JOURNAL OF NUMERICAL ANALYSIS AND APPROXIMATION THEORY
J. Numer. Anal. Approx. Theory, vol. 54 (2025) no. 2, pp. 295–314, doi.org/10.33993/jnaat542-1601
ictp.acad.ro/jnaat

EFFICIENT GLOBAL OPTIMIZATION OF MULTIVARIATE FUNCTIONS WITH UNKNOWN HÖLDER CONSTANTS VIA α-DENSE CURVES

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Abstract. In this paper, we consider the global optimization problem of non-smooth functions over an n-dimensional box, satisfying the Hölder condition. We focus our study on the case when the Hölder constant is not a priori known. We develop and analyze two algorithms. The first one is an extension version of the Piyavskii's method that adaptively construct a linear secants of the Hölder support functions, avoiding the need for a known Hölder constant. The second algorithm employs a reducing transformation approach which consists of generating, in the feasible box, an α-dense curves, effectively converting the multivariate initial problem to a problem of a single variable. We prove the convergence of both algorithms. Their practical efficiency is evaluated through numerical experiments on some test functions and comparison with existing techniques.

MSC. 90C26, 65K05.

Keywords. Global optimization, Hölder condition, covering methods, Piyavskii's method, reducing transformation, α-dense curves.

1. INTRODUCTION

Let us consider the following box constrained global optimization problem of finding at least one point $x^* \in D$ and the corresponding optimal value g^* such that:

$$g^* = g(x^*) = \min_{x \in D} g(x) \tag{1}$$

where the objective function $g : D \to \mathbb{R}$ is non-smooth non convex. It is assumed to satisfy the Hölder condition on the n-dimensional compact box D:

$$D = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n.$$

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Specifically, there exist a Hölder constant $h_g > 0$ and a Hölder exponent $1/\beta$ ($\beta > 1$) such that:

$$|g(x) - g(y)| \leq h_g \| x - y\|^{1/\beta}, \quad \text{for all } x, y \in D. \tag{2}$$

Here, $\|\cdot\|$ denotes the Euclidean norm. A primary focus of this work is when the Hölder constant h_g is not known a priori.

Many real-world optimization problems are complex, involve minimizing continuous functions of n variables possessing multiple extrema over the feasible domain. Frequently, derivative information is either unavailable or its computation is expensive, rendering standard non-linear programming methods ineffective for finding global optima, as they typically converge only to local minima. Global optimization has thus become an active field of research [9]. The challenges are often amplified by local irregularities (non-smoothness) and the potential existence of numerous local minima within the feasible set. Deterministic global optimization strategies, particularly covering methods, offer a rigorous approach to tackle such problems [9, 17]. The first work on covering methods for univariate Lipschitz functions (the case $\beta = 1$ in (2)) was performed by Piyavskii [11], Evtushenko [4], and Shubert [15]. The widely discussed Piyavskii-Shubert algorithm guarantees convergence to the global minimum by constructing a sequence of improving piecewise lower-bounding functions (sub-estimators) based on the known Lipschitz constant [9]. More recently, research has addressed the optimization of less regular functions satisfying the Hölder condition with $\beta > 1$ [5, 10]. Gourdin et al. were among the first to study the global optimization of both univariate and multivariate Hölder functions using such deterministic approaches [5, 8].

The aim of this paper is twofold. First, we develop a novel technique for univariate Hölder optimization ($n = 1$), extending the ideas of Piyavskii. This method utilizes the secants linked to the Hölder support functions between evaluated points. We address both the case where the Hölder constant h_g is known and significantly, the case where the constant h_g is unknown. For the latter, we propose an adaptive estimation scheme for h_g, which is crucial for practical application and performance. Second, we address the multivariate case ($n > 1$). Direct generalization of Piyavskii's algorithm to higher dimensions is computationally challenging, primarily because finding the minimum of the multivariate sub-estimator involves determining intersections of complex hypersurfaces [5, 18]. To overcome this, we propose an approach based on dimension reduction. This method transforms the original n-dimensional problem (1) into a one-dimensional Hölder optimization problems by exploring the box D along an α-dense curve [20, 21, 22]. The univariate algorithm developed in the first part can then be effectively applied to these simpler problems.

The paper is organized as follows. Section 2 presents the univariate algorithm based on Hölder support functions and secant information. Section 3 describes the reducing transformation method for the multivariate case, detailing the use of a specific α-dense curve. Section 4 provides numerical results from experiments on test functions and compares the performance with existing methods. Section 5 concludes the work.

2. UNIVARIATE HÖLDER GLOBAL OPTIMIZATION

Let us begin by considering the problem (1), (2) for a function f with a single variable, i.e.,

$$\min_{x\in[a,b]} f(x),\ \ a,b \in \mathbb{R} \tag{3}$$

where the objective function f satisfying the following Hölder condition

$$|f(x) - f(y)| \le h_f\, |x-y|^{1/\beta}\,, \quad \text{for all } x, y \in [a,b]. \tag{4}$$

with a constant $h_f > 0$ and $\beta > 1$. Let $\varepsilon > 0$ be the desired accuracy with which the global minimum to be searched.

2.1. Sub-estimator function. When minimizing a non-convex function f over a feasible set, the general principle behind most deterministic global optimization methods is to relax the original non-convex problem in order to make the relaxed problem convex by utilizing a sub-estimator of the objective function f. The Hölder condition (4) allows one to construct such a lower bound for f over an interval $[a,b]$. From (4) we have

$$f(y) - h_f\, |x-y|^{1/\beta} \le f(x), \quad \forall x, y \in [a,b].$$

Let x_{i-1} and x_i be two distinct points in [a,b], typically with $x_{i-1} < x_i$. We define the following functions:

$$\begin{cases} U_{i-1}(x) = f(x_{i-1}) - h_f(x - x_{i-1})^{1/\beta}, & \text{for } x \ge x_{i-1}, \\ U_i(x) = f(x_i) - h_f(x_i - x)^{1/\beta}, & \text{for } x \le x_i. \end{cases}$$

By construction setting $y = x_{i-1}$ or $y = x_i$ in the Hölder inequality, these functions are lower bounds for $f(x)$ on their respective domains of definition within $[a,b]$:

$$\begin{cases} U_{i-1}(x) \le f(x), & \forall x \in [a,b] \text{ such that } x \ge x_{i-1}, \\ U_i(x) \le f(x), & \forall x \in [a,b] \text{ such that } x \le x_i. \end{cases}$$

The functions U_{i-1} and U_i are thus sub-estimators of f. Over any subinterval $[x_{i-1}, x_i] \subseteq [a,b]$, both functions are well-defined lower bounds. We can then define a tighter sub-estimator $\psi_i(x)$ on this sub-interval:

$$\psi_i(x) = \max\left\{U_{i-1}(x), U_i(x)\right\}.$$

This function $\psi_i(x)$ is convex (as the maximum of two convex functions, assuming $1/\beta \le 1$) and non-differentiable (typically at the point where $U_{i-1}(x) = U_i(x)$). It also satisfies:

$$\psi_i(x) \le f(x), \quad \forall x \in \left[x_{i-1}, x_i\right], \tag{5}$$

The global minimum of $\psi_i(x)$ over $[x_{i-1}, x_i]$ occurs at the point $\bar{x}$ where U_{i-1} and $U_i(x)$ intersect (assuming such an intersection exists within the interval). This point $\bar{x}$ is given by:

$$\bar{x} = \underset{[x_{i-1},x_i]}{\arg\min}\ \psi_i(x).$$

Thus, in each iteration of Piyavskii's algorithm, we must solve the following non-linear equation to find this intersection point:

$$U_{i-1}(x) - U_i(x) = 0. \tag{6}$$

Determining the unique solution $\bar{x}$ to equation (6) within $[x_{i-1}, x_i]$ is generally straightforward only for specific values of β. For instance, Gourdin et al. [5] provide analytical expressions for this solution when $\beta \in \{2, 3, 4\}$, assuming h_f is known a priori. However, when β is large or not an integer, solving equation (6) can be as complex as a general non-linear local optimization problem. To overcome this difficulty, we propose a new procedure below.

2.2. Procedure approach to the intersection point. Let θ_i be the same midpoint of the intervals $[U_i(x_{i-1}), U_{i-1}(x_{i-1})]$ and $[U_{i-1}(x_i), U_i(x_i)]$ so

$$\theta_i = \tfrac{f(x_{i-1})+f(x_i)-h_f(x_i-x_{i-1})^{1/\beta}}{2}. \tag{7}$$

As $U_{i-1}(x)$ and $U_i(x)$ are monotone continuous functions on the interval $[x_{i-1}, x_i]$ then they are bijective functions. Let U_{i-1}^{-1} and U_i^{-1} be the inverse functions respectively of U_{i-1} and U_i. We denote by $\mu_i = U_{i-1}^{-1}(\theta_i)$ and $\vartheta_i = U_i^{-1}(\theta_i)$. According to the definition of U_{i-1} and U_i we have:

$$\begin{cases} \mu_i = x_{i-1} + (\frac{f(x_{i-1})-\theta_i}{h_f})^{\beta}, \\ \vartheta_i = x_i - (\frac{f(x_i)-\theta_i}{h_f})^{\beta}. \end{cases} \tag{8}$$

Let $S^-(x)$ and $S^+(x)$ be the two straight secants linked to the Hölder support functions U_{i-1} and U_i on the interval $[\mu_i, \vartheta_i] \subset [x_{i-1}, x_i]$ and joining respectively the points (μ_i, θ_i), $(\vartheta_i, U_{i-1}(\vartheta_i))$ and $(\mu_i, U_i(\mu_i))$, (ϑ_i, θ_i), we get

$$S^-(x) = f(x_{i-1}) - h_f(\vartheta_i - x_{i-1})^{1/\beta} + \tfrac{h_f(\mu_i-x_{i-1})^{1/\beta}-h_f(\vartheta_i-x_{i-1})^{1/\beta}}{\vartheta_i-\mu_i}(x - \vartheta_i),$$

$$S^+(x) = f(x_i) - h_f(x_i - \vartheta_i)^{1/\beta} + \tfrac{h_f(x_i-\mu_i)^{1/\beta}-h_f(x_i-\vartheta_i)^{1/\beta}}{\vartheta_i-\mu_i}(x - \vartheta_i).$$

The intersection point of the two secants S^- and S^+ is the point z_i the approximate point $\bar{x}$ of the two Hölder support functions U_{i-1} and U_i (see Fig. 1).

Then

$$z_i = \tfrac{\left[f(x_{i-1})-f(x_i)-h_f(\vartheta_i-x_{i-1})^{1/\beta}+h_f(x_i-\vartheta_i)^{1/\beta}\right](\vartheta_i-\mu_i)}{h_f\left[(x_i-\mu_i)^{1/\beta}-(\mu_i-x_{i-1})^{1/\beta}-(x_i-\vartheta_i)^{1/\beta}+(\vartheta_i-x_{i-1})^{1/\beta}\right]} + v_i. \tag{9}$$

REMARK 1. *If $f(x_{i-1}) = f(x_i)$, the approximate point $\bar{x}$ is immediately the midpoint of the interval $[x_{i-1}, x_i]$. In this case, one can choose:*

$$z_i = \bar{x} = \tfrac{x_{i-1}+x_i}{2}. \tag{10}$$

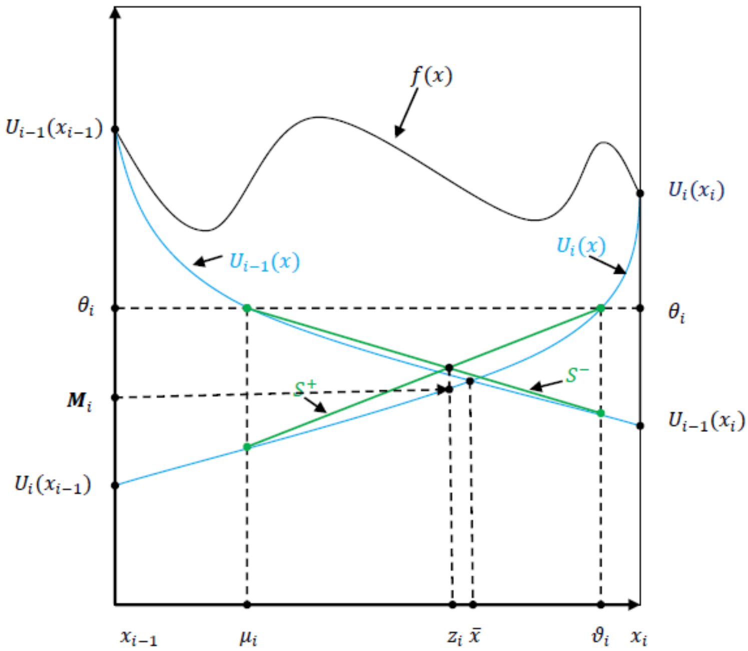

Fig. 1. Illustration of the secant-based approximation for the sub-estimator's intersection point.

Proposition 2. *Let f be a real univariate Hölder function with the constant $h_f > 0$ and $\beta > 1$ defined on the interval $[a, b]$. Let the value $\mathbf{M}_i = \min \{U_{i-1}(z_i), U_i(z_i)\}$ (as a constant lower bound of f on $[x_{i-1}, x_i] \subset [a, b]$). Then we have:*

$$\mathbf{M}_i < f(x), \qquad \forall x \in [x_{i-1}, x_i]. \tag{11}$$

Proof. The value $\mathbf{M}_i$ is given by replacing the variable x in the two functions $U_{i-1}(x)$ and $U_i(x)$ by the expression of z_i. Since we have defined the function

$$\psi_i(x) = \max \{U_{i-1}(x), U_i(x)\},$$

as a sub-estimator function of $f(x)$ over the interval $[x_{i-1}, x_i]$. Then,

$$\psi_i(x) \leq f(x), \ \forall x \in [x_{i-1}, x_i].$$

The function $U_{i-1}(x)$ is strictly decreasing and $U_i(x)$ is strictly increasing over the interval $[x_{i-1}, x_i]$. Thus, it follows

$$\min \{U_{i-1}(x), U_i(x)\} \leq \min_{[x_{i-1}, x_i]} \psi_i(x), \ \forall x \in [x_{i-1}, x_i].$$

In particular, for $x = z_i$ it then follows:

$$\mathbf{M}_i = \min \{U_{i-1}(z_i), U_i(z_i)\} < f(x), \ \forall x \in]x_{i-1}, x_i[.$$

□

Algorithm 1 MSPA with known value of h_f

`Step 1:` Initialization
$[a,b] \subset \mathbb{R}$ a given search, Hölder parameters $h_f, \beta > 1$,
ε accuracy of global minimization
$k \leftarrow 2, x_1 \leftarrow a, x_2 \leftarrow b$
`Step k:` The sampling points $x_1, x_2, ..., x_k$ are ordered such that

$$a = x_1 < x_2 < ... < x_k = b.$$

for $i = 2$ **to** k **do**
  if $f(x_{i-1}) \neq f(x_i)$ **then**
    θ_i as defined in (7), μ_i, ϑ_i as defined in (8), z_i as defined in (9)
  else
    $z_i = \frac{x_{i-1}+x_i}{2}$
  end if
  $\mathbf{M}_i = \min\left\{f(x_{i-1}) - h_f(z_i - x_{i-1})^{1/\beta}, f(x_i) - h_f(x_i - z_i)^{1/\beta}\right\}$
end for

$$\mathbf{M}_\rho = \min\{\mathbf{M}_i \ : \ 2 \leq i \leq k\} \tag{12}$$

$z_\rho = \arg min\{\mathbf{M}_\rho\}$
$x_\rho = z_\rho$
if $|x_\rho - x_{\rho-1}| > \varepsilon$ **then**

$$x_{k+1} = z_\rho \tag{13}$$

  $k \leftarrow k+1$
  go to the Step k
else
  $f_{opt} = \min\{f(x_i) : 1 \leq i \leq k\}$
  Stop
end if
return f_{opt}.

2.3. Convergence result of MPSA.

THEOREM 3. *Let $f(x)$ be a real function defined on a closed interval $[a,b]$, satisfying (4) with $h_f > 0$ and $\beta > 1$. Let x^* be a global minimizer of $f(x)$ over $[a,b]$. Then the sequence $(x_k)_{k\geq 1}$ generated by the MSPA algorithm converges to x^*. i.e.,*

$$\lim_{k\to+\infty} f(x_k) = f(x^*) = \min_{x\in[a,b]} f(x).$$

Proof. The proof is based on the construction of a sequence of points $(x_k)_{k\geq 1}$ generated by Algorithm 1, which converges to a limit point that is the global minimizer of f on $[a,b]$. Let $x_1, x_2, x_3, ...$ be the sampling sequence satisfying (9), (12), (13). Let us consider that $x_m \neq x_{m'}$ for all $m \neq m'$, the set of the elements of the sequence $(x_k)_{k\geq 1}$ is then infinite and therefore has at least one limit point in $[a,b]$. Let $\mathbf{z}$ be any limit point of $(x_k)_{k\geq 1}$ such that $\mathbf{z}\neq a$, $\mathbf{z}\neq b$, then the convergence to $\mathbf{z}$ is bilateral (one can see [10]). Consider the interval $[x_{\mu-1},x_\mu]$ determined by (12) at the $(k+1)$-th iteration. According (9) and (13), we have that the new point x_{k+1} divides the interval $[x_{\mu-1},x_\mu]$ into the subintervals $[x_{\mu-1},x_{k+1}]$ and $[x_{k+1},x_\mu]$, so we can deduce

(14) $$\max\{x_{k+1}-x_{\mu-1},x_\mu-x_{k+1}\}\leq\ |x_\mu-x_{\mu-1}|\,.$$

Consider now an interval $[x_{\rho(k)-1},x_{\rho(k)}]$ which contains $\mathbf{z}$, because $\mathbf{z}$ is a limit point of $(x_k)_{k\geq 1}$ and using (9), (12), (13) and (14), we obtain:

(15) $$\lim_{k\to+\infty}(x_{\rho(k)-1}-x_{\rho(k)})=0.$$

In addition, the value $\mathbf{M}_{\rho(k)}$ that corresponds to $[x_{\rho(k)-1},x_{\rho(k)}]$, is given by

(16) $$\mathbf{M}_{\rho(k)}=\min\left\{f(x_{\rho(k)-1})-h_f(z_\rho-x_{\rho(k)-1})^{1/\beta},f(x_{\rho(k)})-h_f(x_{\rho(k)}-z_\rho)^{1/\beta}\right\},$$

where z_ρ is obtained by replacing i by ρ in (9). As $\mathbf{z}\in[x_{\rho(k)-1},x_{\rho(k)}]$ and from (15) then we have

(17) $$\lim_{k\to+\infty}\mathbf{M}_{\rho(k)}=f(\mathbf{z}).$$

On the other hand, according to (11)

(18) $$\mathbf{M}_{j(k)}\leq f(x),\forall x\in[x_{j(k)-1},x_{j(k)}].$$

From (12), $\mathbf{M}_{\rho(k)}=\min\{\mathbf{M}_j,j=2,...,k\}$, then

$$\mathbf{M}_{\rho(k)}\leq\mathbf{M}_{j(k)},\forall x\in[x_{j(k)-1},x_{j(k)}],$$

and since $[a,b]=\bigcup_{j=2}^{k}[x_{j(k)-1},x_{j(k)}]$, hence

(19) $$\lim_{k\to+\infty}\mathbf{M}_{\rho(k)}\leq\mathbf{M}_{j(k)},\forall x\in[a,b],$$

and from (18), (19) we get

$$\lim_{k\to+\infty}\mathbf{M}_{\rho(k)}\leq f(x),\forall x\in[a,b].$$

Since x^* is the global minimizer of f over $[a,b]$

$$\lim_{k\to+\infty}\mathbf{M}_{\rho(k)}\leq f(x^*)\leq f(\mathbf{z}),$$

thus, we have

$$0\leq f(\mathbf{z})-f(x^*)\leq f(\mathbf{z})-\lim_{k\to+\infty}\mathbf{M}_{\rho(k)}=0,$$

then

$$f(\mathbf{z})=f(x^*).$$

By the condition (4), the function f must be continuous on $[a,b]$ so that

$$f(\mathbf{z})=f(\lim_{k\to+\infty}x_k)=\lim_{k\to+\infty}f(x_k)\ =f(x^*).$$

□

2.4. Estimating of unknown Hölder constant. The algorithm MSPA presented in the section 2 of this paper is applied to a class of Hölder continuous functions defined on the closed and bounded interval $[a, b]$ of $\mathbb{R}$ when suppose a priori knowledge of the Hölder constant $h_f > 0$. However, the constant h_f may be, in most situations, unknown and no procedure is available to obtain a guaranteed overestimate of it. Then, there is a need to find an approximate evaluation of a minimal value of h_f. The algorithm we will extend does not require a priori knowledge of the h_f. To over come this situation, a typical procedure is to look for an approximation of h_f during the course of search ([9], [10], [19]). We consider a global estimate of h_f, for each iteration. Let the sampling points $a = x_1 < x_2 < \cdots < x_n = b$ and the values $f(x_1), f(x_2), ..., f(x_n)$ are also calculated. Let

$$h_f^i = \frac{|f(x_i)-f(x_{i-1})|}{(x_i-x_{i-1})^{1/\beta}}, \quad \text{for} \quad i = 2, ..., k$$

and

$$h_f^k = \max\left\{h_f^i, \quad i = 2, ..., k\right\}.$$

Let $\lambda > 1$ be the multiplicative parameter as an input of the algorithm and ν be a small positive number. The global estimate of h_f then is given by:

$$\widetilde{h}_f = \begin{cases} \lambda h_f^k, & \text{if } f(x_i) \neq f(x_{i-1}), \ \forall i \geq 2 \\ \lambda \nu, & \text{else.} \end{cases}$$

The constant λ acts as a safety margin for estimating the unknown Hölder constant.

- When $\lambda = 1$: The algorithm operates without a safety margin. This can lead to the premature elimination of potentially optimal regions and a failure to detect the global minimum.
- When $\lambda > 1$: Experimental analysis shows that values greater than 1 increase the reliability of the algorithm for the majority of tested functions.
- While a larger λ (or increasing to a $\lambda' > \lambda$) increases the multiplicative safety factor, it also increases the number of function evaluations and the computation time.

The condition $f(x_{i-1}) \neq f(x_i)$, for all i, indicates that the objective function f is not constant over the feasible interval $[a, b]$. Now from the definition of the global estimate $\widetilde{h}_f$, we replace in the formulas of θ_i, μ_i and ϑ_i, the constant h_f by $\widetilde{h}_f$ also the point $(z_i, \mathbf{M}_i(z_i))$ is replaced in the structure of the algorithm MSPA by the point $(\widetilde{z}_i, \mathbf{M}_i(\widetilde{z}_i))$ where:

$$\widetilde{z}_i = \frac{\left[f(x_{i-1})-f(x_i)-\widetilde{h}_f(\vartheta_i-x_{i-1})^{1/\beta}+\widetilde{h}_f(x_i-\vartheta_i)^{1/\beta}\right](\vartheta_i-\mu_i)}{\widetilde{h}_f\left[(x_i-\mu_i)^{1/\beta}-(\mu_i-x_{i-1})^{1/\beta}-(x_i-\vartheta_i)^{1/\beta}+(\vartheta_i-x_{i-1})^{1/\beta}\right]} + v_i \tag{20}$$

$$\mathbf{M}_i(\widetilde{z}_i) = \min\left\{f(x_{i-1}) - \widetilde{h}_f(\widetilde{z}_i - x_{i-1})^{1/\beta}, f(x_i) - \widetilde{h}_f(x_i - \widetilde{z}_i)^{1/\beta}\right\}. \tag{21}$$

Finally, from (20) and (21) we obtain an algorithm noted MSPA_{es} which is based on the use of the estimation constant $\widetilde{h}_f$ during the course of the algorithm.

3. MULTIVARIATE HÖLDER GLOBAL OPTIMIZATION

Let us consider now the multivariate case, i.e., the problem (1), (2) with $x \in \mathbb{R}^n$ and $n \geq 2$.

3.1. Reducing transformation procedure. Directly applying Piyavskii's method to multivariate Hölder optimization is often impractical due to the computational cost of repeatedly minimizing the multivariate sub-estimator function [5, 18]. This sub-estimator is typically the upper envelope of many individual support functions $(g(x_i) - h_g\|x - x_i\|^{1/\beta})$, and finding its minimum involves complex geometric operations (finding intersections of hypersurfaces).

To circumvent this difficulty, we employ a dimension reduction strategy. Such approaches, often utilizing space-filling or α-dense curves to map the n-dimensional domain to a one-dimensional interval, have a history in global optimization, see, e.g., Butz [2], Strongin [16] and Ziadi et al. [7, 14, 20, 21]).

Our proposed multivariate algorithm combines a specific reducing transformation with the specialized univariate Hölder optimization algorithm MSPA developed in Section 2. The key idea is to define a parametric curve $C_\alpha : [0, T] \to D$, where $D \subset \mathbb{R}^n$ is the original search box, such that the curve $C_\alpha(t) = (c_1(t), \ldots, c_n(t))$ is α-dense in D. This property guarantees that for any point $x \in D$, there exists a $t \in [0, T]$ such that $\|C_\alpha(t) - x\| \leq \alpha$.

Definition 4. *Let $\mathbb{J}$ be an interval of $\mathbb{R}$. We say that a parametrized curve of $\mathbb{R}^n$ defined by $C_\alpha : \mathbb{J} \to D = [a_1, b_1] \times \cdots \times [a_n, b_n]$ is α-dense in D, if for all $x \in D$, $\exists t \in \mathbb{J}$ such that*

$$d(x, C_\alpha(t)) \leq \alpha,$$

where d stands for Euclidean distance in $\mathbb{R}^n$.

By restricting the objective function g to this curve, we obtain a univariate function $f : [0, T] \to \mathbb{R}$, defined by:

$$f(t) = g(C_\alpha(t)).$$

Crucially, if g is Hölder continuous and $C_\alpha(t)$ is sufficiently regular (Lipschitz), the resulting function $f(t)$ is also Hölder continuous on the interval $[0, T]$. The original n-dimensional problem (1) is thus effectively reduced to the one-dimensional problem:

$$\text{(P)} \qquad \min_{t \in [0,T]} f(t).$$

This univariate problem can be efficiently solved using the algorithm from Section 2. The quality of the approximation to the original problem depends on the density parameter α of the curve.

Theorem 5. *Let $C_\alpha(t) = (c_1(t), \ldots, c_n(t))$ be a function defined from $[0, T]$ into D. Let $\alpha > 0$ and μ denote the Lebesgue measure, such that:*

1) *$(c_i)_{1\leq i\leq n}$ are continuous and surjective.*
2) *$(c_i)_{2\leq i\leq n}$ are periodic with respective periods $(p_i)_{2\leq i\leq n}$.*
3) *For any interval $I \subset [0, T]$ and for any $i \in \{2, \ldots, n\}$, we have:*

$$\mu(I) \leq p_i \Rightarrow \mu(c_{i-1}(I)) < \alpha.$$

Then, for $t \in [0, T]$, the function $C_\alpha(t)$ is a parametrized $\sqrt{n-1}\alpha$-dense curve in D. (The proof can be found in [22]).

Corollary 6. *Let $C_\alpha(t) = (c_1(t), \ldots, c_n(t)) : \left[0, \frac{\pi}{\alpha_1}\right] \to D$ a function defined by:*

$$c_i(t) = \tfrac{a_i - b_i}{2}\cos(\alpha_i t) + \tfrac{a_i + b_i}{2}, \qquad i = 1, 2, \ldots, n,$$

where $\alpha_1, \ldots, \alpha_n$ are given strictly positive constants satisfying the relationships

$$\alpha_i \geq \tfrac{\pi}{\alpha}(b_{i-1} - a_{i-1})\alpha_{i-1}, \qquad \forall i = 2, \ldots, n.$$

Then the curve defined by the parametric curve $C_\alpha(t)$ is $\sqrt{n-1}\alpha$-dense in D [22].

Remark 7. *According to Corollary 6, the parametrized curve $C_\alpha(t)$ is α-dense in the box D. Moreover, the function C_α is Lipschitz on $\left[0, \frac{\pi}{\alpha_1}\right]$ with constant:*

$$L_\alpha = \tfrac{1}{2}\left(\sum_{i=1}^{n}(b_i - a_i)^2\alpha_i^2\right)^{1/2}.$$

Theorem 8. *The function $f(t) = g(C_\alpha(t))$ for $t \in [0, \frac{\pi}{\alpha_1}]$ is a Hölder function with constant $H_f = h_g L_\alpha^{1/\beta}$ and exponent $\beta > 1$.*

Proof. For t_1 and t_2 in $\left[0, \frac{\pi}{\alpha_1}\right]$, we have

$$|f(t_1) - f(t_2)| = |g(C_\alpha(t_1)) - g(C_\alpha(t_2))| \leq h_g \left\|C_\alpha(t_1) - C_\alpha(t_2)\right\|^{1/\beta}.$$

As the function of the parametric curve C_α is Lipschitz on $\left[0, \frac{\pi}{\alpha_1}\right]$ with the constants L_α, we have

$$\left\|C_\alpha(t_1) - C_\alpha(t_2)\right\| \leq L_\alpha \left|t_1 - t_2\right|,$$

then

$$|f(t_1) - f(t_2)| \leq h_g \left(L_\alpha \left|t_1 - t_2\right|\right)^{1/\beta},$$

hence

$$|f(t_1) - f(t_2)| \leq h_g L_\alpha^{1/\beta} \left|t_1 - t_2\right|^{1/\beta}.$$

□

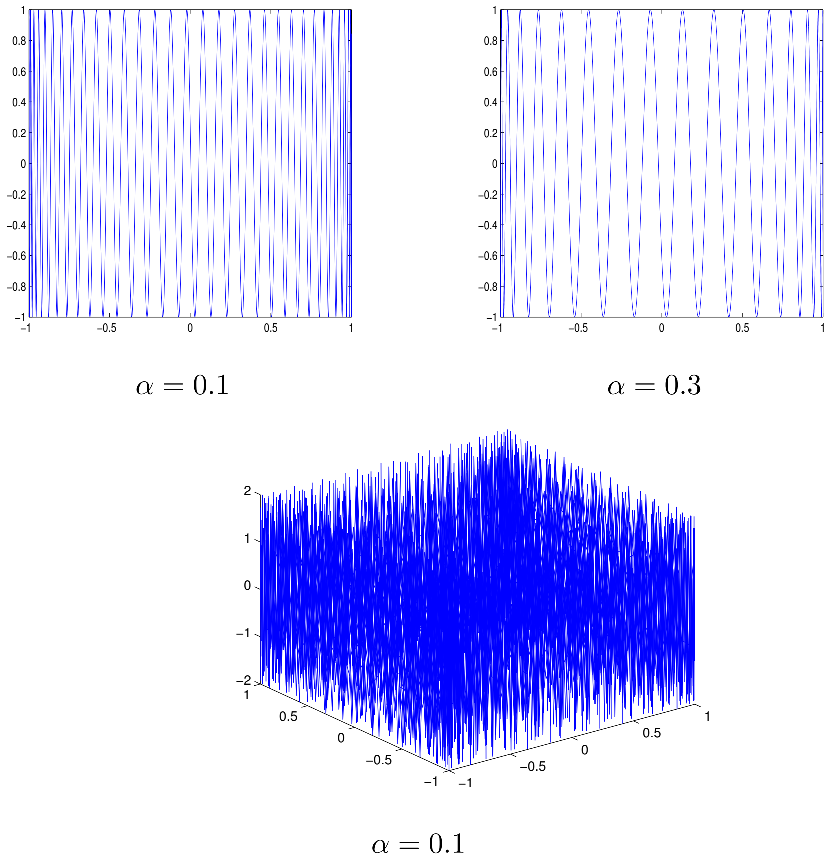

Fig. 2. The densification of the square $[-1,1]^2$ and the cube $[-1,1]^3$ by the support of α-dense curves with different values of α.

3.2. The mixed RT-MSPA method. For determine the global minimum of g on D the mixed RT-MSPA method consists of two steps: the reducing transformation step and the application of the MSPA algorithm to the function $f(t)$ on $[0, \frac{\pi}{\alpha_1}]$.

3.3. Convergence Result of RT-MPSA.

THEOREM 9. *Let g be a Hölder function satisfying the condition (2) over D and $\mathbf{m}$ be the global minimum of g on D. Then the mixed RT-MSPA algorithm converges to the global minimum with an accuracy at least equal to ε.*

Proof. Denote by $\mathbf{m}^*$ the global minimum of f on $[0, \frac{\pi}{\alpha_1}]$, where $f(t) = g(C_\alpha(t))$. On the other hand, let us designate by f_ε the global minimum of the problem (P) obtained by the mixed method RT-MSPA. Let us show that

$$f_\varepsilon - \mathbf{m} \leq \varepsilon.$$

Algorithm 2 RT-MSPA with known value of h_g

Require: $D = [a_1, b_1] \times \cdots \times [a_n, b_n]$ the search box.
The multivariate objective function with known Hölder parameters $h_g > 0$ and $\beta > 1$. $\varepsilon > 0$ small accuracy of the global minimization.
Ensure:
First part: $C_\alpha(t)$ the parametric α-dense curve in D.
$f(t)$ the univariate H_f-Hölder function.
Second part: f_{opt} the best global minimum of f.
First part:
Define $C_\alpha(t)$, $t \in [0, \pi/\alpha_1] \longrightarrow D$
$\alpha = \left(\frac{\varepsilon}{2H_f}\right)^\beta$, $\alpha_1 = 1$.
for $i = 2$ **to** n **do**
$\alpha_i = \frac{\pi}{\alpha}(b_{i-1} - a_{i-1})\alpha_{i-1}$
end for
for $i = 1$ **to** n **do**
$c_i(t) = \frac{a_i - b_i}{2}\cos(\alpha_i t) + \frac{a_i + b_i}{2}$
end for
$C_\alpha(t) = (c_1(t), c_2(t), ..., c_n(t))$ and $f(t) = g(C_\alpha(t))$.
Second part:
Initialization
$[0, \pi]$ the search interval
$k \leftarrow 2, \rho \leftarrow 2, t_1 \leftarrow 0, t_2 \leftarrow \pi$
`Step k:` $t_1, t_2, \ldots, t_k$ are ordered such that $0 = t_1 < t_2 < \cdots < t_k = \pi$.
for $i = 2$ **to** k **do**
$[t_{i-1}, t_i] \subset [0, \pi]$
if $f(t_{i-1}) \neq f(t_i)$ **then**
θ_i, μ_i, ϑ_i and z_i are defined respectively in (5), (6) and (7) (with respect to the variable $t > 0$.)
else
$z_i = \frac{t_{i-1} + t_i}{2}$
end if
$\mathbf{M}_i = \min\left\{f(t_{i-1}) - H_f(z_i - t_{i-1})^{1/\beta}, f(t_i) - H_f(t_i - z_i)^{1/\beta}\right\}$
end for
$\mathbf{M}_\rho = \min\{\mathbf{M}_i \ : \ 2 \leq i \leq k\}$
$t_\rho = z_\rho$
if $|t_\rho - t_{\rho-1}| > \varepsilon$ **then**
$t_{k+1} = z_\rho$
$k \longleftarrow k + 1$
go to the Step k
else
$f_{opt} = min\{f(t_i) : 1 \leq i \leq k\}$
Stop
end if
return f_{opt}

1) As g is continuous on D, there exists a point $\mathbf{x} \in D$ such that $\mathbf{m} = g(\mathbf{x})$. Moreover, there exists $t_0 \in [0, \frac{\pi}{\alpha_1}]$ such that $\|\mathbf{x} - C_\alpha(t_0)\| \leq \left(\frac{\varepsilon}{2h_g}\right)^\beta$ so that

$$\|g(\mathbf{x}) - g(C_\alpha(t_0))\| \leq \tfrac{\varepsilon}{2}.$$

And therefore

$$g(C_\alpha(t_0)) - \mathbf{m} \leq \tfrac{\varepsilon}{2}.$$

But from $\mathbf{m} \leq \mathbf{m}^* \leq g(C_\alpha(t_0))$, we deduce that

$$\mathbf{m}^* - \mathbf{m} \leq \tfrac{\varepsilon}{2}. \tag{22}$$

2) As f is continuous on $[0, \frac{\pi}{\alpha_1}]$, there exists a point $t^* \in [0, \frac{\pi}{\alpha_1}]$ such that $\mathbf{m}^* = f(t^*)$, involving t^* as a global minimizer of f. Then t^* is a limit point of the sequence $(t_k)_{k\geq 1}$ obtained by the mixed algorithm.

Hence $t^* \in [t_{\rho(k)-1}, t_{\rho(k)}]$ and $\lim\limits_{k\to+\infty}(t_{\rho(k)} - t_{\rho(k)-1}) = 0$. i.e.,

$$\forall\varepsilon > 0,\ \exists K \in \mathbb{N} \text{ such that } \forall k \geq K,\ |t_{\rho(k)} - t_{\rho(k)-1}| < \varepsilon.$$

On the other hand, since f_ε is the global minimum obtained after k iterations, we obtain:

$$\exists t_\varepsilon \in [t_{s-1}, t_s] : |t_s - t_{s-1}| \leq \left(\tfrac{\varepsilon}{2H_f}\right)^\beta \text{ and } f_\varepsilon = f(t_\varepsilon)$$

so that

$$\begin{cases} \mathbf{M}_s = \min\left\{f(t_{s-1}) - H_f\,(t_\varepsilon - t_{s-1})^{1/\beta},\ f(t_s) - H_f\,(t_s - t_\varepsilon)^{1/\beta}\right\}, \\ \mathbf{M}_s \leq f(t^*) \leq f(t_\varepsilon) \quad \text{and} \quad t^* \in [t_{s-1}, t_s]. \end{cases}$$

Consequently,

$$f_\varepsilon - \mathbf{m}^* = f(t_\varepsilon) - f(t^*) \leq H_f\,|t_\varepsilon - t^*|^{1/\beta} \leq \frac{\varepsilon}{2}. \tag{23}$$

Finally, from (22) and (23), the result of Theorem 9 is proved. □

4. NUMERICAL EXPERIMENTS

This section details the numerical experiments conducted to evaluate the performance of the proposed MSPA algorithm compared to existing methods, specifically SPA [10] and TPA [3], [6]. The evaluation encompasses both single-variable and multivariate optimization problems.

Two series of standard test functions and their corresponding parameters are listed in Table 1 and Table 4. These functions exhibit diverse properties, such as non-convexity and non-differentiability, and possess multiple local and global minima.

The primary performance criteria used for comparison are the total number of function evaluations (Ev) and the CPU execution time in seconds ($T(s)$). All algorithms were implemented in MATLAB and executed on a PC.

For all single-variable experiments, the desired accuracy for locating the global minimum was set to $\varepsilon = 10^{-4}(b - a)$, where $[a, b]$ denotes the search interval. Detailed results for the known and estimated Hölder constants are presented in Table 2 and Table 3, respectively. In these tables, bold values indicate the best performance achieved among the compared algorithms for each criterion (Ev and $T(s)$).

For multivariate optimization, we implemented the Reducing Transformation (RT) approach combined with the MSPA algorithm, utilizing α-dense curves with a density parameter $\alpha = 0.1$.

When the Hölder constant h_g is known, a theoretical relationship exists between α and the accuracy ε that guarantees convergence. However, deriving α from small values of ε results in extremely small α values, leading to excessive computation time and function evaluations. Consequently, we fixed $\alpha = 0.1$, which proved sufficient to locate the global minimum for all test functions. Similarly, for cases where h_g is unknown, we utilized estimates of the constant h_f with a fixed value of $\alpha = 0.1$ to avoid the computational complexity associated with the dependence of h_f on α.

Regarding the choice of the parameter λ, we adopted a standardized approach to ensure fair comparison. Although theoretically $\lambda > 1$ is sufficient, empirical evidence suggests that methods such as RT-TPAes require $\lambda \geq 1.5$ for convergence. Consequently, we utilized $\lambda = 1.5$ for the majority of experiments across all algorithms (RT-SPAes, RT-MSPAes, and RT-TPAes). While an optimal λ exists for each function, individual tuning is impractical; thus, fixed values were used for broad classes of functions. The only exceptions were specific test problems (noted in Table 6).

Comparative results for multivariate optimization using known and estimated Hölder constants are presented in Table 5 and Table 6, respectively.

4.1. Discussions and remarks. The primary contributions of this paper are the development of MSPA, an enhanced version of Piyavskii's algorithm for finding global minima of univariate Hölder continuous functions, and its extension, RT-MSPA, for solving higher-dimensional problems.

For the univariate problem, the comparison involves two cases based on the availability of the Hölder constant h_f:

- **Known Hölder Constant (h_f):** The standard SPA, TPA, and MSPA algorithms were used directly. The results are presented in Table 2.
- **Unknown Hölder Constant (h_f):** Modified versions of the algorithms, employing an estimation procedure for the Hölder constant, were used. These are denoted as SPA_{es}, TPA_{es} and MSPA_{es}, respectively. These versions utilize an estimate $\widetilde{h}_f$ of the true Hölder constant h_f. The first remark in Table 3 that the results are obtained with the same value of $\lambda = 1.5$ (multiplicative parameter) and $\nu = 10^{-8}$ (tolerance parameter).

In higher-dimensional problems, comparative results are also presented for two cases, based on the availability of the Hölder constant h_g:

$N°$	Function	Interval	Hölder constant h_f	Ref.
1	$\min\left\{\sqrt{\lvert x+4\rvert}-1, \sqrt{\lvert x+1\rvert}-1.005, \sqrt{\lvert x-3\rvert}+0.5\right\}$	$[-5,5]$	2	[19]
2	$\left\lvert 1.5-1.5\sqrt{\lvert 1-x^2\rvert}\right\rvert$	$[-2,2]$	4	*new*
3	$\begin{cases} -\sqrt{2x-x^2} & \text{if } x \le 2 \\ -\sqrt{-x^2+8x-12} & \text{otherwise} \end{cases}$	$[0,6]$	9.798	[5]
4	$\begin{cases} 0.35\sqrt{\lvert x-0.25\rvert} & \text{if } x \le \frac{1}{2} \\ x & \text{otherwise} \end{cases}$	$[0,1]$	1.35	*new*
5	$\begin{cases} 4\sqrt{\lvert x-\frac{1}{2.5}\rvert} & \text{if } x \le \frac{1}{2} \\ 8.5x & \text{otherwise} \end{cases}$	$[0,1]$	12.5	*new*
6	$-\sqrt{1-x^2}$	$[-0.5,0.5]$	$\sqrt{2}$	[18]
7	$-\frac{2\sin x-1}{\sqrt{\lvert x+1\rvert+2}}$	$[-5,5]$	7.8	[3]
8	$\lvert x-0.25\rvert^{\frac{2}{3}}-3\cos\frac{x}{2}$	$[-0.5,0.5]$	4.26	[12]
9	$-\cos(x)e^{(1-\frac{\sqrt{\lvert\sin(\pi x)-0.5\rvert}}{\pi})}$	$[0,1]$	4.3	[12]
10	$5\cos x+\sqrt{0.8\,\lvert x\rvert}$	$[-10,8]$	5.8	*new*
11	$-\sqrt{\frac{9.5}{4}-x^2}-\frac{\sqrt{5}}{2}$	$[-1.5,1.5]$	$\sqrt{3}$	[12]
12	$-\cos\left(x+\frac{\pi}{2}-1\right)\exp\left(1-\frac{1}{\pi}\sqrt{\lvert\sin\pi(x\frac{\pi}{2}-1)-0.5\rvert}\right)$	$[-1.5,1.5]$	7.3	[12]
13	$-\left\lvert\cos\left(\frac{\pi}{2}x\right)\right\rvert\left\lvert\frac{\sqrt{19}-x}{\sqrt{2}-1}\right\rvert^{\frac{1}{2}}$	$[-1.5,1.5]$	5.4	[12]
14	$-\sqrt{1-\sin^2 x}$	$[-1,1]$	1	*new*
15	$-\sqrt{16-\cos^2 x}$	$[-2\pi,2\pi]$	1	*new*
16	$-\sqrt{1-x^2}+\sin x$	$[-1,1]$	2.41	*new*
17	$-\lvert 8-x^3\rvert^{\frac{2}{3}}-\cos x-\sin 3x$	$[-2,2]$	7.73	*new*
18	$\sum_{k=1}^{3}\frac{1}{k}\left\lvert\sin((\frac{3}{k}+1)x+\frac{1}{k})\right\rvert\,\lvert x-k\rvert^{\frac{1}{2}}$	$[0,3]$	6.83	[13]
19	$-\sqrt{25-x^2}-shx$	$[-2.5,2.5]$	8.36	*new*
20	$-\frac{2\cos x-1}{\sqrt{\lvert x+1\rvert+2}}$	$[-5,5]$	7.8	*new*

Table 1. Univariate Hölder test functions.

Problem number		SPA		TPA		MSPA	
$N°$	β	Ev	$T(s)$	Ev	$T(s)$	Ev	$T(s)$
1	2	78	0.1167	**74**	**0.0639**	78	0.0655
2	2	**2005**	**1.5050**	2107	1.5976	2129	1.6245
3	2	4371	9.0077	**2812**	**4.3383**	3965	7.8945
4	2	254	0.1302	**96**	**0.0711**	104	0.0735
5	2	295	0.1940	**79**	**0.0689**	110	0.0957
6	2	3433	4.0167	3061	3.1928	**3013**	**3.1567**
7	2	4403	6.6673	**4185**	**5.9728**	4487	6.9423
8	3/2	87	0.0642	**86**	**0.0636**	89	0.0677
9	2	87	0.0657	86	0.0676	**82**	**0.0650**
10	2	704	0.2988	**46**	**0.0653**	696	0.2894
11	2	2867	2.8312	**1721**	**1.1843**	2149	1.6457
12	2	207	0.1419	216	0.0942	**206**	**0.0905**
13	2	**1179**	0.6164	1333	0.6984	1226	**0.6143**
14	2	2109	1.7937	1701	1.0748	**1527**	**0.8923**
15	2	**3169**	**3.5469**	3233	3.7415	3201	3.6079
16	2	2109	1.9788	1587	0.9685	**1456**	**0.8341**
17	3/2	668	0.6360	**70**	**0.0637**	697	0.2672
18	2	1342	0.7609	874	0.3612	**864**	**0.3611**
19	2	91	0.1198	83	0.0703	**79**	**0.0688**
20	2	2787	2.7692	1835	1.2205	**1809**	**1.2010**

Table 2. Comparison of results between MSPA and the existing algorithms with known value of h_f.

Problem number		SPA$_{es}$		TPA$_{es}$		MSPA$_{es}$	
$N°$	β	Ev	$T(s)$	Ev	$T(s)$	Ev	$T(s)$
1	2	**37**	0.0116	48	0.0104	**37**	**0.0064**
2	2	1215	5.4778	962	3.5355	**667**	**1.8018**
3	2	2271	20.869	**1819**	**13.717**	1877	15.585
4	2	1288	6.4384	546	1.1884	**240**	**0.2420**
5	2	2854	31.324	**575**	**1.2967**	1104	4.8313
6	2	1501	8.6159	1841	12.972	**1438**	**8.3665**
7	2	**1018**	**3.8064**	1224	5.7721	1467	8.4715
8	3/2	22	**0.0025**	23	0.0026	**21**	0.0030
9	2	32	0.0049	31	0.0047	**30**	**0.0044**
10	2	986	3.7106	1062	4.3083	**815**	**2.7356**
11	2	**1937**	**14.211**	2446	22.842	2676	28.106
12	2	**54**	**0.0126**	69	0.0200	66	0.0194
13	2	1193	5.4486	**914**	**3.2404**	1364	7.3024
14	2	1391	7.3437	1707	11.168	**1282**	**6.4865**
15	2	1350	6.930	1540	9.0673	**1045**	**4.3092**
16	2	1995	15.269	**1545**	**9.1268**	1595	10.213
17	3/2	910	3.5283	**638**	**1.7160**	655	2.6685
18	2	578	1.2773	**386**	**0.5728**	437	0.7776
19	2	77	0.0246	74	0.0231	**66**	**0.0191**
20	2	687	1.7730	748	2.1631	**553**	**1.2303**

Table 3. Comparison of results between MSPA$_{es}$ and the existing algorithms with unknown value of h_f.

$N°$	Function	Box	Hölder constant h_g	Ref.
1	$\sum_{k=1}^{3} \frac{1}{k} \left\|\cos((\frac{3}{k}+1)(x+5)+\frac{1}{k})\right\| \|x-y\|^{\frac{1}{3}}$	$[-5,5]^2$	14.77	[18]
2	$-\left\|\cos(x)\cos(y)e^{(1-\frac{\sqrt{x^2+y^2}}{\pi})}\right\|$	$[-1,1]\times[-1,2]$	5.0679	[14]
3	$\sum_{k=1}^{3} \frac{1}{2k} \left\|\cos((\frac{3}{2k}+1)x+\frac{1}{2k})\right\| \|x-y\|^{\frac{2}{3}}$	$[-0.5,0.5]^2$	15.8	[13]
4	$\|x+y-0.25\|^{\frac{2}{3}} - 3\cos\frac{x}{2}$	$[-0.5,0.5]^2$	4.26	[13]
5	$-\cos(x)\sin(y)e^{(1-\frac{\sqrt{x^2+y^2}}{\pi})}$	$[-1,1]\times[-1,2]$	5.0679	*new*
6	$max\left\{\sqrt{\|x\|}, \sqrt{\|y\|}\right\}$	$[-1,1]^2$	1	[1]
7	$\sqrt{\|x\|+\|y\|}$	$[-1,1]^2$	$(\sqrt{2})^{\frac{1}{2}}$	[1]
8	$\sqrt{\|x\|}+\sqrt{\|y\|}$	$[-1,1]^2$	2	[1]
9	$\sqrt{\|x+1\|}+\sqrt{\|y+2\|}+\sqrt{\|z+\sqrt{6}\|}$	$[-1,1]^3$	3	*new*
10	$-10e^{-\sqrt{0.5(\|x\|+\|y\|)}}$	$[-2,12]^2$	$\frac{10}{\sqrt{2}}$	[1]
11	$\|3+\cos(\sqrt{x^2+y^2})\|$	$[-1,1]^2$	$\sqrt{2}$	[14]
12	$\|\sin(\sqrt{x^2+y^2})\|$	$[-1,1]^2$	$\sqrt{2}$	[14]
13	$\frac{1}{2}\sin(\sqrt{\|x-y\|})-\frac{1}{2}\sin(\sqrt{\|x+y\|})$	$[0,1]\times[0,2.71]$	1	[14]
14	$\|\cos(0.5+9.5\sqrt{x^2+y^2})\|$	$[-1,1]^2$	13.43	[14]
15	$min\left\{\sqrt{\|x+0.35\|}, \sqrt{\|y+0.25\|}\right\}$	$[-1,1]^2$	1	*new*

Table 4. Multivariate Hölder test functions.

Problem number		RT-SPA		RT-TPA		RT-MSPA	
$N°$	β	Ev	$T(s)$	Ev	$T(s)$	Ev	$T(s)$
1	3	9131	28.642	**8875**	**26.875**	9055	28.103
2	2	1493	0.8627	1459	0.8278	**1425**	**0.8030**
3	3/2	6298	13.532	6255	**13.316**	**6248**	13.412
4	3/2	1053	0.4863	990	**0.4411**	**986**	0.4489
5	2	2844	2.8212	2770	2.6870	**2692**	**2.600**
6	2	218	0.0806	238	0.0868	**213**	**0.0802**
7	2	214	0.0769	196	0.0920	**186**	**0.0734**
8	2	331	0.1102	**297**	**0.0997**	319	0.1088
9	2	5795	11.432	5536	**10.468**	**5507**	10.501
10	2	4024	5.3531	4266	6.0112	**3724**	**4.6304**
11	2	2046	1.5416	2010	1.4863	**1973**	**1.4524**
12	2	509	0.1726	478	**0.1615**	**475**	0.1625
13	2	6309	13.646	6342	13.534	**6257**	**13.335**
14	2	6972	17.1265	6805	15.8143	**6776**	**15.6564**
15	2	497	0.1703	485	0.1682	**480**	**0.1662**

Table 5. Comparison of results between RT-MSPA and the existing algorithms with known value of h_g.

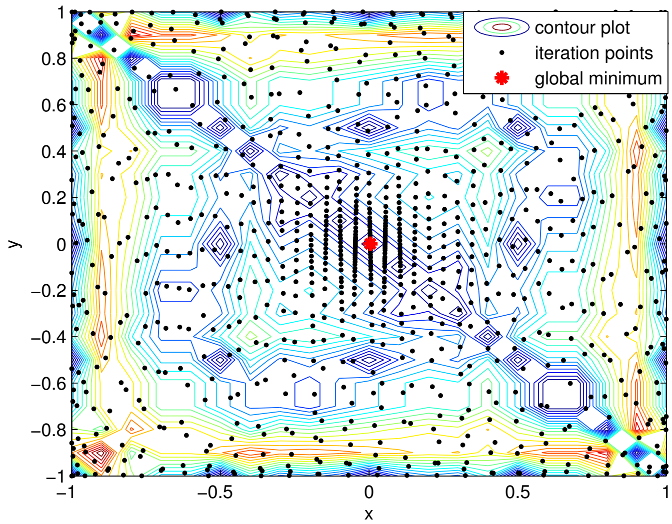

Fig. 3. Illustration of the global minimization process and iteration points generated by RT-MSPA for function 12 (as defined in Table 4) on the domain $[-1, 1]^2$.

- **Known Hölder Constant (h_g):** The corresponding results are shown in Table 5 for the test problems listed in Table 4.

  In this context, Fig. 3 provides a numerical example illustrating how the RT-MSPA algorithm converges, by showing the iteration points produced by this algorithm around the global minimum.

Problem number			RT-SPA$_{es}$		RT-TPA$_{es}$		RT-MSPA$_{es}$	
$N°$	β	λ	Ev	$T(s)$	Ev	$T(s)$	Ev	$T(s)$
1	3	2.1	2938	36.910	2460	**26.040**	**2317**	33.541
		1.07	1194	**6.0336**	/	/	**1147**	8.2836
2	2	1.5	665	1.6772	614	1.4797	**607**	**1.4337**
		1.07	435	0.7450	/	/	**231**	**0.2266**
3	3/2	1.5	**636**	**1.7097**	742	2.2662	732	3.2807
		1.07	466	0.926	/	/	**286**	**0.5239**
4	3/2	1.5	276	0.3303	**267**	**0.3049**	296	0.5407
		1.07	163	0.1200	/	/	**136**	**0.1193**
5	2	1.5	**1595**	9.6273	1765	11.474	1711	11.357
		1.07	1356	6.9890	/	/	**1289**	**6.7921**
6	2	3.1	610	1.4298	674	1.6995	**591**	**1.3795**
7	2	3.5	/	/	534	1.0745	**506**	**1.0562**
8	2	3.1	**521**	**1.0367**	1140	4.7631	529	1.1228
9	2	4.05	130	0.0697	/	/	**33**	**0.0059**
10	2	1.5	**1723**	**11.188**	2355	20.513	1966	15.084
		1.07	1530	8.9580	/	/	**965**	**3.8254**
11	2	14.4	**290**	**0.3300**	310	0.3749	310	0.3992
12	2	3.1	472	0.8553	**459**	**0.7910**	472	0.8762
13	2	1.5	**134**	**0.0739**	147	0.0855	145	0.0873
		1.07	79	0.02688	/	/	**55**	**0.01473**
14	2	1.5	**2792**	**29.536**	3134	36.0570	2923	32.761
15	2	1.5	84	0.0306	**33**	**0.0055**	53	0.0135

Table 6. Comparison of results between RT-MSPA$_{es}$ and the existing algorithms with unknown value of h_g.

- **Unknown Hölder Constant (h_g):** Table 6 presents a comparison between RT-MSPA$_{es}$ and the corresponding estimation-based versions of existing algorithms (RT-SPA$_{es}$ and RT-TPA$_{es}$), where h_g is estimated.

  Moreover, in this case with the estimation of h_g, we observed that using $\nu = 10^{-8}$ and $\lambda = 1.07$ often yielded better results than $\lambda = 1.5$. Notably, the RT-TPA$_{es}$ method failed to produce results when $\lambda = 1.07$.

It should be noted that for the multivariate tests in Table 6 (unknown h_g), two different values for the parameter λ (typically 1.5 and 1.07) were generally used in the estimation procedure to obtain the reported results. The symbol (/) in Table 6 indicates instances where results could not be obtained for $\lambda = 1.07$, particularly for the RT-TPA$_{es}$ method.

Indeed, the TPA method (known h_f) is often faster and requires fewer evaluations according to Table 2, when comparing with SPA and MSPA. However, even in this case, the performance of MSPA remains competitive when compared to TPA. A possible reason for this is that for certain functions with a known Hölder constant h_f, the more aggressive linear approximation used by TPA is more effective at quickly locating the global minimum than the secant-based approximation of MSPA. This can be particularly true for functions where the Hölder exponent β is close to 1, making the function's behavior more linear.

In the case where h_f is unknown a priori, the situation is different (see Table 3). Also, according to Table 5 and Table 6, in the cases where h_g is known and h_g is unknown, the performance of MSPA$_{es}$ and RT-MSPA$_{es}$ is more competitive compared to the other algorithms.

Finally, for problems with a known Hölder constant h_f, MSPA required 68.42% fewer function evaluations than SPA, and 50% fewer than TPA. It also achieved an 80% reduction in execution time compared to SPA, and 50% compared to TPA. When the Hölder constant is unknown, MSPA$_{es}$ maintained strong performance, requiring 78.95% fewer evaluations than SPA$_{es}$, and 60% fewer than TPA$_{es}$, while also reducing execution time by 75% and 55%, respectively. Similarly, RT-MSPA$_{es}$ demonstrated its efficiency for problems with an unknown constant h_g, achieving up to 61.9% fewer function evaluations than RT-SPA$_{es}$, and 85.71% fewer than RT-TPA$_{es}$, along with execution time reductions of 54.55% and 68.18%, respectively.

Based on these comparative studies, the MSPA, MSPA$_{es}$, RT-MSPA, and RT-MSPA$_{es}$ algorithms demonstrate higher efficiency compared to the other techniques evaluated.

5. CONCLUSION

This paper addressed the global optimization of multivariate Hölderian functions defined on a box domain in $\mathbb{R}^n$. A key focus was the challenging yet practical case where the Hölder constant h_g is unknown a priori. To tackle this class of problems, we developed and analyzed two novel algorithms: the MSPA and RT-MSPA. We provided rigorous mathematical proofs establishing the convergence guarantees for both proposed methods. To evaluate their practical efficacy, MSPA and RT-MSPA were implemented and tested on a suite of standard functions commonly used in global optimization literature. The numerical results were compared against those obtained by other well-known search methods. This comparative analysis demonstrated the viability and effectiveness of our approach, in particular, offers a competitive performance in practice. Future work could explore further adaptive strategies within the partitioning framework or extend the approach to handle different types of constraints.

Acknowledgements. The authors would like to thank the anonymous referee for his useful comments and suggestions, which helped to improve the presentation of this paper.

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Received by the editors: August 02, 2025; accepted: December 04, 2025; published online: December 15, 2025.