+1}, \dots, x_d) \in R.$$

Proof. As usual, we consider the linear hull $\left\langle \prod_{\delta=1}^{d} C(I_\delta) \right\rangle$ of the complex product $\prod_{\delta=1}^{d} C(I_\delta)$ as a realization of the tensor product $\otimes_{\delta=1}^{d} C(I_\delta)$. Let $g \in \left\langle \prod_{\delta=1}^{d} C(I_\delta) \right\rangle$. Then $g = \sum_{i=1}^{n} g_{i,1 \cdots g_{i,d}}$ with $g_{i,\delta} \in C(I_\delta)$ and

$$\left(id^1 \otimes \dots \otimes id^{\delta_0-1} \otimes \mu \otimes id^{\delta_0+1} \otimes \dots \otimes id^d\right)(g) =$$

$$= \left(id^1 \otimes \dots \otimes id^{\delta_0-1} \otimes \mu \otimes id^{\delta_0+1} \otimes \dots \otimes id^d\right)\left(\sum_{i=1}^{n} g_{i,1} \cdots g_{i,d}\right)$$

$$= \sum_{i=1}^{n} id^1(g_{i,1}) \otimes \dots \otimes \mu(g_{i,\delta_0}) \otimes \dots \otimes id^d(g_{i,d})$$

$$\in (\otimes_{i\leq\delta\leq\delta_0-1} C(I_\delta)) \otimes R \otimes (\otimes_{\delta_0+1\leq\delta\leq d} C(I_\delta)).$$

□

If as a realization for the last product we also choose the linear hulls of the corresponding complex product, this yields

$$\left(id^1 \otimes \ldots \otimes id^{\delta_0-1} \otimes \mu \otimes id^{\delta_0+1} \otimes \ldots \otimes id^d\right)(g) = \sum_{i=1}^{n} \mu\left(g_{1,\delta_0}\right) \cdot \prod_{\delta=1,\delta\neq\delta_0}^{d} g_{1,d},$$

If we equip $\otimes_{\delta=1}^{d} C\left(I_\delta\right)$ with the ε-norm, that is, if for $g \in \otimes_{\delta=1}^{d} C\left(I_\delta\right)$, $g = \sum_{i=1}^{n} g_{i,1} \cdot \ldots \cdot g_{i,d}$

$$\varepsilon\left(g\right) = \sup_{\substack{\phi_{1\in[C(I_1)]'}\\ \|\phi_1\|\leq 1}} \ldots \sup_{\substack{\phi_d\in[C(I_d)]'\\ \|\phi_d\|\leq 1}} \left|\sum_{i=1}^{n} \prod_{\delta=1}^{d} \phi_\delta\left(g_{1,\delta}\right)\right|,$$

and if we do the same in $\left(\otimes_{1\leq\delta\leq\delta_0-1} C\left(I_\delta\right)\right) \otimes R \otimes \left(\otimes_{\delta_0+|\leq\delta\leq d} C\left(I_\delta\right)\right)$, then $(\varepsilon, \varepsilon)$ are uniform crossnorms with respect to the pair $\left\{\otimes_{\delta=1}^{d} C\left(I_\delta\right), \left(\otimes_{1\leq\delta\leq\delta_0-1} C\left(I_\delta\right)\right) \otimes R \otimes \left(\otimes_{\delta_0+1\leq\delta\leq d} C\left(I_\delta\right)\right)\right\}$ see W. Haussmann and P. Pottinger [5, Theorem 2]. Thus the tensor product operator

$$\begin{aligned} &id^1 \otimes \ldots \otimes id^{\delta_0-1} \otimes \mu \otimes id^{\delta_0+1} \otimes \ldots \otimes id^d : \otimes id_{\delta=1}^{d} C\left(I_\delta\right) \\ &\rightarrow \left(\otimes_{1\leq\delta\leq\delta_0-1} C\left(I_\delta\right)\right) \otimes R \otimes \left(\otimes_{\delta_0+1\leq\delta\leq d} C\left(I_\delta\right)\right) \end{aligned}$$

is continuous.

We now consider the δ_0-th partial mappings $f_{\delta_0}^{\bar{\xi}}$ belonging to f, δ_0 and the fixed $(d-1)$ tuples $\bar{\xi} := \left(x_1, \ldots, x_{\delta_0-1}, x_{\delta_0+1}, \ldots, x_d\right)$. The mapping $f_\mu L : \times_{\delta=1,\delta\neq\delta_0}^{d} I_\delta \rightarrow R$ (where μ is the linear functional from above), given by $F_\mu(\bar{\xi}) := \mu(f_{\delta_0}^{\bar{\xi}})$ is continuous, since for $\bar{\xi} := (x_1, \ldots, x_{\delta_0-1}, x_{\delta_0+1}, \ldots, x_d)$ and $\widehat{\bar{\xi}} = \hat{x}_1, \ldots, \hat{x}_{\delta_0-1}, \hat{x}_{\delta_0+1}, \ldots, \hat{x}_d) \in \times_{\delta=1,\delta\neq\delta_0}^{d} I_\delta$, one has

$$\begin{aligned} \left|f_\mu\left(\bar{\xi}\right) - f_\mu\left(\widehat{\bar{\xi}}\right)\right| &= \left|\mu\left(f_{\delta_0}^{\bar{\xi}}\right) - \mu\left(f_{\delta_0}^{\widehat{\bar{\xi}}}\right)\right| \\ &\leq \|\mu\| \cdot \left\|f_{\delta_0}^{\bar{\xi}} - f_{\delta_0}^{\widehat{\bar{\xi}}}\right\| \left(I_\delta\right) \quad \text{(max norm on } I_{\delta_0}\text{)} \\ &- \|\mu\| \cdot \sup\left\{\left|f\left(x_1, \ldots, x_{\delta_0-1}, x, x_{\delta_0+1}, \ldots, x_d\right) -\right.\right. \\ &\left.\left. - f\left(\hat{x}_1, \ldots, \hat{x}_{\delta_0-1}, x, \hat{x}_{\delta_0+1}, \ldots, \hat{x}_d\right)\right| : x \in I_{\delta_0}\right\}, \end{aligned}$$

where $\|\mu\|$ denotes the norm of μ with respect to $\left(\left(C\left(I_\delta\right), \|\cdot\|_\infty\right), R\left|\cdot\right|\right)$. This and the uniform continuity of f imply the continuity of f_μ.

We now define

$$H_\mu : C\left(\times_{\delta=1}^{d} I_\delta\right) \ni f \longmapsto f_\mu \in C\left(\times_{\delta=1,\delta\neq\delta_0}^{d} I_\delta\right).$$

The fact that H_μ is continuous is a consequence of the following chain of (in)equalities showing that the operator norm $\|H_\mu\|$ is bounded:

$$\begin{aligned}\|H_\mu\| &= \sup\Big\{\|H_\mu(f)\|_\infty : \|f\|_\infty \le 1\Big\}\\ &= \sup\Big\{\|f_\mu\|_\infty : \|f\|_\infty \le 1\Big\}\\ &= \sup\Big\{\sup\Big\{\Big|\mu\left(f_{\delta_0}^{\bar{\xi}}\right)\Big| : \bar{\xi} \in \times_{\delta=1,\delta\neq\delta_0}^{d} I_\delta\Big\} : \|f\|_\infty \le 1\Big\}\\ &\le \sup\Big\{\sup\Big\{\|\mu\| \cdot \Big\|f_{\delta_0}^{\bar{\xi}}\Big\|_\infty : \bar{\xi} \in \times_{\delta=1,\delta\neq\delta_0}^{d} I_\delta\Big\} : \|f\|_\infty \le 1\Big\}\\ &\le \sup\Big\{\|\mu\| \cdot \|f\|_\infty : \|f\|_\infty \le 1\Big\}\\ &= \|\mu\| < \infty.\end{aligned}$$

If $g \in \Big\langle \prod\limits_{\delta=1}^{d} C(I_\delta)\Big\rangle$, $g = \sum\limits_{i=1}^{n} g_{i,1}\cdot\dots\cdot g_{i,d}$ for some $n \in \mathbb{N}$, then

$$\begin{aligned}H_\mu(g) &= H_\mu\left(\sum_{i=1}^{n} g_{i,1}\cdot\dots\cdot g_{i,d}\right) = \sum_{i=1}^{n} H_\mu(g_{i,1}\cdot\dots\cdot g_{i,d})\\ &= \sum_{i=1}^{n}\Big(g_{i,1},\cdots g_{i,d}\Big)_\mu = \sum_{i=1}^{n} g_{i,1}\cdot\dots\cdot g_{i,\delta_0-1}\cdot\mu(g_{i,\delta_0})\cdot g_{i,\delta_0+1}\cdot\dots\cdot g_{i,d}\\ &= \sum_{i=1}^{n}\mu\left(g_{i,\delta_0}\right)\cdot\prod_{\delta=1,\delta\neq\delta_0}^{d} g_{i,\delta}.\end{aligned}$$

Hence the mappings H_μ and $\big(id^1 \otimes \cdots \otimes id^{\delta_0-1} \otimes \mu \otimes id^{\delta_0+1} \otimes \cdots \otimes id^d\big)$ coincide on $\otimes_{\delta=1}^{d} C(I_\delta) = \Big\langle \prod\limits_{\delta=1}^{d} C(I_\delta)\Big\rangle$. Since the Chebyshev norm $\|\cdot\|_\infty$ on $\otimes_{\delta=1,\delta\neq\delta_0}^{d} C(I_\delta)$ induces the ε-norm, both norms also coincide on the completion $\widehat{\otimes}^{\varepsilon}_{1\le\delta\le d,\delta\neq\delta_0} C(I_\delta)$, which is thus isometrically isomorphic to $C\big(\times_{\delta=1,\delta\neq\delta_0}^{d} I_\delta\big)$. From this and from the continuity of both mappings considered above, it follows for each $h \in C(\times_{\delta=1}^{d} I_\delta)$ that

$$\Big\|\Big(id^1\widehat{\otimes}\dots\widehat{\otimes}id^{\delta_0-1}\widehat{\otimes}\mu\widehat{\otimes}id^{\delta_0+1}\widehat{\otimes}\dots\widehat{\otimes}id^d\Big)(h)\Big\|_\varepsilon = \|H_\mu(h)\|_\infty.$$

Here $\|\cdot\|_\varepsilon$ is the ε-norm on $\widehat{\otimes}^{\varepsilon}_{1\le\delta\le d,\delta\neq\delta_0} C(I_\delta)$, and $\|\cdot\|_\infty$ denotes the Chebyshev norm on $C(\times_{\delta=1,\delta\neq\delta_0}^{d} I_\delta)$. Furthermore,

$$\begin{aligned}\|H_\mu(h)\|_\infty = \|h_\mu\|_\infty &= \sup\Big\{\Big|\mu(h_{\delta_0}^{\bar{\xi}})\Big| : \bar{\xi} \in \times_{\delta=1,\delta\neq\delta_0}^{d} I_\delta\Big\}\\ &= \sup\Big\{\Big|\mu\Big(h_{\delta_0}^{(x_1,\dots,x_{\delta_0-1},x_{\delta_0+1},\dots,x_d)}\Big)\Big| : x_\delta \in I_\delta, \delta \neq \delta_0\Big\}.\end{aligned}$$

Lemma 2. *For $1 \le \delta \le d$, let $(X_\delta, \|\cdot\|_\delta)$ be normed vector spaces. If $\mu_\delta : X_\delta \to \mathbb{R}$ are continuous linear functionals, and if $A_\delta : X_\delta \to X_\delta$ are* *continuous linear mappings, then on the space* $\widehat{\otimes}_{\varepsilon 1\leq\delta\leq d}X_\delta$*, the equality*

$$\widehat{\otimes}_{\varepsilon,1\leq\delta\leq d}(\mu_\delta\circ A_\delta)=(\widehat{\otimes}_{1\leq\delta\leq d}\mu_\delta)\circ(\widehat{\otimes}_{1\leq\delta\leq d}A_\delta)$$

holds.

Proof. Let $h\in\otimes_{\delta=1}^d X_\delta$. Then $h=\sum\limits_{i=1}^n x_{i,1}\otimes\cdots\otimes x_{i,d}$, where $n\in\mathbb{N}$ and $x_{i,\delta}\in X_\delta$. Thus

$$\begin{aligned}(\otimes_{\delta=1}^d\mu_\delta)\Big((A_{1\varepsilon\otimes\ldots\otimes A_d})(h)\Big)&=\otimes_{\delta=1}^d\mu_\delta\left(\sum_{i=1}^n A_1(x_{i,1})\otimes\cdots\otimes A_d(x_{i,d})\right)\\&=\sum_{i=1}^n\mu_1\Big(A_1(x_{i,1})\Big)\otimes\cdots\otimes\mu_d(A_d(x_{i,d}))\\&=\sum_{i=1}^n(\mu_1\circ A_1)(x_{i,1})\otimes\cdots\otimes(\mu_d\circ A_d)(x_{i,d})\\&=\sum_{i=1}^n\Big[(\mu_1\circ A_1)\otimes\cdots\otimes(\mu_d\circ A_d)\Big](x_{i,1}\otimes\cdots\otimes x_{i,d})\\&=\otimes_{\delta=1}^d(\mu_\delta\circ A_\delta)\left(\sum_{i=1}^n x_{i,1}\otimes\cdots\otimes x_{i,d}\right)\\&=\otimes_{\delta=1}^d(\mu_\delta\circ A_\delta)(h).\end{aligned}$$

□

Since by Haussmann's and Pottinger's [5, Theorem 2] $(\varepsilon,\varepsilon)$ are uniform cross norms with respect to the couple $(\otimes_{\delta=1}^d X_\delta,\otimes_{\delta=1}^d\mathbb{R}=\mathbb{R})$, the mappings

$$\otimes_{\delta=1}^d A_\delta:\otimes_{1\leq\delta\leq d}^{\varepsilon}X_\delta\to\otimes_{1\leq\delta\leq d}^{\varepsilon}X_\delta$$

and

$$\otimes_{\delta=1}^d\mu_\delta:\otimes_{1\leq\delta\leq d}^{\varepsilon}X_\delta\to\otimes_{1\leq\delta\leq d}^{\varepsilon}\mathbb{R}$$

are continuous. This implies the continuity of

$$\left[\otimes_{\delta=1}^d\mu_\delta\right]\circ\left[\otimes_{\delta=1}^d A_\delta\right].$$

For the same reason we also have continuity of

$$\otimes_{\delta=1}^d(\mu_\delta\circ A_\delta).$$

Together with the observation made at the beginning of the proof this also yields equality of the extensions of the two mappings considered above, *i.e.*,

$$\hat{\otimes}_{\delta=1}^d(\mu_\delta\circ A_\delta)=\widehat{[\otimes_{\delta=1}^d\mu_\delta]\circ\otimes_{\delta=1}^d}A_\delta.$$

An analogous density argument shows the validity of

$$\widehat{[\otimes_{\delta=1}^d\mu_\delta]\circ[\otimes_{\delta=1}^d}A_\delta]=\left[\widehat{\otimes}_{\delta=1}^d\mu_\delta\right]\circ\left[\widehat{\otimes}_{\delta=1}^d A_\delta\right].$$

From this the claim of the lemma immediately follows.

Lemma 3. *For $1 \leq \delta \leq d$ let the normed vector spaces $(X_\delta, \|\cdot\|_\delta)$ be given and let $\mu_\delta : X_\delta \to \mathbb{R}$ be continuous linear functionals. If $h \in \widehat{\otimes}^{\varepsilon}_{1\leq\delta\leq d} X_\delta$, then for each $\delta_0 \in \{1, \ldots, d\}$ one has*

$$\left\| \left(\widehat{\otimes}^d_{\delta=1}\mu_\delta\right)(h)\right\|_\varepsilon \leq \Big(\prod_{\delta=1,\delta\neq\delta_0}^{d} \|\mu_\delta\|\Big)\cdot \left\|\left(id^1\widehat{\otimes}\ldots\widehat{\otimes}^{\delta_0-1}\widehat{\otimes}\mu_{\delta_0}\widehat{\otimes} id^{\delta_0+1}\widehat{\otimes}\ldots\widehat{\otimes} id^d\right)(h)\right\|_\varepsilon .$$

Proof. We write

$$\begin{aligned}\widehat{\otimes}^d_{\delta=1}\mu_\delta &= \widehat{\otimes}^d_{\delta=1}\left(id_R \circ \mu_\delta \circ id^\delta\right) = \\ &= \left[\left(id_R\circ\mu_1\circ id^1\right)\widehat{\otimes}\ldots\widehat{\otimes} id_R\widehat{\otimes}\ldots\widehat{\otimes}\left(id_R\circ\mu_d\circ id^d\right)\right]\circ \\ &\quad\circ\left[id^1\widehat{\otimes}\ldots\widehat{\otimes}\mu_{\delta_0}\circ id^{\delta_0}\widehat{\otimes}\ldots\widehat{\otimes} id^d\right].\end{aligned}$$

□

If $h \in \widehat{\otimes}^{\varepsilon}_{1\leq\delta\leq d} X_\delta$, then

$$\begin{aligned}\left\|\left(\widehat{\otimes}^d_{\delta=1}\mu_\delta\right)(h)\right\|_\varepsilon &\leq \left\|\left(id_R\circ\mu_1\circ id^1\right)\widehat{\otimes}\ldots\widehat{\otimes} id_R\widehat{\otimes}\ldots\widehat{\otimes}\left(id_R\circ\mu_d\circ id^d\right)\right\|_\varepsilon \\ &\quad\cdot\left\|(id^1\widehat{\otimes}\ldots\widehat{\otimes}\mu_{\delta_0}\circ id^{\delta_0}\widehat{\otimes}\ldots\widehat{\otimes} id^d)(h)\right\|_\varepsilon .\end{aligned}$$

The uniform cross norm property of $(\varepsilon, \varepsilon)$ with respect to the couple

$$\Big(X_1\otimes\ldots\otimes X_{\delta_0-1}\otimes\mathbb{R}\otimes X_{\delta_0+1}\otimes\ldots\otimes X_\delta, \otimes^d_{\delta=1}\mathbb{R}\Big)$$

(see W. Haussmann and P. Pottinger [5, Theorem 2]) first implies

$$\begin{aligned}&\left\|\left(\left(id_R\circ\mu_1\circ id^1\right)\otimes\ldots\otimes\left(id_R\circ\mu_d\circ id^d\right)\right)\right\|_\varepsilon = \\ &= \Big\{\prod_{\delta=1,\delta\neq\delta_0}^{d}\left\|id_R\circ\mu_\delta\circ id^\delta\right\|\Big\}\cdot\|id_R\| = \prod_{\delta=1,\delta\neq\delta_0}^{d}\|\mu_\delta\|.\end{aligned}$$

For density reasons the extension of $(id_R\circ\mu_{\delta 1}\circ id^1)\otimes\ldots\otimes id_R\otimes\ldots\otimes id_R\circ\mu_d\circ id^d)$ has the same norm; hence

$$\begin{aligned}&\left\|\left(\widehat{\otimes}^d_{\delta=1}\mu_\delta\right)(h)\right\|_\varepsilon \leq \\ &\leq\left(\prod_{\delta=1,\delta\neq\delta_0}^{d}\|\mu_\delta\|\right)\cdot\left\|\left(id^1\widehat{\otimes}\ldots\widehat{\otimes} id^{\delta_0-1}\widehat{\otimes}\mu_{\delta_0}\widehat{\otimes} id^{\delta_0+1}\widehat{\otimes}\ldots\widehat{\otimes} id^d\right)(h)\right\|_\varepsilon .\end{aligned}$$

Since $\delta_0 \in \{1, \ldots, d\}$ was arbitrarily chosen, the claim of the lemma follows.

Theorem 4 (cf. W. Haussmann and P. Pottinger [5, Theorem 5]). *Consider the normed vector spaces $(X_\delta, \|\cdot\|_\delta)$, $1 \leq \delta \leq d$, and the continuous linear* *functionals* $\mu_\delta : X_\delta \to \mathbb{R}$. *Let* $P^\delta : X_\delta \to X_\delta$ *be continuous linear operators. Then for each* $h \in \widehat{\otimes}^{\varepsilon}_{1\le\delta\le d} X_\delta$ *we have*

$$\left\|\left(\widehat{\otimes}^d_{\delta=1}\mu_\delta\right)\left(h - (\widehat{\otimes}^d_{\delta=1}P^\delta)(h)\right)\right\|_\varepsilon \le \min_{\sigma\in S_d}$$
$$\le \left\{\sum_{\nu=1}^{d}\left\{\sum_{\delta=1}^{d-\nu}\left\|\mu_{\sigma(\delta)}\right\|\right\}\cdot\left\{\prod_{\delta=d-\nu+2}^{d}\left\|\mu_{\sigma(\delta)}\right\|\cdot\left\|P^{\sigma(\delta)}\right\|\right\}\cdot\right.$$
$$\left.\cdot\left\|(id^1\widehat{\otimes}\dots\widehat{\otimes}\mu_{\sigma(d-\nu+1)}\circ\left(id^{\sigma(d-\nu+1)} - P^{\sigma(d-\nu+1)}\right)\widehat{\otimes}\dots\widehat{\otimes}id^d)(h)\right\|_\varepsilon\right\}.$$

Here S_d is the symmetric group of all permutations of $\{1,\dots,d\}$.

Proof. We investigate

$$\widehat{\otimes}^d_{\delta=1}\mu_\delta\Big(h - (\widehat{\otimes}^d_{\delta=1}P^\delta)(h)\Big) = \left[\widehat{\otimes}^d_{\delta=1}\mu_\delta\circ\left(\widehat{\otimes}^d_{\delta=1}id^\delta - \widehat{\otimes}^d_{\delta=1}P^\delta\right)\right](h).$$

Let $\sigma \in S_d$ be an arbitrary permutation. A decomposition of $\widehat{\otimes}^d_{\delta=1}id^\delta - \widehat{\otimes}^d_{\delta=1}P^\delta$ analogous to the one employed by Haussmann and Pottinger together with a density argument yields the equality

$$\begin{aligned}
&\widehat{\otimes}^d_{\delta=1}id^\delta - \widehat{\otimes}^d_{\delta=1}P^\delta =\\
&= \left(id^1\widehat{\otimes}id^2\widehat{\otimes}\dots\widehat{\otimes}id^d\right) - \left(P^1\widehat{\otimes}P^2\widehat{\otimes}\dots\widehat{\otimes}P^d\right)\\
&= \left(id^1\widehat{\otimes}\dots\widehat{\otimes}id^{\sigma(d)-1}\widehat{\otimes}\left(id^{\sigma(d)} - P^{\sigma(d)}\right)\widehat{\otimes}id^{\sigma(d)+1}\widehat{\otimes}\dots\widehat{\otimes}id^d\right)\\
&\quad + id^1\widehat{\otimes}\dots\widehat{\otimes}\left(id^{\sigma(d-1)} - P^{\sigma(d-1)}\right)\widehat{\otimes}id^{\sigma(d-1)+1}\widehat{\otimes}\dots\widehat{\otimes}P^{\sigma(d)}\widehat{\otimes}\dots\widehat{\otimes}id^d\\
&\quad + \dots + P^1\widehat{\otimes}P^2\widehat{\otimes}\dots\widehat{\otimes}\left(id^{\sigma(1)} - P^{\sigma(1)}\right)\widehat{\otimes}P^{\sigma(1)+1}\widehat{\otimes}\dots\widehat{\otimes}P^d\\
&= \sum_{\delta=1}^{d} O_{d-\delta+1}.
\end{aligned}$$

Thus by Lemma 2 for all $h \in \widehat{\otimes}^{\varepsilon}_{1\le\delta\le d}X_\delta$, one obtains

$$\begin{aligned}
&\left(\widehat{\otimes}^d_{\delta=1}\mu_\delta\right)\left(h - \widehat{\otimes}^d_{\delta=1}P^\delta\right)(h) =\\
&= \sum_{\delta=1}^{d}\left((\widehat{\otimes}^d_{\delta=1}\mu_\delta)\circ O_{d-\delta+1}\right)(h)\\
&= \left(\mu_1\circ id^1\right)\widehat{\otimes}\dots\widehat{\otimes}\mu_{\sigma(d)}\circ\left(id^{\sigma(d)} - P^{\sigma(d)}\right)\widehat{\otimes}\dots\widehat{\otimes}\left(\mu_d\circ id^d\right)(h) + \dots +\\
&\quad + \left(\mu_1\circ P^1\right)\widehat{\otimes}\left(\mu_2\circ P^2\right)\widehat{\otimes}\dots\widehat{\otimes}\mu_{\sigma(1)}\circ\left(id^{\sigma(1)} - P^{\sigma(1)}\right)\widehat{\otimes}\dots\widehat{\otimes}\left(\mu_d\circ P^d\right)(h).
\end{aligned}$$

From Lemma 3 we conclude that the difference considered above may be estimated as follows:

$$
\begin{aligned}
&\left\|\left(\widehat{\otimes}_{\delta=1}^{d}\mu_\delta\right)\left(h-\left(\widehat{\otimes}_{\delta=1}^{d}P^\delta\right)(h)\right)\right\|_\varepsilon \leq \\
&\leq \left\|(\mu_1\circ \mathrm{id}^1)\widehat{\otimes}\dots\widehat{\otimes}\left(\mu_{\sigma(d)}\circ(\mathrm{id}^{\sigma(d)}-P^{\sigma(d)})\right)\widehat{\otimes}\dots\widehat{\otimes}(\mu_d\circ id^d)(h)\right\|_\varepsilon+...+\\
&\quad+\left\|(\mu_1\circ P^1)\widehat{\otimes}\dots\widehat{\otimes}\left(\mu_{\sigma(1)}\circ(\mathrm{id}^{\sigma(1)}-P^{\sigma(1)})\right)\widehat{\otimes}\dots\widehat{\otimes}(\mu_d\circ P^d)(h)\right\|_\varepsilon\\
&\leq \left\{\prod_{\delta=1,\delta\neq\sigma(d)}^{d}\|\mu_\delta\|\right\}\left\|\mathrm{id}^1\widehat{\otimes}\dots\widehat{\otimes}(\mu_{\sigma(d)}\circ\left(id^{\sigma(d)}-P^{\sigma(d)})\right)\widehat{\otimes}\dots\widehat{\otimes}id^d(h)\right\|_\varepsilon+\dots+\\
&\quad+\left\{\prod_{\delta=1,\delta\neq\sigma(1)}^{d}\|\mu_\delta\|\cdot\|P^\delta\|\right\}\cdot\left\|id^1\widehat{\otimes}\dots\widehat{\otimes}(\mu_{\sigma(1)})\circ\left(id^{\sigma(1)}-P^{\sigma(1)})\right)\widehat{\otimes}\dots\widehat{\otimes}id^d(h)\right\|_\varepsilon\\
&=\sum_{v=1}^{d}\left\{\prod_{\delta=1}^{d-v}\|\mu_{\sigma(\delta)}\|\right\}\cdot\left\{\prod_{\delta=d-\nu+2}^{d}\|\mu_{\sigma(\delta)}\|\cdot\left\|P^{\sigma(\delta)}\right\|\right\}\cdot\\
&\quad\cdot\left\|id^1\widehat{\otimes}\dots\widehat{\otimes}\left(\mu_{\sigma(d-\nu+1)}\circ(id^{\sigma(d-\nu+1)}-P^{\sigma(d-\nu+1)})\right)\widehat{\otimes}\dots\widehat{\otimes}id^d(h)\right\|_\varepsilon.
\end{aligned}
$$

In the above, we have used the convention that an empty product equals 1. Since this is true for all permutations $\sigma\in S_d$, we may pass to the minimum over all $\sigma\in S_d$ on the right-hand side of the last inequality. We shall show that for all $h\in\otimes_{\delta=1}^{d}X_\delta$ and all $\mu_\delta: X_\delta\to\mathbb{R}$ (μ_δ linear and continuous), the equality

$$\left\|\left(\widehat{\otimes}_{\delta=1}^{d}\mu_\delta\right)(h)\right\|_\varepsilon=\left|\left(\widehat{\otimes}_{\delta=1}^{d}\mu_\delta\right)(h)\right|$$

holds; this will suffice to prove the theorem. Let $h\in\otimes_{\delta=1}^{d}X_\delta$. Hence $h=\sum_{i=1}^{n}x_{i,1}\otimes\dots\otimes x_{i,d}$ for some $n\in\mathbb{N}$. Thus

$$\left(\otimes_{\delta=1}^{d}\mu_\delta\right)(h)=\sum_{i=1}^{n}\mu_1(x_{i,1})\otimes\mu_2(x_{i,2})\otimes\cdots\otimes\mu_d(x_{i,d})$$

and

$$
\begin{aligned}
\left\|(\otimes_{\delta=1}^{d}\mu_\delta)(h)\right\|_\varepsilon &= \sup_{\substack{a_1\in R\\|a_1|\leq 1}}\dots\sup_{\substack{a_d\in R\\|a_d|\leq 1}}\sum_{i=1}^{n}\prod_{\delta=1}^{d}a_\delta\cdot\mu_\delta(x_{i,\delta})|\\
&= \sup_{\substack{a_1\in R\\|a_1|\leq 1}}\dots\sup_{\substack{a_d\in R\\|a_d|\leq 1}}\left|\prod_{\delta=1}^{d}a_\delta\right|\cdot\left|\sum_{i=1}^{n}\prod_{\delta=1}^{d}\mu_\delta(x_{i,\delta})\right|\\
&= \left|\sum_{i=1}^{n}\prod_{\delta=1}^{d}\mu_\delta(x_{i,\delta})\right|
\end{aligned}
$$

$$= \left| \sum_{i=1}^{n} \mu_1(x_{i,1}) \otimes \mu_2(x_{i,2}) \otimes \cdots \otimes \mu_d(x_{i,d}) \right|$$
$$= \left| \left(\otimes_{\delta=1}^{d} \mu_\delta \right)(h) \right|.$$

For density reasons this equality also holds for all $h \in \widehat{\otimes}_{\delta=1}^{d} X_\delta$ and for the extension $\widehat{\otimes}_{\delta=1}^{d} \mu_\delta$ of $\otimes_{\delta=1}^{d} \mu_\delta$. Thus Theorem 4 is proved. □

In the sequel we shall discuss the case where $(X_\delta, \|\cdot\|_\delta) = (C(I_\delta), \|\cdot\|_\infty)$.

Theorem 5. *For $1 \le \delta \le d$, let continuous linear functionals $\mu_\delta : C(I_\delta) \to \mathbb{R}$, and continuous linear operators $P^\delta : C(I_\delta) \to C(I_\delta)$ be given. If $h \in C\left(\times_{\delta=1}^{d} I_\delta \right)$, then*

$$\left| \left(\widehat{\otimes}_{\delta=1}^{d} \mu_\delta \right) \left(h - (\widehat{\otimes}_{\delta=1}^{d} P^\delta)(h) \right) \right| \le$$
$$\le \min_{\sigma \in S_d} \left\{ \sum_{\nu=1}^{d} \left\{ \prod_{\delta=1}^{d-\nu} \|\mu_{\sigma(\delta)}\| \right\} \cdot \left\{ \prod_{\delta=d-\nu+2}^{d} \|\mu_{\sigma(\delta)}\| \cdot \left\| P^{\sigma(\delta)} \right\| \right\} \cdot \right.$$
$$\left. \cdot \sup_{\substack{x_\delta \in I_\delta \\ \delta \ne \sigma(d-\nu+1)}} \left| \left(\mu_{\sigma(d-\nu+1} \circ \left(id^{\sigma(d-\nu+1)} - P^{\sigma(d-\nu+1)} \right) \right) \left(h^{\bar{\xi}}_{\sigma(d-\nu+1)} \right) \right| \right\}.$$

Here $h^{\bar{\xi}}_{\sigma(d-\nu+1)}$ is the partial mapping belonging to fixed $\bar{\xi} \in \times_{\delta=1, \delta \ne \sigma(d-\nu+1)}^{d} I_\delta$.

Proof. Recall Theorem 4 This yields our claim if we disregard the term

$$\left\| id^1 \widehat{\otimes} \ldots \widehat{\otimes} \left(\mu_{\sigma(d-\nu+1)} \right) \circ \left(id^{\sigma(d-\nu+1)} - P^{\sigma(d-\nu+1)} \right) \right) \widehat{\otimes} \ldots \widehat{\otimes} id^d \, (h) \right\|_\varepsilon.$$

Using Theorem 1 the above may be replaced by

$$\sup_{\substack{x_\delta \in I_\delta \\ 1 \le \delta \le d \\ \delta \ne \sigma(d-\nu+1)}} \left| \left(\mu_{\sigma(d-v+1)} \circ \left(\mathrm{id}^{\sigma(d-v+1)} - P^{\sigma(d-v+1)} \right) \right) \left(h^{\bar{\xi}}_{\sigma(d-\nu+1)} \right) \right|$$

where $\overline{\xi}$ is a point in $X_{\delta=1, \delta \ne \sigma-(d-v+1)}^{d} I_\delta$. Plugging this upper bound into the estimate of Theorem 4 gives our claim. □

If we neglect to pass to the min over $\sigma \in S_d$ and use $\sigma = \mathrm{id}$ in the proof of Theorem 5 we obtain the somewhat weaker

Corollary 6. *Under the assumptions of Theorem 5 the following are true:*

(i)

$$\left| (\hat{\otimes}_{\delta=1}^{d} \mu_\delta)(h - \left(\otimes_{\delta=1}^{d} P^\delta \right)(h)) \right| \le \sum_{\nu=1}^{d} \left\{ \prod_{\delta=1}^{d-\nu} \|\mu_\delta\| \right\} \cdot \left\{ \prod_{\delta=d-\nu+2}^{d} \|\mu_\delta\| \cdot \|P^\delta\| \right\}.$$
$$\sup_{\substack{x_\delta \in I_\delta \\ \delta \ne d-\nu+1}} \left| \left(\mu_{d-\nu+1} \circ \left(id^{d-\nu+1} - P^{d-\nu+1} \right) \right) (h^{\overline{\xi}}_{d-\nu+1}) \right|.$$

(ii) *If, moreover,* $\|\mu_\delta\| = 1$ *and if for some constant* $A \geq 1$ *the inequality* $\|P^\delta\| \leq A$ *holds for* $1 \leq \delta \leq d$*, then the inequality of* (i) *simplifies further to*

$$\Big|(\widehat{\otimes}_{\delta=1}^{d}\mu_\delta)\Big(h - (\widehat{\otimes}_{\delta=1}^{d}P^\delta)(h)\Big)\Big| \leq \sum_{v=1}^{d} A^{d-v} \sup_{\substack{x_\delta \in I_\delta \\ \delta \neq \nu}} \Big|\Big(\mu_v \circ (id^v - P^v)\Big)h_\nu^{\bar{\xi}}\Big|$$

$$\leq A^{d-1} \sum_{v=1}^{d} \sup_{\substack{x_\delta \in I_\delta \\ \delta \neq \nu}} \Big|\mu_v \circ \Big(id^v - P^v\Big)\Big)(h_\nu^{\bar{\xi}})\Big| .$$

A particularly important consequence of Corollary 6 is given in the following theorem. It shows how certain univariate inequalities may be directly used when striving for error estimates on approximation by the tensor product of d univariate operators. For the definition of the (higher order) modulus of continuity $\omega_{r_\delta}(f;\cdot)$ and that of the partial moduli $\omega_{rv}(h;0,\ldots,0,\ldots,0)$, see, *e.g.*, the books by Timan [8] and Schumaker [7].

Theorem 7. *Let linear operators* $P^\delta : C(I_\delta) \to C(I_\delta)$*,* $1 \leq \delta \leq d$*, be given such that for* $f \in C(I_\delta)$ *and* $x \in I_\delta$*,*

$$|f(x) - P^\delta(f;x)| \leq \Gamma_\delta(x) \cdot \omega_{r_\delta}\Big(f;\Delta_\delta(x)\Big), \quad r_\delta \in \mathbb{N}_0 \text{ fixed,}$$

and with bounded functions Γ_δ *and nonnegative real-valued functions* Δ_δ*. Then for any* $h \in C(\times_{\delta=1}^{d} I_\delta)$ *and* $\xi = (x_1,\ldots,x_d) \in \times_{\delta=1}^{d} I_\delta$*, there holds:*

$$\Big|h(\xi) - \widehat{\otimes}_{\delta=1}^{d} P^\delta(h,\xi)\Big| \leq A^{d-1} \sum_{v=1}^{d} \Gamma_v(x_\nu) \cdot \omega_{r_v}(h;0,\ldots,0,\Delta_v(x_\nu),0,\ldots,0).$$

Here A may be chosen as $\max\Big\{1, \|P^\delta\| : 1 \leq \delta \leq d\Big\}$.

Proof. From Corollary 6 (ii) it follows that with

$$\widehat{\otimes}_{\delta=1}^{d}\mu_\delta = \varepsilon_\xi$$

(point evaluation functional) that

$$\Big|h(\xi) - \widehat{\otimes}_{\delta=1}^{d} P^\delta(h,\xi)\Big| \leq \sum_{v=1}^{d} A^{d-v} \sup_{\substack{x_\delta \in I_\delta \\ \delta \neq v}} \Big|(\mathrm{id}^v - P^v)\left(h_\nu^{\bar{\xi}}; x_v\right)\Big|,$$

where $A \geq 1$ is such that $\|P^\delta\| \leq A$ for $1 \leq \delta \leq d$.

Note that the constant A indeed exists because the Γ_δ are bounded and $\omega_r(f,\delta) \leq 2^r\|f\|_\infty$. By the above assumption on P^δ, $1 \leq \delta \leq d$, and the fact that for fixed $\bar{\xi} = (x_1,\ldots,x_{v-1},x_{v+1},\ldots,x_d)$ the function $h_\nu^{\bar{\xi}}$ is given by

$$h_\nu^{\bar{\xi}} : I_\nu \ni x \mapsto h(x_1,\ldots,x_{v-1},x,x_{v+1},\ldots,x_d) \in \mathbb{R},$$

it is seen that

$$\sup_{\substack{x_\delta\in I_\delta\\ \delta\neq\nu}}\left|\left(\mathrm{id}^v-P^v\right)\langle h_\nu^{\overline{\xi}},x_v\rangle\right|\le\sup_{\substack{x_\delta\in I_\delta\\ \delta\neq\nu}}\left|\Gamma_v(x_\nu)\cdot\omega_{r_\nu}\left(h_\nu^{\overline{\xi}};\Delta_v(x_\nu)\right)\right|$$

$$=\Gamma_\nu\left(x_\nu\right)\cdot\omega_{r_\nu}\Big(h;0,\ldots,0,\triangle_\nu\left(x_\nu\right),0,\ldots,0\Big).$$

Hence because $A\ge1$,

$$\left|h(\xi)-\widehat{\otimes}_{\delta=1}^dP^\delta(h,\xi)\right|\le A^{d-1}\sum_{v=1}^d\Gamma_v(x_\nu)\cdot\omega_{r_\nu}\Big(h;0,\ldots,0,\Delta_v(x_\nu),0,\ldots,0\Big).$$

□

Corollary 8. *If d operators P^δ, $1\le\delta\le d$, are given as in Theorem 7 , and if h is a function in $C^{r_1,\ldots,r_d}(\times_{\delta=1}^dI_\delta)$, $\xi\in\times_{\delta=1}^dI_\delta$, then for $0\le\alpha_\delta\le r_\delta$, $1\le\delta\le d$, we have*

$$\left|h(\xi)-\widehat{\otimes}_{\delta=1}^dP^\delta(h,\xi)\right|\le$$

$$\le A^{d-1}\cdot\sum_{v=1}^d\Gamma_v(x_\nu)\cdot\Delta_v^{\alpha_v}(x_\nu)\cdot\cdot\omega_{r_\nu-\alpha_v}\Big(\big(\tfrac{\partial}{\partial x_v}\big)^{\alpha_v}h;0,\ldots,0,\Delta_v(x_\nu),0,\ldots,0\Big).$$

Proof. The inequality to be used in order to get from Theorem 7 to the inequality of the corollary is

$$\omega_r\left(h;0,\ldots,0,\varepsilon,0,\ldots,0\right)\le\epsilon^\alpha\cdot\omega_{r-\alpha}\left(\big(\tfrac{\partial}{\partial x_v}\big)^\alpha h;0,\ldots,0,\varepsilon,0,\ldots,0\right),\ 0\le\alpha\le r,$$

where ε figures in the v-th component of

$$(0,\ldots,0,\varepsilon,0,\ldots,0).$$

□

3. EXAMPLES: POINTWISE INEQUALITIES FOR PRODUCTS OF POSITIVE LINEAR OPERATORS

In the above we mainly considered continuous linear mappings $\mu_\delta:C(I_\delta)\to\mathbb{R}$ and $P^\delta:C(I_\delta)\to C(I_\delta)$. We shall assume throughout this section that μ_δ is a point evaluation functional and that P^δ is positive. The assertions proved here will be based upon a special instance of a theorem by the author (see [1, Theorem 4.6]) and of an improvement of part of the same theorem due to Păltănea [6]. We summarize as follows.

Theorem 9. *Let $L:C[a,b]\to C[a,b]$ be a positive linear operator with $L(e_0)=e_0$, and let $f\in C[a,b]$, $x\in[a,b]$.*

(i) *For each $h,e>0$ one has*

$$\left|L(f,x)-f(x)\right|\le\max\left\{1,L(|e_1-x|\,;x)\cdot h^{-1}\right\}\cdot\left(1+h\cdot\varepsilon^{-1}\right)\cdot\omega_1(f;\varepsilon),\qquad(1)$$

where $e_1:[a,b]\ni\mapsto t\in\mathbb{R}$.

(ii) *For each* $0 < h \le \frac{1}{2}(b-a)$ *there also holds*

(2)
$$\Big|L(f,x)-f(x)\Big| \le h^{-1}\cdot|L(e_1-x;x)|\cdot\omega_1(f;h)+\Big[1+\tfrac{1}{2}\cdot h^{-2}L\Big(\,(e_1-x)^2\,;x\Big)\Big]\cdot\omega_2(f;h).$$

These inequalities will now be combined with the results from Section 2 The following theorem gives an estimate in terms of first order partial moduli of continuity.

THEOREM 10. *Let positive linear operators* $P^\delta : C(I_\delta) \to C(I_\delta)$ *be given such that* $P^\delta(e_0) = e_0$, $1 \le \delta \le d$. *Then for* $k \in C(X_{\delta=1}^d I_\delta)$ *and* $\xi = (x_1,\dots,x_d) \in X_{\delta=1}^d I_\delta$ *the following inequality holds:*

$$\Big|k(\xi) - \widehat{\otimes}_{\delta=1}^d P^\delta(k,\xi)\Big| \le \sum_{v=1}^d \alpha(P^v;h_v,\varepsilon_v;x_\nu)\cdot\omega_1(k;0,\dots,0,\varepsilon_v,0,\dots,0),$$

where $(h_v,\varepsilon_v) > (0,0)$ *may be arbitrarily chosen, and the function* α *is given by*

$$\alpha(P;h,\varepsilon;x) = \max\Big\{1, P(\,|e_1-x|\,;x)\cdot h^{-1}\Big\}\,(1+h\varepsilon^{-1}).$$

Proof. Because $\mu_\delta = \varepsilon_{x_\delta}$, with $\|\varepsilon_{x_\delta}\| = 1$, $\|P^\delta\| = 1$, and $\widehat{\otimes}_{\delta=1}^d \mu_\delta = \varepsilon_\xi$ with $\xi = (x_1,\dots,x_d)$, Corollary 6 (ii) shows that

$$\Big|k(\xi) - \widehat{\otimes}_{\delta=1}^d P^\delta(k,\xi)\Big| \le \sum_{v=1}^d \sup_{\substack{x_\delta\in I_\delta\\ \delta\ne\nu}} \Big|\Big(\varepsilon_{x_v}\circ(\mathrm{id}^v - P^v)\Big)(h_v^{\hat{\xi}})\Big|.$$

For $\bar{\xi}$ fixed, the expression

$$\Big(\varepsilon_{x_\nu}\circ(id^v - P^\nu)\Big)(k_\nu^{\bar{\xi}})$$

is a univariate difference which may be estimated from above using (1). Note that for each coordinate we may choose a separate couple $(h_\nu, e_\nu) > (0,0)$. Hence

$$\begin{aligned}
&\Big|\Big(\varepsilon_{x_\nu}\circ(id^v - P^\nu)\Big)(k_\nu^{\bar{\xi}})\Big| \le\\
&\le \max\Big\{1, P^\nu\Big(|e_1-x_\nu|;x_\nu\Big)\cdot h_\nu^{-1}\Big\}\cdot\Big(1+h_\nu e_\nu^{-1}\Big)\cdot\omega_1(k_\nu^{\bar{\xi}}, e_\nu)\\
&=: \alpha(P^\nu;h_\nu,\varepsilon_\nu;x_\nu)\cdot\omega_1(k_\nu^{\bar{\xi}},\varepsilon_\nu).
\end{aligned}$$

(Here e_1 simultaneously denotes the functions $I_\nu \ni x_\nu \mapsto x_\nu \in R$, $1 \le \nu \le d$). Thus

$$\Big|k(\xi) - \widehat{\otimes}_{\delta=1}^d P^\delta(k,\xi)\Big| \le \sum_{\nu=1}^d \sup_{\substack{x_\delta\in I_\delta\\ \delta\ne\nu}} \Big\{\alpha\Big(P^\nu;h_\nu,\varepsilon_\nu;x_\nu\Big)\cdot\omega_1(k_\nu^{\bar{\xi}},\varepsilon_\nu)\Big\}.$$

Since the function $\alpha(P^\nu; h_\nu, \varepsilon_\nu; x_\nu)$ does not depend on $\bar{\xi} \in \times_{\delta=1,\delta\neq\nu}^d I_\delta$, the latter sum may be rewritten as

$$\sum_{\nu=1}^{d} \alpha\Big(P^\nu; h_\nu, \varepsilon_\nu; x_\nu\Big) \cdot \sup_{\substack{x_\delta \in I_\delta \\ \delta\neq\nu}} \omega_1(k_\nu^{\bar{\xi}}, \varepsilon_\nu) =$$
$$= \sum_{\nu=1}^{d} \alpha\Big(P^\nu; h_\nu, \varepsilon_\nu; x_\nu\Big) \cdot \omega_1(k; 0, \ldots, 0, \varepsilon_\nu, 0, \ldots, 0),$$

which is the upper bound of Theorem 9 in terms of a sum of first order partial moduli of continuity. □

We also have

THEOREM 11. *Under the assumptions of Theorem 10 the following is true:*

$$\Big|k(\xi) - \widehat{\otimes}_{\delta=1}^d P^\delta(k, \xi)\Big| \leq \sum_{\nu=1}^{d} \alpha(P^\nu; h_\nu; x_\nu) \cdot \omega_2(k; 0, \ldots, 0, h_\nu, 0, \ldots, 0)$$
$$+ \sum_{\nu=1}^{d} \beta(P^\nu; h_\nu; x_\nu) \cdot \omega_1(k; 0, \ldots, 0, h_\nu, 0, \ldots, 0).$$

Here $0 < h_\nu \leq \frac{1}{2}[b_\nu - a_\nu]$ may be arbitrarily chosen, and the functions α and β are given by:

$$\alpha(P; h; x) = 1 + \tfrac{1}{2} \cdot h^{-2} \cdot P\left((e_1 - x)^2; x\right), \text{ and}$$
$$\beta(P; h; x) = h \cdot \left|P\left(e_1 - x; x\right)\right|.$$

Proof. As in the proof of Theorem 10 one observes that for all $k \in C(\times_{\delta=1}^d I_\delta)$ we have

$$\Big|k(\xi) - \widehat{\otimes}_{\delta=1}^d P^\delta(k, \xi))\Big| \leq \sum_{v=1}^{d} \sup_{\substack{x_\delta \in I_\delta \\ \delta\neq\nu}} \Big\{\Big|\Big(\varepsilon_{x_v} \circ (id^v - P^v)\Big)(k_\nu^{\bar{\xi}})\Big|\Big\}.$$

The univariate differences figuring in the sups may now be estimated using (2) from which we get

$$\Big|\varepsilon_{x_v} \circ (id^v - P^v)(k_\nu^{\bar{\xi}})\Big| \leq \Big[1 + \tfrac{1}{2} h_v^{-2} \cdot P^v\left((e_1 - x_\nu)^2; x_\nu\right)\Big] \cdot \omega_2\left(k_\nu^{\bar{\xi}}, h_\nu\right)$$
$$+ h_v^{-1} \cdot \Big|P^v\left(e_1 - x_\nu; x_\nu\right)\Big| \cdot \omega_1\left(k_\nu^{\bar{\xi}}, h_\nu\right)$$
$$=: \alpha\left(P^v; h_v; x_v\right) \cdot \omega_2\left(k_\nu^{\bar{\xi}}, h_\nu\right) + \beta\left(P^v; h_v; x_v\right) \cdot \omega_1\left(k_\nu^{\bar{\xi}}, h_\nu\right).$$

Note again that for each coordinate a separate h_v may be chosen. Since both α and β do not depend on $\bar{\xi}$ we may write

$$\begin{aligned}\left|k(\xi)-\widehat{\otimes}_{\delta=1}^{d}P^{\delta}(k,\xi)\right| &\leq \sum_{v=1}^{d}\alpha\left(P^{v};h_v;x_v\right)\cdot\sup_{\substack{x_\delta\in I_\delta\\ \delta\neq\nu}}\omega_2\left(k_{\nu}^{\bar{\xi}},h_\nu\right)\\ &\quad+\sum_{v=1}^{d}\beta\Big(P^{v};h_v;x_v\Big)\cdot\sup_{\substack{x_\delta\in I_\delta\\ \delta\neq\nu}}\omega_1\left(k_{\nu}^{\bar{\xi}},h_\nu\right)\\ &=\sum_{v=1}^{d}\alpha\Big(P^{v};h_v;x_v\Big)\cdot\omega_2(k;0,...,0,h_v,0,\ldots,0)\\ &\quad+\sum_{v=1}^{d}\beta\Big(P^{v};h_v;x_v\Big)\cdot\omega_1(k;0,...,0,h_v,0,\ldots,0).\end{aligned}$$

□

Corollary 12. *If in addition to the assumptions of Theorem 9 the operators P^{δ} satisfy $P^{\delta}(e_1)=e_1, 1\leq\delta\leq d$, then the inequality of Theorem 11 simplifies to*

$$\left|k(\xi)-\widehat{\otimes}_{\delta=1}^{d}P^{\delta}(k,\xi)\right|\leq\sum_{v=1}^{d}\alpha(P^{v};h_v;x_v)\cdot\omega_2(k;0,...,0,h_v,0,\ldots,0).$$

Proof. Since $P^{\delta}(e_i)=e_1$ for $i=0,1,\ 0\leq\delta\leq d$, we have $\beta\left(P^{v};h_v;x_v\right)=0$ for $1\leq v\leq d$. □

Corollary 13. *If the operators P^{δ} satisfy the assumptions of Corollary 12 and if $f\in C_{\delta=1}^{1,\ldots,1}(\times_{\delta=1}^{d}I_\delta)$, then for $\xi\in\times_{\delta=1}^{d}I_\delta$ we have*

$$\begin{aligned}\left|k(\xi)-\widehat{\otimes}_{\delta=1}^{d}P^{\delta}(k,\xi)\right|&\leq\tfrac{3}{2}\cdot\sum_{v=1}^{d}\left(P^{v}\left((e_v-x_v)^2;x_v\right)\right)^{\frac12}\cdot\\ &\quad\cdot\omega_1\left(\tfrac{\partial}{\partial x_v};k;0,\ldots,0,\left(P^{v}\left((e_1-x_v)^2;x_v\right)\right)^{\frac12},0,\ldots,0\right).\end{aligned}$$

Proof. Here we use the inequality

$$\omega_2\left(k;0,\ldots,0,h_\nu,0,\ldots,0\right)\leq h_\nu\cdot\omega_1\left(\tfrac{\partial}{\partial x_\nu}k;0,\ldots,0,h_\nu,0,\ldots,0\right).$$

Making appropriate choices for h_ν, $1\leq\nu\leq d$, gives the above inequality. □

4. CONCLUDING REMARK

All fundamental estimates given in Section 2 and Section 3 are those concerning the differences $\left|k(\xi)-\widehat{\otimes}_{\delta=1}^{d}P^{\delta}(k,\xi)\right|$, where $k\in C(\times_{\delta=1}^{d},I_\delta)$. It is also possible to modify the assumptions made in Theorem 7 by assuming that similar inequalities hold in order to arrive at somewhat improved estimates for subspaces of smooth functions.

Furthermore, no assertions were made concerning the pointwise degree of simultaneous approximation of partial derivatives. While this is also possible, we decline to do so for the sake of brevity. Related material can be found in the author's "Habilitationsschrift" [2].

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