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# A STANCU TYPE EXTENSION OF THE CAMPITI-METAFUNE OPERATOR

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**Abstract.** We consider an extension of the Campiti-Metafune operator using a Stancu type technique. We study some properties of the new obtained operator.

**MSC.** 41A10, 41A35, 41A36, 47A58.

**Keywords.** Campiti-Metafune operator, Stancu operator, Bernstein operator, moments of the operator.

## 1. INTRODUCTION

In 1982, D.D. Stancu [6], introduced a new Bernstein type operator given by

$$L_{n,r}(f;x) = \sum_{k=0}^{n-r} b_{n-r,k}(x)\left[(1-x)f\left(\tfrac{k}{n}\right) + xf\left(\tfrac{k+r}{n}\right)\right], \tag{1}$$

where $b_{n,k}$ denote the basis Bernstein polynomials of degree $n$,

$$b_{n,k} = \tbinom{n}{k}x^k(1-x)^{n-k},\ k = 0, 1, \dots, n,$$

for $f \in C[0,1]$, $n, r \in \mathbb{N}$ such that $n > 2r$.

In 1996, M. Campiti and G. Metafune [3], introduced and studied a new Bernstein type operator that now bears their names. For introducing the operator, we need two sequences $\lambda = (\lambda_n)_{n\geq 1}$ and $\rho = (\rho_n)_{n\geq 1}$, and the numbers $\alpha_{n,k}$ defined by

$$\alpha_{n,0} = \lambda_n,\ \alpha_{n,n} = \rho_n,\ \alpha_{n+1,k} = \alpha_{n,k} + \alpha_{n,k-1}, \qquad \text{for } k = \overline{1,n},\ n \in \mathbb{N}. \tag{2}$$

The Campiti-Metafune operator $A_n : C([0,1]) \to C([0,1])$, is given by [3]

$$(A_nf)(x) = \sum_{k=0}^{n} \alpha_{n,k}x^k(1-x)^{n-k}f(\tfrac{k}{n}). \tag{3}$$

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Remark 1. *If $\lambda$ and $\rho$ are constant sequences of term 1, then $\alpha_{n,k} = \binom{n}{k}$, for $k = \overline{0,n}$ and $A_n$ becomes Bernstein operator $B_n$, given by*

$$(B_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f(\tfrac{k}{n}).$$

For the sequences $(\lambda, \rho)$ we consider the following choices, as in [1]:

(i) Case $\lambda = \delta_m$, $m \in \mathbb{N}$, $\rho = 0$, where $\delta_m = (\delta_{n,m})_{n \geq 1}$, $\delta_{n,m} = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$.

We denote $A_n^{(\delta_m, 0)}$ by $L_{m,n}$ and it is called *the elementary left operator of order* $m$ associated to $A_n$.

The coefficients of the operators $L_{m,n}$ are denoted by $l_{m,n,k}$ and they are defined by

$$l_{m,n,k} = \begin{cases} 0, & (n < m) \text{ or } \begin{array}{c} (n = m,\ k \geq 1) \\ (n > m,\ k = 0) \\ (n > m,\ k \geq n - m + 1) \end{array} \\ 1, & n = m,\ k = 0 \\ \binom{n-m-1}{k-1}, & n > m, 1 \leq k \leq n - m. \end{cases}.$$

(ii) Case $\lambda = 0$, $\rho = \delta_m$, $m \in \mathbb{N}$. We denote $A_n^{(0, \delta_m)}$ by $R_{m,n}$ and it is called *elementary right operator of order* $m$ associated to $A_n$.

The coefficients of the operators $R_{m,n}$ are denoted by $r_{m,n,k}$ and they are defined by

$$r_{m,n,k} = \begin{cases} 0, & (n < m) \text{ or } \begin{array}{c} (n = m,\ k \leq n - 1) \\ (n > m,\ k \leq m - 1) \\ (n > m,\ k = n) \end{array} \\ 1, & n = m,\ k = n \\ \binom{n-m-1}{k-m}, & n > m, m \leq k \leq n - 1. \end{cases}.$$

Consequently, for any $(m, n) \in \mathbb{N} \times \mathbb{N}$ we get

$$(L_{m,n} f)(x) = \sum_{k=1}^{n-m} a_{m,n,k}(x) f(\tfrac{k}{n}), \tag{4}$$

with

$$a_{m,n,k}(x) = \binom{n-m-1}{k-1} x^k (1-x)^{n-k}, \tag{5}$$

for $m < n$ and

$$(L_{n,n} f)(x) = (1-x)^n f(0).$$

We have

$$(R_{m,n} f)(x) = \sum_{k=m}^{n-1} b_{m,n,k}(x) f(\tfrac{k}{n}), \tag{6}$$

with

$$b_{m,n,k}(x) = \binom{n-m-1}{k-m} x^k (1-x)^{n-k}, \tag{7}$$

for $m < n$ and

$$(R_{n,n} f)(x) = x^n f(1).$$

Using these elementary operators, the operator $A_n$ is decomposed as

$$A_n = \sum_{m=1}^{n} \lambda_m L_{m,n} + \sum_{m=1}^{n} \rho_m R_{m,n}. \tag{8}$$

REMARK 2. *For the particular case of* $\lambda = \rho = 1$, *we get*

$$B_n = \sum_{m=1}^{n} (L_{m,n} + R_{m,n}).$$

## 2. A STANCU TYPE EXTENSION OF THE CAMPITI-METAFUNE OPERATOR

Now we introduce *the Stancu type extension of the Campiti-Metafune operator*, based on an idea from [2], also used, for example, in [4], [5]. Using the Stancu type operator (1) and the operator (8), we get

$$(A_n^S f)(x) := \sum_{m=1}^{n} \lambda_m (L_{m,n,r}^S f)(x) + \sum_{m=1}^{n} \rho_m (R_{m,n,r}^S f)(x) \tag{9}$$

with

$$(L_{m,n,r}^S f)(x) = \sum_{k=1}^{n-m-r} a_{m,n-r,k}(x) [(1-x) f(\tfrac{k}{n}) + x f(\tfrac{k+r}{n})]$$

and

$$(R_{m,n,r}^S f)(x) = \sum_{k=m}^{n-r-1} b_{m,n-r,k}(x) [(1-x) f(\tfrac{k}{n}) + x f(\tfrac{k+r}{n})],$$

where $a_{m,n-r,k}(x)$ and $b_{m,n-r,k}(x)$ are given by (5) and (7), for $f \in C[0,1]$ and $n, r \in \mathbb{N}$ such that $n > 2r$.

REMARK 3. *For* $r = 0$, $A_n^S$ *is obtained as the Campiti-Metafune operator* $A_n$.

We are going to calculate the moments of the new operators and to study some approximation properties.

## 3. PROPERTIES OF THE CAMPITI-METAFUNE OPERATOR

We give first some results regarding the Campiti-Metafune operator that will be used in the sequel in order to prove some properties of the new constructed operator.

LEMMA 4 ([1]). *The operators defined by* (4) *and* (6) *verify the following relations:*

1°

$$(L_{m,n}e_0)(x) = \begin{cases} x(1-x)^m, & n > m; \\ (1-x)^n, & n = m; \end{cases}$$

$$(R_{m,n}e_0)(x) = \begin{cases} (1-x)x^m, & n > m; \\ x^n, & n = m; \end{cases}$$

2°

$$(L_{m,n}e_1)(x) = \begin{cases} \frac{x(1-x)^m(1+(n-m-1)x)}{n}, & n > m; \\ 0, & n = m; \end{cases}$$

$$(R_{m,n}e_1)(x) = \begin{cases} \frac{(1-x)x^m(m+(n-m-1)x)}{n}, & n > m; \\ x^n, & n = m; \end{cases}$$

3°

$$(L_{m,n}e_2)(x) = \begin{cases} x(1-x)^m \frac{1+3(n-m-1)x+(n-m-1)(n-m-2)x^2}{n^2}, & n > m; \\ 0, & n = m; \end{cases}$$

$$(R_{m,n}e_2)(x) = \begin{cases} (1-x)x^m \frac{m^2+(1+2m)(n-m-1)x+(n-m-1)(n-m-2)x^2}{n^2}, & n > m; \\ x^n, & n = m. \end{cases}$$

Now we study some properties for the new operator, $A_n^S$ introduced in (9).

Theorem 5. *For every $x \in [0,1]$, $n, r \in \mathbb{N}$ such that $n > 2r$, we have the following results:*

i)

$$\begin{aligned}(A_n^S e_0)(x) &= \sum_{m=1}^{n} \lambda_m (L_{m,n,r}^S e_0)(x) + \sum_{m=1}^{n} \rho_m (R_{m,n,r}^S e_0)(x) \\ &= \sum_{m=1}^{n-r-1} (\lambda_m x(1-x)^m + \rho_m x^m(1-x)) + \lambda_{n-r}(1-x)^{n-r} + \rho_{n-r}x^{n-r}\end{aligned}$$

ii)

$$(A_n^S e_1)(x) := \sum_{m=1}^{n} \lambda_m (L_{m,n,r}^S e_1)(x) + \sum_{m=1}^{n} \rho_m (R_{m,n,r}^S e_1)(x)$$

iii)

$$(A_n^S e_2)(x) := \sum_{m=1}^{n} \lambda_m (L_{m,n,r}^S e_2)(x) + \sum_{m=1}^{n} \rho_m (R_{m,n,r}^S e_2)(x)$$

*with*

$$(L_{m,n,r}^S e_0)(x) = \begin{cases} x(1-x)^m, & n > m+r; \\ (1-x)^{n-r}, & n = m+r; \end{cases}$$

$$(R_{m,n,r}^S e_0)(x) = \begin{cases} (1-x)x^m, & n > m+r; \\ x^{n-r}, & n = m+r; \end{cases},$$

*and*

$$(L^S_{m,n,r}e_1)(x) = \begin{cases} \frac{x(1-x)^m(1+(n-r-m-1)x)}{n-r} + \frac{r}{n}x^2(1-x)^m, & n > m+r; \\ \frac{r}{n}x(1-x)^{n-r}, & n = m+r; \end{cases}$$

$$(R^S_{m,n,r}e_1)(x) = \begin{cases} \frac{(1-x)x^m(m+(n-r-m-1)x)}{n-r} + \frac{r}{n}x(1-x)x^m, & n > m+r; \\ x^{n-r} + \frac{r}{n}x^{n-r+1}, & n = m+r; \end{cases}$$

*and*

$$(L^S_{m,n,r}e_2)(x)(x) =$$

$$= \begin{cases} x(1-x)^m \frac{\{1+3(n-r-m-1)x+(n-r-m-1)(n-r-m-2)x^2\}}{(n-r)^2} & \\ \quad + \frac{2r}{n}x^2\frac{(1-x)^m(1+(n-r-m-1)x)}{n-r} + \frac{r^2}{n^2}x^3(1-x)^m, & n > m+r; \\ \frac{r^2}{n^2}x^2(1-x)^{n-r}, & n = m+r; \end{cases}$$

$$(R^S_{m,n,r}e_2)(x)(x) =$$

$$= \begin{cases} (1-x)x^m \frac{m^2+(1+2m)(n-r-m-1)x+(n-r-m-1)(n-r-m-2)x^2}{(n-r)^2} & \\ \quad + \frac{2r}{n}x\frac{(1-x)x^m(m+(n-r-m-1)x)}{n-r} + \frac{r^2x^2}{n^2}(1-x)x^m, & n > m+r; \\ x^{n-r} + \frac{2r}{n}x^{n-r+1} + \frac{r^2x^2}{n^2}x^{n-r}, & n = m+r; \end{cases}$$

*Proof.*

$$(A^S_n e_0)(x) := \sum_{m=1}^{n} \lambda_m (L^S_{m,n,r}e_0)(x) + \sum_{m=1}^{n} \rho_m (R^S_{m,n,r}e_0)(x) \tag{10}$$

with

$$(L^S_{m,n,r}e_0)(x) = \sum_{k=1}^{n-m-r} a_{m-r,n,k}(x) = (L_{m,n-r}e_0)(x)$$
$$= \begin{cases} x(1-x)^m, & n > m+r; \\ (1-x)^{n-r}, & n = m+r; \end{cases}$$

and

$$(R^S_{m,n,r}e_0)(x) = \sum_{k=m}^{n-r-1} b_{m-r,n,k}(x) = (R_{m,n-r}e_0)(x)$$
$$= \begin{cases} (1-x)x^m, & n > m+r; \\ x^{n-r}, & n = m+r; \end{cases} .$$

So, we have

$$(A_n^S e_0)(x) := \sum_{m=1}^{n} \lambda_m (L_{m,n-r} e_0)(x) + \sum_{m=1}^{n} \rho_m (R_{m,n-r} e_0)(x) = (A_{n-r} e_0)(x)$$

By (10) we get

$$(A_n^S e_0)(x) = \sum_{m=1}^{n-r-1} (\lambda_m x(1-x)^m + \rho_m x^m (1-x)) + \lambda_{n-r}(1-x)^{n-r} + \rho_{n-r} x^{n-r}$$

ii) We have

$$(A_n^S e_1)(x) := \sum_{m=1}^{n} \lambda_m (L_{m,n,r}^S e_1)(x) + \sum_{m=1}^{n} \rho_m (R_{m,n,r}^S e_1)(x)$$

with

$$\begin{aligned}(L_{m,n,r}^S e_1)(x) &= \sum_{k=1}^{n-m-r} a_{m,n-r,k}(x)[(1-x)\tfrac{k}{n} + x\tfrac{k+r}{n}] \\ &= (L_{m,n-r} e_1)(x) + \tfrac{r}{n} x (L_{m.n-r} e_0)(x)\end{aligned}$$

and

$$\begin{aligned}(R_{m,n,r}^S e_1)(x) &= \sum_{k=m}^{n-r-1} b_{m,n-r,k}(x)[(1-x)\tfrac{k}{n} + x\tfrac{k+r}{n}] \\ &= (R_{m,n-r} e_1)(x) + \tfrac{r}{n} x (R_{m.n-r} e_0)(x)\end{aligned}$$

and by Lemma 4, we get

$$\begin{aligned}(L_{m,n,r}^S e_1)(x)(x) &= (L_{m,n-r} e_1) + \tfrac{r}{n} x (L_{m.n-r} e_0)(x) \\ &= \begin{cases} \frac{x(1-x)^m (1+(n-r-m-1)x)}{n-r} + \frac{r}{n} x^2 (1-x)^m, & n > m+r; \\ \frac{r}{n} x (1-x)^{n-r}, & n = m+r; \end{cases}\end{aligned}$$

and

$$\begin{aligned}(R_{m,n,r}^S e_1)(x) &= (R_{m,n-r} e_1)(x) + \tfrac{r}{n} x (R_{m.n-r} e_0)(x) \\ &= \begin{cases} \frac{(1-x)x^m (m+(n-r-m-1)x)}{n-r} + \frac{r}{n} x (1-x) x^m, & n > m+r; \\ x^{n-r} + \frac{r}{n} x^{n-r+1}, & n = m+r; \end{cases}\end{aligned}$$

iii) We have

$$(A_n^S e_2)(x) := \sum_{m=1}^{n} \lambda_m (L_{m,n,r}^S e_2)(x) + \sum_{m=1}^{n} \rho_m (R_{m,n,r}^S e_2)(x)$$

with

$$(L_{m,n,r}^S e_2)(x)(x) = \sum_{k=1}^{n-m-r} a_{m,n-r,k}(x) \Big[ (1-x)\tfrac{k^2}{n^2} + x \tfrac{(k+r)^2}{n^2} \Big]$$

$$= (L_{m,n-r}e_2)(x) + \tfrac{2r}{n}x(L_{m.n-r}e_1)(x) + \tfrac{r^2x^2}{n^2}(L_{m.n-r}e_0)(x)$$

and

$$(R^S_{m,n,r}e_2)(x)(x) = \sum_{k=1}^{n-m-r} a_{m,n-r,k}(x)\Big[(1-x)\tfrac{k^2}{n^2} + x\tfrac{(k+r)^2}{n^2}\Big]$$
$$= (R_{m,n-r}e_2)(x) + \tfrac{2r}{n}x(R_{m.n-r}e_1)(x) + \tfrac{r^2x^2}{n^2}(R_{m.n-r}e_0)(x).$$

By Lemma 4, we get

$$(L^S_{m,n,r}e_2)(x)(x) =$$
$$= (L_{m,n-r}e_2)(x) + \tfrac{2r}{n}x(L_{m.n-r}e_1)(x) + \tfrac{r^2x^2}{n^2}(L_{m.n-r}e_0)(x)$$
$$= \begin{cases} x(1-x)^m \frac{1+3(n-r-m-1)x+(n-r-m-1)(n-r-m-2)x^2}{(n-r)^2} & \\ \quad + \tfrac{2r}{n}x\frac{x(1-x)^m(1+(n-r-m-1)x)}{n-r} + \tfrac{r^2}{n^2}x^3(1-x)^m, & n > m+r; \\ \tfrac{r^2}{n^2}x^2(1-x)^{n-r}, & n = m+r; \end{cases}$$

and

$$(R^S_{m,n,r}e_2)(x)(x) =$$
$$= (R_{m,n-r}e_2)(x) + \tfrac{2r}{n}x(R_{m.n-r}e_1)(x) + \tfrac{r^2x^2}{n^2}(R_{m.n-r}e_0)(x)$$
$$= \begin{cases} (1-x)x^m \frac{m^2+(1+2m)(n-r-m-1)x+(n-r-m-1)(n-r-m-2)x^2}{(n-r)^2} & \\ \quad + \tfrac{2r}{n}x\frac{(1-x)x^m(m+(n-r-m-1)x)}{n-r} + \tfrac{r^2x^2}{n^2}(1-x)x^m, & n > m+r; \\ x^{n-r} + \tfrac{2r}{n}x^{n-r+1} + \tfrac{r^2x^2}{n^2}x^{n-r}, & n = m+r; \end{cases}$$

□

THEOREM 6. *For every $f \in C[0,1]$, we have*

$$\|L^S_{m,n,r}\| \le \|f\| \quad \textit{and} \ \|R^S_{m,n,r}\| \le \|f\|.$$

*Proof.* Considering the expression of $L^S_{m,n,r}f$ and $R^S_{m,n,r}f$, and Lemma 4, we get

$$|(L^S_{m,n,r}f)(x)| = \left|\sum_{k=1}^{n-m-r} a_{m,n-r,k}(x)\Big[(1-x)f(\tfrac{k}{n}) + xf(\tfrac{k+r}{n})\Big]\right|$$
$$\le \sum_{k=1}^{n-m-r} a_{m,n-r,k}(x)\left|\Big[(1-x)f(\tfrac{k}{n}) + xf(\tfrac{k+r}{n})\Big]\right|$$
$$\le \sum_{k=1}^{n-m-r} a_{m,n-r,k}(x)\Big[(1-x)\left|f(\tfrac{k}{n})\right| + x\left|f(\tfrac{k+r}{n})\right|\Big]$$

$$
\begin{aligned}
&\le \|f\| \sum_{k=1}^{n-m-r} a_{m,n-r,k}(x) \\
&\le \|f\|(L^S_{m,n,r}e_0)(x) \\
&\le \|f\|,
\end{aligned}
$$

$$
\begin{aligned}
|(R^S_{m,n,r}f)(x)| &= \left| \sum_{k=m}^{n-r-1} b_{m,n-r,k}(x)\Big[(1-x)f(\tfrac{k}{n}) + xf(\tfrac{k+r}{n})\Big] \right| \\
&\le \sum_{k=1}^{n-m-r} b_{m,n-r,k}(x) \left|[(1-x)f(\tfrac{k}{n}) + xf(\tfrac{k+r}{n})]\right| \\
&\le \sum_{k=1}^{n-m-r} b_{m,n-r,k}(x)\Big[(1-x)\left|f(\tfrac{k}{n})\right| + x\left|f(\tfrac{k+r}{n})\right|\Big] \\
&\le \|f\|(R^S_{m,n,r}e_0)(x) \\
&\le \|f\|.
\end{aligned}
$$

□

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