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LAURENT OPERATOR-BASED REPRESENTATION OF DISCRETE SOLUTIONS IN THE NEWMARK SCHEME WITH NON-HOMOGENEOUS TERMS

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Abstract. This paper investigates representation results for second-order evolution equations arising in structural dynamics, discretized using the Newmark time integration scheme. More precisely, the discrete solution is expressed in terms of bi-infinite Toeplitz or Laurent operators. A spectral analysis of the associated discrete operators is discussed, and a convergence analysis is performed under relaxed regularity assumptions on the source term. Furthermore, we examine the errors introduced by some truncation strategies, including one that is commonly used in engineering practice.

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1. INTRODUCTION

It is well-known that the time integration schemes play a central role in the numerical simulation of several problems in solid mechanics. The accurate time evolution of mechanical systems subject to transient loads, vibrations, or dynamic interactions is essential in a wide range of engineering applications, including aerospace, civil infrastructure, and mechanical design. The challenge lies in developing numerical methods that are not only robust and computationally efficient but also capable of accurately capturing the dynamic response of complex structures over long time intervals.

In this paper, we focus on a fundamental problem in structural dynamics. Specifically, we consider the second-order system of differential equations given by:

$$\begin{cases} M\ddot{\mathcal{U}} + C\dot{\mathcal{U}} + K\mathcal{U} = F \\ \operatorname*{ess\,lim}_{t\to-\infty} \mathcal{U}(t) = 0 \\ \operatorname*{ess\,lim}_{t\to-\infty} \dot{\mathcal{U}}(t) = 0 \end{cases} \tag{1.1}$$

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where the vector-valued functions $t \mapsto \ddot{\mathcal{U}}(t)$, $t \mapsto \dot{\mathcal{U}}(t)$, and $t \mapsto \mathcal{U}(t)$ represent the n-dimensional acceleration, velocity, and displacement vectors, respectively, with the dot symbol denoting the time derivative. The function $t \mapsto F(t)$ represents the external force. The matrices M, K, and C are symmetric and positive definite. In this setting, let $\omega_1^2, \ldots, \omega_n^2 > 0$ be the real, positive eigenvalues (possibly repeated) of $M^{-1}K$, arranged in non-increasing order. It is known that the eigenvectors of $M^{-1}K$ are orthogonal with respect to the Hermitian product induced by M, i.e., $\langle \bullet, M \bullet \rangle$. Concerning the matrix C, we assume that the positive eigenvalues $M^{-1}C$ are written as $2\xi_1\omega_1, \ldots, 2\xi_n\omega_n$, where $\xi_1, \xi_2, \ldots, \xi_n > 0$.

Furthermore, the following assumptions are implicitly considered in the subsequent sections:

(H1):

(1) The function $t \mapsto F(t)$ is Lebesgue-measurable on $\mathbb{R}$ and satisfies $\operatorname*{ess\,lim}_{t \to -\infty} F(t) = 0$.

(2) For every integer $1 \le i \le n$, the eigenspaces associated with the eigenvalues ω_i^2 and $2\xi_i\omega_i$ coincide. In other words, there exist orthogonal projections $\mathcal{P}_i$, $1 \le i \le n$, with respect to the product $\langle \bullet, M \bullet \rangle$ such that:

$$M = \sum_{i=1}^{n} M\mathcal{P}_i, \quad K = \sum_{i=1}^{n} \omega_i^2 M\mathcal{P}_i, \quad \text{and} \quad C = \sum_{i=1}^{n} 2\xi_i\omega_i M\mathcal{P}_i. \tag{1.2}$$

Over the decades, numerous time integration methods have been proposed to solve numerically (1.1), each developped to specific modeling goals, such as energy conservation, high-frequency damping, unconditional stability, or second-order accuracy. Classical schemes such as the Newmark family [15], the Wilson-θ method [1], the Hilber–Hughes–Taylor (HHT-α) method [13] and the generalized-α method [7] have been widely adopted in industrial codes due to their simplicity and effectivenessw. Furthermore, these time integration schemes have also served as building blocks for multi-time-step methods, which allow different parts of a structure or different physical subdomains to be integrated using different time resolutions [8, 12, 16, 19, 20, 4, 5].

Although it is one of the earliest time integration schemes developed for structural analysis, the *Newmark family of methods* remains among the most widely used techniques for solving second-order differential systems in structural dynamics, such as the system described in (1.1). Introduced by Newmark in the 1950s [15], this class of methods encompasses both explicit and implicit variants, with two parameters that offer control over numerical dissipation and stability. Most accuracy analyses in the literature have been conducted under the assumption of smooth (at leat two differentiable) or vanishing nonhomogeneous terms [6, 10, 2]. The method's adaptability, stability properties, and reliable performance continue to justify its widespread adoption in modern commercial solvers.

This paper aims to provide new representation results for the discrete solution in terms of Laurent operators (or bi-infinite Toeplitz operators). These representations allow us to derive some spectral properties of the associated operators, along with convergence results under relaxed smoothness assumptions on the source term F in (1.1). Furthermore, we analyze the errors introduced by two truncation procedures, one of which is commonly employed in engineering applications.

The rest of the paper is organized as follows. Section 2 reviews the Newmark scheme in the form of a block matrix formulation. Section 3 presents the representation of the discrete solution using Laurent operators and discusses related spectral properties, convergence results, and some truncation errors.

2. REVIEW ON THE MATRIX FORMULATION OF THE DISCRETIZED EQUATION

In this section, we present the matrix formulation of the discretized equation associated with (1.1), together with the main assumptions on the Newmark parameters and classical results that play a key role in the analysis developed in the subsequent sections.

To begin, we choose a time step h, such that the approximate solution of (1.1) is described at discrete time instances $t_l = lh$, where l is an integer, i.e., $l \in \mathbb{Z}$. Furthermore, let $\ddot{\mathcal{U}}_l$, $\dot{\mathcal{U}}_l$, $\mathcal{U}_l$, and F_l denote the acceleration, velocity, displacement, and external force, respectively, at time $t_l = lh$.

In this paper, F_l is considered an approximation of $F(lh)$ and does not necessarily coincide with its exact value. For a fixed $l \in \mathbb{Z}$, the discretized equation at the time t_l writes:

$$M\ddot{\mathcal{U}}_l + C\dot{\mathcal{U}}_l + K\mathcal{U}_l = F_l \tag{2.1}$$

The approximated quantities $\mathcal{U}_l$ and $\dot{\mathcal{U}}_l$, computed using the Newmark scheme with parameters γ and β, are given by:

$$\begin{cases} \mathcal{U}_l = \mathcal{U}_{l-1} + h\dot{\mathcal{U}}_{l-1} + h^2\left(\frac{1}{2} - \beta\right)\ddot{\mathcal{U}}_{l-1} + h^2\beta\ddot{\mathcal{U}}_l, \\ \dot{\mathcal{U}}_l = \dot{\mathcal{U}}_{l-1} + h(1-\gamma)\ddot{\mathcal{U}}_{l-1} + h\gamma\ddot{\mathcal{U}}_l. \end{cases} \tag{2.2}$$

Furthermore, the discrete sequences $(\mathcal{U}_l)_{l\in\mathbb{Z}}$ and $(\dot{\mathcal{U}}_l)_{l\in\mathbb{Z}}$ should satisfy the decay condition:

$$\lim_{l\to-\infty} \mathcal{U}_l = 0, \quad \lim_{l\to-\infty} \dot{\mathcal{U}}_l = 0. \tag{2.3}$$

It is more convenient to rewrite the previous time discretized equations in a block matrix representation between two times t_k and t_j (with $-\infty < k < j$) as :

$$
\begin{bmatrix}
\mathbb{M} & & & & \\
\mathbb{N} & \mathbb{M} & & & \\
 & \cdot & \cdot & & \\
 & & \cdot & \cdot & \\
 & & & \mathbb{N} & \mathbb{M}
\end{bmatrix}
\begin{bmatrix}
\mathbb{U}_k \\ \mathbb{U}_{k+1} \\ \cdot \\ \cdot \\ \mathbb{U}_j
\end{bmatrix}
=
\begin{bmatrix}
\mathbb{P}_k - \mathbb{N}\mathbb{U}_{k-1} \\ \mathbb{P}_{k+1} \\ \cdot \\ \cdot \\ \mathbb{P}_j
\end{bmatrix}
\tag{2.4}
$$

where we have:

$$
\begin{gathered}
\mathbb{M} = \begin{bmatrix} M & C & K \\ -\gamma h I_n & I_n & 0 \\ -\beta h^2 I_n & 0 & I_n \end{bmatrix}, \quad \mathbb{N} = \begin{bmatrix} 0 & 0 & 0 \\ -(1-\gamma) h I_n & -I_n & 0 \\ -(\frac{1}{2}-\beta) h^2 I_n & -h I_n & -I_n \end{bmatrix} \\
\mathbb{P}_l = \begin{bmatrix} F_l \\ 0 \\ 0 \end{bmatrix} \quad , \quad \mathbb{U}_l = \begin{bmatrix} \ddot{\mathcal{U}}_l \\ \dot{\mathcal{U}}_l \\ \mathcal{U}_l \end{bmatrix}
\end{gathered}
\tag{2.5}
$$

For every $k \leq l \leq j$. Here, the matrix I_n stands for the $n \times n$-identity matrix.

We consider the following assumption (**H2**): The Newmark parameters (γ, β, h) satisfy:

(1) $\gamma \geq \frac{1}{2}$.
(2) $h > 0$.
(3) $\tilde{M} = M + h^2(\beta - \frac{\gamma}{2})K$ is positive definite.
(4) The inequality

$$
\left(\gamma + \frac{1}{2}\right)^2 - 4\beta < \min_{1 \leq i \leq n} \left[\frac{4(1-\xi_i^2)}{\omega_i^2 h^2} + \frac{2\xi_i}{\omega_i h}(2\gamma - 1) \right]
\tag{2.6}
$$

is commonly used in practice for physical purposes (see, for instance, Section 7.2.7 in [10]). In addition, we assume $0 < \xi_i < 1$.

Throughout the remainder of this paper, assumptions (**H1**) and (**H2**) are considered to be satisfied, even if not explicitly mentioned and the time step belongs to the set for which the above assumption holds. The following results are easy to establish by arguments similar to those in [17] and [18].

Lemma 1. *Let $\mathcal{D} = \{z \in \mathbb{C} \mid \quad |z| < 1\}$. The spectrum of $\mathbb{N}\mathbb{M}^{-1}$ is a subset of $\mathcal{D}$. More precisely, the spectrum is given by:*

$$
\{0, z_1, \bar{z}_1, \cdots, z_n, \bar{z}_n\}
\tag{2.7}
$$

For every $1 \leq i \leq n$, the real and imaginary parts of z_i are given by:

(2.8)
$$\begin{cases} \Re(z_i) = \frac{1}{2}\left(\left(\gamma_i' + \frac{1}{2}\right)\frac{\Omega_i^2}{1+\beta_i'\Omega_i^2} - 2\right), \\ \Im(z_i) = -\frac{1}{2}\sqrt{\frac{4\Omega_i^2}{1+\beta_i'\Omega_i^2} - \left(\frac{\Omega_i^2}{1+\beta_i'\Omega_i^2}\right)^2\left(\gamma_i' + \frac{1}{2}\right)^2}. \end{cases}$$

where $\Re(z)$ *and* $\Im(z)$ *denote the real and imaginary parts of a complex number* z*, and* $\Omega_i = h\omega_i$*.*

Additionally, the modified parameters γ_i' *and* β_i' *are defined as:*

(2.9)
$$\gamma_i' = \gamma + 2\frac{\xi_i}{\Omega_i}, \quad \text{and} \quad \beta_i' = \beta + 2\frac{\gamma\xi_i}{\Omega_i}.$$

In this case, the magnitude of z_i *is given by:*

(2.10)
$$|z_i| = \sqrt{1 - \left(\gamma_i' - \frac{1}{2}\right)\frac{\Omega_i^2}{1+\beta_i'\Omega_i^2}}.$$

□

REMARK 2. *If we define the* ***resolvent*** $\mathcal{R}$ *of* $\mathbb{N}\mathbb{M}^{-1}$ *as*

(2.11)
$$\mathcal{R}(z) = \left(zI_{3n} - \mathbb{N}\mathbb{M}^{-1}\right)^{-1} = \mathbb{M}\left(z\mathbb{M} - \mathbb{N}\right)^{-1}$$

where $z \in \mathbb{C}$ *and* I_{3n} *is the* $3n \times 3n$*-identity matrix, it can be observed that* $\mathcal{R}(z)$ *is well-defined on* $\mathbb{C} \setminus \mathcal{D}$ *under the hypothesis of Lemma 1.*

LEMMA 3. *If* $(z\mathbb{M} - \mathbb{N})^{-1}$ *is written as* 3×3 *block matrix where each entry is a* $n \times n$*-matrix. Let* $\begin{bmatrix} X_{11}(z) & X_{21}(z) & X_{31}(z) \end{bmatrix}^\mathsf{T}$ *be the first column of the previous matrix, then:*

(2.12)
$$\begin{cases} X_{11}(z) &= \frac{(1+z)^2}{z}Q^{-1}(z) \\ X_{21}(z) &= h\frac{(1+z)(\gamma(1+z)-1)}{z}Q^{-1}(z) \\ X_{31}(z) &= h^2\frac{\beta(1+z)^2-(1+z)(\gamma+\frac{1}{2})+1}{z}Q^{-1}(z) \end{cases}$$

where:

(2.13)
$$\begin{aligned} Q(z) =& (1+z)^2 M + h^2\left(\beta(1+z)^2 - (\gamma + \frac{1}{2})(1+z) + 1\right)K \\ &+ h\left(\gamma(1+z)^2 - (1+z)\right)C \end{aligned}$$

□

REMARK 4. *Using the decomposition* (1.2), *one can verify that:*

$$Q(z) = \sum_{i=1}^{n} \Lambda_i(\Omega_i)(z - z_i)(z - \bar{z}_i) M \mathcal{P}_i, \tag{2.14}$$

where we define

$$\Lambda_i(\Omega_i) := 1 + \beta_i' \Omega_i^2. \tag{2.15}$$

□

3. LAURENT OPERATORS AND REPRESENTATION RESULTS FOR THE NEWMARK SCHEME

This section focuses on the representation of various quantities in (1.1) in terms of Laurent operators, as well as the behavior of the error when dealing with less regular nonhomogeneous terms compared to those studied in the literature. To begin, we introduce the following classical definitions:

DEFINITION 5. *Let $p \geq 1$ and k be a positive integer, let $U \subset \mathbb{R}$ be an open set, and let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. We define the following spaces along with their associated norms:*

- $\mathrm{C}_0(U, \mathbb{K}^k)$*: The space of continuous functions from U to $\mathbb{K}^k$, equipped with the supremum norm:*

$$\|f\|_{\mathrm{C}_0} = \sup_{t \in U} \|f(t)\|.$$

- $\mathrm{L}^p(U, \mathbb{K}^k)$*: The space of p-integrable functions from U to $\mathbb{K}^k$, i.e.,*

$$\mathrm{L}^p(U, \mathbb{K}^k) = \begin{cases} \left\{ f : U \to \mathbb{K}^k \;\middle|\; \int_U \|f(t)\|^p \, dt < \infty \right\}, & 1 \leq p < \infty, \\ \left\{ f : U \to \mathbb{K}^k \;\middle|\; \operatorname*{ess\,sup}_{t \in U} \|f(t)\| < \infty \right\}, & p = \infty. \end{cases}$$

The associated norm is:

$$\|f\|_{\mathrm{L}^p} = \begin{cases} \left(\int_U \|f(t)\|^p \, dt\right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \operatorname*{ess\,sup}_{t \in U} \|f(t)\|, & p = \infty. \end{cases}$$

- *For sequence spaces, we define:*

$$l_p(\mathbb{K}^k) = \begin{cases} \left\{ (X_l)_{l \in \mathbb{Z}} \subset \mathbb{K}^k \;\middle|\; \sum_{l \in \mathbb{Z}} \|X_l\|^p < \infty \right\}, & 1 \leq p < \infty, \\ \left\{ (X_l)_{l \in \mathbb{Z}} \subset \mathbb{K}^k \;\middle|\; \sup_{l \in \mathbb{Z}} \|X_l\| < \infty \right\}, & p = \infty. \end{cases}$$

The corresponding norm for $X = (X_l)_{l \in \mathbb{Z}}$ is:

$$\|X\|_{l_p} = \begin{cases} \left(\sum_{l\in\mathbb{Z}} \|X_l\|^p\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \sup_{l\in\mathbb{Z}} \|X_l\|, & p = \infty. \end{cases}$$

We also define the space $c_0^-(\mathbb{K}^k)$, consisting of all sequences in $l_\infty(\mathbb{K}^k)$ that converge to zero at $-\infty$:

$$c_0^-(\mathbb{K}^k) = \left\{ (X_l)_{l\in\mathbb{Z}} \in l_\infty(\mathbb{K}^k) \;\middle|\; \lim_{l\to-\infty} X_l = 0 \right\},$$

□

Remark 6. *In the preceding Definition 5, the norm notations are written without explicit reference to the index k or the domain U for the sake of simplicity; their precise meaning should be inferred from the context.*

Definition 7. *Let $a \in L^\infty\left((-\frac{\pi}{h}, \frac{\pi}{h}), \mathbb{C}\right)$ and $0 < p \le \infty$. The **Laurent operator** (or **bi-infinite Toeplitz operator**) $T(a)$ on $l_p(\mathbb{C}^n)$ is defined as follows: for every sequence $\tilde{X} = (\ldots, X_{-1}, X_0, X_1, \ldots) \in l_p(\mathbb{C}^n)$,*

$$(T(a)\tilde{X})_m = \sum_{l=-\infty}^{\infty} c_{m-l}(a) X_l, \quad \text{for all } m \in \mathbb{Z}, \tag{3.1}$$

where $(c_l(a))_{l\in\mathbb{Z}}$ are the Fourier coefficients of the function a, defined by:

$$c_l(a) = \frac{h}{2\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} a(t) e^{-i_c l h t}\, dt, \quad \text{for all } l \in \mathbb{Z}. \tag{3.2}$$

where i_c is the imaginary unit.

□

When $1 \le p \le 2$, it follows that the operator $T(a)$ is bounded as a consequence of the Riesz-Thorin interpolation theorem (see, for instance, Theorem 1.3.4 in [11]).

Definition 8. *Let $k > 0$ be an integer and $\tilde{X} = (X_\ell)_{\ell\in\mathbb{Z}}$ be a sequence of elements in $\mathbb{C}^k$. We define the following operators:*

(i) ***Discrete Fourier Transform:*** *The operator $\mathfrak{F}_h$ on $l_2(\mathbb{C}^k)$ is given by*

$$\mathfrak{F}_h \tilde{X}(\theta) = \sum_{\ell=-\infty}^{\infty} X_\ell e^{i_c \ell h \theta}, \quad \forall \theta \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]. \tag{3.3}$$

(ii) ***Piecewise Constant Interpolation:*** *The interpolation operator $\mathcal{I}_h$ is defined by*

$$\mathcal{I}_h \tilde{X}(t) = \sum_{\ell=-\infty}^{\infty} X_\ell \chi_{[\ell h, (\ell+1)h)}(t), \tag{3.4}$$

where $\chi_{[\ell h,(\ell+1)h)}(t)$ *is the characteristic function of the interval* $[\ell h,(\ell+1)h)$.

□

REMARK 9. *Let* $\mathbb{K}=\mathbb{R}$ *or* $\mathbb{C}$, *and* $r>1$. *It is straightforward to show that the operator* $\mathcal{I}_h$ *is continuous from* $l_r(\mathbb{K}^n)$ *into* $\mathrm{L}^r(\mathbb{R},\mathbb{K}^n)$. *Moreover, its adjoint operator* $\mathcal{I}_h^\star$ *is given by:*

$$\mathcal{I}_h^\star X=\left(\int_{lh}^{(l+1)h} X(s)\,ds\right)_{l\in\mathbb{Z}},\quad \text{for } X\in\mathrm{L}^{r^\star}(\mathbb{R},\mathbb{K}^n),$$

and we have:

$$\|\mathcal{I}_h\|_{\mathcal{L}(l_r(\mathbb{K}^n),\mathrm{L}^r(\mathbb{R},\mathbb{K}^n))}=\|\mathcal{I}_h^\star\|_{\mathcal{L}\left(\mathrm{L}^{r^\star}(\mathbb{R},\mathbb{K}^n),l_{r^\star}(\mathbb{K}^n)\right)}=h^{\frac{1}{r}}=h^{\frac{r^\star-1}{r^\star}} \tag{3.5}$$

where $\frac{1}{r}+\frac{1}{r^\star}=1$.

□

We are now in a position to state the main result concerning the representation of the discrete solution in terms of Laurent operators. This will be followed by a series of corollaries presenting some convergence results and analyzing the errors introduced by some truncation procedures.

THEOREM 10. *Assume that* $(F_\ell)_{\ell\in\mathbb{Z}}\in c_0^-(\mathbb{R}^n)$. *Let* $w\in\mathcal{D}$ *with* $\Im w\neq 0$. *Define the functions:*

(3.6)
$$\begin{cases} g_w^u(\theta)=\beta h^2+\displaystyle\sum_{\ell=1}^{\infty} h^2\frac{\Im\left[(-1)^\ell w^{\ell-1}\left(\beta(1+w)^2-(1+w)\left(\gamma+\frac{1}{2}\right)+1\right)\right]}{\Im w}\\ \qquad\cdot e^{i_c\ell h\theta},\\ g_w^v(\theta)=\gamma h+\displaystyle\sum_{\ell=1}^{\infty} h\frac{\Im\left[(-1)^\ell w^{\ell-1}(1+w)\left(\gamma(1+w)-1\right)\right]}{\Im w}e^{i_c\ell h\theta},\\ g_w^a(\theta)=1+\displaystyle\sum_{\ell=1}^{\infty}\frac{\Im\left[(-1)^\ell w^{\ell-1}(1+w)^2\right]}{\Im w}e^{i_c\ell h\theta}. \end{cases}$$

Each of these series defines an uniformly convergent function. Moreover, we have:

$$\begin{cases} \tilde{\mathcal{U}}=\sum_{i=1}^{n}\dfrac{1}{\Lambda_i(\Omega_i)}T(g_{z_i}^u)\mathcal{P}_i\widetilde{M^{-1}F},\\ \dot{\tilde{\mathcal{U}}}=\sum_{i=1}^{n}\dfrac{1}{\Lambda_i(\Omega_i)}T(g_{z_i}^v)\mathcal{P}_i\widetilde{M^{-1}F},\\ \ddot{\tilde{\mathcal{U}}}=\sum_{i=1}^{n}\dfrac{1}{\Lambda_i(\Omega_i)}T(g_{z_i}^a)\mathcal{P}_i\widetilde{M^{-1}F}, \end{cases} \tag{3.7}$$

where $\Lambda_i(\Omega_i)$ *is defined in* (2.15)*, and the notations are as follows:*

$$\tilde{\mathcal{U}} = (\mathcal{U}_\ell)_{\ell\in\mathbb{Z}}, \quad \tilde{\dot{\mathcal{U}}} = (\dot{\mathcal{U}}_\ell)_{\ell\in\mathbb{Z}}, \quad \tilde{\ddot{\mathcal{U}}} = (\ddot{\mathcal{U}}_\ell)_{\ell\in\mathbb{Z}}, \quad \widetilde{M^{-1}F} = \left(M^{-1}F_\ell\right)_{\ell\in\mathbb{Z}},$$

and, for any $1 \le i \le n$ *and any bi-infinite sequence* $\tilde{X} = (X_\ell)_{\ell\in\mathbb{Z}}$*,*

$$\mathcal{P}_i\tilde{X} := (\mathcal{P}_i X_\ell)_{\ell\in\mathbb{Z}}. \tag{3.8}$$

□

In the proof of Theorem 10, we need the following lemma:

Lemma 11. *Under the assumptions of Theorem 10, the discrete block solution* $(\mathbb{U}_m)_{m\in\mathbb{Z}}$*, defined in* (2.5)*, is given for all* $m \in \mathbb{Z}$ *by:*

$$\mathbb{U}_m = \sum_{l=-\infty}^{m} (-1)^{m-l}\,\mathbb{M}^{-1}\left(\mathbb{N}\mathbb{M}^{-1}\right)^{m-l}\mathbb{P}_l. \tag{3.9}$$

□

Proof. First, observe that the right-hand side of (3.9) is well defined. Indeed, by Lemma 2.8, the eigenvalues of the matrix $\mathbb{N}\mathbb{M}^{-1}$ lie strictly inside the unit disk, and the sequence $(F_\ell)_{\ell\in\mathbb{Z}}$ is bounded.

It is straightforward to verify that the constructed solution satisfies the recurrence relation

$$\mathbb{M}\mathbb{U}_m + \mathbb{N}\mathbb{U}_{m-1} = \mathbb{P}_m, \quad \text{for all } m \in \mathbb{Z},$$

as well as the decay condition (2.3). By the uniqueness of the discrete solution, the result follows.

□

Proof of Theorem 10. Let $\epsilon > 0$. We introduce a small perturbation by replacing the sequence $(F_\ell)_{\ell\in\mathbb{Z}}$ with the exponentially damped sequence $(F_\ell^\epsilon)_{\ell\in\mathbb{Z}}$, defined by

$$F_\ell^\epsilon = e^{-\epsilon\ell^2}F_\ell, \quad \forall \ell \in \mathbb{Z}.$$

We denote by $\mathbb{P}_\ell^\epsilon$ and $\mathbb{U}_\ell^\epsilon$ the perturbed block vectors corresponding to the discrete data and the discrete solution, respectively, as defined in (2.5):

$$\mathbb{P}_\ell^\epsilon = \begin{bmatrix} F_\ell^\epsilon \\ 0 \\ 0 \end{bmatrix}, \quad \mathbb{U}_\ell^\epsilon = \begin{bmatrix} \ddot{\mathcal{U}}_\ell^\epsilon \\ \dot{\mathcal{U}}_\ell^\epsilon \\ \mathcal{U}_\ell^\epsilon \end{bmatrix}. \tag{3.10}$$

and from (3.9), we have for every $m \in \mathbb{Z}$:

$$\mathbb{U}_m^\epsilon = \sum_{l=-\infty}^{m} (-1)^{m-l}\,\mathbb{M}^{-1}\left(\mathbb{N}\mathbb{M}^{-1}\right)^{m-l}\mathbb{P}_l^\epsilon. \tag{3.11}$$

Since the eigenvalues of the matrix $\mathbb{N}\mathbb{M}^{-1}$ lie strictly inside the unit disk, we conclude that

$$\lim_{\epsilon\to 0} \mathbb{U}^{\epsilon}_m = \mathbb{U}_m, \quad \forall m \in \mathbb{Z}.$$

Let $\tilde{\mathbb{P}}^{\epsilon} = (\mathbb{P}^{\epsilon}_l)_{l\in\mathbb{Z}}$. Using the Dunford–Taylor integral representation [14] and the definition of the resolvent in Remark 2, we can express $\mathbb{U}^{\epsilon}_m$ for every integer m as:

(3.12)

$$\begin{aligned}
\mathbb{U}^{\epsilon}_m &= \sum_{l=-\infty}^{m} (-1)^{m-l}\, \mathbb{M}^{-1} \left(\mathbb{N}\mathbb{M}^{-1}\right)^{m-l} \mathbb{P}^{\epsilon}_l \\
&= \frac{h}{2\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-i_c m h\theta} \sum_{l=-\infty}^{m} (-1)^{m-l}\, \mathbb{M}^{-1} \left(\mathbb{N}\mathbb{M}^{-1}\right)^{m-l} e^{i_c(m-l)h\theta} \mathfrak{F}_h \tilde{\mathbb{P}}^{\epsilon}(\theta)\, d\theta \\
&= \frac{h}{(2\pi)^2 i_c} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-i_c m h\theta} \left[\int_{\mathcal{C}} \left(\sum_{l=0}^{\infty} (-1)^l e^{i_c l h\theta} z^l \right) (z\mathbb{M} - \mathbb{N})^{-1}\, dz \right] \mathfrak{F}_h \tilde{\mathbb{P}}^{\epsilon}(\theta)\, d\theta
\end{aligned}$$

where $\mathcal{C}$ is a positively oriented circle of radius strictly less than one, whose interior contains all the roots z_i described in (2.8). Hence, by applying Lemma 3 and using the definition of the resolvent in Remark 2, we obtain:

$$\text{(3.13)} \quad \begin{cases}
\mathcal{U}^{\epsilon}_m = \dfrac{h}{(2\pi)^2 i_c} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-i_c m h\theta} \left[\int_{\mathcal{C}} \dfrac{X_{13}(z)}{1 + e^{i_c h\theta} z}\, dz \right] \mathfrak{F}_h \tilde{F}^{\epsilon}(\theta)\, d\theta, \\
\dot{\mathcal{U}}^{\epsilon}_m = \dfrac{h}{(2\pi)^2 i_c} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-i_c m h\theta} \left[\int_{\mathcal{C}} \dfrac{X_{12}(z)}{1 + e^{i_c h\theta} z}\, dz \right] \mathfrak{F}_h \tilde{F}^{\epsilon}(\theta)\, d\theta, \\
\ddot{\mathcal{U}}^{\epsilon}_m = \dfrac{h}{(2\pi)^2 i_c} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-i_c m h\theta} \left[\int_{\mathcal{C}} \dfrac{X_{11}(z)}{1 + e^{i_c h\theta} z}\, dz \right] \mathfrak{F}_h \tilde{F}^{\epsilon}(\theta)\, d\theta
\end{cases}$$

A direct application of the residue theorem, combined with the identity (2.14), yields the desired result for the sequence $(F^{\epsilon}_{\ell})_{\ell}$. Taking the limit as $\epsilon \to 0$, we obtain the corresponding result for $(F_{\ell})_{\ell}$, since the series $\sum_{\ell} c_{\ell}(g^r_{z_i})$ converges absolutely. Here, $c_{\ell}(g^r_{z_i})$ denotes the Fourier coefficients of the uniformly convergent functions $g^r_{z_i}$, with $r \in \{u, v, a\}$.

□

By extending the definition of the operator $\mathcal{P}_i$ to the space $l_2(\mathbb{C}^n)$ as given in (3.8), the following result follows from Theorem 10:

Corollary 12. *We have the following:*

(i) *For every $r \in \{u, v, a\}$, consider the operator*

$$\mathcal{T}_r := \sum_{i=1}^{n} \frac{1}{\Lambda_i(\Omega_i)} T(g^r_{z_i})\, \mathcal{P}_i$$

acting on the Hilbert space $l_2(\mathbb{C}^n)$, equipped with the Hermitian inner product

$$\langle \tilde{X}, \tilde{Y} \rangle_h := \sum_{\ell \in \mathbb{Z}} \langle X_\ell, M Y_\ell \rangle_{\mathbb{C}^n}, \quad \text{for } \tilde{X} = (X_\ell)_\ell,\ \tilde{Y} = (Y_\ell)_\ell \in l_2(\mathbb{C}^n).$$

Then, the spectrum of $\mathcal{T}_r$ is given by

$$\sigma(\mathcal{T}_r) = \bigcup_{i=1}^{n} \frac{1}{\Lambda_i(\Omega_i)} g^r_{z_i}\left(\left[-\frac{\pi}{h}, \frac{\pi}{h}\right]\right). \tag{3.14}$$

(ii) *The operator $\mathcal{T}_r$ defines a bijection from $l_2(\mathbb{C}^n)$ onto the following spaces:*

$$\begin{cases} l_2(\mathbb{C}^n), & \text{if } r = u, \\ \left\{ (X_m)_m \in l_2(\mathbb{C}^n) \ \middle|\ \begin{array}{c} X_m = \sum\limits_{\substack{l \in \mathbb{Z} \\ l \neq m}} \dfrac{(-1)^{m-l}}{m-l} Y_l, \\ (Y_l)_l \in l_2(\mathbb{C}^n) \end{array} \right\}, & \text{if } r = v, \\ \left\{ (X_m)_m \in l_2(\mathbb{C}^n) \ \middle|\ \begin{array}{c} X_m = \dfrac{\pi^2}{6} Y_m + \sum\limits_{\substack{l \in \mathbb{Z} \\ l \neq m}} \dfrac{(-1)^{m-l}}{(m-l)^2} Y_l, \\ (Y_l)_l \in l_2(\mathbb{C}^n) \end{array} \right\}, & \text{if } r = a. \end{cases} \tag{3.15}$$

This holds under the condition $\gamma > \frac{1}{2}$. Moreover, if $\gamma = \frac{1}{2}$ and $\beta < \frac{1}{4}$, the operator defines a bijection from $l_2(\mathbb{C}^n)$ onto:

$$\begin{cases} l_2(\mathbb{C}^n), & \text{if } r = u, \\ \left\{ (X_m)_m \in l_2(\mathbb{C}^n) \ \middle|\ \begin{array}{c} X_m = \sum\limits_{\substack{l \in \mathbb{Z} \\ l \neq m}} \dfrac{(-1)^{m-l}}{(m-l)^3} Y_l, \\ (Y_l)_l \in l_2(\mathbb{C}^n) \end{array} \right\}, & \text{if } r = v, \\ \left\{ (X_m)_m \in l_2(\mathbb{C}^n) \ \middle|\ \begin{array}{c} X_m = \dfrac{\pi^2}{6} Y_m + \sum\limits_{\substack{l \in \mathbb{Z} \\ l \neq m}} \dfrac{(-1)^{m-l}}{(m-l)^2} Y_l, \\ (Y_l)_l \in l_2(\mathbb{C}^n) \end{array} \right\}, & \text{if } r = a. \end{cases} \tag{3.16}$$

□

Proof of Corollary 12. For a fixed $r \in \{u, v, a\}$, observe that for each $1 \leq i \leq n$, the operator $\mathcal{T}_r$ is stable on the subspace

$$\operatorname{Ran}\left(\frac{1}{\Lambda_i(\Omega_i)} T(g^r_{z_i}) \mathcal{P}_i\right),$$

since the $\mathcal{P}_i$'s are orthogonal projections in the sense of (3.8), with respect to the Hermitian inner product defined in the present lemma. It is clear that the restriction of $\mathcal{T}_r$ to this subspace coincides with $\frac{1}{\Lambda_i(\Omega_i)}T(g^r_{z_i})\mathcal{P}_i$.

Therefore, by applying Lemma 1 (I.128) from Bourbaki [3], we deduce:

$$\sigma\left(\mathcal{T}_r\right) = \bigcup_{i=1}^{n} \sigma\left(\frac{1}{\Lambda_i(\Omega_i)}T(g^r_{z_i})\mathcal{P}_i\right).$$

Moreover, the operator $\frac{1}{\Lambda_i(\Omega_i)}\,T(g^r_{z_i})\,\mathcal{P}_i$, when restricted to its range, is unitarily equivalent to the multiplication operator

$$M_{g^r_{z_i}/\Lambda_i(\Omega_i)} : f \mapsto \frac{1}{\Lambda_i(\Omega_i)}\,g^r_{z_i}\cdot f,$$

acting on the Hilbert space

$$\mathrm{L}^2\left(\left(-\frac{\pi}{h},\frac{\pi}{h}\right);\mathrm{Ran}(\mathcal{P}_i)\right).$$

Following the arguments in [9], and using the fact that $g^r_{z_i}$ is continuous on the interval $\left[-\frac{\pi}{h},\frac{\pi}{h}\right]$, we conclude that the spectrum of $M_{g^r_{z_i}/\Lambda_i(\Omega_i)}$ is given by

$$\sigma\left(M_{g^r_{z_i}/\Lambda_i(\Omega_i)}\right) = \left\{\frac{1}{\Lambda_i(\Omega_i)}\,g^r_{z_i}(\theta)\,\middle|\,\theta\in\left[-\frac{\pi}{h},\frac{\pi}{h}\right]\right\}.$$

This yields the desired spectral characterization and concludes the proof of the first assertion.

The second statement is proved as follows: We focus only on (3.15), since (3.16) can be established by analogy. Replacing $(\ddot{\mathcal{U}}_l)_{l\in\mathbb{Z}}$, $(\dot{\mathcal{U}}_l)_{l\in\mathbb{Z}}$, $(\mathcal{U}_l)_{l\in\mathbb{Z}}$, and $(F_l)_{l\in\mathbb{Z}}$ by $\tilde{X} = (X_l)_{l\in\mathbb{Z}}$, $\tilde{Y} = (Y_l)_{l\in\mathbb{Z}}$, $\tilde{Z} = (Z_l)_{l\in\mathbb{Z}}$, and $\tilde{W} = (W_l)_{l\in\mathbb{Z}}$, respectively, where all these sequences belong to $l_2(\mathbb{C}^n)$, in (2.1) and (2.2), we obtain, by applying the discrete Fourier transform $\mathfrak{F}_h$:

(3.17)
$$\begin{cases} M\mathfrak{F}_h\tilde{X}(\theta) + C\mathfrak{F}_h\tilde{Y}(\theta) + K\mathfrak{F}_h\tilde{Z}(\theta) = \mathfrak{F}_h\tilde{W}(\theta),\\ (1-e^{i_c h\theta})\mathfrak{F}_h\tilde{Z}(\theta) = he^{i_c h\theta}\mathfrak{F}_h\tilde{Y}(\theta) + h^2\left(\beta + \left(\frac{1}{2}-\beta\right)e^{i_c h\theta}\right)\mathfrak{F}_h\tilde{X}(\theta),\\ (1-e^{i_c h\theta})\mathfrak{F}_h\tilde{Y}(\theta) = h\left((1-\gamma)e^{i_c h\theta}+\gamma\right)\mathfrak{F}_h\tilde{X}(\theta). \end{cases}$$

Given that $\mathfrak{F}_h$ is an isomorphism between $l_2(\mathbb{C}^n)$ and $\mathrm{L}^2\left(\left(-\frac{\pi}{h},\frac{\pi}{h}\right),\mathbb{C}^n\right)$, it follows from (3.17) that the operator $\mathcal{T}_r$ is injective for $r\in\{u,v,a\}$. Moreover, we can easily verify that $\mathfrak{F}_h\tilde{Z}$, $\mathfrak{F}_h\tilde{Y}$, and $\mathfrak{F}_h\tilde{X}$ belong to the function spaces

$$\mathrm{L}^2\left(\left(-\frac{\pi}{h},\frac{\pi}{h}\right),\mathbb{C}^n\right),$$

$$\left\{\theta f \mid f\in\mathrm{L}^2\left(\left(-\frac{\pi}{h},\frac{\pi}{h}\right),\mathbb{C}^n\right)\right\},$$

and

$$\left\{\theta^2 f \mid f \in \mathrm{L}^2\left(\left(-\frac{\pi}{h}, \frac{\pi}{h}\right), \mathbb{C}^n\right)\right\},$$

respectively. One can notice by computing the Fourier coefficients that the premiage of the previous spaces by $\mathfrak{F}_h$ are given by (3.15). The surjectivity is a consequence of the assumption $\gamma > \frac{1}{2}$, (3.17) and the fact that $\mathfrak{F}_h$ is an isomorphism.

In the case of (3.16), one can verify that $\mathfrak{F}_h \tilde{Z}$, $\mathfrak{F}_h \tilde{Y}$, and $\mathfrak{F}_h \tilde{X}$ belong to the following function spaces:

$$\mathrm{L}^2\left(\left(-\frac{\pi}{h}, \frac{\pi}{h}\right), \mathbb{C}^n\right),$$

$$\left\{\left(\frac{\pi^2}{h^2} - \theta^2\right)\theta f \,\middle|\, f \in \mathrm{L}^2\left(\left(-\frac{\pi}{h}, \frac{\pi}{h}\right), \mathbb{C}^n\right)\right\},$$

and

$$\left\{\theta^2 f \,\middle|\, f \in \mathrm{L}^2\left(\left(-\frac{\pi}{h}, \frac{\pi}{h}\right), \mathbb{C}^n\right)\right\}.$$

This concludes the proof. □

Remark 13. *In Corollary 12, the spectrum $\sigma(\mathcal{T}_r)$ provides, from a physical point of view, essential information about the system's behavior in the frequency domain for each mode. Specifically, it describes how the system amplifies or attenuates various frequency components (gain) and introduces corresponding phase shifts, all within the discrete framework. Moreover, analyzing the Fourier coefficients in (3.6) offers insight into which frequencies contribute most significantly to the spectrum.*

It follows from Corollary 12 that if $\gamma > \frac{1}{2}$ or $\gamma = \frac{1}{2}$ with $\beta < \frac{1}{4}$, then $\mathcal{T}_u$ is an isomorphism, and we have $0 \notin \mathcal{T}_u$. Moreover, the spaces in (3.15) and (3.16) associated with the operators $\mathcal{T}_v$ and $\mathcal{T}_a$ are not closed in $l_2(\mathbb{C}^n)$.

In [6], a discretization error was established for twice-differentiable functions F. The following result extends this analysis for positive coefficients ξ_i to the cases of bounded continuous functions and integrable functions.

Corollary 14. *With the same notations as in Theorem 10, we claim the following:*

i. If $F \in \mathrm{C}_0(\mathbb{R}, \mathbb{R}^n)$ is bounded and $F_l \in F([lh, (l+1)h])$, then there exists $h_0 > 0$ such that for every $0 < h \leq h_0$ and for every $x, y \in \mathbb{R}$ with $y > x$, we have:

(3.18)
$$\left\|\mathcal{I}_h\tilde{\mathcal{U}}(y)-\sum_{i=1}^{n}(G_i*\mathcal{P}_iM^{-1}F)(y)\right\|+\left\|\mathcal{I}_h\tilde{\dot{\mathcal{U}}}(y)-\sum_{i=1}^{n}\frac{d}{dt}(G_i*\mathcal{P}_iM^{-1}F)(y)\right\|$$
$$+\left\|\mathcal{I}_h\tilde{\ddot{\mathcal{U}}}(y)-\sum_{i=1}^{n}\frac{d^2}{dt^2}(G_i*\mathcal{P}_iM^{-1}F)(y)\right\|$$
$$\leq \mathfrak{C}\left[\max_{1\leq i\leq n} e^{-\xi_i\omega_i(y-x)}\sup_{s\leq x+h}\|F(s)\|+h\|F\|_{\mathrm{L}^\infty}+\psi_{F,[x-h,y+h]}(h)\right].$$

where:

- $\mathfrak{C}$ *is independent of* h, F, x, *and* y.
- *The function* $h\mapsto\psi_{F,[x,y]}(h)$ *denotes the modulus of continuity of* F *over* $[x,y]$.
- *The convolution product* $G_i*\mathcal{P}_iM^{-1}F$ *is defined as:*

$$G_i*\mathcal{P}_iM^{-1}F(t)=\frac{1}{\omega_{di}}\int_{-\infty}^{t}\exp(-\xi_i\omega_i(t-s))\sin(\omega_{di}(t-s))\mathcal{P}_iM^{-1}F(s)\,ds$$

where the kernel G_i *is given by:*

$$G_i(t)=\frac{1}{\omega_{di}}\exp(-\xi_i\omega_it)\sin(\omega_{di}t)\,\chi_{\mathbb{R}_+}(t),$$

and $\omega_{di}=\omega_i\sqrt{1-\xi_i^2}$ *is the damped natural frequency.*

ii. If $F\in\mathrm{L}^p(\mathbb{R},\mathbb{R}^n)$, *with* $1<p\leq 2$, *and*

$$F_l=\frac{1}{h}\int_{lh}^{(l+1)h}F(s)\,ds,\quad \forall l\in\mathbb{Z},$$

then, as $h\to 0$, *we have the following convergences:*

$$\begin{aligned}
&\mathcal{I}_h\tilde{\mathcal{U}}\to\sum_{i=1}^{n}G_i*\mathcal{P}_iM^{-1}F && \text{in } \mathrm{L}^{p^\star}(\mathbb{R},\mathbb{R}^n)\\
&\mathcal{I}_h\tilde{\dot{\mathcal{U}}}\to\sum_{i=1}^{n}\frac{d}{dt}\left(G_i*\mathcal{P}_iM^{-1}F\right) && \text{in } \mathrm{L}^{p}(\mathbb{R},\mathbb{R}^n)+\mathrm{L}^{p^\star}(\mathbb{R},\mathbb{R}^n)\\
&\mathcal{I}_h\tilde{\ddot{\mathcal{U}}}\to\sum_{i=1}^{n}\frac{d^2}{dt^2}\left(G_i*\mathcal{P}_iM^{-1}F\right) && \text{in } \mathrm{L}^{p}(\mathbb{R},\mathbb{R}^n)+\mathrm{L}^{p^\star}(\mathbb{R},\mathbb{R}^n)
\end{aligned}\tag{3.19}$$

with $1=\frac{1}{p}+\frac{1}{p^\star}$.

□

Proof. i- We restrict ourselves to the case $\mathcal{I}_h\tilde{\mathcal{U}}$, as the remaining cases can be treated analogously. Let $0<\epsilon<1$ and $\mathfrak{C}>0$ an arbitrary constant independent of h, F, x, and y but may depends on ϵ. Moreover, we choose h_0' sufficently small such that:

$$\left|\frac{1}{\Lambda_i(\Omega_i)} - 1\right| \leq \epsilon, \quad \text{for all } 1 \leq i \leq n \text{ and } 0 < h < h_0' \tag{3.20}$$

It suffices to prove that the inequality (3.18) holds on the subspace $\mathrm{Ran}(\mathcal{P}_i)$ for each $1 \leq i \leq n$, assuming that h satisfies (3.20). More precisely, our goal is to establish the following estimate:

$$\begin{aligned} &\left\|\frac{1}{\Lambda_i(\Omega_i)}\mathcal{I}_h\left(T(g^u_{z_i})\mathcal{P}_i\widetilde{M^{-1}F}\right)(y) - \left(G_i * \mathcal{P}_i M^{-1}F\right)(y)\right\| \\ &\leq \mathfrak{C}\left[e^{-\xi_i\omega_i(y-x)}\sup_{s\leq x+h}\|F(s)\| + h\|F\|_{\mathrm{L}^\infty} + \psi_{F,[x-h,y+h]}(h)\right] \end{aligned} \tag{3.21}$$

To simplify the presentation, we assume without loss of generality that $\mathcal{P}_i M^{-1}F = F$. Furthermore, we use the notation $f(h) = \mathcal{O}(h^r)$, with $r \geq 0$, to indicate that $|f(h)| \leq \mathfrak{C}|h^r|$ for all $h \leq h_0$, where h_0 is sufficiently small. We also define the sequence $\mathbf{c} = (c_l)_l$ by:

$$c_l = \frac{1}{\Lambda_i(\Omega_i)}\frac{h}{2\pi}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} g^u_{z_i}(t)\, e^{-i_c lht}\, dt, \quad \text{for all } l \in \mathbb{Z}. \tag{3.22}$$

Let m be a positive integer such that $mh \leq y < mh + h$. First, we have:

$$\begin{cases} 1 + z_i = \mathcal{O}(h), \\ \dfrac{h}{\Im(-z_i)} = \dfrac{1}{\omega_{di}} + \mathcal{O}(h) \\ \frac{1}{\Lambda_i(\Omega_i)} = 1 + \mathcal{O}(h) \end{cases} \tag{3.23}$$

Thus,

$$\frac{1}{\Lambda_i(\Omega_i)} \cdot \frac{h}{\Im(-z_i)}\left[\beta(1+z_i)^2 - (1+z_i)\left(\gamma + \tfrac{1}{2}\right) + 1\right] - \frac{1}{\omega_{di}} = \mathcal{O}(h). \tag{3.24}$$

Moreover, taking into account (2.8) and (2.10), we have for every $l \in \mathbb{Z}$:

(3.25)

$$\begin{aligned} &\Im\left((-z_i)^l\right) - \exp\left(-\xi_i\omega_i lh\right)\sin\left(\omega_{di} lh\right) \\ &= \left[\exp\left(\frac{l}{2}\ln\left(1 - \left(\gamma_i' - \tfrac{1}{2}\right)\frac{\Omega_i^2}{\Lambda_i(\Omega_i)}\right)\right) - \exp\left(-\xi_i\omega_i lh\right)\right]\Im\left(\exp\left(i_c l\arg(-z_i)\right)\right) \\ &\quad + \exp\left(-\xi_i\omega_i lh\right)\left[\Im\left(\exp\left(i_c l\arg(-z_i)\right)\right) - \sin\left(\omega_{di} lh\right)\right]. \end{aligned}$$

By the mean value theorem, there exists $t \in [0,1]$ such that

$$\begin{aligned}(3.26)\quad &\left|\exp\left(\frac{l}{2}\ln\left(1-\left(\gamma_i'-\tfrac{1}{2}\right)\frac{\Omega_i^2}{\Lambda_i(\Omega_i)}\right)\right)-\exp\left(-\xi_i\omega_i lh\right)\right|\\ &\leq \exp(-\alpha_t l)\cdot\left|-\tfrac{l}{2}\ln\left(1-\left(\gamma_i'-\tfrac{1}{2}\right)\frac{\Omega_i^2}{\Lambda_i(\Omega_i)}\right)-\xi_i\omega_i lh\right|,\end{aligned}$$

where

$$(3.27)\qquad \alpha_t = t\cdot\left(-\tfrac{1}{2}\ln\left(1-\left(\gamma_i'-\tfrac{1}{2}\right)\frac{\Omega_i^2}{\Lambda_i(\Omega_i)}\right)\right)+(1-t)\cdot\xi_i\omega_i h.$$

since $\gamma \geq \frac{1}{2}$ and taking into account (3.20), we have :

$$(3.28)\qquad \begin{cases}\alpha_t \geq (1-\epsilon)\,\xi_i\omega_i h,\\ \exp(-\alpha_t l) \leq \exp\left(-(1-\epsilon)\,\xi_i\omega_i lh\right)\end{cases}$$

Using the expression (2.8) and the Taylor expansion, we obtain:

$$(3.29)\qquad \begin{cases}\left|-\tfrac{l}{2}\ln\left(1-\left(\gamma_i'-\tfrac{1}{2}\right)\frac{\Omega_i^2}{\Lambda_i(\Omega_i)}\right)-\xi_i\omega_i lh\right| = l\,\mathcal{O}(h^2),\\ \left|\Im\left(\exp\left(i_c l\arg(-z_i)\right)\right)-\sin\left(\omega_{di} lh\right)\right| = l\,\mathcal{O}(h^2).\end{cases}$$

The constants implicit in the $\mathcal{O}(\cdot)$ notation in (3.29) are, of course, independent of l. Thus, using (3.25), (3.28) and (3.29):

$$(3.30)\qquad \left|\Im\left((-z_i)^l\right)-\exp\left(-\xi_i\omega_i lh\right)\sin\left(\omega_{di} lh\right)\right| \leq \mathfrak{C}\cdot\exp\left(-(1-\epsilon)\,\xi_i\omega_i lh\right)\cdot lh^2.$$

Since

$$\sum_{l=-\infty}^{m-1} h|(-z_i)^{m-l}| = \mathcal{O}(1),$$

combining (3.24) and (3.30), we obtain:

$$\begin{aligned}(3.31)\quad &\left\|\sum_{l=-\infty}^{m} c_{m-l}F_l - \frac{h}{\omega_{di}}\sum_{l=-\infty}^{m}\exp\left(-\xi_i\omega_i(m-l)h\right)\sin\left(\omega_{di}(m-l)h\right)F_l\right\|\\ &\leq \mathfrak{C}\,h\,\|F\|_{\mathrm{L}^\infty}\end{aligned}$$

Moreover, it is not difficult to show, taking into account that $|y-s-(m-l)h| \leq h$ for $lh \leq s < (l+1)h$, and using the mean value theorem, that:

(3.32)
$$\left\| \frac{h}{\omega_{di}} \sum_{l=-\infty}^{m} \exp\left(-\xi_i\omega_i(m-l)h\right) \sin\left(\omega_{di}(m-l)h\right) F_l - \frac{1}{\omega_{di}} \int_{-\infty}^{y} \exp(-\xi_i\omega_i(y-s)) \sin(\omega_{di}(y-s)) \mathcal{I}_h \tilde{F}(s)\, ds \right\| \leq \mathfrak{C}\, h\, \|F\|_{\mathrm{L}^\infty}.$$

and

$$\left\| \int_{-\infty}^{y} \exp(-\xi_i\omega_i(y-s)) \sin(\omega_{di}(y-s)) \left[\mathcal{I}_h\tilde{F}(s) - F(s)\right] ds \right\| \leq \mathfrak{C} \left(\psi_{F,[x-h,y+h]}(h) + \exp(-\xi_i\omega_i(y-x)) \sup_{s\leq x+h} \|F(s)\| \right). \tag{3.33}$$

Here, $\tilde{F} = (F_\ell)_{\ell\in\mathbb{Z}}$. Thus, we obtain the desired result for the first claim.

ii- Let $p^\star$ denote the conjugate exponent of p, i.e., such that $\frac{1}{p} + \frac{1}{p^\star} = 1$. We focus on the first case related to $\mathcal{I}_h\tilde{\mathcal{U}}$, and we only need to prove that

$$\mathcal{I}_h\left(T(g^u_{z_i})\, \mathcal{P}_i\, \widetilde{M^{-1}F}\right) \to G_i * \mathcal{P}_i M^{-1} F$$

in $\mathrm{L}^{p^\star}(\mathbb{R},\mathbb{R}^n)$, for a fixed $1 \leq i \leq n$, since $\Lambda_i(\Omega_i) \to 1$ as $h \to 0$. For this purpose and without loss of generality, we assume $\mathcal{P}_i M^{-1} F = F$ for simplicity.

First, observe that $\mathcal{I}_h\tilde{F} \to F$ in $\mathrm{L}^p(\mathbb{R},\mathbb{R}^n)$, where $\tilde{F} = (F_\ell)_{\ell\in\mathbb{Z}}$. Indeed, one can check that $\mathcal{I}_h\tilde{F}$ is uniformly bounded with respect to h in $\mathrm{L}^p(\mathbb{R},\mathbb{R}^n)$, with

$$\|\mathcal{I}_h\tilde{F}\|_{\mathrm{L}^p} \leq \|F\|_{\mathrm{L}^p}.$$

More precisely, using (3.5), we have

$$\begin{aligned} \|\mathcal{I}_h\tilde{F}\|_{\mathrm{L}^p} &= \frac{1}{h}\, \|\mathcal{I}_h \circ \mathcal{I}_h^\star F\|_{\mathrm{L}^p} \\ &\leq \frac{1}{h}\, \|\mathcal{I}_h\|_{\mathcal{L}(l_p(\mathbb{R}^n),\mathrm{L}^p(\mathbb{R},\mathbb{R}^n))} \|\mathcal{I}_h^\star\|_{\mathcal{L}(\mathrm{L}^p(\mathbb{R},\mathbb{R}^n),l_p(\mathbb{R}^n))} \|F\|_{\mathrm{L}^p} \\ &\leq \|F\|_{\mathrm{L}^p}. \end{aligned} \tag{3.34}$$

Moreover, the convergence holds for functions in $\mathrm{C}_0(\mathbb{R},\mathbb{R}^n)$ with compact support. By density of this space in $\mathrm{L}^p(\mathbb{R},\mathbb{R}^n)$ and by the contractivity property in (3.34), we deduce the desired result.

Let G_i^h be the piecewise constant kernel defined by

$$G_i^h(t) = G_i(lh) \quad \text{for } lh \leq t < (l+1)h, \quad l \in \mathbb{Z},$$

Define also the function h_i^u by

$$h_i^u(\theta) = \sum_{l=1}^{\infty} h\, G_i^h(lh)\, e^{-i_c lh\theta}, \tag{3.35}$$

Taking into account the fact that $G_i(0)=0$, we write:

(3.36)
$$\begin{aligned}\mathcal{I}_h\circ T(g^u_{z_i})\tilde F - G_i*\mathcal{I}_h\tilde F =&\mathcal{I}_h\circ T(g^u_{z_i})\tilde F - G^h_i*\mathcal{I}_h\tilde F + G^h_i*\mathcal{I}_h\tilde F - G_i*\mathcal{I}_h\tilde F\\ =&\mathcal{I}_h\circ\Big(T(g^u_{z_i})-T(h^u_i)\Big)\circ\frac{1}{h}\mathcal{I}^\star_h F+\\ &\Big(G^h_i-G_i\Big)*\Big(\mathcal{I}_h\circ\frac{1}{h}\mathcal{I}^\star_h\Big)F\end{aligned}$$

Moreover, the operators $\mathcal{I}_h\circ\Big(T(g^u_{z_i})-T(h^u_i)\Big)\circ\frac{1}{h}\mathcal{I}^\star_h$ and $\Big(G^h_i-G_i\Big)*\Big(\mathcal{I}_h\circ\frac{1}{h}\mathcal{I}^\star_h\Big)$ are well defined on $\mathrm{L}^1(\mathbb{R},\mathbb{R}^n)$ and $\mathrm{L}^2(\mathbb{R},\mathbb{R}^n)$. Furthermore, we have the following estimates:

For every $X\in\mathrm{L}^1(\mathbb{R},\mathbb{R}^n)$, we have:

$$\begin{aligned}\Big\|\mathcal{I}_h\circ\Big(T(g^u_{z_i})-T(h^u_i)\Big)\circ\frac{1}{h}\mathcal{I}^\star_h X\Big\|_{\mathrm{L}^\infty}&\le\sup_{l\in\mathbb{Z}}\Big|\frac{1}{h}c_l-G^h_i(lh)\Big|\cdot\|X\|_{\mathrm{L}^1},\\ \Big\|\Big(G^h_i-G_i\Big)*\Big(\mathcal{I}_h\circ\frac{1}{h}\mathcal{I}^\star_h\Big)X\Big\|_{\mathrm{L}^\infty}&\le\|G^h_i-G_i\|_{\mathrm{L}^\infty}\cdot\|X\|_{\mathrm{L}^1}.\end{aligned}$$

For every $X\in\mathrm{L}^2(\mathbb{R},\mathbb{R}^n)$, we have:

$$\begin{aligned}\Big\|\mathcal{I}_h\circ\Big(T(g^u_{z_i})-T(h^u_i)\Big)\circ\frac{1}{h}\mathcal{I}^\star_h X\Big\|_{\mathrm{L}^2}&\le\Big\|g^u_{z_i}-h^u_i\Big\|_{\mathrm{L}^\infty}\cdot\|X\|_{\mathrm{L}^2},\\ \Big\|\Big(G^h_i-G_i\Big)*\Big(\mathcal{I}_h\circ\frac{1}{h}\mathcal{I}^\star_h\Big)X\Big\|_{\mathrm{L}^2}&\le\Big\|\mathfrak{F}G^h_i-\mathfrak{F}G_i\Big\|_{\mathrm{L}^\infty}\cdot\|X\|_{\mathrm{L}^2},\end{aligned}$$

where $\mathfrak{F}$ denotes the Fourier transform.

As a consequence, using the Riesz–Thorin interpolation theorem, we obtain the following estimate:

$$\begin{aligned}\Big\|\mathcal{I}_h\circ T(g^u_{z_i})\tilde F - G_i*\mathcal{I}_h\tilde F\Big\|_{\mathrm{L}^{p^\star}}\le\Bigg(&\sup_{l\in\mathbb{Z}}\Big|\frac{1}{h}c_l-G^h_i(lh)\Big|^{\frac{2}{p}-1}\Big\|g^u_{z_i}-h^u_i\Big\|_{\mathrm{L}^\infty}^{2(1-\frac{1}{p})}\\ &+\Big\|G^h_i-G_i\Big\|_{\mathrm{L}^\infty}^{\frac{2}{p}-1}\Big\|\mathfrak{F}G^h_i-\mathfrak{F}G_i\Big\|_{\mathrm{L}^\infty}^{2(1-\frac{1}{p})}\Bigg)\cdot\|F\|_{\mathrm{L}^p}\end{aligned}\tag{3.37}$$

It is not difficult to show, using the asymptotic expansion as $h\to0$ and following similar arguments as in the previous case, that the right-hand side of (3.37) tends to zero. Moreover, applying once again the Riesz–Thorin interpolation theorem on $X\mapsto G_i*X$, we deduce that

$$G_i*\mathcal{I}_h\tilde F\to G_i*F\quad\text{in}\quad\mathrm{L}^{p^\star}(\mathbb{R},\mathbb{R}^n),$$

taking into account the convergence $\mathcal{I}_h\tilde F\to F$ in $\mathrm{L}^p(\mathbb{R},\mathbb{R}^n)$ established above. Thus, we have $\mathcal{I}_h\circ T(g^u_{z_i})\tilde F\to G_i*F$ in $\mathrm{L}^{p^\star}(\mathbb{R},\mathbb{R}^n)$

In the case of $\mathcal{I}_h\tilde{\tilde{\mathcal{U}}}$, we prove, as in the previous case, that:

$$\mathcal{I}_h\left(T(g_{z_i}^v)\,\tilde{F}\right) \to \frac{d}{dt}\left(G_i * F\right)$$

in $\mathrm{L}^p(\mathbb{R},\mathbb{R}^n)+\mathrm{L}^{p^\star}(\mathbb{R},\mathbb{R}^n)$.

Taking as before the following notations:

$$\dot{G}_i^h(t)=\frac{d}{dt}G_i(mh) \quad \text{for } mh \le t < (m+1)h, \quad m\in\mathbb{Z},$$

and

$$h_i^v(\theta)=\sum_{l=1}^{\infty} h\,\dot{G}_i^h(lh)\,e^{-i_c lh\theta}, \tag{3.38}$$

It is sufficient to observe that $\left(\frac{d}{dt}G_i\right)*F=\frac{d}{dt}\left(G_i*F\right)$ almost everywhere. In particular, at every $t\in\mathbb{R}$, we have:

(3.39)

$$\begin{aligned}\mathcal{I}_h\circ T(g_{z_i}^v)\,\tilde{F}(t)-\left(\frac{d}{dt}G_i\right)*\mathcal{I}_h\tilde{F}(t) &= \mathcal{I}_h\circ\left(T(g_{z_i}^v)-T(h_i^v)\right)\circ\frac{1}{h}\mathcal{I}_h^\star F(t)\\ &+\left(\dot{G}_i^h-\frac{d}{dt}G_i\right)*\left(\mathcal{I}_h\circ\frac{1}{h}\mathcal{I}_h^\star\right)F(t)\\ &-\left(t-\left\lfloor\frac{t}{h}\right\rfloor h\right)\,\dot{G}_i^h(0)\,\mathcal{I}_h\circ\frac{1}{h}\mathcal{I}_h^\star F(t)\end{aligned}$$

where $\lfloor\frac{t}{h}\rfloor$ denotes the integer part of $\frac{t}{h}$. As in the previous case, the first two terms on the right-hand side of (3.39) tend to zero in $\mathrm{L}^{p^\star}(\mathbb{R},\mathbb{R}^n)$.

Denoting the last term on the right-hand side of (3.39) by the map $E_h(t)$, we obtain:

$$\left(\frac{d}{dt}G_i\right)*\mathcal{I}_h\tilde{F}-E_h \to \left(\frac{d}{dt}G_i\right)*F \tag{3.40}$$

In $\mathrm{L}^p(\mathbb{R},\mathbb{R}^n)+\mathrm{L}^{p^\star}(\mathbb{R},\mathbb{R}^n)$, since $E_h\to 0$ in $\mathrm{L}^p(\mathbb{R},\mathbb{R}^n)$ and $\left(\frac{d}{dt}G_i\right)*\mathcal{I}_h\tilde{F}\to\left(\frac{d}{dt}G_i\right)*F$ in $\mathrm{L}^{p^\star}(\mathbb{R},\mathbb{R}^n)$. Thus, we conclude the desired convergence result.

The final case, concerning $\mathcal{I}_h\tilde{\tilde{\tilde{\mathcal{U}}}}$, can be handled analogously, with only minor modifications. It suffices to observe that, at every Lebesgue point $t\in\mathbb{R}$ of the map F, the identity

$$\left(\frac{d^2}{dt^2}G_i\right)*F(t)+\frac{d}{dt}G_i(0)\cdot F(t)=\frac{d^2}{dt^2}\left(G_i*F\right)(t)$$

holds.

□

In many structural dynamics applications, a cut-off frequency is employed for modal truncation, allowing the solution to be computed in the subspace generated by the projection operators $\{\mathcal{P}_1, \ldots, \mathcal{P}_q\}$, where $1 \le q \le n$.

The cut-off frequency is characterized by the existence of an integer q such that, for all $i \ge q$, the approximation

$$\omega_i^2\, G_i * \left(\mathcal{P}_i M^{-1} F\right) \approx \mathcal{P}_i M^{-1} F$$

holds in the asymptotic regime where ω_i is sufficiently large.

This means that the contribution of higher modes becomes negligible, and the dynamics can be captured by the first q modes. Moreover, using the notation of Corollary 14 and assuming additionally that the weak derivative $\dot{F} \in \mathrm{L}^\infty(\mathbb{R}, \mathbb{R}^n)$, one can show that

$$\|\mathcal{P}_i M^{-1} F - \omega_i^2 G_i * \mathcal{P}_i M^{-1} F\|_{\mathrm{L}^\infty} \le \frac{1}{\omega_i \xi_i \sqrt{1-\xi_i^2}} \|\mathcal{P}_i M^{-1} \dot{F}\|_{\mathrm{L}^\infty}.$$

The discretized version of the above inequality is given by the following:

Corollary 15. *Assume that* $F \in \mathrm{C}_0(\mathbb{R}, \mathbb{R}^n)$*,* $\dot{F} \in \mathrm{L}^\infty(\mathbb{R}, \mathbb{R}^n)$ *and* $F_l \in F([lh, (l+1)h])$*. For a fixed* $1 \le i \le n$*, we have the following estimate:*

$$\left\| \mathcal{P}_i \widetilde{M^{-1}F} - \frac{\omega_i^2}{\Lambda_i(\Omega_i)} T(g^u_{z_i}) \mathcal{P}_i \widetilde{M^{-1}F} \right\|_{l_\infty} \le \frac{\mathfrak{C}}{\omega_i \xi_i} g(\Omega_i) \|\mathcal{P}_i M^{-1} \dot{F}\|_{\mathrm{L}^\infty} \tag{3.41}$$

where:

$$g(\Omega_i) = \frac{\Lambda_i(\Omega_i)}{\sqrt{\Lambda_i(\Omega_i) - \frac{1}{4}\Omega_i^2 \left(\gamma_i' + \frac{1}{2}\right)^2}} \tag{3.42}$$

and $\mathfrak{C}$ *depends only on* γ *and* β*. Recall that* γ_i' *is given by* (2.9)*. Moreover, we have the asymptotic behavior:*

$$\lim_{\Omega_i \to 0} g(\Omega_i) = \frac{1}{\sqrt{1-\xi_i^2}}. \tag{3.43}$$

□

Proof. Let $\mathfrak{C}$ be an arbitrary constant that depends only on γ and β and $m \in \mathbb{Z}$, we put:

$$c_l = \frac{h}{2\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} g^u_{z_i}(t) e^{-i_c l h t}\, dt, \quad \text{for all } l \in \mathbb{Z}. \tag{3.44}$$

and for arbitrary integer q, $-\infty < q \le m$:

$$A_q = \omega_i^2 \sum_{l=-\infty}^{q} c_{m-l} \tag{3.45}$$

Using Abel's summation by parts, we obtain:

$$\omega_i^2 \sum_{l=-\infty}^{m} c_{m-l}\mathcal{P}_i M^{-1} F_l = A_m \mathcal{P}_i M^{-1} F_m - \sum_{l=-\infty}^{m-1} A_l \mathcal{P}_i M^{-1}(F_{l+1} - F_l) \tag{3.46}$$

Using the expression (2.8) on can establish:

$$A_m = \Lambda_i(\Omega_i) \tag{3.47}$$

Moreover, taking into account that $|z_i| < 1$, we have for $q < m$:

$$\begin{aligned} |A_q| &= \Omega_i^2 \left| \frac{\Im\left[\frac{(-z_i)^{m-q}}{1+z_i}\left(\beta(1+z_i)^2 - (1+z_i)\left(\gamma+\frac{1}{2}\right)+1\right)\right]}{\Im(-z_i)} \right| \\ &\leq \mathfrak{C}\frac{\Omega_i^2}{|\Im(z_i)|}\cdot\frac{|z_i|^{m-q}}{|1+z_i|} \\ &= \mathfrak{C}\cdot\frac{(\Lambda_i(\Omega_i))^{\frac{3}{2}}}{\sqrt{\Lambda_i(\Omega_i) - \frac{1}{4}\Omega_i^2\left(\gamma_i'+\frac{1}{2}\right)^2}}\cdot|z_i|^{m-q} \end{aligned} \tag{3.48}$$

Combining this with the fact that $\dot{F} \in \mathrm{L}^\infty(\mathbb{R}, \mathbb{R}^n)$ and using again that $|z_i| < 1$ with (2.10), we have for every $l < m$:

$$\begin{cases} \left\|\mathcal{P}_i M^{-1} F_{l+1} - \mathcal{P}_i M^{-1} F_l\right\| \leq 2h \left\|\mathcal{P}_i M^{-1}\dot{F}\right\|_{\mathrm{L}^\infty} = 2\frac{\Omega_i}{\omega_i}\left\|\mathcal{P}_i M^{-1}\dot{F}\right\|_{\mathrm{L}^\infty} \\ \Omega_i \sum_{q=-\infty}^{m-1} |z_i|^{m-q} = \frac{\Omega_i |z_i|}{1-|z_i|} \leq \frac{(\Lambda_i(\Omega_i))^{\frac{1}{2}}}{\xi_i} \end{cases} \tag{3.49}$$

Combining equations (3.46), (3.47), (3.48), and (3.49), we obtain the desired result. This completes the proof.

□

The following corollary provides an explicit convergence rate for the truncated operator representation introduced in Theorem 10. It can be interpreted as the error introduced by considering only the N recent time steps.

Corollary 16. *With the notations of Theorem 10, assume that* $\tilde{F} = (F_\ell)_{\ell\in\mathbb{Z}} \in l_2(\mathbb{R}^n)$. *Let* $N \in \mathbb{N}^*$, *and define the following truncated functions:*

(3.50)

$$\begin{cases} g^u_{w,N}(\theta) = \beta h^2 + \sum_{l=1}^{N} h^2 \frac{\Im\left[(-1)^l w^{l-1}\left(\beta(1+w)^2 - (1+w)\left(\gamma+\frac{1}{2}\right)+1\right)\right]}{\Im w} \\ \quad \cdot e^{i_c l h\theta}, \\ g^v_{w,N}(\theta) = \gamma h + \sum_{l=1}^{N} h \frac{\Im\left[(-1)^l w^{l-1}(1+w)\left(\gamma(1+w)-1\right)\right]}{\Im w} e^{i_c l h\theta} \\ g^a_{w,N}(\theta) = 1 + \sum_{l=1}^{N} \frac{\Im\left[(-1)^l w^{l-1}(1+w)^2\right]}{\Im w} e^{i_c l h\theta} \end{cases}$$

Next, define the associated truncated sequences:

$$\begin{cases} \tilde{\mathcal{U}}_N = \sum_{i=1}^{n} \frac{1}{\Lambda_i(\Omega_i)} T(g^u_{z_i,N}) \mathcal{P}_i \widetilde{M^{-1}F}, \\ \dot{\tilde{\mathcal{U}}}_N = \sum_{i=1}^{n} \frac{1}{\Lambda_i(\Omega_i)} T(g^v_{z_i,N}) \mathcal{P}_i \widetilde{M^{-1}F}, \\ \ddot{\tilde{\mathcal{U}}}_N = \sum_{i=1}^{n} \frac{1}{\Lambda_i(\Omega_i)} T(g^a_{z_i,N}) \mathcal{P}_i \widetilde{M^{-1}F}. \end{cases} \tag{3.51}$$

Then, for every integer $r > 0$, there exists a constant $h_0 > 0$ such that for every $0 < h < h_0$, the following estimate holds:

$$\|\tilde{\mathcal{U}}_N - \tilde{\mathcal{U}}\|_{l_2} + \|\dot{\tilde{\mathcal{U}}}_N - \dot{\tilde{\mathcal{U}}}\|_{l_2} + \|\ddot{\tilde{\mathcal{U}}}_N - \ddot{\tilde{\mathcal{U}}}\|_{l_2} \leq \mathfrak{C} \frac{L_N + 1}{(hN)^r} \|\tilde{F}\|_{l_2}, \tag{3.52}$$

where L_N denotes the Lebesgue constant (see Chapter II, Section 12 of [21] for its definition), and $\mathfrak{C}$ is a constant independent of h, N and $\tilde{F}$ but depends on r. □

Proof. Throughout this proof, let $\mathfrak{C} > 0$ be an arbitrary constant, independent of h, N and $\tilde{F}$, but possibly depending on r. Since the projectors $\mathcal{P}_i$ are orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle_h$ introduced in Corollary 12, and since M is positive definite, we deduce, taking into account that $\frac{1}{\Lambda_i(\Omega_i)} \leq 1$, that:

$$\left\|\tilde{\mathcal{U}}_N - \tilde{\mathcal{U}}\right\|_{l_2} \leq \mathfrak{C} \sum_{i=1}^{n} \left\|g^u_{z_i} - g^u_{z_i,N}\right\|_{\mathrm{L}^\infty} \left\|\tilde{F}\right\|_{l_2}. \tag{3.53}$$

Moreover, for each $1 \leq i \leq n$, using inequality (13.25) from [21] and the fact that $g^u_{z_i}$ is h-periodic, we obtain:

$$\left\|g^u_{z_i} - g^u_{z_i,N}\right\|_{\mathrm{L}^\infty} \leq \mathfrak{C} \frac{L_N + 1}{(hN)^r} \left\|\frac{d^r}{d\theta^r} g^u_{z_i}\right\|_{\mathrm{L}^\infty}. \tag{3.54}$$

Furthermore, using the fact that $\gamma \geq \frac{1}{2}$ and $|z_i| < 1$, we have:

$$\begin{aligned}
\left\| \frac{d^r}{d\theta^r} g^u_{z_i} \right\|_{\mathrm{L}^\infty} &\leq \mathfrak{C} \frac{h}{|\Im(z_i)|} \sum_{l=1}^{\infty} (lh)^r \exp\left(-\frac{\xi_i \omega_i (l-1)h}{\Lambda_i(\Omega_i)}\right) \\
&\leq \mathfrak{C} \frac{h}{|\Im(z_i)|} \exp\left(\frac{2\xi_i\Omega_i}{\Lambda_i(\Omega_i)}\right) \int_0^\infty s^r \exp\left(-\frac{\xi_i\omega_i s}{\Lambda_i(\Omega_i)}\right) ds \quad (3.55) \\
&= \mathfrak{C} \frac{h}{|\Im(z_i)|} \exp\left(\frac{2\xi_i\Omega_i}{\Lambda_i(\Omega_i)}\right) \left[\frac{\Lambda_i(\Omega_i)}{\xi_i\omega_i}\right]^r \Gamma(r) \quad (3.56) \\
&\leq \mathfrak{C} \frac{h}{|\Im(z_i)|} \exp\left(2\xi_i\Omega_i\right) \left[\frac{\Lambda_i(\Omega_i)}{\xi_i\omega_i}\right]^r \quad (3.57)
\end{aligned}$$

where $r \mapsto \Gamma(r)$ denotes the Gamma function.

In the other cases, i.e., when considering $g^v_{z_i}$ or $g^a_{z_i}$, we use the following identity from (2.8):

$$|1 + z_i| = \frac{h\omega_i}{\sqrt{\Lambda_i(\Omega_i)}}.$$

and we obtain the following estimates:

$$\left\| \frac{d^r}{d\theta^r} g^v_{z_i} \right\|_{\mathrm{L}^\infty} \leq \mathfrak{C} \frac{h\omega_i}{|\Im(z_i)|} \exp\left(2\xi_i\Omega_i\right) \left[\frac{\Lambda_i(\Omega_i)}{\xi_i\omega_i}\right]^r, \quad (3.58)$$

$$\left\| \frac{d^r}{d\theta^r} g^a_{z_i} \right\|_{\mathrm{L}^\infty} \leq \mathfrak{C} \frac{h\omega_i^2}{|\Im(z_i)|} \exp\left(2\xi_i\Omega_i\right) \left[\frac{\Lambda_i(\Omega_i)}{\xi_i\omega_i}\right]^r. \quad (3.59)$$

Thus we obtain:

$$\begin{aligned}
&\|\tilde{\mathcal{U}}_N - \tilde{\mathcal{U}}\|_{l_2} + \|\dot{\tilde{\mathcal{U}}}_N - \dot{\tilde{\mathcal{U}}}\|_{l_2} + \|\ddot{\tilde{\mathcal{U}}}_N - \ddot{\tilde{\mathcal{U}}}\|_{l_2} \\
&\quad \leq \mathfrak{C} \frac{(L_N+1)}{(hN)^r} \left(\sum_{i=1}^{n} \frac{h(1+\omega_i)^2 \exp(2\xi_i\Omega_i)}{|\Im(z_i)|} \left[\frac{\Lambda_i(\Omega_i)}{\xi_i\omega_i}\right]^r \right) \|\tilde{F}\|_{l_2}, \quad (3.60)
\end{aligned}$$

Using the Taylor approximations in (3.23), we obtain the desired result for $0 < h < h_0$, for some $h_0 > 0$ satisfying the assumption (**H2**). ☐

4. CONCLUSIONS

In conclusion, this paper investigates some representation results for the discrete solutions of second-order evolution equations, discretized using the Newmark scheme, with the solutions expressed through bi-infinite Toeplitz or Laurent operators. This approach provides some insights into the spectral properties of the associated operators. Additionally, we establish convergence results under relaxed regularity assumptions, requiring only either continuous or integrable source terms. These results, although derived in the context of the Newmark scheme, can be adapted to other time discretization methods that preserve similar structural properties such the convolution-type recurrence relations.

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