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ON THE ENHANCED STRUCTURES AND ROUGH PATH INTEGRALS FOR TIME-CHANGED PATHS

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Abstract. In this paper, we study the effects of a discontinuous time change Φ on a continuous path x. While rough integration in this setting poses significant challenges, we construct an enhancement for the time-changed path $x \circ \Phi$, providing a structured framework for discontinuous-time rough paths. We then apply our approach to Lévy processes, demonstrating its effectiveness in specific stochastic models, including the variance gamma process and the normal inverse Gaussian process.

A key focus of our work is the study of rough path integrals under time changes. We establish conditions under which the Lyons integral of the continuously time-changed rough path $X \circ \Phi$ coincides with the Φ-time-changed rough path integral driven by the original rough path X. As a central result, we extend Terry Lyons' Itô formula to a time-changed Itô formula, offering a novel perspective on rough path calculus under time deformations.

These findings enhance the theoretical understanding of rough integration under time changes, with potential applications in stochastic analysis and mathematical finance.

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Keywords. Rough paths, time change, Itô formula, Lévy process, variance gamma process, normal inverse Gaussian process.

1. INTRODUCTION

The theory of rough paths, pioneered by Terry Lyons, provides a powerful framework for analyzing differential equations driven by highly irregular signals, including paths with low regularity and stochastic processes with jumps. A fundamental aspect of rough path theory is the construction of pathwise integrals, enabling the robust study of differential equations beyond classical Itô and Stratonovich calculus.

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A natural question arises when considering time-changed rough paths, where the driving signal undergoes a time transformation. Time changes are fundamental in stochastic analysis, appearing in models where volatility varies dynamically or where time is distorted due to external influences. This motivates the study of rough integration in the context of time-changed paths.

In the case of discontinuous time changes, additional challenges emerge. Unlike the continuous-time setting, rough paths enhanced over a discontinuous time change require additional structure. We address this issue by constructing a suitable enhancement of the time-changed path $x \circ \Phi$, ensuring a meaningful interpretation within the rough path framework.

We then apply our results to Lévy processes, which serve as natural candidates for studying rough integration under discontinuous time changes. Specifically, we provide explicit enhancements for key examples such as the variance gamma process and the normal inverse Gaussian process, demonstrating the applicability of our theoretical framework to real-world stochastic models.

A central focus of this work is the relationship between different formulations of rough path integration under time changes. We investigate whether the Lyons integral of a time-changed rough path $X \circ \Phi$ coincides with the Φ-time-changed integral of X. Under suitable conditions, assuming the time change Φ is continuous, we establish this equivalence and derive a novel time-changed Itô formula, extending Lyons' results to time-deformed rough paths.

This paper is structured as follows: In Section 2, we review the necessary background on rough paths and time changes.

In Section 3, we construct enhancements for discontinuous time-changed paths, apply our results to Lévy processes, and provide concrete examples. In Section 4, we study rough path integrals under time changes and establish conditions under which different integration formulations coincide.

2. PRELIMINARIES

2.1. Rough Paths. In this section, with reference to [4, 6, 7, 8, 9], we gather notations, definitions, and recapitulate some fundamental concepts.

Let (E, d) be a metric space. A path $x : [0, T] \to E$ is said to be:

(i) α-Hölder continuous if

$$|x|_{\alpha\text{-Höl},[0,T]} := \sup_{0 \le s < t \le T} \frac{d(x_s, x_t)}{|t-s|^{\alpha}} < \infty.$$

(ii) of finite p-variation for some $p > 0$ if

$$|x|_{p\text{-var},[0,T]} := \sup_{t_i \in D([0,T])} \left(\sum_i d(x_{t_i}, x_{t_{i+1}})^p \right)^{\frac{1}{p}} < \infty,$$

where $D([0, T])$ denotes the set of all finite dissections of $[0, T]$.

Let V be a Euclidean space. For each $n \in \mathbb{N}$, we denote by $T^n(V)$ the truncated tensor algebra:

$$T^n(V) = \mathbb{R} \oplus V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n}.$$

A *multiplicative functional* in $T^n(V)$ is a map $\mathbb{X} = (1, \mathbb{X}^1, \ldots, \mathbb{X}^n)$ defined on the simplex

$$\Delta = \{(s,t) \in [0,T]^2 \mid 0 \le s \le t \le T, T > 0\}$$

with values in $T^n(V)$, satisfying the *Chen identity*:

$$\mathbb{X}_{s,u} = \mathbb{X}_{s,t} \otimes \mathbb{X}_{t,u}, \quad \text{for all } s \le t \le u.$$

A multiplicative functional $\mathbb{X} = (1, \mathbb{X}^1, \ldots, \mathbb{X}^n)$ in $T^n(V)$ has finite total p-variation if

$$\sup_D \sum_l \left|\mathbb{X}^i_{t_{l-1}t_l}\right|^{\frac{p}{i}} < +\infty, \quad i = 1, \ldots, n,$$

where $\sup_D$ runs over the set D of all finite dissections of $[0,T]$.

Multiplicative functionals with finite total p-variation in $T^{[p]}(V)$ are called *rough paths* (of roughness p), and their set is denoted by $\Omega_p(V)$. The space $\Omega_p(V)$ is endowed with the p-variation metric:

$$d_p(\mathbb{X}, \mathbb{Y}) = \max_{0 \le i \le [p]} \sup_D \left(\sum_l \left|\mathbb{X}^i_{t_{l-1}t_l} - \mathbb{Y}^i_{t_{l-1}t_l}\right|^{\frac{p}{i}} \right)^{\frac{i}{p}}.$$

An *enhanced path* $\mathbb{X}$ of degree n associated with a path x in some Banach space is a multiplicative functional $\mathbb{X}_{s,t} = (1, \mathbb{X}^1_{s,t}, \ldots, \mathbb{X}^n_{s,t})$ from Δ into $T^n(V)$ such that $\mathbb{X}^1_{s,t} = x_{s,t}$. The path x is called the *trace* of X, and $\mathbb{X}$ is called the *enhancement* of x.

Definition 1. *A control function, is a continuous non-negative function w on Δ_T which is super-additive in the sense that*

$$w(s,t) + w(t,u) \leqslant w(s,u) \quad \forall s \leqslant t \leqslant u, \quad (s,t) \in [0,T]^2,$$

and for which $w(t,t) = 0$ for all $t \in [0,T]$.

Lemma 2 ([8, Lemma 1.10, p. 7]). *Let ω be a control. Let $x : J \to E$ be a continuous path. Assume that, for some $p \ge 1$ and for all $(s,t) \in \Delta_T$, one has*

$$|x_s - x_t|^p \le \omega(s,t).$$

Then, for all $(s,t) \in \Delta_T$,

$$\|x\|_{p\text{-}var;[s,t]} \le \omega(s,t)^{1/p}.$$

When the conclusion of this lemma holds, we say that the p-variation of x is controlled by ω.

A rough path $\mathbb{X} \in \Omega_p(V)$ is called a smooth rough path if $t \longrightarrow x_t \equiv \mathbb{X}^1_{0,t}$ is a continuous path with finite variation and $\mathbb{X}^i_{s,t}$ is the i-th iterated integral of the path x_t over the interval $[s,t]$, for $i = 1, \dots, [p]$.

That is,

$$\mathbb{X}^i_{s,t} = \int_{s<t_1<\cdots<t_i<t} dx_{t_1} \otimes ... \otimes dx_{t_i} \quad \forall (s,t) \in \Delta.$$

A multiplicative functional $\mathbb{X} \in \Omega_p(V)$ is called geometric rough path if there is a sequence $\mathbb{X}(n)$ of smooth rough path in $\Omega_p(V)$ such that

$$d_p(\mathbb{X}(n), \mathbb{X}) \longrightarrow 0 \quad as \quad n \longrightarrow +\infty.$$

We denote by $G\Omega_p(V)$ the set of all geometric rough paths of roughness p in $T^{[p]}(V)$.

A function $\mathbb{X} : \Delta \longrightarrow T^{[p]}(V)$ is called an almost rough path (of roughness p) if it is of finite p-variation, $\mathbb{X}^0_{s,t} = 1$, and, for some control ω and some constant $\theta > 1$,

$$\left|(\mathbb{X}_{s,t} \otimes \mathbb{X}_{t,u})^i - \mathbb{X}^i_{s,u}\right| \leqslant \omega(s,u)^\theta,$$

for all $(s,t),(t,u) \in \Delta$ and $i = 1, ..., [p]$.

In [9, p. 41], it was shown that if $\mathbb{X}$ a θ-almost p-rough path controlled by ω, then there exists a unique p-rough path $\widehat{\mathbb{X}} : \Delta \longrightarrow T^{[p]}(V)$ and a constant K which depends only on p,θ and $\omega(0,T)$ such that

$$\sup_{0\leqslant s<t\leqslant T} \frac{\left\|\widehat{\mathbb{X}}^i_{s,t} - \mathbb{X}^i_{s,t}\right\|}{w(s,t)^\theta} \;<\; K$$

and the p-variation of $\widehat{\mathbb{X}}$ is controlled by $K\omega$.

2.2. Rough Path Integration. In the following, we assume that:

- $\alpha : V \to L(V,W)$ is a function that maps elements of V linearly to W-valued one-forms on V.
- α possesses all continuous and bounded derivatives $d^k\alpha$ up to order $[p]$.
- We denote $\alpha^i = d^{i-1}\alpha$ for $i = 1, 2, \dots, [p]$.

2.2.1. *Integration of Rough Paths of Degree Two.*

Definition 3. *Let $\mathbb{X} \in \Omega_p(V)$. The integral of the one-form α against the rough path $\mathbb{X}$, denoted by*

$$\int \alpha(\mathbb{X})d\mathbb{X},$$

is the unique p-rough path in $T^2(W)$ associated with the almost rough path $\mathbb{Y}$, where:

$$\mathbb{Y}^1_{s,t} = \alpha^1(\mathbb{X}_s)(\mathbb{X}^1_{s,t}) + \alpha^2(\mathbb{X}_s)(\mathbb{X}^2_{s,t}),$$

and

$$\mathbb{Y}^2_{s,t} = \alpha^1(\mathbb{X}_s) \otimes \alpha^1(\mathbb{X}_s)(\mathbb{X}^2_{s,t}).$$

With $\mathbb{X}_s \equiv \mathbb{X}_{0,s}$.

2.2.2. *Itô's Formula.*

LEMMA 4 ([9, Lemma 5.4.1, p. 135]). *Let $\mathbb{X} \in \Omega_p(V)$, and define*

$$\mathbb{Y}_{s,t} = (1, \mathbb{X}^1_{s,t}, \mathbb{X}^2_{s,t} + A_{s,t}).$$

Then $\mathbb{Y} \in \Omega_p(V)$ if and only if A is additive and possesses finite $\frac{p}{2}$-variation. We denote $\mathbb{Y}$ by $\mathbb{X}^A$.

THEOREM 5. *Let $\mathbb{X} \in \Omega_p(V)$, and let A be a continuous path in $V^{\otimes 2}$ which possesses finite $\frac{p}{2}$ variation. Then*

$$\begin{aligned}
\int_s^t \alpha(\mathbb{X}^A) d\mathbb{X}^{A^1} &= \int_s^t \alpha(\mathbb{X}) d\mathbb{X}^1 + \int_s^t \alpha^2(\mathbb{X}_r)(dA_r) \int_s^t \alpha(\mathbb{X}^A) d\mathbb{X}^{A^2} \\
&= \int_s^t \alpha(\mathbb{X}) d\mathbb{X}^2 + \int_s^t \alpha^1(\mathbb{X}_r) \otimes \alpha^1(\mathbb{X}_r)(dA_r) \\
&\quad + \int_s^t \left[\int_s^r \alpha^2(\mathbb{X}_r)(dA_r) \right] \otimes d\mathbb{Z}_r + \int_s^t \mathbb{Z}_{s,r} \otimes \alpha^2(\mathbb{X}_r)(dA_r) \\
&\quad + \int_s^t \left(\int_s^u \alpha^2(\mathbb{X}_r) dA_r \right) \otimes \alpha^2(\mathbb{X}_u) dA_u.
\end{aligned}$$

Here the integrals involving A are Young's integrals, and $\mathbb{Z}_{s,t} = \int_s^t \alpha(\mathbb{X}) d\mathbb{X}^1$.

2.2.3. *Stochastic Integration.*

THEOREM 6. *Let $(W_t)_{t \geq 0}$ be a d-dimensional Brownian motion, and let*

$$\mathbb{X}_{s,t} = (1, \mathbb{X}^1_{s,t}, \mathbb{X}^2_{s,t})$$

be the geometric rough path associated with W. Then:

$$\left(\int \alpha(\mathbb{X}) d\mathbb{X} \right)_{s,t} \equiv \int_s^t \alpha(\mathbb{X}) d\mathbb{X} = \left(1, \int_s^t \alpha(\mathbb{X}) d\mathbb{X}^1, \int_s^t \alpha(\mathbb{X}) d\mathbb{X}^2 \right),$$

where:

$$Z_{s,t} = \int_s^t \alpha(\mathbb{X}) d\mathbb{X}^1 = \int_s^t \alpha(W_r) \circ dW_r,$$

and

$$\int_s^t \alpha(\mathbb{X}) d\mathbb{X}^2 = \int_{s<t_1<t_2<t} \circ dZ_{t_1} \otimes \circ dZ_{t_2}.$$

Here, the integrals are Stratonovich integrals.

2.2.4. *Integration Against Geometric Rough Paths.*

DEFINITION 7. *Let $\mathbb{X} \in G\Omega_p(V)$. The integral $\int \alpha(\mathbb{X}) d\mathbb{X}$ is the unique rough path in $\Omega_p(W)$ associated with the quasi α-differential almost rough path $\mathbb{Y}$, where:*

$$\mathbb{Y}^i_{s,t} = \sum_{l=(l_1,\ldots,l_i), 1 \leq l_j \leq [p]} \alpha^{l_1} \otimes \cdots \otimes \alpha^{l_i}(\mathbb{X}_s) \left(\sum_{\pi \in \Pi_l} \pi^{-1} \mathbb{X}^{|l|}_{s,t} \right).$$

2.3. Time-Changed Young Integration and the Rough Path Approach. It is well known (see, for example, [1]) that if x and y are, respectively, α-Hölder and β-Hölder continuous functions on $[0,T]$ such that $\alpha+\beta>1$, and if Φ is an increasing γ-Hölder continuous function from $[0,T]$ into itself satisfying $(\alpha+\beta)\gamma>1$, then for any $0\le a\le b\le T$, we have the following identity for the Young integral:

$$\int_a^b (x\circ\Phi)(u)d(y\circ\Phi)(u)=\int_{\Phi(a)}^{\Phi(b)} x(u)dy(u).$$

This result provides a fundamental relationship between integration and time changes in the Young integration framework.

Terry Lyons extended these ideas by developing a general theory of rough paths, which gives meaning to integrals of the form

$$Z_t=Z_0+\int_0^t \alpha(x_s)dx_s,$$

where x is a path of finite p-variation in a Banach space V. To define such an integral rigorously, one needs to lift x to a path X of finite p-variation in the free nilpotent group of V. This lifting procedure enables the application of rough path integration techniques, which generalize Young's integration by incorporating iterated integrals.

In this paper, we consider a path x with finite p-variation and a time change Φ, and we investigate the integral:

$$\int_a^b (x\circ\Phi)(u)d(x\circ\Phi)(u),\quad \text{for } 0\le a\le b\le T,$$

where the integral is interpreted in the rough path sense using Lyons' theory.

3. ENHANCED TIME-CHANGED PATHS

In [7], T. Lyons developed a theory that provides a rigorous framework for solving the integral equation

$$Y_t=Y_0+\int_0^t f(x_s)dx_s, \tag{1}$$

where x is a continuous path of finite p-variation in a Banach space V with $p>1$. This requires lifting x to an enhanced path $\mathbb{X}$ of finite p-variation in the free nilpotent group of V, allowing the integral to be defined using rough path techniques.

Building on this foundation, Peter Friz [3] extended Lyons' approach to handle cases where the driving process x is a discontinuous path of finite p-variation in a Banach space V. In this work, we further explore this setting by considering the integral equation (1) under a time change Φ.

Given a time change Φ, we define the transformed process $x_{\Phi(t)}$ and denote it by x_t^{Φ}. Our goal is to provide a rigorous interpretation of the equation:

$$Y_t = Y_0 + \int_0^t f(x_s^{\Phi}) dx_s^{\Phi}. \tag{2}$$

To achieve this, we employ Lyons' rough path integration framework, which necessitates constructing an enhancement of the time-changed process x^{Φ}. The development that follows is dedicated to establishing this enhancement and its properties.

Definition 8. *A time change Φ is a family $\Phi_s, s \geq 0$, of stopping times such that the map $s \mapsto \Phi_s$ is almost surely increasing and right-continuous.*

3.1. Finite discontinuities. Let V be a Banach space and let $x : [0, T] \to V$ be a continuous path. Assume that x has a geometric enhancement $\mathbb{X}$, that is,

$$\mathbb{X}_{s,t} = \left(1, \mathbb{X}^1_{s,t}, \mathbb{X}^2_{s,t}\right),$$

where:

$$\mathbb{X}^1_{s,t} = x_t - x_s,$$

and

$$\mathbb{X}^{2,i,j}_{s,t} = \lim_{n \to +\infty} \sum_{s < \frac{l}{n} < t} (x^i_{\frac{l}{n}} - x^i_s) \otimes (x^j_{\frac{l+1}{n}} - x^j_{\frac{l}{n}}).$$

Theorem 9. *Let $s, t \in [0, T]$ and let Φ be a time change with a finite number of discontinuities $a_1, a_2, \ldots, a_r$ in $[s, t]$. Assume that $a_k \neq \frac{l}{n}$ for all $k = 1, \ldots, r$. Then,*

$$\mathbb{X}^{\Phi,2}_{s,t} = \mathbb{X}^2_{\Phi(s),\Phi(t)} - \sum_{k=1}^{r} \left[\mathbb{X}^2_{\Phi(a_k^-),\Phi(a_k^+)} + (x_{\Phi(a_k^-)} - x_{\Phi(s)}) \otimes (x_{\Phi(a_k^+)} - x_{\Phi(a_k^-)})\right].$$

Proof. By definition,

$$\mathbb{X}^{\Phi,2,i,j}_{s,t} = \int_s^t \left(x^{\Phi,i}(u) - x^{\Phi,i}(s)\right) \otimes \mathrm{d}x^{\Phi,j}(u).$$

Taking the limit as $n \to +\infty$ over a partition,

$$\begin{aligned}
\mathbb{X}^{\Phi,2,i,j}_{s,t} = \lim_{n \to +\infty} \Bigg[&\sum_{l/n < a_1} \left(x^i_{\Phi(l/n)} - x^i_{\Phi(s)}\right) \otimes \left(x^j_{\Phi((l+1)/n)} - x^j_{\Phi(l/n)}\right) \\
&+ \sum_{k=1}^{r-1} \sum_{a_k < l/n < a_{k+1}} \left(x^i_{\Phi(l/n)} - x^i_{\Phi(s)}\right) \otimes \left(x^j_{\Phi((l+1)/n)} - x^j_{\Phi(l/n)}\right) \\
&+ \sum_{l/n > a_r} \left(x^i_{\Phi(l/n)} - x^i_{\Phi(s)}\right) \otimes \left(x^j_{\Phi((l+1)/n)} - x^j_{\Phi(l/n)}\right) \Bigg].
\end{aligned}$$

Using integration properties,

$$\mathbb{X}_{s,t}^{\Phi,2} = \int_{\Phi(s)}^{\Phi(a_1^-)} (x_u - x_{\Phi(s)}) \otimes \mathrm{d}x_u + \sum_{k=1}^{r-1} \int_{\Phi(a_k^+)}^{\Phi(a_{k+1}^-)} (x_u - x_{\Phi(s)}) \otimes \mathrm{d}x_u$$
$$+ \int_{\Phi(a_r^+)}^{\Phi(t)} (x_u - x_{\Phi(s)}) \otimes \mathrm{d}x_u.$$

Rewriting the sums using enhancement properties,

$$\mathbb{X}_{s,t}^{\Phi,2} = \mathbb{X}^2_{\Phi(s),\Phi(a_1^-)} + \sum_{k=1}^{r-1} \mathbb{X}^2_{\Phi(a_k^+),\Phi(a_{k+1}^-)} + \mathbb{X}^2_{\Phi(a_r^+),\Phi(t)}$$
$$+ \sum_{k=1}^{r-1} \left(x_{\Phi(a_k^+)} - x_{\Phi(s)}\right) \otimes \left(x_{\Phi(a_{k+1}^-)} - x_{\Phi(a_k^+)}\right)$$
$$+ \left(x_{\Phi(a_r^+)} - x_{\Phi(s)}\right) \otimes \left(x_{\Phi(t)} - x_{\Phi(a_r^+)}\right).$$

Using the repeated Chen identity,

$$\mathbb{X}^2_{\Phi(s),\Phi(t)} = \mathbb{X}^2_{\Phi(s),\Phi(a_1^-)} + \sum_{k=1}^{r} \mathbb{X}^2_{\Phi(a_k^-),\Phi(a_k^+)} + \sum_{k=1}^{r-1} \mathbb{X}^2_{\Phi(a_k^+),\Phi(a_{k+1}^-)} + \mathbb{X}^2_{\Phi(a_r^+),\Phi(t)}$$
$$+ \sum_{k=1}^{r} \left(x_{\Phi(a_k^-)} - x_{\Phi(s)}\right) \otimes \left(x_{\Phi(a_k^+)} - x_{\Phi(a_k^-)}\right).$$

Rearranging the terms completes the proof. □

3.2. Countable discontinuities.

THEOREM 10. *Let $x : [0,T] \to V$ be a continuous path admitting a geometric rough path lift $\mathbb{X} = (1, \mathbb{X}^1, \mathbb{X}^2) \in G\Omega_p(V)$ with $2 \le p < 3$. Let Φ be an increasing càdlàg time change with countably many discontinuities $(a_k)_{k\in\mathbb{N}}$ in (s,t). Assume that there exists a control ω such that*

$$|\mathbb{X}^1_{u,v}| \le \omega(u,v)^{\frac{1}{p}}, \qquad |\mathbb{X}^2_{u,v}| \le \omega(u,v)^{\frac{2}{p}},$$

and

$$\sum_{s<a_k<t} \omega(\Phi(a_k^-), \Phi(a_k^+))^{\frac{2}{p}} < \infty.$$

Then the second level of the enhanced time-changed path $x \circ \Phi$ is given by

$$\mathbb{X}_{s,t}^{\Phi,2} = \mathbb{X}^2_{\Phi(s),\Phi(t)} - \sum_{s<a_k<t} \left[\mathbb{X}^2_{\Phi(a_k^-),\Phi(a_k^+)} + (x_{\Phi(a_k^-)} - x_{\Phi(s)}) \otimes (x_{\Phi(a_k^+)} - x_{\Phi(a_k^-)})\right],$$

and $\mathbb{X}^\Phi$ defines a geometric rough path of roughness p.

Proof. The proof proceeds by approximation. We construct a sequence $(\Phi_n)_{n\ge1}$ of increasing càdlàg time changes, each having only finitely many discontinuities, such that $\Phi_n \to \Phi$ pointwise. For each Φ_n, we apply Theorem 9 (finite jump case) to obtain an explicit expression for the second level $\mathbb{X}^{\Phi_n,2}$.

We then show that the sequence of rough paths $(\mathbb{X}^{\Phi_n})_{n\geq 1}$ is Cauchy with respect to the p-variation metric and hence converges to a limiting rough path, denoted by $\mathbb{X}^{\Phi}$. Finally, we identify this limit and verify that it satisfies the announced formula, and that it retains the geometric rough path structure.

Step 1: Approximation of Φ. Let (a_k) be the discontinuity points of Φ in (s,t), ordered increasingly. Define

$$\Delta\Phi(a_k) := \Phi(a_k^+) - \Phi(a_k^-).$$

Since Φ is increasing,

$$\sum_{k=1}^{\infty} \Delta\Phi(a_k) \leq \Phi(t) - \Phi(s) < \infty.$$

For each $n \in \mathbb{N}$, define

$$\Phi_n(u) := \Phi(u) - \sum_{k>n:\, a_k \leq u} \Delta\Phi(a_k).$$

Then Φ_n is increasing, right-continuous, has finitely many jumps $(a_1,\dots,a_n)$, and

$$\Phi_n(u) \uparrow \Phi(u), \qquad \forall u.$$

Step 2: Finite jump formula. By the finite discontinuity case, for each n,

$$\mathbb{X}^{\Phi_n,2}_{s,t} = \mathbb{X}^2_{\Phi_n(s),\Phi_n(t)} - \sum_{k=1}^{n} \Big[\mathbb{X}^2_{\Phi(a_k^-),\Phi(a_k^+)} + (x_{\Phi(a_k^-)} - x_{\Phi_n(s)}) \otimes (x_{\Phi(a_k^+)} - x_{\Phi(a_k^-)})\Big].$$

Step 3: Convergence of the second level. We show that $(\mathbb{X}^{\Phi_n,2})$ is Cauchy uniformly. Let $m > n$. Then

$$\mathbb{X}^{\Phi_m,2}_{s,t} - \mathbb{X}^{\Phi_n,2}_{s,t} = A_{m,n} + B_{m,n} + C_{m,n},$$

where:

$$\begin{aligned}
A_{m,n} :=& \mathbb{X}^2_{\Phi_m(s),\Phi_m(t)} - \mathbb{X}^2_{\Phi_n(s),\Phi_n(t)},\\
B_{m,n} :=& -\sum_{k=n+1}^{m} \Big[\mathbb{X}^2_{\Phi(a_k^-),\Phi(a_k^+)} + (x_{\Phi(a_k^-)} - x_{\Phi_m(s)}) \otimes (x_{\Phi(a_k^+)} - x_{\Phi(a_k^-)})\Big],\\
C_{m,n} :=& (x_{\Phi_m(s)} - x_{\Phi_n(s)}) \otimes \sum_{k=1}^{n} (x_{\Phi(a_k^+)} - x_{\Phi(a_k^-)}).
\end{aligned}$$

Estimate of $A_{m,n}$. By continuity of $(u,v) \mapsto \mathbb{X}^2_{u,v}$,

$$A_{m,n} \to 0.$$

Estimate of $B_{m,n}$. Using the control,

$$\begin{aligned}
|\mathbb{X}^2_{\Phi(a_k^-),\Phi(a_k^+)}| \leq& \omega(\Phi(a_k^-),\Phi(a_k^+))^{2/p},\\
|x_{\Phi(a_k^+)} - x_{\Phi(a_k^-)}| \leq& \omega(\Phi(a_k^-),\Phi(a_k^+))^{1/p}.
\end{aligned}$$

Hence

$$|B_{m,n}| \le C \sum_{k=n+1}^{m} \omega(\Phi(a_k^-), \Phi(a_k^+))^{2/p}.$$

By assumption, the series converges, so $B_{m,n} \to 0$.

Estimate of $C_{m,n}$. We have

$$|C_{m,n}| \le |x_{\Phi_m(s)} - x_{\Phi_n(s)}| \cdot \sum_{k=1}^{n} |x_{\Phi(a_k^+)} - x_{\Phi(a_k^-)}|.$$

Using the control,

$$\sum_{k=1}^{\infty} |x_{\Phi(a_k^+)} - x_{\Phi(a_k^-)}| \le \sum_{k=1}^{\infty} \omega\Big(\Phi(a_k^-), \Phi(a_k^+)\Big)^{1/p}.$$

Since $\omega^{1/p} \le 1 + \omega^{2/p}$ for small ω, the series converges. Thus the sum is bounded, and since $\Phi_n(s) \to \Phi(s)$,

$$C_{m,n} \to 0.$$

Step 4: Convergence in d_p. The same estimates applied to increments over partitions show that

$$d_p(\mathbb{X}^{\Phi_n}, \mathbb{X}^{\Phi_m}) \to 0.$$

Since $(\Omega_p(V), d_p)$ is complete, there exists a limit $\mathbb{X}^{\Phi}$.

Step 5: Identification of the limit. Passing to the limit,

$$\mathbb{X}^{\Phi,2}_{s,t} = \mathbb{X}^2_{\Phi(s),\Phi(t)} - \sum_{s<a_k<t} \Big[\mathbb{X}^2_{\Phi(a_k^-),\Phi(a_k^+)} + \Big(x_{\Phi(a_k^-)} - x_{\Phi(s)}\Big) \otimes \Big(x_{\Phi(a_k^+)} - x_{\Phi(a_k^-)}\Big)\Big].$$

Step 6: Chen identity. Each $\mathbb{X}^{\Phi_n}$ satisfies Chen's identity. Passing to the limit preserves it, hence $\mathbb{X}^{\Phi}$ is a geometric rough path. □

The summability condition ensures that the correction terms define a path of finite $p/2$-variation.

THEOREM 11. *Let $x : [0,T] \to V$ be a continuous path admitting a geometric rough path lift $X = (1, X^1, X^2) \in G\Omega_p(V)$ with $2 \le p < 3$. Assume that there exists a control ω such that*

$$|X^1_{s,t}| \le \omega(s,t)^{\frac{1}{p}}, \qquad |X^2_{s,t}| \le \omega(s,t)^{\frac{2}{p}}.$$

Let Φ be an increasing càdlàg time change with jump times (a_k) in (s,t). Assume that

$$\sum_{s<a_k<t} \omega\big(\Phi(a_k^-), \Phi(a_k^+)\big)^{\frac{2}{p}} < \infty.$$

Define

$$A_{s,t} := \sum_{s<a_k<t} X^2_{\Phi(a_k^-),\Phi(a_k^+)},$$

$$\Gamma_{s,t} := \sum_{s<a_k<t} \Big(x_{\Phi(a_k^-)} - x_{\Phi(s)}\Big) \otimes \Big(x_{\Phi(a_k^+)} - x_{\Phi(a_k^-)}\Big),$$

and

$$\widetilde{X}^2_{s,t} := X^2_{\Phi(s),\Phi(t)} - A_{s,t} - \Gamma_{s,t}.$$

Then the path

$$X^\Phi_{s,t} := \Big(1, X^1_{\Phi(s),\Phi(t)}, \widetilde{X}^2_{s,t}\Big)$$

defines a geometric rough path of finite p-variation.

Proof. **Step 1: Additivity.** For $s < u < t$, the sums defining $A_{s,t}$ and $\Gamma_{s,t}$ split over disjoint sets of indices, hence

$$A_{s,t} = A_{s,u} + A_{u,t}, \qquad \Gamma_{s,t} = \Gamma_{s,u} + \Gamma_{u,t}.$$

Thus both A and Γ are additive.

Step 2: $p/2$-variation of A. Let $D = \{t_i\}$ be a partition of $[0,T]$. Then

$$\sum_i |A_{t_i,t_{i+1}}|^{\frac{p}{2}} = \sum_i \left| \sum_{t_i<a_k<t_{i+1}} X^2_{\Phi(a_k^-),\Phi(a_k^+)} \right|^{\frac{p}{2}}.$$

Using the triangle inequality and the control on X^2, we obtain

$$\sum_i |A_{t_i,t_{i+1}}|^{\frac{p}{2}} \le \sum_i \left(\sum_{t_i<a_k<t_{i+1}} \omega\Big(\Phi(a_k^-), \Phi(a_k^+)\Big)^{\frac{2}{p}} \right)^{\frac{p}{2}}.$$

Since $\frac{p}{2} \ge 1$, there exists a constant $C_p > 0$ such that

$$\Big(\sum x_k\Big)^{\frac{p}{2}} \le C_p \sum x_k^{\frac{p}{2}}.$$

Hence

$$\sum_i |A_{t_i,t_{i+1}}|^{\frac{p}{2}} \le C_p \sum_k \omega\Big(\Phi(a_k^-), \Phi(a_k^+)\Big).$$

By assumption, this sum is finite, so A has finite $\frac{p}{2}$-variation.

Step 3: $p/2$-variation of Γ. We estimate

$$|\Gamma_{s,t}| \le \sum_{s<a_k<t} |x_{\Phi(a_k^-)} - x_{\Phi(s)}| \cdot |x_{\Phi(a_k^+)} - x_{\Phi(a_k^-)}|.$$

Using the control,

$$|x_u - x_v| \le \omega(u,v)^{\frac{1}{p}},$$

we obtain

$$|\Gamma_{s,t}| \le \sum_{s<a_k<t} \omega\Big(\Phi(s), \Phi(a_k^-)\Big)^{\frac{1}{p}} \cdot \omega\Big(\Phi(a_k^-), \Phi(a_k^+)\Big)^{\frac{1}{p}}.$$

Since ω is a control,

$$\omega\Big(\Phi(s), \Phi(a_k^-)) \le \omega\Big(\Phi(s), \Phi(t)\Big),$$

hence

$$|\Gamma_{s,t}| \le \omega\Big(\Phi(s), \Phi(t)\Big)^{\frac{1}{p}} \sum_{s<a_k<t} \omega\Big(\Phi(a_k^-), \Phi(a_k^+)\Big)^{\frac{1}{p}}.$$

Using Hölder's inequality and the summability assumption, we deduce that Γ has finite $\frac{p}{2}$-variation.

Step 4: Control of $\widetilde{X}^2$. We have

$$\widetilde{X}^2_{s,t} = X^2_{\Phi(s),\Phi(t)} - A_{s,t} - \Gamma_{s,t}.$$

Each term has finite $\frac{p}{2}$-variation, hence so does $\widetilde{X}^2$.

Step 5: Chen identity. Since X satisfies Chen's identity and A, Γ are additive, we obtain

$$\widetilde{X}^2_{s,t} = \widetilde{X}^2_{s,u} + \widetilde{X}^2_{u,t} + X^1_{\Phi(s),\Phi(u)} \otimes X^1_{\Phi(u),\Phi(t)}.$$

Thus X^Φ satisfies Chen's identity.

Conclusion. The path X^Φ has finite p-variation and satisfies Chen's identity, hence it is a geometric rough path. □

3.3. Algebraic structure in the case $p \ge 3$.

THEOREM 12. *Let $x : [0,T] \to V$ be a continuous path admitting a geometric rough path enhancement X. Let $s < t$ and let Φ be a time change with finitely many discontinuities $a_1, \dots, a_r$ in (s,t). Then, for any $k \ge 1$, the enhanced time-changed rough path satisfies*

(3)

$$X^{\Phi,k}_{s,t} = X^k_{\Phi(s),\Phi(t)} - \sum_{m=1}^{r} \left[X^k_{\Phi(a_m^-),\Phi(a_m^+)} + \sum_{j=1}^{k-1} X^j_{\Phi(s),\Phi(a_m^-)} \otimes X^{k-j}_{\Phi(a_m^-),\Phi(a_m^+)} \right].$$

Proof. By definition, the iterated integrals of the time-changed path satisfy

$$X^{\Phi,k}_{s,t} = \int_s^t X^{\Phi,k-1}_{s,u} \otimes dx^\Phi_u.$$

Approximating the integral by Riemann sums over partitions (t_l^n) of $[s,t]$, we write

$$\begin{aligned} X^{\Phi,k}_{s,t} = \lim_{n\to\infty} \Bigg[& \sum_{t_l^n < a_1} X^{k-1}_{\Phi(s),\Phi(t_l^n)} \otimes \Big(x_{\Phi(t_{l+1}^n)} - x_{\Phi(t_l^n)} \Big) \\ & + \sum_{m=1}^{r-1} \sum_{a_m < t_l^n < a_{m+1}} X^{k-1}_{\Phi(s),\Phi(t_l^n)} \otimes \Big(x_{\Phi(t_{l+1}^n)} - x_{\Phi(t_l^n)} \Big) \\ & + \sum_{t_l^n > a_r} X^{k-1}_{\Phi(s),\Phi(t_l^n)} \otimes \Big(x_{\Phi(t_{l+1}^n)} - x_{\Phi(t_l^n)} \Big) \Bigg]. \end{aligned}$$

Passing to the limit yields

$$X^{\Phi,k}_{s,t} = \int_{\Phi(s)}^{\Phi(a_1^-)} X^{k-1}_{\Phi(s),u} \otimes dx_u + \sum_{m=1}^{r-1} \int_{\Phi(a_m^+)}^{\Phi(a_{m+1}^-)} X^{k-1}_{\Phi(s),u} \otimes dx_u + \int_{\Phi(a_r^+)}^{\Phi(t)} X^{k-1}_{\Phi(s),u} \otimes dx_u.$$

Using the multiplicative (Chen) property of the rough path, we write for $u \in [\Phi(a_m^+), \Phi(a_{m+1}^-)]$:

$$X^{k-1}_{\Phi(s),u} = \sum_{j=0}^{k-1} X^{j}_{\Phi(s),\Phi(a_m^-)} \otimes X^{k-1-j}_{\Phi(a_m^-),u}. \tag{4}$$

Substituting this decomposition into the integrals, we obtain

$$\begin{aligned} X^{\Phi,k}_{s,t} &= X^{k}_{\Phi(s),\Phi(a_1^-)} + \sum_{m=1}^{r-1} X^{k}_{\Phi(a_m^+),\Phi(a_{m+1}^-)} + X^{k}_{\Phi(a_r^+),\Phi(t)} \\ &\quad + \sum_{m=1}^{r-1} \sum_{j=1}^{k-1} X^{j}_{\Phi(s),\Phi(a_m^-)} \otimes X^{k-j}_{\Phi(a_m^-),\Phi(a_{m+1}^-)} \\ &\quad + \sum_{j=1}^{k-1} X^{j}_{\Phi(s),\Phi(a_r^-)} \otimes X^{k-j}_{\Phi(a_r^-),\Phi(t)}. \end{aligned}$$

Now, applying Chen's identity to the partition

$$s < a_1^- < a_1^+ < \ldots < a_r^- < a_r^+ < t,$$

we have

$$\begin{aligned} X^{k}_{\Phi(s),\Phi(t)} &= X^{k}_{\Phi(s),\Phi(a_1^-)} + \sum_{m=1}^{r} X^{k}_{\Phi(a_m^-),\Phi(a_m^+)} + \sum_{m=1}^{r-1} X^{k}_{\Phi(a_m^+),\Phi(a_{m+1}^-)} + X^{k}_{\Phi(a_r^+),\Phi(t)} \\ &\quad + \sum_{m=1}^{r} \sum_{j=1}^{k-1} X^{j}_{\Phi(s),\Phi(a_m^-)} \otimes X^{k-j}_{\Phi(a_m^-),\Phi(a_m^+)}. \end{aligned}$$

Rearranging terms yields exactly

$$X^{\Phi,k}_{s,t} = X^{k}_{\Phi(s),\Phi(t)} - \sum_{m=1}^{r} \left[X^{k}_{\Phi(a_m^-),\Phi(a_m^+)} + \sum_{j=1}^{k-1} X^{j}_{\Phi(s),\Phi(a_m^-)} \otimes X^{k-j}_{\Phi(a_m^-),\Phi(a_m^+)} \right].$$

Finally, since this representation is obtained by subtracting local jump contributions from a Chen decomposition, it follows that X^{Φ} inherits Chen's identity from X. □

The previous results provide an explicit construction of the enhanced time-changed path. We now show that this construction is in fact canonical.

3.4. Uniqueness of the Enhanced Time-Changed Path. The constructions in Theorem 9–Theorem 11 provide an explicit enhancement of the time-changed path $x^\Phi := x \circ \Phi$. A natural question is whether this enhancement is uniquely determined by the original rough path $\mathbb{X}$ and the time change Φ.

Continuous time changes. If Φ is continuous, then the enhancement is given by

$$\mathbb{X}^\Phi_{s,t} = \mathbb{X}_{\Phi(s),\Phi(t)}.$$

In this case, $\mathbb{X}^\Phi$ is a geometric rough path and coincides with the canonical lift of $x \circ \Phi$. Uniqueness follows from the universal limit property of geometric rough paths: the lift is obtained as the limit of smooth approximations, and any other candidate with the same trace and finite p-variation must coincide with it (see [4, Proposition 7.10, p. 132]).

Discontinuous time changes. When Φ has jumps, the path x^Φ is no longer continuous, and the resulting object is not a geometric rough path in the classical sense. Nevertheless, one can still define an enhancement satisfying Chen's identity and suitable p-variation bounds. The formulas in Theorem 9 and Theorem 10 are dictated by the following structural requirements:

1) **Chen's identity:** the enhanced path must satisfy the multiplicative (Chen) relation on all intervals.
2) **Consistency away from jumps:** on any interval containing no discontinuity of Φ, the enhancement must reduce to the canonical composition $\mathbb{X}_{\Phi(s),\Phi(t)}$.
3) **Finite p-variation:** the second level must satisfy the $p/2$-variation bounds ensured by Theorem 11.

Uniqueness argument. These conditions determine the enhancement uniquely. Indeed, consider two candidates $\mathbb{X}^\Phi$ and $\widetilde{\mathbb{X}}^\Phi$ satisfying the above properties, and define their difference at level two:

$$A_{s,t} := \mathbb{X}^{\Phi,2}_{s,t} - \widetilde{\mathbb{X}}^{\Phi,2}_{s,t}.$$

Then A satisfies:

- $A_{s,t} = 0$ whenever (s,t) contains no discontinuity of Φ,
- an additive (Chen-type) relation: $A_{s,t} = A_{s,u} + A_{u,t}$,
- finite $p/2$-variation.

Hence A is supported on the jump intervals and must be a purely atomic additive functional. The $p/2$-variation constraint forces the total contribution of these atoms to be summable, and the Chen relation then uniquely determines their values. This yields $A \equiv 0$.

Relation with Lévy rough paths. This uniqueness property is consistent with the construction of rough paths over Lévy processes developed by P. K. Friz and Atul Shekhar, where a canonical lift is obtained from the Lévy–Khintchine triplet under a Blumenthal–Getoor condition. In that framework, the summability of jumps ensures both existence and uniqueness of the enhancement. In our setting, Theorem 17 shows that the condition $\beta < 1/p$

guarantees the required summability of the jump corrections, and hence the uniqueness of the time-changed enhancement follows as a consequence.

Conclusion. The enhancement constructed in Theorem 9–Theorem 11 is therefore the unique rough path over x^Φ satisfying Chen's identity, finite p-variation bounds, and consistency with the original rough path away from the discontinuities of Φ.

3.5. Application on Lévy Processes.

Proposition 13 ([2, Proposition 3.5, p. 75]). *If X is a Lévy process, then the quantity*

$$J\left([0,t] \times A\right) = \#\left\{s \in [0,t] : \Delta X_s \in A\right\}$$

defines a Poisson random measure on $\mathbb{R}_+ \times (\mathbb{R} \setminus \{0\})$ with intensity measure $dt \otimes d\nu$, where

$$\nu(A) = \mathbb{E}\left[J\left([0,1] \times A\right)\right].$$

3.5.1. *Poisson Integral.* Let J be a Poisson random measure with intensity measure $dt \otimes d\nu$ on $\mathbb{R}_+ \times (\mathbb{R}^d \setminus \{0\})$.

Let $f : \mathbb{R}^d \to \mathbb{R}^n$ be a measurable function, and let $A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ be such that

$$\nu(A) < +\infty.$$

Then, for all $t \geq 0$ and $\omega \in \Omega$, the Poisson integral of f is defined by:

$$\int_A f(Z)J(t, dZ) = \sum_{Z \in A} f(Z)J(t, \{Z\}).$$

Remark 14. 1) If J is associated with a Lévy process, then

$$\int_A f(Z)J(t, dZ) = \sum_{0 \leq s \leq t} f(\Delta X_s)1_A(\Delta X_s).$$

2) If ν is the Lévy measure and $0 \notin \overline{A}$, then $\nu(A) < +\infty$, ensuring that $\int_A f(Z)J(t, dZ)$ is well-defined.

Theorem 15. *Let A be a Borel set such that $0 \notin \overline{A}$. Then:*

1) For all $t \geq 0$, the integral $\int\limits_A f(Z)J(t, dZ)$ follows a compensated Poisson law characterized by:

$$\mathbb{E}\left[\exp\left(i\left\langle u, \int_A f(Z)J(t, dZ)\right\rangle\right)\right] = \exp\left(t\int_A \left(e^{i\langle u, f(Z)\rangle} - 1\right)\nu(dZ)\right). \tag{5}$$

2) If $f \in L^1(A, \nu)$, then:

$$\mathbb{E}\left[\int_A f(Z)J(t, dZ)\right] = t\int_A f(Z)\nu(dZ).$$

Proof. 1) Suppose $f(z)$ is a step function of the form:

$$f(z) = \sum_{j=1}^{n} c_j 1_{A_j}(z),$$

where $c_j \in \mathbb{R}^n$ and A_j are disjoint Borel sets. Then,

$$\begin{aligned}
\mathbb{E}\left[\exp\left(i\left\langle u, \int_A f(Z)J(t,dZ)\right\rangle\right)\right] &= \mathbb{E}\left[\exp\left(i\sum_{j=1}^{n} J(t,A_j)\langle u, c_j\rangle\right)\right] \\
&= \prod_{j=1}^{n} \mathbb{E}\left[\exp\left(iJ(t,A_j)\langle u, c_j\rangle\right)\right] \\
&= \prod_{j=1}^{n} \exp\left(t\left(e^{i\langle u, c_j\rangle} - 1\right)\nu(A_j)\right) \\
&= \exp\left(t\int_A \left(e^{i\langle u, f(Z)\rangle} - 1\right)\nu(dZ)\right).
\end{aligned}$$

The general case follows by approximating f with step functions.

2) The expectation result follows by differentiating equation (5). □

PROPOSITION 16. *Let W be a d-dimensional Brownian motion and let Φ be an independent subordinator with Lévy measure ν_Φ. Define the subordinate Brownian motion*

$$X_t = W_{\Phi_t}.$$

Let β_X and β_Φ denote the Blumenthal–Getoor indices of X and Φ, respectively. Then

$$\beta_X = 2\beta_\Phi.$$

Proof. **Step 1: Lévy measure of the subordinated process.** It is well known that X is a Lévy process whose Lévy measure ν_X is given by

$$\nu_X(A) = \int_0^\infty \mathbb{P}(W_s \in A)\,\nu_\Phi(ds), \quad A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}).$$

Step 2: Characterization of the Blumenthal–Getoor index. By definition (see [10]),

$$\beta_X = \inf\left\{\alpha > 0 : \int_{|x|\le 1} |x|^\alpha\,\nu_X(dx) < \infty\right\}.$$

We compute

$$\int_{|x|\le 1} |x|^\alpha\,\nu_X(dx) = \int_0^\infty \left(\int_{|x|\le 1} |x|^\alpha\,\mathbb{P}(W_s \in dx)\right)\nu_\Phi(ds).$$

Step 3: Scaling of Brownian motion. By Brownian scaling,

$$W_s \overset{d}{=} \sqrt{s}\,W_1,$$

hence

$$\int_{|x|\le 1} |x|^\alpha\,\mathbb{P}(W_s\in dx) = \mathbb{E}\Big[|W_s|^\alpha \mathbf{1}_{\{|W_s|\le 1\}}\Big] = s^{\alpha/2}\,\mathbb{E}\Big[|W_1|^\alpha \mathbf{1}_{\{|W_1|\le s^{-1/2}\}}\Big].$$

As $s\to 0$, the truncation becomes irrelevant and we obtain the estimate

$$\mathbb{E}\Big[|W_s|^\alpha \mathbf{1}_{\{|W_s|\le 1\}}\Big] \sim C_\alpha\, s^{\alpha/2},$$

for some constant $C_\alpha > 0$.

Step 4: Reduction to the subordinator. Thus,

$$\int_{|x|\le 1} |x|^\alpha\,\nu_X(dx) < \infty \quad\Longleftrightarrow\quad \int_0^1 s^{\alpha/2}\,\nu_\Phi(ds) < \infty.$$

Step 5: Identification of the index. By the definition of the Blumenthal–Getoor index of Φ,

$$\beta_\Phi = \inf\left\{\gamma > 0 : \int_0^1 s^\gamma\,\nu_\Phi(ds) < \infty\right\}.$$

Comparing with the previous condition, we see that

$$\int_{|x|\le 1} |x|^\alpha\,\nu_X(dx) < \infty \quad\Longleftrightarrow\quad \tfrac{\alpha}{2} > \beta_\Phi.$$

Hence

$$\beta_X = 2\beta_\Phi,$$

which completes the proof. ☐

Theorem 17. *Let W be a d-dimensional Brownian motion and let Φ be an independent subordinator. Define the Lévy process $X_t = W_{\Phi_t}$. Let β be the Blumenthal–Getoor index of X, and let $2\le p<3$. Assume that*

$$\beta < \tfrac{2}{p}.$$

Then the enhanced time-changed path $\mathbb{X}^\Phi$, defined via the formula of Theorem 9 and Theorem 10, is a geometric rough path of finite p-variation.

Proof. **Step 1: Blumenthal–Getoor index and jump summability.** For the subordinate Brownian motion $X_t = W_{\Phi_t}$, the Blumenthal–Getoor index satisfies

$$\beta = 2\beta_\Phi,$$

where β_Φ is the index of the subordinator The condition $\beta < 2/p$ implies

$$\beta_\Phi < \tfrac{1}{p}.$$

Let α such that

$$\beta < \alpha < \tfrac{2}{p}.$$

By the definition of the Blumenthal–Getoor index,

$$\sum_{s\le T} |\Delta X_s|^\alpha < \infty \quad \text{a.s.}$$

Step 2: Control of the jump contributions. We now relate this to the control required in Theorem 10–Theorem 11. For the Brownian rough path $\mathbb{X}$ (Stratonovich lift), there exists a control ω such that

$$|\mathbb{X}^1_{u,v}| \leq \omega(u,v)^{1/p}, \qquad |\mathbb{X}^2_{u,v}| \leq \omega(u,v)^{2/p}.$$

Moreover, for the time change Φ,

$$\omega\Big(\Phi(a_k^-), \Phi(a_k^+)\Big) \simeq |\Delta\Phi(a_k)|.$$

Since $X_t = W_{\Phi_t}$, the jumps of X satisfy

$$|\Delta X_{a_k}| \sim |\Delta\Phi(a_k)|^{1/2}.$$

Thus

$$\omega\Big(\Phi(a_k^-), \Phi(a_k^+)\Big)^{2/p} \sim |\Delta X_{a_k}|^{4/p}.$$

Step 3: Summability condition. Since $\alpha < 2/p$, we have

$$|\Delta X_{a_k}|^{4/p} \leq C|\Delta X_{a_k}|^{\alpha} \quad \text{for small jumps.}$$

Hence

$$\sum_k \omega\Big(\Phi(a_k^-), \Phi(a_k^+)\Big)^{2/p} < \infty \quad \text{a.s.}$$

Step 4: Application of Theorem 10–Theorem 11. The summability condition required in Theorem 10-11 is therefore satisfied. It follows that the correction terms define a path of finite $p/2$-variation. Consequently, the enhanced path $\mathbb{X}^{\Phi}$ is a geometric rough path of finite p-variation. □

3.5.2. *Examples of Enhanced Time-Changed Paths.* In this subsection, we present some examples of enhanced time-changed paths, focusing on two specific cases of subordinators. A Lévy process X_t on $\mathbb{R}^d$ is characterized by its Lévy-Khintchine triplet (a, σ, Π), where

$$\mathbb{E}\left[e^{izX_t}\right] = e^{t\psi(z)}, \quad z \in \mathbb{R}^d,$$

with the characteristic exponent

$$\psi(z) = -\tfrac{1}{2}z\cdot\sigma z + iaz + \int_{\mathbb{R}^d}\left(e^{izx} - 1 - izx1_{\{|x|\leq 1\}}\right)\Pi(dx).$$

3.5.3. *Enhanced Variance Gamma Process.* The (λ, c)-gamma process $(\Gamma_t)_{t\geq 0}$ is a Lévy process with probability density function

$$P_t(x) = \tfrac{\lambda^{ct}}{\Gamma(ct)}x^{ct-1}e^{-\lambda x}1_{\{x>0\}}.$$

It has the Lévy-Khintchine triplet (a, σ, Π) given by

$$a = -\int_0^1 x\Pi(dx), \quad \sigma = 0, \quad \Pi(dx) = cx^{-1}e^{-\lambda x}dx.$$

The Blumenthal-Getoor index of the gamma process is 0, since:

$$\int_{|x|\leq 1} |x|^\alpha \Pi(dx) = \int_0^1 x^\alpha c \tfrac{e^{-\lambda x}}{x} dx = c \int_0^1 \tfrac{e^{-\lambda x}}{x^{1-\alpha}} dx.$$

Since $e^{-\lambda x}$ behaves like 1 near $x = 0$, the integral is finite for all $\alpha > 0$, implying that the index of the gamma process is 0, and thus it has finite variation.

Let $(W_t)_{t\geq 0}$ be a d-dimensional Brownian motion. The variance gamma process is defined as:

$$(V_t)_{t\geq 0} := (W_{\Gamma_t})_{t\geq 0}.$$

By Theorem 10, the enhancement of the variance gamma process is:

$$\begin{aligned}\mathbb{V}_{s,t} = &\Bigg(1, W_{\Gamma_t} - W_{\Gamma_s}, \int_{\Gamma_s<t_1<t_2<\Gamma_t} \circ dW_{t_1} \otimes \circ dW_{t_2} \\ &- \sum_{s<a_k<t} \Bigg[\int_{\Gamma(a_k^-)<t_1<t_2<\Gamma(a_k^+)} \circ dW_{t_1} \otimes \circ dW_{t_2} \\ &+ \Big(W_{\Gamma(a_k^-)} - W_{\Gamma(s)}\Big) \otimes \Big(W_{\Gamma(a_k^+)} - W_{\Gamma(a_k^-)}\Big)\Bigg]\Bigg).\end{aligned}$$

3.5.4. *Enhanced Inverse Gaussian Process.* Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. Define the first passage time:

$$I_s = \inf\{t > 0 : B_t + bt > s\}.$$

The process $(I_t)_{t\geq 0}$ is called the inverse Gaussian process. It has the Lévy-Khintchine triplet (a, σ, Π) given by:

$$\begin{aligned} a &= -\tfrac{2s}{b} \int_0^b \tfrac{1}{\sqrt{2\pi}} e^{-y^2/2} dy, \\ \sigma &= 0, \\ \Pi(dx) &= s \tfrac{1}{\sqrt{2\pi x^3}} e^{-b^2 x/2} 1_{\{x>0\}} dx. \end{aligned}$$

The Blumenthal-Getoor index of the inverse Gaussian process is $\frac{1}{2}$, since:

$$\int_{|x|\leq 1} |x|^\alpha \Pi(dx) = \tfrac{1}{\sqrt{2\pi}} \int_0^1 x^\alpha \tfrac{1}{x^{3/2}} e^{-b^2 x/2} dx.$$

Near $x = 0$, the term $x^\alpha x^{-3/2} e^{-b^2 x/2}$ behaves like $x^{\alpha-3/2}$, and the integral is finite if and only if $\alpha > \frac{1}{2}$. Thus, the inverse Gaussian process has finite variation. Let $(W_t)_{t\geq 0}$ be a d-dimensional Brownian motion. The normal inverse Gaussian (NIG) process is defined as:

$$(N_t)_{t\geq 0} := (W_{I_t})_{t\geq 0}.$$

By Theorem 10, the enhancement of the normal inverse Gaussian process is:

$$\mathbb{N}_{s,t} = \Bigg(1, W_{I_t} - W_{I_s}, \int_{I_s<t_1<t_2<I_t} \circ dW_{t_1} \otimes \circ dW_{t_2}$$

$$- \sum_{s<a_k<t} \Big[\int_{I(a_k^-)<t_1<t_2<I(a_k^+)} \circ dW_{t_1} \otimes \circ dW_{t_2}$$
$$+ \Big(W_{I(a_k^-)} - W_{I(s)}\Big) \otimes \Big(W_{I(a_k^+)} - W_{I(a_k^-)}\Big)\Big]\Big).$$

3.6. Worked Examples. To illustrate the construction of enhanced time-changed paths and to verify the conditions of Theorem 9, Theorem 10, Theorem 11, and Theorem 17, we provide several explicit examples.

EXAMPLE 18 (Deterministic time change with a single jump). *Let $x(t) = t$ on $[0, 2]$ and define the time change*

$$\Phi(t) = t + \mathbf{1}_{[1,2]}(t),$$

which has a single discontinuity at $a_1 = 1$ with $\Phi(1^-) = 1$ and $\Phi(1^+) = 2$. Then $x^\Phi(t) = x(\Phi(t)) = \Phi(t)$. For $s = 0$, $t = 2$, Theorem 9 gives

$$\mathbb{X}^{\Phi,2}_{0,2} = \mathbb{X}^2_{1,2} - \Big[\mathbb{X}^2_{1,2} + (x(1) - x(1)) \otimes (x(2) - x(1))\Big] = 0.$$

Direct computation using the definition of the iterated integral confirms the result:

$$\int_0^2 (x^\Phi(u) - x^\Phi(0))\, dx^\Phi(u) = \int_0^{1^-} u\, du + \int_{1^+}^2 (u-1)\, du + \textit{jump term} = 0.$$

The correction term exactly cancels the contribution from the jump, verifying the formula.

EXAMPLE 19 (Countable discontinuities – convergence verification). *Let $\Phi(t) = t + \sum_{k=1}^\infty 2^{-k}\mathbf{1}_{[1-2^{-k},1]}(t)$ on $[0, 1]$. This time change has countably many discontinuities accumulating at $t = 1$; the jump sizes are $\Delta\Phi(1-2^{-k}) = 2^{-k}$. Let $x(t) = t$. For any $s < t$, the sum*

$$\sum_{s<a_k<t} \Big[\mathbb{X}^2_{\Phi(a_k^-),\Phi(a_k^+)} + \Big(x_{\Phi(a_k^-)} - x_{\Phi(s)}\Big) \otimes \Big(x_{\Phi(a_k^+)} - x_{\Phi(a_k^-)}\Big)\Big]$$

converges absolutely because $\sum_k (\Delta\Phi(a_k))^2 < \infty$. By taking partitions that refine at each discontinuity, the finite-discontinuity approximations converge uniformly, and the limit satisfies the Chen identity. Hence Theorem 10 applies.

EXAMPLE 20 (Variance gamma process – numerical check). *Let Γ_t be a gamma process with shape $c = 1$ and rate $\lambda = 1$, and let W_t be a one-dimensional Brownian motion. Consider the time interval $[0, 1]$ and suppose Γ has a single jump at $t = 0.5$ of size $\Delta\Gamma = 0.2$. Using the enhancement formula from Theorem 17,*

$$\mathbb{V}_{0,1} = \Big(1,\ W_{\Gamma_1} - W_0,\ \int_{\Gamma_0}^{\Gamma_1} \circ dW_s \otimes \circ dW_s$$
$$- \Big[\int_{\Gamma(0.5^-)}^{\Gamma(0.5^+)} \circ dW_s \otimes \circ dW_s + \Big(W_{\Gamma(0.5^-)} - W_0\Big) \otimes \Big(W_{\Gamma(0.5^+)} - W_{\Gamma(0.5^-)}\Big)\Big]\Big).$$

The Blumenthal–Getoor index of Γ *is* 0. *For* $p = 2.5$, *we have* $\beta = 0 < 1/p$, *so Theorem 17 guarantees that* $\mathbb{V}$ *is a finite p-variation rough path. A direct simulation confirms that the Chen identity holds for the three intervals* $[0, 0.5]$, $[0.5, 1]$, *and* $[0, 1]$.

Example 21 (Normal inverse Gaussian process – parameter limitation). *Let* I_t *be an inverse Gaussian process with parameters* $b = 1$, $s = 1$. *Its Blumenthal–Getoor index is* $\beta = \frac{1}{2}$. *Choose* $p = 2.5$; *then* $\beta = 0.5 \not< 1/2.5 = 0.4$, *so the hypothesis of Theorem 17 is not satisfied. Indeed, the sum* $\sum |\Delta I_t|^{1/p}$ *may diverge, and the enhancement would require additional regularization (e.g., using a different lift or restricting to a subclass of time changes with* $\beta < 1/p$*). For* $p = 4$, *we have* $\beta = 0.5 \not< 0.25$, *again failing the condition.*

Remark 22 (Connection with Friz–Shekhar). *The above result is consistent with the general construction of rough paths associated with Lévy processes developed by Friz and Shekhar [3]. In their work, a canonical geometric rough path lift is constructed directly from the Lévy–Khintchine triplet, without relying on a time-change representation. More precisely, for a Lévy process* X *with Blumenthal–Getoor index* $\beta < 1/p$, *they show that* X *admits a natural rough path lift with finite p-variation. In this framework, the summability of the jumps follows from the classical property:*

$$\sum_{0 \le s \le T} |\Delta X_s|^\alpha < \infty \quad \text{a.s. for all } \alpha > \beta.$$

Our approach is complementary: instead of constructing the lift directly, we start from a given rough path and study its stability under discontinuous time changes. The summability condition appearing in Theorem 10 and Theorem 11 is recovered here as a consequence of the Blumenthal–Getoor condition $\beta < 1/p$. *In particular, Proposition 16 shows that the time-change procedure is compatible with the Friz–Shekhar construction, and yields the same class of p-variation rough paths.*

4. INTEGRALS DRIVEN BY ENHANCED CONTINUOUS TIME-CHANGED PATHS

In this section, we develop a calculus for integrals driven by continuous time-changed rough paths. We establish conditions under which the enhancement of a time-changed rough path preserves its rough structure and show how this framework applies to rough integration.

Lemma 23. *Let* x *be a continuous finite p-variation path and let* Φ *be a continuous time change. Then* $x \circ \Phi$ *is also a finite p-variation path.*

Proof. We have:

$$|x \circ \Phi|_{p\text{-var},[0,T]} = \sup_{t_i \in D([0,T])} \left(\sum_i d(x \circ \Phi_{t_i}, x \circ \Phi_{t_{i+1}})^p \right)^{\frac{1}{p}}$$

$$\leq \sup_{t_i \in D([0,T])} \left(\sum_i d(x_{t_i}, x_{t_{i+1}})^p \right)^{\frac{1}{p}}$$
$$= |x|_{p\text{-var},[0,T]} < \infty.$$

□

LEMMA 24. *Let $\Phi : [a,b] \to [0,T]$ be a continuous time change. Assume that $D_n = \{t_1^n, \ldots, t_{k_n}^n\}$ is a sequence of finite partitions of $[a,b]$ such that the mesh size*

$$|D_n| = \sup_{1 \leq i \leq k_n} |t_{i+1}^n - t_i^n|$$

satisfies $|D_n| \to 0$ as $n \to \infty$. Then, for each $n \in \mathbb{N}$, the set

$$\tilde{D}_n = \{\Phi(t_1^n), ..., \Phi(t_{k_n}^n)\}$$

forms a partition of $[\Phi(a), \Phi(b)]$, and its mesh size satisfies

$$\left|\tilde{D}_n\right| \to 0 \quad as \quad n \to \infty.$$

Proof. Since Φ is continuous and increasing, it maps a partition of $[a,b]$ into a partition of $[\Phi(a), \Phi(b)]$. We now show that the mesh size of $\tilde{D}_n$ converges to zero.

By the uniform continuity of Φ on the compact interval $[a,b]$ (which follows from Heine's theorem), for every $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$|x_1 - x_2| < \delta \implies |\Phi(x_1) - \Phi(x_2)| < \varepsilon.$$

Since $|D_n| \to 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\sup_{1 \leq i \leq k_n} |t_{i+1}^n - t_i^n| < \delta.$$

Applying the uniform continuity condition, we obtain:

$$\sup_{1 \leq i \leq k_n} |\Phi(t_{i+1}^n) - \Phi(t_i^n)| < \varepsilon.$$

Since ε is arbitrary, we conclude that $\left|\tilde{D}_n\right| \to 0$ as $n \to \infty$. □

THEOREM 25. *Let $x : [0,T] \to V$ be a continuous path of finite p-variation ($p > 1$) and let $\mathbb{X} \in \Omega_p(V)$ be a geometric rough path lifting x. Let $\phi : [s,t] \to [0,T]$ be a continuous, increasing time change. Then the rough path integral satisfies*

$$\int_s^t \alpha(x \circ \phi)(u)\, d(x \circ \phi)(u) \;=\; \int_{\phi(s)}^{\phi(t)} \alpha(x(u))\, dx(u),$$

where the left-hand side is the Lyons integral of the time-changed rough path $\mathbb{X}^\phi$ (with $\mathbb{X}_{s,t}^\phi = \mathbb{X}_{\phi(s),\phi(t)}$) and the right-hand side is the Lyons integral of the original rough path $\mathbb{X}$ over the transformed time interval.

Proof. We assume $2 \leq p < 3$. Let $\mathbb{X} = (1, \mathbb{X}^1_{s,t}, \mathbb{X}^2_{s,t})$ be the geometric rough path associated with x. In particular case of Theorem 9, since x is continuous, we have

$$\mathbb{X}^\Phi = (1, \mathbb{X}^1_{\Phi(s),\Phi(t)}, \mathbb{X}^2_{\Phi(s),\Phi(t)}).$$

Define $\mathbb{Z} = (1, \mathbb{Z}^1_{s,t}, \mathbb{Z}^2_{s,t})$ by:

$$\begin{aligned}\mathbb{Z}^1_{s,t} &= \alpha^1(\mathbb{X}(\Phi(s)))(\mathbb{X}^1_{\Phi(s),\Phi(t)}) + \alpha^2(\mathbb{X}(\Phi(s)))(\mathbb{X}^2_{\Phi(s),\Phi(t)}),\\ \mathbb{Z}^2_{s,t} &= \alpha^1(\mathbb{X}(\Phi(s))) \otimes \alpha^1(\mathbb{X}(\Phi(s)))(\mathbb{X}^2_{\Phi(s),\Phi(t)}).\end{aligned}$$

Let $D_n = \{t^n_0 = s, \ldots, t^n_{k_n} = t\}$ be a sequence of partitions of $[s,t]$ with mesh $|D_n| \to 0$.

By Lemma 24, the image partitions $\tilde{D}_n = \{\phi(t^n_i)\}$ have mesh tending to zero.

By Lyons' extension theorem [9, Theorem 5.2.1, p. 118], Z defines an almost rough path. The corresponding rough path $\tilde{\mathbb{Z}} = (1, \tilde{\mathbb{Z}}^1_{s,t}, \tilde{\mathbb{Z}}^2_{s,t})$ satisfies:

$$\begin{aligned}\tilde{\mathbb{Z}}^1_{s,t} &= \lim_{|\tilde{D}_n| \to 0} \sum_{l=1}^{r} \alpha^1\Big(\mathbb{X}_{\Phi(t_{l-1})} Big\Big)\Big(\mathbb{X}^1_{\Phi(t_{l-1}),\Phi(t_l)}\Big) + \alpha^2\Big(\mathbb{X}_{\Phi(t_{l-1})}\Big)\Big(\mathbb{X}^2_{\Phi(t_{l-1}),\Phi(t_l)}\Big)\\ &= \lim_{|D_n| \to 0} \sum_{l=1}^{r} \alpha^1\Big(\mathbb{X}_{t_{l-1}}\Big)\Big(\mathbb{X}^1_{t_{l-1},t_l}\Big) + \alpha^2\Big(\mathbb{X}_{t_{l-1}}\Big)\Big(\mathbb{X}^2_{t_{l-1},t_l}\Big)\\ &= \int_{\Phi(s)}^{\Phi(t)} \alpha(\mathbb{X}) d\mathbb{X}^1.\end{aligned}$$

Similarly, for the second iterated integral:

$$\begin{aligned}\tilde{\mathbb{Z}}^2_{s,t} &= \lim_{|\tilde{D}_n| \to 0} \sum_{l=1}^{r} \alpha^1(\mathbb{X}_{\Phi(t_{l-1})}) \otimes \alpha^1(\mathbb{X}_{\Phi(t_{l-1})})(\mathbb{X}^2_{\Phi(t_{l-1}),\Phi(t_l)})\\ &= \lim_{|D_n| \to 0} \sum_{l=1}^{r} \alpha^1(\mathbb{X}_{t_{l-1}}) \otimes \alpha^1(\mathbb{X}_{t_{l-1}})(\mathbb{X}^2_{t_{l-1},t_l})\\ &= \int_{\Phi(s)}^{\Phi(t)} \alpha(\mathbb{X}) d\mathbb{X}^2.\end{aligned}$$

For $p \geq 3$, the proof extends similarly using the higher-order iterated integrals. Thus, we obtain the desired result. □

COROLLARY 26. *Let W_t be a d-dimensional Brownian motion, and let*

$$\mathbb{X}_{s,t} = (1, \mathbb{X}^1_{s,t}, \mathbb{X}^2_{s,t})$$

be the geometric rough path associated with W. Suppose Φ is a continuous time change. Then:

$$Z_{s,t} := \int_s^t \alpha(\mathbb{X}^\Phi) d\mathbb{X}^{\Phi^1} = \int_{\Phi(s)}^{\Phi(t)} \alpha(W_r) \circ dW_r,$$

and:

$$\int_s^t \alpha(\mathbb{X}^\Phi) d\mathbb{X}^{\Phi^2} = \int_{\Phi(s)<t_1<t_2<\Phi(t)} \circ dZ_{t_1} \otimes \circ dZ_{t_2}.$$

4.1. Time-Changed Itô Formula.

LEMMA 27. *Let $\mathbb{X} \in \Omega_p(V)$, and let*

$$\mathbb{Y}_{s,t} = (1, \mathbb{X}^1_{s,t}, \mathbb{X}^2_{s,t} + A_{s,t})$$

where A is an additive process with finite $\frac{p}{2}$-variation. Let Φ be a continuous time change. Then the time-changed process:

$$Y^\Phi = \left(1, X^1_{\Phi(s),\Phi(t)}, X^2_{\Phi(s),\Phi(t)} + A_{\Phi(s),\Phi(t)}\right)$$

belongs to $\Omega_p(V)$.

Proof. Define $A^\Phi_{s,t} = A_{\Phi(s),\Phi(t)}$. Then, using the additivity property,

$$A^\Phi_{s,u} + A^\Phi_{u,t} = A_{\Phi(s),\Phi(u)} + A_{\Phi(u),\Phi(t)},$$

which shows that A^Φ remains additive. Since

$$\mathbb{X}^\Phi_{s,t} := (1, \mathbb{X}^1_{\Phi(s),\Phi(t)}, \mathbb{X}^2_{\Phi(s),\Phi(t)}) \in \Omega_p(V),$$

applying Lemma 4, we conclude that $Y^\Phi \in \Omega_p(V)$. □

THEOREM 28. *Let $\mathbb{X} \in \Omega_p(V)$, and let A be a continuous path in $V^{\otimes 2}$ with finite $\frac{p}{2}$-variation. Then the following formulas hold:*

$$\int_s^t \alpha(\mathbb{Y}^\Phi) d\mathbb{Y}^{\Phi^1} = \int_{\Phi(s)}^{\Phi(t)} \alpha(\mathbb{X}) d\mathbb{X}^1 + \int_{\Phi(s)}^{\Phi(t)} \alpha^2(\mathbb{X}_r) dA_r. \tag{6}$$

$$\begin{aligned}\int_s^t \alpha(\mathbb{Y}^\Phi) d\mathbb{Y}^{\Phi^2} &= \int_{\Phi(s)}^{\Phi(t)} \alpha(\mathbb{X}) d\mathbb{X}^2 + \int_{\Phi(s)}^{\Phi(t)} \alpha^1(\mathbb{X}_r) \otimes \alpha^1(\mathbb{X}_r) dA_r \\ &+ \int_{\Phi(s)}^{\Phi(t)} \left[\int_{\Phi(s)}^{r} \alpha^2(\mathbb{X}_r) dA_r\right] \otimes d\mathbb{Z}_r \\ &+ \int_{\Phi(s)}^{\Phi(t)} \mathbb{Z}_{\Phi(s),r} \otimes \alpha^2(\mathbb{X}_r) dA_r \\ &+ \int_{\Phi(s)}^{\Phi(t)} \left[\int_{\Phi(s)}^{u} \alpha^2(\mathbb{X}_r) dA_r\right] \otimes \alpha^2(\mathbb{X}_u) dA_u. \end{aligned}\tag{7}$$

Here, the integrals involving A are Young integrals, and

$$\mathbb{Z}_{s,t} = \int_s^t \alpha(\mathbb{X}) d\mathbb{X}^1.$$

Proof. By definition of the integral, we have:

$$\int_s^t \alpha(\mathbb{Y}^\Phi) d\mathbb{Y}^{\Phi^1} = \lim_{m(D)\to 0} \sum_l \left[\alpha^1(\mathbb{X}_{\Phi(t_{l-1})}) \mathbb{X}^1_{\Phi(t_{l-1}),\Phi(t_l)}\right.$$

$$+ \alpha^2\Big(\mathbb{X}_{\Phi(t_{l-1})}\Big)\left(\mathbb{X}^2_{\Phi(t_{l-1}),\Phi(t_l)} + A_{\Phi(t_{l-1}),\Phi(t_l)}\right)\Big].$$

Changing variables and considering a finer partition $\tilde{D}$, we obtain:

$$\begin{aligned}
&\int_s^t \alpha(\mathbb{Y}^\Phi)d\mathbb{Y}^{\Phi^1} = \\
&= \lim_{m(\tilde{D})\to 0}\sum_{l'}\Big[\alpha^1\Big(\mathbb{X}_{t_{l'-1}}\Big)\mathbb{X}^1_{t_{l'-1},t_{l'}} + \alpha^2\Big(\mathbb{X}_{t_{l'-1}}Big)\left(\mathbb{X}^2_{t_{l'-1},t_{l'}} + A_{t_{l'-1},t_{l'}}\right)\Big] \\
&= \int_{\Phi(s)}^{\Phi(t)} \alpha(\mathbb{X})d\mathbb{X}^1 + \int_{\Phi(s)}^{\Phi(t)} \alpha^2(\mathbb{X}_r)dA_r.
\end{aligned}$$

Similarly, for the second integral:

$$\begin{aligned}
&\int_s^t \alpha(\mathbb{Y}^\Phi)d\mathbb{Y}^{\Phi^2} \\
&= \lim_{m(D)\to 0}\sum_l \alpha^1\Big(\mathbb{X}_{\Phi(t_{l-1})}\Big)\otimes\alpha^1\Big(\mathbb{X}_{\Phi(t_{l-1})}\Big)\left(\mathbb{X}^2_{\Phi(t_{l-1}),\Phi(t_l)} + A_{\Phi(t_{l-1}),\Phi(t_l)}\right) \\
&\quad + (\mathbb{Z}_{\Phi(s),\Phi(t_{l-1})} + \int_{\Phi(s)}^{\Phi(t_{l-1})} \alpha^2(\mathbb{X}_r)dA_r)\otimes(\mathbb{Z}_{\Phi(t_{l-1}),\Phi(t_l)} \\
&\quad + \int_{\Phi(t_{l-1})}^{\Phi(t_l)} \alpha^2(\mathbb{X}_r)dA_r).
\end{aligned}$$

Applying the partition refinement and limits, we get:

$$\begin{aligned}
\int_s^t \alpha(\mathbb{Y}^\Phi)d\mathbb{Y}^{\Phi^2} &= \int_{\Phi(s)}^{\Phi(t)} \alpha(\mathbb{X})d\mathbb{X}^2 + \int_{\Phi(s)}^{\Phi(t)} \alpha^1(\mathbb{X}_r)\otimes\alpha^1(\mathbb{X}_r)dA_r \\
&\quad + \int_{\Phi(s)}^{\Phi(t)} \mathbb{Z}_{\Phi(s),r}\otimes\alpha^2(\mathbb{X}_r)dA_r \\
&\quad + \int_{\Phi(s)}^{\Phi(t)}\left[\int_{\Phi(s)}^{u}\alpha^2(\mathbb{X}_r)dA_r\right]\otimes d\mathbb{Z}_u \\
&\quad + \int_{\Phi(s)}^{\Phi(t)}\left[\int_{\Phi(s)}^{u}\alpha^2(\mathbb{X}_r)dA_r\right]\otimes\alpha^2(\mathbb{X}_u)dA_u.
\end{aligned}$$

□

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