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RECONSTRUCTION INVERSION FORMULAS FOR THE LAGUERRE GABOR TRANSFORM

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Abstract. In this paper, we define and study the Gabor transform $\mathcal{G}_\psi$ in the context of the Laguerre hypergroup. We prove some of its basic properties, such as Plancherel theorem, inversion formula and Calderón's reproducing inversion formula. Next, using the harmonic analysis related to Laguerre hypergroup, we examine spaces of Sobolev type for which we make explicit kernels reproducing. Exploiting the aforesaid theory, we introduce and study the extremal function associated with the Gabor transform $\mathcal{G}_\psi$. Finally, by utilizing the reproducing kernels we establish important estimates for this extremal function.

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1. INTRODUCTION

The Gabor transform is a foundational method in signal processing that provides a time-frequency representation of signals, helping to analyze and interpret their evolving frequency content. In [25], the author defined the classical Gabor transform by using translation, convolution and modulation operators of a single Gaussian to represent a one dimensional signal. The Gabor transform has been found to be very useful in many physical and engineering applications, including wave propagation, signal processing and quantum optics [4]. Many authors developed the theory of the Gabor transform and found many interesting results see for example [7, 8, 9, 15, 17, 24]. In particular, Gröchenig [7] extended Gabor theory to the setup of locally compact abelian groups. Moreover, Hleili [8], proved some uncertainty principles for the windowed linear canonical transform and investigated the localization operators associated with this transform. Recently, in [15], the author studied the Dunkl–Gabor transform on $\mathbb{R}^d$ and gave the practical real inversion formulas for this transform using the theory of reproducing kernels.

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Tikhonov regularization is widely applied across diverse disciplines to address ill-posed problems, enhance numerical stability, and mitigate overfitting. It plays a crucial role in a variety of applications, particularly in machine learning, as well as in signal and image processing. Over the years, the theory has been extensively developed and refined by numerous researchers (see, for example [5, 23]). Recent advances in approximation theory have introduced more sophisticated techniques to address challenges in high-dimensional and noisy data settings. While classical methods like Tikhonov regularization remain foundational, newer approaches such as compressed sensing, variational regularization, kernel based learning, and neural network approximators offer greater flexibility and improved performance in complex inverse problems and machine learning tasks. Recent developments in approximation theory can be found in [13].

In this paper we are interested in the Laguerre hypergroup $\mathbb{K} = [0, +\infty[\times\mathbb{R}$ which is the fundamental manifold of the radial function space for the Heisenberg group ([3, 11]. The dual of a hypergroup is the space of all bounded continuous and multiplicative functions χ such that $\overline{\chi} = \chi$. The dual of the Laguerre hypergroup $\hat{\mathbb{K}}$ can be topologically identified with the so-called Heisenberg fan [6], i.e., the subset embedded in $\mathbb{R}^2$ given by

$$\bigcup_{j\in\mathbb{N}} \Big\{(\mu,\lambda) \in \mathbb{R}^2;\ \lambda = |\mu|(2j+\alpha+1),\ \mu \neq 0\Big\} \cup \Big\{(0,\lambda) \in \mathbb{R}^2;\ \lambda \geqslant 0\Big\}.$$

Moreover, the subset $\Big\{(0,\lambda) \in \mathbb{R}^2;\ \lambda \geqslant 0\Big\}$ has zero Plancherel measure, therefore it will be usually disregarded. Following [19], in this paper, we identify the dual of the Laguerre hypergroup by $\hat{\mathbb{K}} = \mathbb{R} \times \mathbb{N}$.

The Fourier Laguerre transform $\mathcal{F}_\alpha$ of a suitable function $f : \mathbb{K} \longrightarrow \mathbb{C}$ is given by

$$\forall(\mu,m) \in \hat{\mathbb{K}},\ \mathcal{F}_\alpha(f)(\mu,m) = \int_{\mathbb{K}} f(r,x)\varphi_{(-\mu,m)}(r,x)d\nu_\alpha(r,x),$$

where ν_α is the weighted Lebesgue measure on $\mathbb{K}$, given by

$$d\nu_\alpha(r,x) = \tfrac{r^{2\alpha+1}drdx}{\pi\Gamma(\alpha+1)},\ \alpha \geqslant 0,$$

$\varphi_{(\mu,m)}$ is the function infinitely differentiable on $\mathbb{R}^2$, even with respect to the first variable defined by

$$\varphi_{(\mu,m)}(r,x) = e^{i\mu x}\mathcal{L}_m^\alpha(|\mu|^2 r^2),$$

and $\mathcal{L}_m^\alpha$ is the Laguerre polynomial of degree m and order α.

The Fourier transform $\mathcal{F}_\alpha$ has rich calculus and is applicable in many areas of mathematical sciences. Many authors exploited the theory of the Fourier transform and found many interesting results see for example [1, 2, 10, 16, 18, 19].

Our purpose in this work consists to study the Laguerre Gabor transform $\mathcal{G}_\psi$ and to introduce Sobolev type spaces and for which we present their reproducing kernels. Next, by utilizing the theory of reproducing kernels, we establish some important results for the Laguerre Gabor transform $\mathcal{G}_\psi$ and we give interesting estimates for the extremal function.

The remainder of this paper is arranged as follows. Section 2 contains some basic facts about the Laguerre hypergroup. The section Section 3 is devoted to study the Laguerre Gabor transform $\mathcal{G}_\psi$, for which we give a Plancherel formula, inversion formula and a Calderón's reproducing formula. In the last section, using the aforesaid theory, we give best approximate inversion formulas for the Laguerre Gabor transform $\mathcal{G}_\psi$.

2. PRELIMINARIES

In this section, we recall some important properties and results of the translation operators and the Fourier transform on the Laguerre hypergroup, which are useful in our present work. For more details, see [19]. We denote by

- $L^p(\mathbb{K}), p \in [1, +\infty]$, the spaces of complex-valued functions f, measurable on $\mathbb{K}$, such that

$$\|f\|_{p,\nu_\alpha} = \begin{cases} \left(\int_{\mathbb{K}} |f(r,x)|^p d\nu_\alpha(r,x)\right)^{\frac{1}{p}}, & p \in [1, +\infty[; \\ \text{ess sup}_{(r,x)\in\mathbb{K}} |f(r,x)|, & p = +\infty. \end{cases}$$

- $\mathcal{C}_e(\mathbb{K})$, the space of continuous functions on $\mathbb{R}^2$, even with respect to the first variable.
- $\mathcal{C}_{e,c}(\mathbb{K})$, the subspace of $\mathcal{C}_e(\mathbb{K})$ formed by functions with compact support.
- $\mathcal{L}_m^\alpha$ the Laguerre function defined on $]0, +\infty[$ by

$$\mathcal{L}_m^\alpha(r) = \frac{e^{\frac{-r}{2}} \mathcal{L}_m^\alpha(r)}{\mathcal{L}_m^\alpha(0)},$$

  where $\mathcal{L}_m^\alpha$ is the Laguerre polynomial of degree m and order α.
- $\hat{\mathbb{K}} = \mathbb{R} \times \mathbb{N}$ equipped with the weighted Lebesgue measure γ_α on $\hat{\mathbb{K}}$ given by

$$\int_{\hat{\mathbb{K}}} h(\mu, m) d\gamma_\alpha(\mu, m) = \sum_{m=0}^{+\infty} \mathcal{L}_m^\alpha(0) \int_{\mathbb{R}} h(\mu, m)|\mu|^{\alpha+1} d\mu.$$

- $L^p(\hat{\mathbb{K}}), p \in [1, +\infty]$, the spaces of complex-valued functions h, measurable on $\hat{\mathbb{K}}$, such that

$$\|h\|_{p,\gamma_\alpha} = \begin{cases} \left(\int_{\hat{\mathbb{K}}} |h(\mu,m)|^p d\gamma_\alpha(\mu,m)\right)^{\frac{1}{p}}, & p \in [1, +\infty[; \\ \text{ess sup}_{(\mu,m)\in\hat{\mathbb{K}}} |h(\mu,m)|, & p = +\infty. \end{cases}$$

Consider the following partial differential operators system

$$\begin{cases} D_1 = \frac{\partial}{\partial x}, \\ \mathcal{D}_2 = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r} + r^2\frac{\partial^2}{\partial x^2}, \\ (r,x) \in]0,+\infty[\times\mathbb{R} \text{ and } \alpha \geqslant 0. \end{cases}$$

For $\alpha = n-1$, $n \in \mathbb{N}^*$, the operator $\mathcal{D}_2$ is the radial part of the sub-Laplacian on the Heisenberg group $\mathbb{H}_n$.

For $(\mu, m) \in \hat{\mathbb{K}}$, the initial problem (see [19])

$$\begin{cases} D_1 u(r,x) = i\mu u(r,x), & (r,x) \in \mathbb{K}; \\ \mathcal{D}_2 u(r,x) = -4|\mu|\Big(m + \frac{\alpha+1}{2}\Big)u(r,x), & (r,x) \in \mathbb{K}; \\ u(0,0) = 1, \ \frac{\partial u}{\partial r}(0,x) = 0, & \text{for all } x \in \mathbb{R}, \end{cases}$$

has a unique solution $\varphi_{(\mu,m)}$ given by

$$\varphi_{(\mu,m)}(r,x) = e^{i\mu x}\mathcal{L}_m^\alpha(|\mu|^2 r^2).$$

For all $(\mu, m) \in \hat{\mathbb{K}}$, the function $\varphi_{(\mu,m)}$ is infinitely differentiable on $\mathbb{R}^2$, even with respect to the first variable and satisfies

$$\sup_{(r,x)\in\mathbb{K}} |\varphi_{(\mu,m)}(r,x)| = 1. \tag{1}$$

The harmonic analysis on the Laguerre hypergroup $\mathbb{K}$ is generated by the singular operator

$$\mathcal{L}_\alpha = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r} + r^2\frac{\partial^2}{\partial x^2},$$

and the norm

$$\mathcal{N}(r,x) = |(r,x)| = (r^4 + x^2)^{\frac{1}{4}}, \ (r,x) \in \mathbb{K}.$$

Also, we introduce the operator $\Lambda = \Lambda_1^2 - \Big(2\Lambda_2 + 2\frac{\partial}{\partial\mu}\Big)^2$ defined on $\hat{\mathbb{K}}$, where $\Lambda_1 = \frac{1}{|\mu|}\Big(m\Delta_+\Delta_- + (\alpha+1)\Delta_+\Big)$ and $\Lambda_2 = \frac{-1}{2|\mu|}\Big((\alpha+m+1)\Delta_+ + m\Delta_-\Big)$.

The difference operators Δ_+, Δ_- are given for a suitable function h on $\hat{\mathbb{K}}$, by

$$\Delta_+ h(\mu,m) = h(\mu,m+1) - h(\mu,m),$$

$$\Delta_- h(\mu,m) = \begin{cases} h(\mu,m) - h(\mu,m-1), & \text{if } m \geqslant 1; \\ h(\mu,0), & \text{if } m = 0. \end{cases}$$

We introduce also the quasinorm

$$N(\mu,m) = |\mu|\Big(m + \frac{\alpha+1}{2}\Big), \ (\mu,m) \in \hat{\mathbb{K}}.$$

These operators satisfy some basic properties which can be found in [2, 19], namely one has

$$\mathcal{L}_\alpha\varphi_{(\mu,m)}(r,x) = -N(\mu,m)\varphi_{(\mu,m)}(r,x),$$

$$\Lambda\varphi_{(\mu,m)}(r,x) = \mathcal{N}^4(r,x)\varphi_{(\mu,m)}(r,x).$$

For $(r,x),(s,y)\in\mathbb{K}$ and $\theta\in[0,2\pi[, t\in[0,1]$, let

$$((r,x),(s,y))_{\theta,t}=\Big(\sqrt{r^2+s^2+2rst\cos(\theta)},x+y+rst\sin(\theta)\Big).$$

The generalized translation operators $\mathcal{T}^{(\alpha)}_{(r,x)}$ on the Laguerre hypergroup are given for $f\in\mathcal{C}_{e,c}(\mathbb{K})$ by

$$T^{(\alpha)}_{(r,x)}(s,y)=\begin{cases}\frac{1}{2\pi}\displaystyle\int_0^{2\pi}f(((r,x),(s,y))_{\theta,1})d\theta, & \text{if } \alpha=0;\\ \frac{\alpha}{\pi}\displaystyle\int_0^{2\pi}\int_0^1 f(((r,x),(s,y))_{\theta,t})t(1-t^2)^{\alpha-1}dtd\theta, & \text{if } \alpha>0.\end{cases}$$

The generalized translation operators $\mathcal{T}^{(\alpha)}_{(r,x)}$ on the Laguerre hypergroup satisfy the following properties:

(i) For all $f\in L^p(\mathbb{K}), p\in[1,+\infty]$ and $(r,x)\in\mathbb{K}$, the function $\mathcal{T}^{(\alpha)}_{(r,x)}(f)$ belongs to $L^p(\mathbb{K})$ and we have

$$\|\mathcal{T}^{(\alpha)}_{(r,x)}(f)\|_{p,\nu_\alpha}\leqslant\|f\|_{p,\nu_\alpha}. \tag{2}$$

(ii) For all $(r,x)\in\mathbb{K}$ and $f\in L^1(\mathbb{K})$, we get

$$\int_{\mathbb{K}}\mathcal{T}^{(\alpha)}_{(r,x)}(f)(s,y)d\nu_\alpha(s,y)=\int_{\mathbb{K}}f(s,y)d\nu_\alpha(s,y). \tag{3}$$

We denote by

• $\mathcal{S}_e(\mathbb{K})$, the space of functions $f:\mathbb{R}^2\longrightarrow\mathbb{C}$, even with respect to the first variable, C^∞ on $\mathbb{R}^2$ and rapidly decreasing together with their derivatives, i.e., for all $k,p,q\in\mathbb{N}$, we have

$$N_{k,p,n}(f)=\sup_{(r,x)\in\mathbb{K}}\Big((1+r^2+x^2)^k\Big|\tfrac{\partial^{p+q}}{\partial r^p\partial x^q}f(r,x)\Big|\Big)<\infty.$$

Equipped with the topology defined by the semi-norms $N_{k,p,n}$, $\mathcal{S}_e(\mathbb{K})$ is a Fréchet space.

• $\mathcal{S}(\hat{\mathbb{K}})$, the space of functions $h:\hat{\mathbb{K}}\longrightarrow\mathbb{C}$ such that

(i) For all $m,n,p,q,\ell\in\mathbb{N}$, the function

$$\mu\longmapsto\mu^p\Big(|\mu|(m+\tfrac{\alpha+1}{2})\Big)^q\Lambda_1^n\Big(\Lambda_2+\tfrac{\partial}{\partial\mu}\Big)^\ell h(\mu,m),$$

is bounded and continuous on $\mathbb{R}$, C^∞ on $\mathbb{R}^*$ such that the left and the right derivatives at zero exist.

(ii) For all $k,p,q\in\mathbb{N}$, we have

$$\mathcal{M}_{k,p,q}(h)=\sup_{(\mu,m)\in\mathbb{R}^*\times\mathbb{N}}\Big((1+\mu^2(1+m^2))^k\Big|\Lambda_1^p\Big(\Lambda_2+\tfrac{\partial}{\partial\mu}\Big)^q h(\mu,m)\Big|\Big)<\infty.$$

Equipped with the topology defined by the semi-norms $\mathcal{M}_{k,p,q}$, $\mathcal{S}(\hat{\mathbb{K}})$ is a Fréchet space.

For $f \in L^1(\mathbb{K})$, the Fourier–Laguerre transform $\mathcal{F}_\alpha$ is defined by

$$\forall(\mu, m) \in \hat{\mathbb{K}},\ \mathcal{F}_\alpha(f)(\mu, m) = \int_{\mathbb{K}} f(r, x)\varphi_{(-\mu,m)}(r, x)d\nu_\alpha(r, x). \tag{4}$$

For every $f \in L^1(\mathbb{K})$, the function $\mathcal{F}_\alpha(f)$ is bounded on $\hat{\mathbb{K}}$ and satisfies

$$\|\mathcal{F}_\alpha\|_{\infty,\gamma_\alpha} \leqslant \|f\|_{1,\nu_\alpha}.$$

Theorem 1 (Inversion formula). *Let $f \in L^1(\mathbb{K})$ such that $\mathcal{F}_\alpha(f) \in L^1(\hat{\mathbb{K}})$, then for almost every $(r, x) \in \mathbb{K}$*

$$f(r, x) = \int_{\hat{\mathbb{K}}} \mathcal{F}_\alpha(f)(\mu, m)\varphi_{(\mu,m)}(r, x)d\gamma_\alpha(\mu, m). \tag{5}$$

Theorem 2 (Plancherel theorem). *The Fourier transform $\mathcal{F}_\alpha$ can be extended to an isometric isomorphism from $L^2(\mathbb{K})$ onto $L^2(\hat{\mathbb{K}})$. In particular, for every $f \in L^2(\mathbb{K})$*

$$\|\mathcal{F}_\alpha(f)\|_{2,\gamma_\alpha} = \|f\|_{2,\nu_\alpha}.$$

Corollary 3. *For all functions f and h in $L^2(\mathbb{K})$, we have*

$$\int_{\mathbb{K}} f(r, x)\overline{h(r, x)}d\nu_\alpha(r, x) = \int_{\hat{\mathbb{K}}} \mathcal{F}_\alpha(f)(\mu, m)\overline{\mathcal{F}_\alpha(h)(\mu, m)}d\gamma_\alpha(\mu, m). \tag{6}$$

Theorem 4. *The generalized Fourier transform $\mathcal{F}_\alpha$ is a topological isomorphism from $\mathcal{S}_e(\mathbb{K})$ onto $\mathcal{S}(\hat{\mathbb{K}})$. The inverse mapping is given by*

$$\forall(r, x) \in \mathbb{K},\ \mathcal{F}_\alpha^{-1}(f)(r, x) = \int_{\hat{\mathbb{K}}} f(\mu, m)\varphi_{(\mu,m)}(r, x)d\gamma_\alpha(\mu, m).$$

3. THE LAGUERRE-GABOR TRANSFORM

Let $\phi, \psi \in \mathcal{S}(\hat{\mathbb{K}})$. We define the convolution product $\phi * \psi$ of ϕ and ψ by

$$\phi * \psi(\mu, m) = \mathcal{F}_\alpha\Big(\mathcal{F}_\alpha^{-1}(\phi)\mathcal{F}_\alpha^{-1}(\psi)\Big)(\mu, m), \quad (\mu, m) \in \hat{\mathbb{K}}. \tag{7}$$

This definition extends to $\phi \in L^p(\hat{\mathbb{K}}), p = 1, 2$ and $\psi \in L^2(\hat{\mathbb{K}})$.
The convolution $*$ verifies the following properties:

Lemma 5. 1) *For all $\phi \in L^1(\hat{\mathbb{K}})$ and for all $\psi \in L^2(\hat{\mathbb{K}})$, the function $\phi * \psi$ belongs to $L^2(\hat{\mathbb{K}})$ and we have*

$$\mathcal{F}_\alpha^{-1}(\phi * \psi) = \mathcal{F}_\alpha^{-1}(\phi)\mathcal{F}_\alpha^{-1}(\psi).$$

2) *Let $\phi, \psi \in L^2(\hat{\mathbb{K}})$. Then the function $\phi * \psi$ belongs to $L^2(\hat{\mathbb{K}})$ if and only if $\mathcal{F}_\alpha^{-1}(\phi)\mathcal{F}_\alpha^{-1}(\psi)$ belongs to $L^2(\mathbb{K})$ and we have*

$$\mathcal{F}_\alpha^{-1}(\phi * \psi) = \mathcal{F}_\alpha^{-1}(\phi)\mathcal{F}_\alpha^{-1}(\psi), \quad \textit{in the} \quad L^2 \ - \textit{case}.$$

3) *Let $\phi, \psi \in L^2(\hat{\mathbb{K}})$. Then*

$$\int_{\hat{\mathbb{K}}} |\phi * \psi(\mu, m)|^2 d\gamma_\alpha(\mu, m) = \int_{\mathbb{K}} |\mathcal{F}_\alpha^{-1}(\phi)(r, x)|^2 |\mathcal{F}_\alpha^{-1}(\psi)(r, x)|^2 d\nu_\alpha(r, x), \tag{8}$$

where both sides are finite or infinite.

Proof. 1) For $\phi \in L^1(\hat{\mathbb{K}})$, the function $\mathcal{F}_\alpha^{-1}(\phi)$ belongs to $L^\infty(\mathbb{K})$ and for $\psi \in L^2(\hat{\mathbb{K}})$, $\mathcal{F}_\alpha^{-1}(\psi) \in L^2(\mathbb{K})$, then we deduce that $\mathcal{F}_\alpha^{-1}(\phi)\mathcal{F}_\alpha^{-1}(\psi) \in L^2(\mathbb{K})$. Hence the result follows from (7) and Theorem 2.

2) The result follows from (7) and Theorem 2.

3) Let $\phi, \psi \in L^2(\hat{\mathbb{K}})$. For $\phi * \psi \in L^2(\hat{\mathbb{K}})$, the function $\mathcal{F}_\alpha^{-1}(\phi)\mathcal{F}_\alpha^{-1}(\psi)$ belongs to $L^2(\mathbb{K})$. Then the result can be deduced according to (7) and Theorem 2. □

Definition 6. *Let $\psi \in L^2(\hat{\mathbb{K}})$ and $(r,x) \in \mathbb{K}$. The modulation of ψ by (r,x) is the function defined by*

$$\psi_{(r,x)}(\mu,m) = \mathcal{F}_\alpha\Big(\sqrt{\mathcal{T}^{(\alpha)}_{(r,x)}|\mathcal{F}_\alpha^{-1}(\psi)|^2}\Big)(\mu,m),\ (\mu,m) \in \hat{\mathbb{K}}.$$

On view of (3) and Theorem 2, we get

$$\|\psi_{(r,x)}\|_{2,\gamma_\alpha} = \|\psi\|_{2,\gamma_\alpha}. \tag{9}$$

Definition 7. *Let $\psi \in L^2(\hat{\mathbb{K}})$. For a function $\phi \in L^2(\hat{\mathbb{K}})$, we define the Laguerre Gabor transform by*

$$\mathcal{G}_\psi(\phi)(\mu,m,r,x) = \phi * \psi_{(r,x)}(\mu,m),\ (\mu,m) \in \hat{\mathbb{K}}. \tag{10}$$

Proposition 8. *Let $\phi, \psi \in L^2(\hat{\mathbb{K}})$, then*

$$\mathcal{G}_\psi(\phi)(\mu,m,r,x) = \int_{\mathbb{K}} \mathcal{F}_\alpha^{-1}(\phi)(s,y)\sqrt{\mathcal{T}^{(\alpha)}_{(r,x)}|\mathcal{F}_\alpha^{-1}(\psi)|^2(s,y)}\,\varphi_{(\mu,m)}(s,y)d\nu_\alpha(s,y).$$

Proof. The result follows from (7), the definition of $\mathcal{F}_\alpha$ and the fact that $\mathcal{F}_\alpha^{-1}(\psi_{(r,x)})(s,y) = \sqrt{\mathcal{T}^{(\alpha)}_{(r,x)}|\mathcal{F}_\alpha^{-1}(\psi)|^2(s,y)}$. □

We denote by $L^p(\hat{\mathbb{K}} \times \mathbb{K}), p \in [1,+\infty]$, the space of measurable functions on $\hat{\mathbb{K}}_+ \times \mathbb{K}$ satisfying for $p \in [1,+\infty[$

$$\|\phi\|_{p,\gamma_\alpha\otimes\nu_\alpha} = \Big(\int_{\mathbb{K}}\int_{\hat{\mathbb{K}}} |\phi(\mu,m,r,x)|^p d\gamma_\alpha(\mu,m)d\nu_\alpha(r,x)\Big)^{\frac{1}{p}} < \infty,$$

and for $p = +\infty$

$$\|\phi\|_{\infty,\gamma_\alpha\otimes\nu_\alpha} = \sup_{\substack{(r,x)\in\mathbb{K},\\ (\mu,m)\in\hat{\mathbb{K}}}} |\phi(\mu,m,r,x)| < \infty.$$

Theorem 9 (Plancherel formula). *Let $\psi \in L^2(\hat{\mathbb{K}})\backslash\{0\}$. Then for every $\phi \in L^2(\hat{\mathbb{K}})$, we have*

$$\|\mathcal{G}_\psi(\phi)\|_{2,\gamma_\alpha\otimes\nu_\alpha} = \|\phi\|_{2,\gamma_\alpha}\|\psi\|_{2,\gamma_\alpha}.$$

Proof. Let $\psi \in L^2(\hat{\mathbb{K}})$. In view of (10) and (8), we get

$$\int_{\mathbb{K}}\int_{\hat{\mathbb{K}}} |\mathcal{G}_\psi(\phi)(\mu,m,r,x)|^2 d\gamma_\alpha(\mu,m)d\nu_\alpha(r,x) =$$
$$= \int_{\mathbb{K}}\int_{\hat{\mathbb{K}}} |\phi * \psi_{(r,x)}(\mu,m)|^2 d\gamma_\alpha(\mu,m)d\nu_\alpha(r,x)$$
$$= \int_{\mathbb{K}}\int_{\mathbb{K}} |\mathcal{F}_\alpha^{-1}(\phi)(s,y)|^2|\mathcal{F}_\alpha^{-1}(\psi_{(r,x)})(s,y)|^2 d\nu_\alpha(s,y)d\nu_\alpha(r,x).$$

Now, using the fact that $\mathcal{F}_\alpha^{-1}(\psi_{(r,x)})(s,y) = \sqrt{\mathcal{T}^{(\alpha)}_{(r,x)}|\mathcal{F}_\alpha^{-1}(\psi)|^2(s,y)}$, the relation (3), Theorem 2 and Fubini-Tonelli theorem, we obtain

$$\int_{\mathbb{K}}\int_{\hat{\mathbb{K}}} |\mathcal{G}_\psi(\phi)(\mu,m,r,x)|^2 d\gamma_\alpha(\mu,m)d\nu_\alpha(r,x) =$$
$$= \int_{\mathbb{K}}\int_{\mathbb{K}} |\mathcal{F}_\alpha^{-1}(\phi)(s,y)|^2 \mathcal{T}^{(\alpha)}_{(r,x)}|\mathcal{F}_\alpha^{-1}(\psi)|^2(s,y)d\nu_\alpha(s,y)d\nu_\alpha(r,x)$$
$$= \|\phi\|_{2,\gamma_\alpha}\|\psi\|_{2,\gamma_\alpha}.$$

Which gives the desired result. □

THEOREM 10 (Inversion formula). *Let $\psi \in L^2(\hat{\mathbb{K}})\backslash\{0\}$. For every $\phi \in L^1(\hat{\mathbb{K}}) \cap L^2(\hat{\mathbb{K}})$ such that $\mathcal{F}_\alpha^{-1}(\phi) \in L^1(\mathbb{K})$, we have*

$$\phi(\mu,m) = \frac{1}{\|\psi\|^2_{2,\gamma_\alpha}} \int_{\mathbb{K}} \mathcal{G}_\psi(\phi)(.,.,r,x) * \psi_{(r,x)}(s,y)d\nu_\alpha(r,x),\ (\mu,m) \in \hat{\mathbb{K}}.$$

Proof. In view of Theorem 5 (1), the function $\mathcal{G}_\psi(\phi)(.,.,r,x)$ belongs to $L^2(\hat{\mathbb{K}})$. Then by (7), we deduce that

$$\mathcal{G}_\psi(\phi)(.,.,r,x) * \psi_{(r,x)}(\mu,m) =$$
$$= \int_{\mathbb{K}} \mathcal{F}_\alpha^{-1}(\mathcal{G}_\psi(\phi)(.,.,r,x))(s,y)\mathcal{F}_\alpha^{-1}(\psi_{(r,x)})(s,y)\varphi_{(-\mu,m)}(s,y)d\nu_\alpha(s,y).$$

Now, by Theorem 5 (1), we obtain

$$\mathcal{F}_\alpha^{-1}(\mathcal{G}_\psi(\phi)(.,.,r,x))(s,y)$$
$$= \mathcal{F}_\alpha^{-1}(\phi)(s,y)\mathcal{F}_\alpha^{-1}(\psi_{(r,x)})(s,y) = \mathcal{F}_\alpha^{-1}(\phi)(s,y)\sqrt{\mathcal{T}^{(\alpha)}_{(r,x)}|\mathcal{F}_\alpha^{-1}(\psi)|^2(s,y)}.$$

Hence,

$$\mathcal{G}_\psi(\phi)(.,.,r,x) * \psi_{(r,x)}(\mu,m)$$
$$= \int_{\mathbb{K}} \mathcal{F}_\alpha^{-1}(\phi)(s,y)\mathcal{T}^{(\alpha)}_{(r,x)}|\mathcal{F}_\alpha^{-1}(\psi)|^2(s,y)\varphi_{(-\mu,m)}(s,y)d\nu_\alpha(s,y).$$

Finally, using Fubini's theorem, definition of $\mathcal{F}_\alpha$, Theorem 2 and (3), we get

$$\int_{\mathbb{K}} \mathcal{G}_\psi(\phi)(.,.,r,x) * \psi_{(r,x)}(\mu,m)d\nu_\alpha(r,x) =$$
$$= \int_{\mathbb{K}} \mathcal{T}^{(\alpha)}_{(r,x)}|\mathcal{F}^{-1}_\alpha(\psi)|^2(s,y)\Big(\int_{\mathbb{K}} \mathcal{F}^{-1}_\alpha(\phi)(s,y)\varphi_{(-\mu,m)}(s,y)d\nu_\alpha(s,y)\Big)d\nu_\alpha(r,x)$$
$$=\phi(\mu,m)\|\psi\|^2_{2,\gamma_\alpha}.$$

And the proof of this theorem is completed. □

In the following we establish reproducing inversion formula of Calderón's type for the Laguerre-Gabor transform $\mathcal{G}_\psi$.

Theorem 11. *Let $\psi \in L^2(\hat{\mathbb{K}})\backslash\{0\}$ such that $\mathcal{F}^{-1}_\alpha(\psi) \in L^\infty(\mathbb{K})$. Then, for every $\phi \in L^2(\hat{\mathbb{K}})$ and $k \in \mathbb{N}^*$, the function ϕ_k given by*

$$\phi_k(\mu,m) = \frac{1}{\|\psi\|^2_{2,\gamma_\alpha}} \int_{B_k^+} \mathcal{G}_\psi(\phi)(.,.,r,x) * \psi_{(r,x)}(s,y)d\nu_\alpha(r,x),$$

belongs to $L^2(\hat{\mathbb{K}})$ and satisfies

$$\lim_{k\longrightarrow+\infty} \|\phi_k - \phi\|_{2,\gamma_\alpha} = 0, \tag{11}$$

where $B_k^+ = \{(s,y) \in \mathbb{K},\ |(s,y)| \leqslant k\}$.

Proof. According to Theorem 5 (2), the function $\mathcal{G}_\psi(\phi)(.,.,r,x)$ belongs to $L^2(\hat{\mathbb{K}})$, then by (7), we get

$$\mathcal{G}_\psi(\phi)(.,.,r,x) * \psi_{(r,x)}(\mu,m) =$$
$$= \int_{\mathbb{K}} \mathcal{F}^{-1}_\alpha(\mathcal{G}_\psi(\phi)(.,.,r,x))(s,y)\mathcal{F}^{-1}_\alpha(\psi_{(r,x)})(s,y)\varphi_{(-\mu,m)}(s,y)d\nu_\alpha(s,y).$$

Now, by Theorem 5 (2), we obtain

$$\mathcal{F}^{-1}_\alpha(\mathcal{G}_\psi(\phi)(.,.,r,x))(s,y) = \tag{12}$$
$$= \mathcal{F}^{-1}_\alpha(\phi)(s,y)\mathcal{F}^{-1}_\alpha(\psi_{(r,x)})(s,y) = \mathcal{F}^{-1}_\alpha(\phi)(s,y)\sqrt{\mathcal{T}^{(\alpha)}_{(r,x)}|\mathcal{F}^{-1}_\alpha(\psi)|^2(s,y)}.$$

Thus,

$$\mathcal{G}_\psi(\phi)(.,.,r,x) * \psi_{(r,x)}(\mu,m) =$$
$$= \int_{\mathbb{K}} \mathcal{F}^{-1}_\alpha(\phi)(s,y)\mathcal{T}^{(\alpha)}_{(r,x)}|\mathcal{F}^{-1}_\alpha(\psi)|^2(s,y)\varphi_{(-\mu,m)}(s,y)d\nu_\alpha(s,y),$$

and

$$\phi_k(\mu,m) =$$
$$= \frac{1}{\|\psi\|^2_{2,\gamma_\alpha}} \int_{B_k^+}\int_{\mathbb{K}} F^{-1}_\alpha(\phi)(s,y)\mathcal{T}^{(\alpha)}_{(r,x)}|\mathcal{F}^{-1}_\alpha(\psi)|^2(s,y)\varphi_{(-\mu,m)}(s,y)d\nu_\alpha(s,y)d\nu_\alpha(r,x)$$
$$= \int_{\mathbb{K}} \Phi_k(s,y)\mathcal{F}^{-1}_\alpha(\phi)(s,y)\varphi_{(-\mu,m)}(s,y)d\nu_\alpha(s,y), \tag{13}$$

where

$$\Phi_k(s,y)=\frac{1}{\|\psi\|^2_{2,\gamma_\alpha}}\int_{B_k^+}\mathcal{T}^{(\alpha)}_{(r,x)}|\mathcal{F}_\alpha^{-1}(\psi)|^2(s,y)d\nu_\alpha(r,x).$$

From (3) and Theorem 2, we deduce that

$$\|\Phi_k\|_{\infty,\nu_\alpha}\leqslant 1.$$

Applying Hölder's inequality, we obtain

$$|\Phi_k(s,y)|^2\leqslant\frac{\nu_\alpha(B_k^+)}{\|\psi\|^4_{2,\gamma_\alpha}}\int_{B^+(k)}|\mathcal{T}^{(\alpha)}_{(r,x)}(|\mathcal{F}_\alpha^{-1}(\psi)|^2)(s,y)|^2d\nu_\alpha(r,x).$$

Invoking (2), the above expression becomes

$$\begin{aligned}\|\Phi_k\|^2_{2,\nu_\alpha}&\leqslant\frac{\nu^2_\alpha(B_k^+)}{\|\psi\|^4_{2,\gamma_\alpha}}\int_{\mathbb{K}}|\mathcal{F}_\alpha^{-1}(\psi)(s,y)|^4d\nu_\alpha(s,y)\\&\leqslant\frac{\nu^2_\alpha(B_k^+)\|\mathcal{F}_\alpha^{-1}(\psi)\|^2_{\infty,\nu_\alpha}}{\|\psi\|^2_{2,\gamma_\alpha}}.\end{aligned}$$

Hence $\Phi_k\in L^\infty(\mathbb{K})\cap L^2(\mathbb{K})$. Therefore by (13), we have

$$\phi_k=\mathcal{F}_\alpha(\Phi_k\mathcal{F}_\alpha^{-1}(\phi)).$$

Then by Theorem 2, it follows that $\phi_k\in L^2(\hat{\mathbb{K}})$ and

$$\|\phi_k-\phi\|^2_{2,\gamma_\alpha}=\int_{\mathbb{K}}|\mathcal{F}_\alpha^{-1}(\phi)(s,y)|^2(1-\Phi_k(s,y))^2d\nu_\alpha(s,y).$$

On the other hand from (3), we get

$$\lim_{k\longrightarrow+\infty}\Phi_k(s,y)=1,$$

and

$$\forall(s,y)\in\mathbb{K},\ |\mathcal{F}_\alpha^{-1}(\phi)(s,y)|^2(1-\Phi_k(s,y))^2\leqslant|\mathcal{F}_\alpha^{-1}(\phi)(s,y)|^2.$$

Then, the expression (11) follows from the dominated convergence theorem. □

4. THE EXTREMAL FUNCTION ASSOCIATED WITH THE LAGUERRE-GABOR TRANSFORM

In this section, building on the ideas of Saitoh [20, 22] and by utilizing the theory of the Fourier transform $\mathcal{F}_\alpha$, we give the important estimates for the extremal function related to the Laguerre-Gabor transform.

In the next, we will use the integral $\int_{\mathbb{K}}\frac{d\nu_\alpha(r,x)}{(1+(r^4+x^2)^{\frac12})^\delta}$. This integral is finite if and only if $\delta>\alpha+2$.

Set $r=\rho\cos^{\frac12}\theta$ and $x=\rho^2\sin\theta$, we get

$$\begin{aligned}\int_{\mathbb{K}}\frac{d\nu_\alpha(r,x)}{(1+(r^4+x^2)^{\frac12})^\delta}&=\frac{1}{\pi\Gamma(\alpha+1)}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\cos^\alpha\theta\Big(\int_0^{+\infty}\frac{\rho^{2\alpha+3}d\rho}{(1+\rho^2)^\delta}\Big)d\theta\\&=\frac{(\alpha+1)\Gamma(\delta-\alpha-2)\Gamma(\frac{\alpha+1}{2})}{2\sqrt{\pi}\Gamma(\delta)\Gamma(\frac{\alpha+2}{2})}.\end{aligned}$$

In the following we suppose that $\delta > \alpha + 2$. We denote by $\mathcal{H}_\delta$ the Sobolev type space of functions $\phi \in L^2(\hat{\mathbb{K}})$ such that $(1+(r^4+x^2)^{\frac{1}{2}})^{\frac{\delta}{2}}\mathcal{F}_\alpha^{-1}(\phi) \in L^2(\mathbb{K})$. The space $\mathcal{H}_\delta$ provided with inner product

$$\langle \phi, \psi\rangle_\delta = \int_{\mathbb{K}} (1 + (r^4 + x^2)^{\frac{1}{2}})^\delta \mathcal{F}_\alpha^{-1}(\phi)(r, x)\overline{\mathcal{F}_\alpha^{-1}(\psi)(r, x)}d\nu_\alpha(r, x),$$

and the norm $\|\phi\|_\delta = \sqrt{\langle \phi, \phi\rangle_\delta}$.

Proposition 12. *Let $\delta > \alpha + 2$. Then the function $\mathcal{K}_\delta$ defined by*

$$\mathcal{K}_\delta(\mu, m, s, y) = \int_{\mathbb{K}} \frac{\varphi_{(s,y)}(r,x)\varphi_{(-\mu,m)}(r,x)}{(1+(r^4+x^2)^{\frac{1}{2}})^\delta} d\nu_\alpha(r, x), \tag{14}$$

is a reproducing kernel of the Hilbert space $(\mathcal{H}_\delta, \langle , \rangle_\delta)$. That is

(i) *For every $(s, y) \in \hat{\mathbb{K}}$, the function $(\mu, m) \longmapsto \mathcal{K}_\delta(\mu, m, s, y)$ belongs to $\mathcal{H}_\delta$.*

(ii) *For every $\phi \in \mathcal{H}_\delta$, and $(s, y) \in \hat{\mathbb{K}}$, we have the reproducing property,*

$$\langle \phi, \mathcal{K}_\delta(., ., s, y)\rangle_\delta = \phi(s, y).$$

Proof. For $\delta > \alpha+2$, the function $(r, x) \longmapsto \frac{1}{(1+(r^4+x^2)^{\frac{1}{2}})^{\frac{\delta}{2}}}$ belongs to $L^2(\mathbb{K})$. Then for $\phi \in \mathcal{H}_\delta$, $\mathcal{F}_\alpha^{-1}(\phi) \in L^1(\mathbb{K}) \cap L^2(\mathbb{K})$.

In view of (1), we deduce that the function $(r, x) \longmapsto \frac{\varphi_{(s,y)}(r,x)}{(1+(r^4+x^2)^{\frac{1}{2}})^\delta}$ belongs to $L^1(\mathbb{K}) \cap L^2(\mathbb{K})$. Consequently, the kernel $\mathcal{K}_\delta(., ., ., .)$ is well defined and we have

$$\mathcal{K}_\delta(\mu, m, s, y) = \mathcal{F}_\alpha((1 + (r^4 + x^2)^{\frac{1}{2}})^{-\delta}\varphi_{(s,y)}(r, x))(\mu, m), \quad (\mu, m) \in \hat{\mathbb{K}}.$$

By Theorem 2, it follows that the function $\mathcal{K}_\delta(., ., s, y)$, belongs to $L^2(\hat{\mathbb{K}})$ and we have

$$\mathcal{F}_\alpha^{-1}(\mathcal{K}_\delta(., ., s, y))(r, x) = (1 + (r^4 + x^2)^{\frac{1}{2}})^{-\delta}\varphi_{(s,y)}(r, x), \quad (r, x) \in \mathbb{K}. \tag{15}$$

Invoking (1), we obtain

$$\begin{aligned}\|\mathcal{K}_\delta(., ., s, y)\|_\delta^2 &= \int_{\mathbb{K}} (1 + (r^4 + x^2)^{\frac{1}{2}})^{-2\delta}|\varphi_{(s,y)}(r, x)|^2 d\nu_\alpha(r, x)\\ &\leqslant \int_{\mathbb{K}} (1 + (r^4 + x^2)^{\frac{1}{2}})^{-2\delta}d\nu_\alpha(r, x) < \infty.\end{aligned}$$

This proves that for all $(s, y) \in \hat{\mathbb{K}}$, $\mathcal{K}_\delta(., ., s, y) \in \mathcal{H}_\delta$.

(ii) Let $\phi \in \mathcal{H}_\delta$. By (15), we obtain

$$\langle \phi, \mathcal{K}_\delta(., ., s, y)\rangle_\delta = \int_{\mathbb{K}} \mathcal{F}_\alpha^{-1}(\phi)(r, x)\varphi_{(-s,y)}(r, x)d\nu_\alpha(r, x) = \phi(s, y).$$

This completes the proof of the proposition. □

Proposition 13. *Let $\delta > \alpha + 2$ and $\psi \in L^2(\hat{\mathbb{K}})$. The mapping $\mathcal{G}_\psi$ is a bounded linear operator from $\mathcal{H}_\delta$ into $L^2(\hat{\mathbb{K}} \times \mathbb{K})$. Moreover, for all $\phi \in \mathcal{H}_\delta$,*

$$\|\mathcal{G}_\psi(\phi)\|_{2,\gamma_\alpha\otimes\nu_\alpha} \leqslant \|\psi\|_{2,\gamma_\alpha}\|\phi\|_\delta.$$

Proof. From Theorem 9, the mapping $\mathcal{G}_\psi(\phi)$ belongs to $L^2(\hat{\mathbb{K}}\times\mathbb{K})$, and

$$\|\mathcal{G}_\psi(\phi)\|_{2,\gamma_\alpha\otimes\nu_\alpha} = \|\phi\|_{2,\gamma_\alpha}\|\psi\|_{2,\gamma_\alpha}.$$

Moreover, for all $\phi\in\mathcal{H}_\delta$ and from Theorem 2, we have
$\|\phi\|_\delta^2 \geqslant \int_{\mathbb{K}} |\mathcal{F}_\alpha^{-1}(\phi)(r,x)|^2 d\nu_\alpha(r,x) = \|\phi\|_{2,\gamma_\alpha}^2$, which gives the result. $\square$

Let $\sigma>0$. We denote by $\langle .,.\rangle_{\delta,\sigma}$ the inner product defined on the space $\mathcal{H}_\delta$ by

$$\langle \phi,h\rangle_{\delta,\sigma} = \sigma\langle\phi,h\rangle_\delta + \langle\mathcal{G}_\psi(\phi),\mathcal{G}_\psi(h)\rangle_{\gamma_\alpha\otimes\nu_\alpha},$$

and $\mathcal{H}_{\delta,\sigma}$ the space $(\mathcal{H}_\delta,\langle .,.\rangle_{\delta,\sigma})$ which is a Hilbert space.

THEOREM 14. *Let $\psi\in L^2(\hat{\mathbb{K}})$ and let $\sigma>0$. Then for $\delta>\alpha+2$, the Hilbert space $\mathcal{H}_{\delta,\sigma}$ has the following reproducing Kernel*

$$\mathcal{K}_{\delta,\sigma}(\mu,m,s,y) = \int_{\mathbb{K}} \frac{\varphi_{(-\mu,m)}(r,x)\varphi_{(s,y)}(r,x)}{\sigma(1+(r^4+x^2)^{\frac12})^\delta+\|\psi\|_{2,\gamma_\alpha}^2} d\nu_\alpha(r,x), \tag{16}$$

that is

(i) *For every $(s,y)\in\hat{\mathbb{K}}$, the function $\mathcal{K}_{\delta,\sigma}(.,.,s,y)$ belongs to $\mathcal{H}_{\delta,\sigma}$.*

(ii) *For every $\phi\in\mathcal{H}_{\delta,\sigma}$, and $(s,y)\in\hat{\mathbb{K}}$, we have the reproducing property,*

$$\langle\phi,\mathcal{K}_{\delta,\sigma}(.,.,s,y)\rangle_{\delta,\sigma} = \phi(s,y).$$

Proof. In view of (1), we have

$$\Big|\frac{\varphi_{(s,y)}(r,x)}{\sigma(1+(r^4+x^2)^{\frac12})^\delta+\|\psi\|_{2,\gamma_\alpha}^2}\Big| \leqslant \frac{1}{\sigma(1+(r^4+x^2)^{\frac12})^\delta}.$$

Since $\delta>\alpha+2$, then for all $(s,y)\in\hat{\mathbb{K}}$, the function
$(r,x)\longmapsto \dfrac{\varphi_{(s,y)}(r,x)}{\sigma(1+(r^4+x^2)^{\frac12})^\delta+\|\psi\|_{2,\gamma_\alpha}^2}$ belongs to $L^1(\mathbb{K})\cap L^2(\mathbb{K})$.

We conclude that the function $\mathcal{K}_{\delta,\sigma}(.,.,.,.)$ is well defined and

$$\mathcal{K}_{\delta,\sigma}(\mu,m,s,y) = \mathcal{F}_\alpha\Big(\frac{\varphi_{(s,y)}(r,x)}{\sigma(1+(r^4+x^2)^{\frac12})^\delta+\|\psi\|_{2,\gamma_\alpha}^2}\Big)(\mu,m), \quad (\mu,m)\in\hat{\mathbb{K}}. \tag{17}$$

By Theorem 2, it follows that the function $\mathcal{K}_{\delta,\sigma}(.,.,s,y)$, belongs to $L^2(\hat{\mathbb{K}})$ and we have

$$\begin{aligned} 2\Big|(1+(r^4+x^2)^{\frac12})^{\frac{\delta}{2}}\mathcal{F}_\alpha^{-1}(\mathcal{K}_{\delta,\sigma}(.,.,s,y))(r,x)\Big| &= \\ = \Big|(1+(r^4+x^2)^{\frac12})^{\frac{\delta}{2}}\frac{\varphi_{(s,y)}(r,x)}{\sigma(1+(r^4+x^2)^{\frac12})^\delta+\|\psi\|_{2,\gamma_\alpha}^2}\Big| \\ \leqslant \frac{1}{\sigma(1+(r^4+x^2)^{\frac12})^{\frac{\delta}{2}}}. \end{aligned}$$

This shows that for every $(s,y)\in\hat{\mathbb{K}}$, the function $\mathcal{K}_{\delta,\sigma}(.,.,s,y)$ belongs to $\mathcal{H}_{\delta,\sigma}$.

(ii) Let $\phi \in \mathcal{H}_{\delta,\sigma}$. By (17), we have

(18)

$$
\begin{aligned}
&2\langle \phi, \mathcal{K}_{\delta,\sigma}(.,.,s,y)\rangle_{\delta} = \\
&= \int_{\mathbb{K}} (1+(r^4+x^2)^{\frac{1}{2}})^{\delta} \mathcal{F}_{\alpha}^{-1}(\phi)(r,x) \times \overline{\mathcal{F}_{\alpha}^{-1}(\mathcal{K}_{\delta,\sigma}(.,.,s,y))(r,x)} d\nu_{\alpha}(r,x) \\
&= \int_{\mathbb{K}} (1+(r^4+x^2)^{\frac{1}{2}})^{\delta} \mathcal{F}_{\alpha}^{-1}(\phi)(r,x) \times \frac{\varphi_{(-s,y)}(r,x)}{\sigma(1+(r^4+x^2)^{\frac{1}{2}})^{\delta}+\|\psi\|^2_{2,\gamma_\alpha}} d\nu_{\alpha}(r,x).
\end{aligned}
$$

Now, using (7), (17) and the fact that $\mathcal{F}_{\alpha}^{-1}(\psi_{(r,x)})(s,y) = \sqrt{\mathcal{T}^{(\alpha)}_{(r,x)}|\mathcal{F}_{\alpha}^{-1}(\psi)|^2(s,y)}$, we obtain

$$
\begin{aligned}
&2\mathcal{G}_{\psi}(\mathcal{K}_{\delta,\sigma}(.,.,s,y))(\mu,m,u,v) = \\
&= \mathcal{K}_{\delta,\sigma}(.,.,s,y) * \psi_{(u,v)}(\mu,m) = \mathcal{F}_{\alpha}\Big(\mathcal{F}_{\alpha}^{-1}(\mathcal{K}_{\delta,\sigma}(.,.,s,y))\mathcal{F}_{\alpha}^{-1}(\psi_{(u,v)})\Big)(\mu,m)
\end{aligned}
$$

(19)

$$
= \mathcal{F}_{\alpha}\left(\frac{\varphi_{(s,y)}(r,x)}{\sigma(1+(r^4+x^2)^{\frac{1}{2}})^{\delta}+\|\psi\|^2_{2,\gamma_\alpha}} \sqrt{\mathcal{T}^{(\alpha)}_{(u,v)}|\mathcal{F}_{\alpha}^{-1}(\psi)|^2(r,x)}\right)(\mu,m),
$$

and

$$
\mathcal{G}_{\psi}(\phi)(\mu,m,u,v) = \mathcal{F}_{\alpha}\Big(\mathcal{F}_{\alpha}^{-1}(\phi)(r,x)\sqrt{\mathcal{T}^{(\alpha)}_{(u,v)}|\mathcal{F}_{\alpha}^{-1}(\psi)|^2(r,x)}\Big)(\mu,m). \tag{20}
$$

Now, by (19), (20), (3), (6) and Theorem 2, we get

$$
\begin{aligned}
&2\langle \mathcal{G}_{\psi}(\phi), \mathcal{G}_{\psi}(\mathcal{K}_{\delta,\sigma}(.,.,s,y))\rangle_{\gamma_\alpha\otimes\nu_\alpha} = \\
&= \int_{\mathbb{K}}\int_{\hat{\mathbb{K}}} \mathcal{G}_{\psi}(\phi)(\mu,m,u,v)\overline{\mathcal{G}_{\psi}(\mathcal{K}_{\delta,\sigma}(.,.,s,y))(\mu,m,u,v)} d\gamma_{\alpha}(\mu,m) d\nu_{\alpha}(u,v) \\
&= \int_{\mathbb{K}}\int_{\mathbb{K}} \mathcal{F}_{\alpha}^{-1}(\phi)(r,x) \frac{\varphi_{(-s,y)}(r,x)}{\sigma(1+(r^4+x^2)^{\frac{1}{2}})^{\delta}+\|\psi\|^2_{2,\gamma_\alpha}} \\
&\quad \times \mathcal{T}^{(\alpha)}_{(u,v)}|\mathcal{F}_{\alpha}^{-1}(\psi)|^2(r,x) d\nu_{\alpha}(r,x) d\nu_{\alpha}(u,v)
\end{aligned}
$$

(21)

$$
= \int_{\mathbb{K}} \|\psi\|^2_{2,\gamma_\alpha} \mathcal{F}_{\alpha}^{-1}(\phi)(r,x) \frac{\varphi_{(-s,y)}(r,x)}{\sigma(1+(r^4+x^2)^{\frac{1}{2}})^{\delta}+\|\psi\|^2_{2,\gamma_\alpha}} d\nu_{\alpha}(r,x).
$$

In view of (18) and (21), we obtain

$$
\begin{aligned}
2\langle \phi, \mathcal{K}_{\delta,\sigma}(.,.,s,y)\rangle_{\delta,\sigma} &= \sigma\langle \phi, \mathcal{K}_{\delta,\sigma}(.,.,s,y)\rangle_{\delta} + \langle \mathcal{G}_{\psi}(\phi), \mathcal{G}_{\psi}(\mathcal{K}_{\delta,\sigma}(.,.,s,y))\rangle_{\gamma_\alpha\otimes\nu_\alpha} \\
&= \int_{\mathbb{K}} \mathcal{F}_{\alpha}^{-1}(\phi)(r,x)\varphi_{(-s,y)}(r,x) d\nu_{\alpha}(r,x) = \phi(s,y).
\end{aligned}
$$

This completes the proof of the theorem. □

THEOREM 15. *Let $\delta > \alpha + 2$ and let $\psi \in L^2(\hat{\mathbb{K}})$. Then for every $g \in L^2(\hat{\mathbb{K}} \times \mathbb{K})$ and for every $\sigma > 0$, there exists a unique function $\phi^*_{\sigma,g}$, where the* *infimum*

$$\inf_{\phi\in\mathcal{H}_\delta}\Big\{\sigma\|\phi\|_\delta^2+\|g-\mathcal{G}_\psi(\phi)\|^2_{2,\gamma_\alpha\otimes\nu_\alpha}\Big\}, \tag{22}$$

is attained. Moreover the extremal function $\phi^*_{\sigma,g}$ *is given by*

$$2\phi^*_{\sigma,g}(s,y)= \tag{23}$$
$$=\int_{\mathbb{K}}\int_{\mathbb{K}}\frac{\varphi_{(-s,y)}(r,x)\mathcal{F}_\alpha^{-1}(g(.,.,u,v))(r,x)\sqrt{\mathcal{T}^{(\alpha)}_{(u,v)}|\mathcal{F}_\alpha^{-1}(\psi)|^2(r,x)}}{\sigma(1+(r^4+x^2)^{\frac12})^\delta+\|\psi\|^2_{2,\gamma_\alpha}}\times d\nu_\alpha(r,x)d\nu_\alpha(u,v).$$

Proof. The existence and unicity of the extremal function $\phi^*_{\sigma,g}$ satisfying relation (22) is given by [12, 14, 21]. On the other hand from Proposition 13 and Theorem 14, we have

$$\phi^*_{\sigma,g}(s,y)=\langle g,\mathcal{G}_\psi(\mathcal{K}_{\delta,\sigma}(.,.,s,y))\rangle_{\gamma\alpha\otimes\nu_\alpha}. \tag{24}$$

In view of (19), we have

$$2\mathcal{G}_\psi(\mathcal{K}_{\delta,\sigma}(.,.,s,y))(\mu,m,u,v)=$$
$$=\int_{\mathbb{K}}\frac{\varphi_{(-\mu,m)}(r,x)\varphi_{(s,y)}(r,x)}{\sigma(1+(r^4+x^2)^{\frac12})^\delta+\|\psi\|^2_{2,\gamma_\alpha}}\sqrt{\mathcal{T}^{(\alpha)}_{(u,v)}|\mathcal{F}_\alpha^{-1}(\psi)|^2(r,x)d\nu_\alpha(r,x)}.$$

Therefore,

$$2\phi^*_{\sigma,g}(s,y)=\int_{\mathbb{K}}\int_{\hat{\mathbb{K}}}\int_{\mathbb{K}}g(\mu,m,u,v)\frac{\varphi_{(\mu,m)}(r,x)\varphi_{(-s,y)}(r,x)}{\sigma(1+(r^4+x^2)^{\frac12})^\delta+\|\psi\|^2_{2,\gamma_\alpha}}$$
$$\times\sqrt{\mathcal{T}^{(\alpha)}_{(u,v)}|\mathcal{F}_\alpha^{-1}(\psi)|^2(r,x)}d\nu_\alpha(r,x)d\nu_\alpha(u,v)d\gamma_\alpha(\mu,m).$$

Hence, by Fubibi's theorem, we get the desired result. □

Lemma 16. *Let* $\delta>\alpha+2$ *and let* $\psi\in L^2(\hat{\mathbb{K}})$*. Then for all* $g\in L^2(\hat{\mathbb{K}}\times\mathbb{K})$ *and for* $\sigma>0$*, we have*

(1) $\forall(s,y)\in\hat{\mathbb{K}},\ |\phi^*_{\sigma,g}(s,y)|\leqslant\frac{C_{\alpha,\delta}}{2\sqrt{\sigma}}\|g\|_{2,\gamma_\alpha\otimes\nu_\alpha}$.

(2) $\|\phi^*_{\sigma,g}\|_\delta\leqslant\frac{1}{2\sigma}\|g\|_{2,\gamma_\alpha\otimes\nu_\alpha}$.

Proof. (1) From Theorem 9 and (24), we have

$$\begin{aligned}2|\phi^*_{\sigma,g}(s,y)|&\leqslant\|g\|_{2,\gamma_\alpha\otimes\nu_\alpha}\|\mathcal{G}_\psi(\mathcal{K}_{\delta,\sigma}(.,.,s,y))\|_{2,\gamma_\alpha\otimes\nu_\alpha}\\&\leqslant\|g\|_{2,\gamma_\alpha\otimes\nu_\alpha}\|\psi\|_{2,\gamma_\alpha}\|\mathcal{K}_{\delta,\sigma}(.,.,s,y)\|_{2,\gamma_\alpha}.\end{aligned}$$

Again, according to Theorem 2 and (17), we get

$$\begin{aligned}2|\phi^*_{\sigma,g}(s,y)|&\leqslant\|g\|_{2,\gamma_\alpha\otimes\nu_\alpha}\|\psi\|_{2,\gamma_\alpha}\|\mathcal{F}_\alpha^{-1}(\mathcal{K}_{\delta,\sigma}(.,.,s,y))\|_{2,\nu_\alpha}\\&\leqslant\|g\|_{2,\gamma_\alpha\otimes\nu_\alpha}\|\psi\|_{2,\gamma_\alpha}\left(\int_{\mathbb{K}}\frac{d\nu_\alpha(r,x)}{(\sigma(1+(r^4+x^2)^{\frac12})^\delta+\|\psi\|^2_{2,\gamma_\alpha})^2}\right)^{\frac12}.\end{aligned}$$

Then, the result follows from the fact

$$\Big(\sigma(1+(r^4+x^2)^{\frac{1}{2}})^\delta+\|\psi\|^2_{2,\gamma_\alpha}\Big)^2 \geqslant 4\sigma(1+(r^4+x^2)^{\frac{1}{2}})^\delta\|\psi\|^2_{2,\gamma_\alpha}.$$

(2) The function

$(r,x)\longmapsto \int_{\mathbb{K}} \frac{\mathcal{F}_\alpha^{-1}(g(.,.,u,v))(r,x)\sqrt{\mathcal{T}^{(\alpha)}_{(u,v)}|\mathcal{F}_\alpha^{-1}(\psi)|^2(r,x)}}{\sigma(1+(r^4+x^2)^{\frac{1}{2}})^\delta+\|\psi\|^2_{2,\gamma_\alpha}}d\nu_\alpha(u,v)$ belongs to $L^1(\mathbb{K})\cap L^2(\mathbb{K})$. Then, by (23), we deduce that the function $\phi^*_{\sigma,g}\in L^2(\hat{\mathbb{K}})$ and

$$\mathcal{F}_\alpha^{-1}(\phi^*_{\sigma,g})(r,x)=\int_{\mathbb{K}} \frac{\mathcal{F}_\alpha^{-1}(g(.,.,u,v))(r,x)\sqrt{\mathcal{T}^{(\alpha)}_{(u,v)}|\mathcal{F}_\alpha^{-1}(\psi)|^2(r,x)}}{\sigma(1+(r^4+x^2)^{\frac{1}{2}})^\delta+\|\psi\|^2_{2,\gamma_\alpha}}d\nu_\alpha(u,v).$$

Applying Hölder's inequality, (3) and Theorem 2, the last expression becomes,

$$|\mathcal{F}_\alpha^{-1}(\phi^*_{\sigma,g})(r,x)|^2=\|\psi\|^2_{2,\gamma_\alpha}\int_{\mathbb{K}} \frac{|\mathcal{F}_\alpha^{-1}(g(.,.,u,v))(r,x)|^2}{(\sigma(1+(r^4+x^2)^{\frac{1}{2}})^\delta+\|\psi\|^2_{2,\gamma_\alpha})^2}d\nu_\alpha(u,v).$$

Hence,

$$\begin{aligned}2\|\phi^*_{\sigma,g}\|^2_\delta &\leqslant \int_{\mathbb{K}} \frac{\|\psi\|^2_{2,\gamma_\alpha}(1+(r^4+x^2)^{\frac{1}{2}})^\delta}{(\sigma(1+(r^4+x^2)^{\frac{1}{2}})^\delta+\|\psi\|^2_{2,\gamma_\alpha})^2}\times\\ &\quad\times\Big(\int_{\mathbb{K}}|\mathcal{F}_\alpha^{-1}(g(.,.,u,v))(r,x)|^2d\nu_\alpha(u,v)\Big)d\nu_\alpha(r,x)\\ &\leqslant \tfrac{1}{4\sigma}\int_{\mathbb{K}}\int_{\mathbb{K}}|\mathcal{F}_\alpha^{-1}(g(.,.,u,v))(r,x)|^2d\nu_\alpha(u,v)d\nu_\alpha(r,x),\end{aligned}$$

and Theorem 2 completes the proof. □

Theorem 17. *Let $\delta>\alpha+2$ and let $\psi\in L^2(\hat{\mathbb{K}})$. Then for every $\phi\in\mathcal{H}_\delta$, the function $\phi^*_{\sigma,\mathcal{G}_\psi(\phi)}$ belongs to $\mathcal{H}_\delta$ and verifies*

$$\lim_{\sigma\longrightarrow 0^+}\|\phi^*_{\sigma,\mathcal{G}_\psi(\phi)}-\phi\|_\delta=0.$$

Moreover, the family of functions $(\phi^*_{\sigma,\mathcal{G}_\psi(\phi)})_{\sigma>0}$ converges uniformly to ϕ as $\sigma\longrightarrow 0^+$.

Proof. Let $\phi\in\mathcal{H}_\delta$. By (12) and (23), we have

$$\phi^*_{\sigma,\mathcal{G}_\psi(\phi)}(s,y)=\|\psi\|^2_{2,\gamma_\alpha}\int_{\mathbb{K}}\frac{\varphi_{(-s,y)}(r,x)\mathcal{F}_\alpha^{-1}(\phi)(r,x)}{\sigma(1+(r^4+x^2)^{\frac{1}{2}})^\delta+\|\psi\|^2_{2,\gamma_\alpha}}d\nu_\alpha(r,x),\ \ (s,y)\in\hat{\mathbb{K}}.$$

On the other hand, $\mathcal{F}_\alpha^{-1}(\phi)\in L^1(\mathbb{K})\cap L^2(\mathbb{K})$, then from the last expression and Eq. (5), we get

(25)

$$\phi^*_{\sigma,\mathcal{G}_\psi(\phi)}(s,y)-\phi(s,y)=-\sigma\int_{\mathbb{K}}\frac{\varphi_{(-s,y)}(r,x)(1+(r^2+x^2)^{\frac{1}{2}})^\delta\mathcal{F}_\alpha^{-1}(\phi)(r,x)}{\sigma(1+(r^4+x^2)^{\frac{1}{2}})^\delta+\|\psi\|^2_{2,\gamma_\alpha}}d\nu_\alpha(r,x).$$

Hence,

$$\mathcal{F}_\alpha^{-1}(\phi^*_{\sigma,\mathcal{G}_\psi(\phi)} - \phi)(r,x) = -\sigma \frac{(1+(r^4+x^2)^{\frac{1}{2}})^\delta \mathcal{F}_\alpha^{-1}(\phi)(r,x)}{\sigma(1+(r^4+x^2)^{\frac{1}{2}})^\delta + \|\psi\|^2_{2,\gamma_\alpha}}.$$

It arives that

$$\|\phi^*_{\sigma,\mathcal{G}_\psi(\phi)} - \phi\|_\delta^2 = \int_{\mathbb{K}} \sigma^2 \frac{(1+(r^4+x^2)^{\frac{1}{2}})^{3\delta} |\mathcal{F}_\alpha^{-1}(\phi)(r,x)|^2}{\left(\sigma(1+(r^4+x^2)^{\frac{1}{2}})^\delta + \|\psi\|^2_{2,\gamma_\alpha}\right)^2} d\nu_\alpha(r,x).$$

Now, using the fact $\sigma^2 \frac{(1+(r^4+x^2)^{\frac{1}{2}})^{3\delta} |\mathcal{F}_\alpha^{-1}(\phi)(r,x)|^2}{\left(\sigma(1+(r^4+x^2)^{\frac{1}{2}})^\delta + \|\psi\|^2_{2,\gamma_\alpha}\right)^2} \leqslant (1+(r^4+x^2)^{\frac{1}{2}})^\delta |\mathcal{F}_\alpha^{-1}(\phi)(r,x)|^2$, dominated convergence theorem and the fact that $\phi \in \mathcal{H}_\delta$, we deduce that

$$\lim_{\sigma \longrightarrow 0^+} \|\phi^*_{\sigma,\mathcal{G}_\psi(\phi)} - \phi\|_\delta = 0.$$

On the other hand, in view of (25), we have

$$|\phi^*_{\sigma,\mathcal{G}_\psi(\phi)}(s,y) - \phi(s,y)| \leqslant \sigma \int_{\mathbb{K}} \frac{(1+(r^4+x^2)^{\frac{1}{2}})^\delta |\mathcal{F}_\alpha^{-1}(\phi)(r,x)|}{\sigma(1+(r^4+x^2)^{\frac{1}{2}})^\delta + \|\psi\|^2_{2,\gamma_\alpha}} d\nu_\alpha(r,x).$$

Using the fact that $\frac{(1+(r^4+x^2)^{\frac{1}{2}})^\delta |\mathcal{F}_\alpha^{-1}(\phi)(r,x)|}{\sigma(1+(r^4+x^2)^{\frac{1}{2}})^\delta + \|\psi\|^2_{2,\gamma_\alpha}} \leqslant |\mathcal{F}_\alpha^{-1}(\phi)(r,x)|$ and dominated convergence theorem, we get

$$\lim_{\sigma \longrightarrow 0^+} \sup_{(s,y)\in\hat{\mathbb{K}}} |\phi^*_{\sigma,\mathcal{G}_\psi(\phi)}(s,y) - \phi(s,y)| = 0.$$

□

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