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DYNAMICS OF A MODEL OF NONLOCAL DISPERSION VIRAL INFECTION THAT INCLUDES HUMORAL IMMUNITY, INTRACELLULAR DELAY, AND VIRUS-TO-CELL AND CELL-TO-CELL TRANSMISSIONS

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Abstract. This paper investigates a delayed spatiotemporal viral infection model with two types of transmissions, namely, virus-to-cell transmission, and cell-to-cell transmission, and a special focus on the cell-mediated immunity. First, we establish that the model system has a global attractor and that the model admits a unique positive solution which is globally defined. The basic reproduction numbers of immunity $\Re_1$ and infection $\Re_0$ are then determined. The immunity-related reproduction number $\Re_1$ is defined only when $\Re_0 > 1$, which serves as a threshold parameter. Indeed, it is obtained that the infection-free steady state is globally asymptotically stable when $\Re_0 < 1$, the immunity-free infected steady state is globally asymptotically stable when $\Re_1 \leq 1 < \Re_0$, and the infected-immune steady state is globally asymptotically stable when $\Re_1 > 1$. Finally, numerical simulation is performed in order to validate our theoretical results.

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1. INTRODUCTION

Viral dynamics is a well-established branch of applied mathematics that aims to describe the temporal evolution of infected cells and viral load through mathematical modeling. During viral infection, the specific immune response plays a crucial role in controlling disease progression. This response includes both cell-mediated immunity, driven by cytotoxic T lymphocytes (CTLs), and humoral immunity, mediated by antibodies produced by B cells. Since the pioneering work of Nowak and Bangham [1], a wide range of mathematical models have been developed to investigate immune responses against infected cells [9]–[15].

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To better understand the interplay between virus dynamics and immune response, we focus on a transmission mechanism that incorporates both virus-to-cell and cell-to-cell infection pathways. In this context, Jiazhe and Rui [2] proposed the following basic viral dynamic model:

(1)
$$
\begin{aligned}
S'(\iota) &= \lambda - \beta S(\iota)I(\iota) - \alpha S(\iota)V(\iota) - \beta_S S(\iota),\\
V'(\iota) &= \beta e^{-\mu\tau_V} S(\iota-\tau_V)I(\iota-\tau_V) + \alpha e^{-\mu\tau_V} S(\iota-\tau_V)V(\iota-\tau_V) - \beta_V V(\iota),\\
I'(\iota) &= \beta_1 V(\iota) - \gamma_1 I(\iota)R(\iota) - \beta_I I(\iota),\\
R'(\iota) &= \zeta I(\iota)R(\iota) - \beta_R R(\iota).
\end{aligned}
$$

Here, S, I, V, and R represent the densities of uninfected cells, infected cells, free virus particles, and B cells, respectively. The parameters describe biological processes such as infection rates, natural death rates, immune response activation, and viral production. The delay τ_V accounts for the intracellular period between viral entry and the release of new virions, while $e^{-\mu\tau_V}$ represents the survival probability during this latent phase.

It is well known that viruses can spread over relatively long distances within biological tissues, for instance through blood circulation or intercellular transport mechanisms. Therefore, the classical diffusion operator, typically modeled by the Laplacian, may not adequately capture such long-range movements, as it only describes local interactions based on infinitesimal displacements. In contrast, nonlocal diffusion operators provide a more realistic framework by incorporating spatial interactions over a finite range.

Specifically, nonlocal diffusion is often modeled by an integral operator of the form

$$\int_\Omega J(\eta-\vartheta)\big(\Psi(\iota,\vartheta)-\Psi(\iota,\eta)\big)\,d\vartheta,$$

where $J(\cdot)$ is a dispersal kernel describing the probability of movement from location ϑ to η. Unlike classical diffusion, this operator accounts for long-range dispersal and allows individuals (or particles) to move directly between distant locations. From a biological perspective, this is particularly relevant for viral propagation, where transport mechanisms are not purely local.

Another important distinction concerns boundary conditions. In classical diffusion models, no-flux (Neumann) boundary conditions are imposed through local derivatives. However, in the nonlocal framework, the dispersal process inherently involves interactions across the entire domain Ω. As a result, boundary effects are implicitly incorporated into the integral operator, provided that the kernel J is properly defined (for instance, with compact support or normalization conditions). In this work, we complement the nonlocal formulation with homogeneous Neumann-type conditions to ensure consistency with biological assumptions of no net outward flux.

Moreover, to enhance biological realism, we incorporate time delays representing the latent period between infection and viral production, as well as the activation delay of the immune response. Such delays have been widely studied in the literature [17, 22, 24], mainly in the context of virus-to-cell transmission. However, their combined effect with cell-to-cell transmission and nonlocal dispersal remains less understood.

Motivated by the above considerations, we propose the following delayed viral dynamic model with nonlocal diffusion:

$$(2)\quad \begin{aligned} \tfrac{\partial S}{\partial \iota} =& \lambda_1(\eta) - \beta_S(\eta)S - \beta(\eta)SI - \alpha(\eta)SV, \\ \tfrac{\partial V}{\partial \iota} =& \beta(\eta)S(\iota-\tau_v,\eta)I(\iota-\tau_v,\eta)e^{-\mu\tau_V} \\ &+ \alpha(\eta)S(\iota-\tau_v,\eta)V(\iota-\tau_v,\eta)e^{-\mu\tau_V} - \beta_V(\eta)V, \\ \tfrac{\partial I}{\partial \iota} =& K_I\int_\Omega J(\eta-\epsilon)(I(\iota,\epsilon)-I(\iota,\eta))\,d\epsilon + \beta_1(\eta)V - \beta_I(\eta)I - \gamma_1(\eta)IR, \\ \tfrac{\partial R}{\partial \iota} =& K_R\int_\Omega J(\eta-\epsilon)(R(\iota,\epsilon)-R(\iota,\eta))\,d\epsilon \\ &+ \zeta(\eta)I(\iota-\tau_R,\eta)R(\iota-\tau_R,\eta) - \beta_R(\eta)R. \end{aligned}$$

with $\iota > 0$ and $\eta \in \Omega$. The associated initial and boundary conditions are given by

$$(3)\quad \begin{aligned} S(l,\eta) =& S_0(l,\eta), I(l,\eta) = I_0(l,\eta), V(l,\eta) = V_0(l,\eta), R(l,\eta) = R_0(l,\eta), \\ & (l,\eta) \in [-\tau, 0] \times \bar{\Omega} \end{aligned}$$

and

$$\tfrac{\partial I}{\partial n} = \tfrac{\partial R}{\partial n} = 0, \quad \eta \in \partial\Omega,\ \iota > 0,$$

where K_R and K_I are the diffusion coefficients. τ_R is the period of time needed for antigenic stimulation generating antibody response, $\bar{\Omega} \subset \mathbb{R}^n$ is a bounded set with smooth boundary (represents the position).

Let $\tau = \max\{\tau_v, \tau_R\}$. Every parameter should be positive, space-dependent, and Hölder continuous on $\bar{\Omega}$.

The paper is structured as follows. We will examine the well-posedness of (2) and (3) in Section 2. In Section 3, we utilize Kuratowski's measure of non-compactness to demonstrate the existence of a global compact attractor for the semiflow. The purpose of Section 4 is to determine the basic reproduction numbers of immunity $\Re_1$ and infection $\Re_0$. Keep in mind that for $\Re_1$ to be defined, $\Re_0$ must be greater than **1**. Section 5: A triple dynamics is the primary outcome. In Section 6, when $\Re_1 \leq 1 < \Re_0$, the immunity-free infected steady state is globally asymptotically stable, when $\Re_1 > 1$, the infected-immune steady state is globally asymptotically stable. These are the general terms used to describe the infection-free steady state and immunity-free infected steady state. Section 7 is configured to demonstrate the use of numerical simulation.

2. PRELIMINARIES

We make the following assumptions on the system parameters (2)–(3):

(P0) $\mathbf{J}(\eta)$ is Lipschitz continuous on $\overline{\Omega}$.

(P1) $\mathbf{J}$ is symmetric, non-negative and normalized: $\mathbf{J}(-z) = \mathbf{J}(z)$ on $\mathbb{R}^n$, $\int_\Omega \mathbf{J}(z)\,dz = 1$, and $\mathbf{J}(z) > 0$ on $\overline{\Omega}$.

Let $\mathbb{X} = C([-\tau, 0] \times \overline{\Omega}, \mathbb{R})$, $\quad X = \mathbb{X}^4$, equipped with the supremum norm $\|\cdot\|_\infty$. Denote by $\mathbb{X}_+$ and X_+ the associated positive cones. Throughout this work, all model parameters are assumed to be nonnegative, bounded, and continuous on $\overline{\Omega}$. For notational convenience, given any function $\mathcal{H} \in C(\overline{\Omega}, \mathbb{R})$, we define $\mathcal{H}^+ := \sup_{\eta\in\overline{\Omega}} \mathcal{H}(\eta)$, $\quad \mathcal{H}^- := \inf_{\eta\in\overline{\Omega}} \mathcal{H}(\eta)$, with $\mathcal{H} \in \{\lambda_1, \beta, \alpha, \mu, \beta_{V_1}, \beta_{V_2}, \beta_{V_3}, \beta_{V_4}\}$.

We now provide a few standard results taking into account the model's well-posedness (2)–(3). Model (2)–(3) can be rewritten in the following abstract form

$$\frac{d\mathbb{U}}{d\iota}(\iota) = A[\mathbb{U}](\iota) + F[\mathbb{U}](\iota)\ , \mathbb{U}(\iota) \in X_+, \tag{4}$$

with

$$AU = \begin{pmatrix} A_1\mathbb{U}_1 \\ A_2\mathbb{U}_2 \\ A_3\mathbb{U}_3 \\ A_4\mathbb{U}_4 \end{pmatrix} = \begin{pmatrix} -\beta_S(v)S(\iota,\cdot) \\ -\beta_V(\eta)V(\iota,\cdot) \\ K_I\int_\Omega J(\eta-y)(I(\iota,y)-I(\iota,\eta))dy - \beta_I(\eta)I(\iota,\cdot) \\ K_R\int_\Omega J(\eta-y)(R(\iota,y)-R(\iota,\eta))dy - \beta_R(\eta)R(\iota,\cdot) \end{pmatrix},$$

and

$$F\mathbb{U} = \begin{pmatrix} \lambda_1(\eta) - \beta(\eta)S(\iota,\cdot)I(\iota,\cdot) - \alpha(\eta)S(\iota,\cdot)V(\iota,\cdot) \\ \beta S(\iota-\tau_v,.)I(\iota-\tau_v,\cdot)e^{-\mu\tau_V} + \alpha S(\iota-\tau_v,\cdot)V(\iota-\tau_v,\cdot)e^{-\mu\tau_V} \\ \beta_1(\eta)V(\iota,\cdot) - \gamma_1(\eta)I(\iota,\eta)R(\iota,\cdot) \\ \zeta(\eta)I(\iota-\tau_R,\cdot)R(\iota-\tau_R,\cdot) \end{pmatrix}.$$

The following linear operator should be defined on $C(\bar{\Omega})$.

$$A_I[\Phi](\cdot) := K_I\int_\Omega J(\eta-y)(\Phi(y)-\Phi(\cdot))dy - \beta_I(\cdot)\Phi(\cdot);\ \Phi \in C(\bar{\Omega}).$$

Some properties on A are given by the following lemma.

Lemma 1. *A generates a strictly contractive and uniformly continuous semigroup $\{e^{A\iota}\}_{\iota\geq 0}$.*
Additionally, $e^{A\iota}X_+ \subset X_+\ \forall\ \iota \geq 0$.

Proof. It is possible to break down operator A as $A = A_1 + A_2$, with

$$A_1\mathbb{U} = \begin{pmatrix} -\beta_S S(\iota,\cdot) \\ -\beta_V V(\iota,\cdot) \\ -\beta_I I(\iota,\cdot) \\ -\beta_R R(\iota,\cdot) \end{pmatrix}, A_2\mathbb{U} = \begin{pmatrix} 0 \\ 0 \\ K_I\int_\Omega J(\eta-\epsilon)(I(\iota,\epsilon)-I(\iota,\cdot))d\epsilon \\ K_R\int_\Omega J(\eta-\epsilon)(R(\iota,\epsilon)-R(\iota,\cdot))d\epsilon \end{pmatrix},$$

for $\mathbb{U} \in X$. Therefore, a uniformly continuous and strictly contractive semigroup $\{e^{A_\iota}\}_{\iota\geq 0}$ on X is generated by A_1. We may deduce that A_2 is a Lipschitz function since J is bounded. A generates a C_0 -semigroup $\{e^{A\iota}\}_{\iota\geq 0}$, by Theorem 1.2 in [4] and Corollary VI.1.11 in [5].

□

Theorem 2. *Suppose that* $\mathbb{U}_0 = (\varphi_1(b,\eta), \varphi_2(b,\eta), \varphi_3(b,\eta)) \in X$. *Hence, there exists* $\tau_0 > 0$ *such that* (2)-(3) *has unique positive solution* $\mathbb{U}(\cdot, \mathbb{U}_0) \in X$, *such that either* $\tau_0 = \infty$ *or* $\lim_{\iota \to \tau_0-} \sup \|\mathbb{U}(\iota, \mathbb{U}_0)\| = \infty$.
Moreover, $\mathbb{U}(\iota, \mathbb{U}_0) \in X_+$ *for* $\iota \in (0, \tau_0)$ *if* $\mathbb{U}_0 \in X_+$.

Proof. The solution of system (2)-(3) can be written as follows

$$\mathbb{U}(\iota) = \mathbb{U}_0 e^{A\iota} + \int_0^\iota e^{A(\iota - s)} F(\mathbb{U}(s)) ds. \tag{5}$$

The function $F : X \to X$ is locally Lipschitz on X. As established by Lemma 3.1 in [6], Proposition 4.16 in [7] guarantees that system (2) has a unique mild solution, denoted by $\mathbb{U}(., \mathbb{U}_0)$, where τ_0 verifies either $\tau_0 = \infty$ or $\limsup_{\iota \to \tau_0^-} \|\mathbb{U}(\iota, \mathbb{U}_0)\| = \infty$.

Now, we show $S(\iota, \eta) > 0$ for $\iota \in (0, \tau_0)$ and $\eta \in \bar{\Omega}$. We suppose that $\exists$ $\widetilde{\tau} \in (0, \tau_0)$and $\eta_0 \in \Omega$ verifying that $S(\widetilde{\tau}, \eta_0) = 0$ and $S(\iota, \eta_0) > 0$ for $\iota < \widetilde{\tau}$. Next, we can specify

$$\tau_1 = \inf \Big\{ \iota \in (0, \tau_0); S(\iota, \eta_0) = 0 \Big\}.$$

Since τ_0 satisfies $\tau_0 = \infty$ or $\limsup_{\iota \to \tau_0^-} \|\mathbb{U}(\iota, \mathbb{U}_0)\| = \infty.$, then we deduce that $\tau_1 \in (0, \tau_0)$ and $S(\widetilde{\tau}, \eta_0) = 0$ with $\frac{\partial S(\widetilde{\tau}, \eta)}{\partial \iota} \leq 0$. The first equation of (2) yields

$$\frac{\partial S(\tau_1, \eta_0)}{\partial \iota} = \lambda_1(\eta_0) > 0,$$

that contradicts the assumptions. Hence, $S(\iota, \eta) > 0$ for $\iota \in (0, \tau_0)$, $\eta \in \bar{\Omega}$.

We put $A_{h_r}(\mathbb{U}) = \mathbb{U}(\iota, \cdot) + h_r \mathbb{U}(\iota, \cdot)$ for a sufficiently large positive h_r. For $\mathbb{U} \in C((0, \tau_0), X) \cap ß(0, \frac{1}{h_r})$, obviously $A_{h_r}(\mathbb{U})$ is positive. In conjunction with the open ball $ß(0, \frac{1}{h_r})$ in X, with radius $\frac{1}{h_r}$ and center 0. Hence, (2) can be reworded as this:

$$\frac{d\mathbb{U}}{d\iota}(\tau) = A_{h_r}[\mathbb{U}](\iota) + F_{h_r}[\mathbb{U}](\iota).$$

with $A_{h_r}\mathbb{U}(\iota, \cdot) = \mathbb{U}(\iota, \cdot) - h_r \mathbb{U}(\iota, \cdot)$. Then,

$$U(\iota) = U_0(\eta) e^{A_{h_r}\iota} + \int_0^\iota e^{A_{h_r}(\iota - s)} F_{h_r}(U(s)) ds.$$

Hence, $\mathbb{U}(\iota, \mathbb{U}_0) \in X_+$ for $\iota \in (0, \tau_0)$ and $\mathbb{U}_0 \in X_+$. □

Theorem 3. *The solution of* (2)–(3) *is globally defined.*

Proof. Let $\mathbb{U} = (S, I, V, R)$ be the solution associated to $\mathbb{U}_0 = (S_0, I_0, V_0, R_0)$. Since $\frac{dS}{dt}(\iota, \eta) \leq \lambda(\eta) - \beta_S S(\iota, \eta)$, $\check{S}$ is a subsolution of

$$\begin{cases} \frac{\partial \check{S}}{\partial \iota}(\iota, \eta) = \lambda_1(\eta) - \beta_S \check{S}(\iota, \eta), & \eta \in \Omega, \quad \iota > 0, \\ \check{S}(0, \eta) = \max_{s \in [-\tau, 0]} S_0(s, \eta), & \eta \in \Omega. \end{cases} \tag{6}$$

The unique positive steady state of (6), represented by $\check{S}_f$, is globally attractive and well known. By the comparison theorem, we get

$$\limsup_{\iota\to\infty} S(\iota,\eta) \leq \limsup_{\iota\to\infty} \check{S}(\iota,\eta) = \check{S}_f(\eta) = \frac{\lambda_1(\eta)}{\beta_s(\eta)} \text{ uniformly for } \eta \in \Omega.$$

Consequently, $\exists K > 0$ that depends on $\mathbb{U}_0$ such that

$$\|S\| \leq K,\ \iota > 0.$$

Next, we let A_I be the generator of the uniformly continuous semigroup $\{T_1(\iota)\}_{\iota\geq 0}$

$$\begin{aligned} I(\iota,\cdot) &= T_1(\iota)I_0 + \int_0^\iota T_1(\iota-\sigma)(\beta_1 V(\sigma,\cdot) - \gamma_1(x)I(\sigma,\cdot)R(\sigma,\cdot))d\sigma, \\ &\leq T_1(\iota)I_0 + \int_0^\iota T_1(\iota-\sigma)\beta_1 V(\sigma,\cdot)d\sigma. \end{aligned}$$

By taking the norm on both sides, we obtain

$$\|I(\iota,\cdot)\| \leq e^{-\lambda\iota}\|I_0\| + \|\beta_1\| \int_0^\iota e^{-\lambda(\iota-\sigma)}\|V(\sigma,\cdot)\|d\sigma.$$

We assume that the eigenvalue for the operator A_I is $-\lambda$. According to Proposition 1 in [8], $\exists c > 0$ fulfills $\lambda > c > 0$. The second equation of (2)–(3) suggests

$$V(\iota,\cdot) = e^{-\beta_V\iota}V_0 + \int_0^\iota e^{-\beta_V(\iota-\sigma)}(e^{-\mu\tau_V}S(\sigma-\tau_v,\cdot)(\beta I(\sigma-\tau_v,\cdot)+\alpha V(\sigma-\tau_v,\cdot))d\sigma.$$

Therefore,

$$\begin{aligned} &\|V(\iota;.)\| \leq \\ &\leq e^{-\beta_V\iota}\|V_0\| \\ &\quad + \int_0^\iota e^{-\beta_V(\iota-\sigma)}e^{-\mu\tau_V}\|S(\sigma-\tau_v,\cdot)\|(\|\beta\|\|I(\sigma-\tau_v,\cdot)\| + \|\alpha\|\|V(\sigma-\tau_v,\cdot)\|)d\sigma, \\ &\leq e^{-\beta_V\iota}\|V_0\| + Ke^{-\mu\tau_V}\int_0^\iota e^{-\beta_V(\iota-\sigma)}(\|\beta\|\|I(\sigma-\tau_v,\cdot)\| + \|\alpha\|\|V(\sigma-\tau_v,\cdot)\|)d\sigma, \\ &\leq e^{-\beta_V\iota}\|V_0\| + K\beta^* e^{-\mu\tau_V}\int_0^\iota e^{-\beta_V(\iota-\sigma)}(\|I(\sigma-\tau_v,\cdot)\| + \|V(\sigma-\tau_v,\cdot)\|)d\sigma, \end{aligned}$$

where $\beta^* = \max\{\|\beta\|, \|\alpha\|\}$. Using the substitution $\sigma^* = \sigma - \tau_v$ and dropping tilde for simplicity, thus,

$$\begin{aligned} &\|V(\iota,\cdot)\| \leq \\ &\leq e^{-\beta_V\iota}\|V_0\| + K\beta^* e^{-\mu\tau_V}\int_{\tau_v}^{\iota-\tau_v} e^{-\beta_V(\iota-\tau_v-\sigma)}(\|I(\sigma,\cdot)\| + \|V(\sigma,\cdot)\|)d\sigma \end{aligned}$$

$$\leq e^{-\beta_V \iota}\|V_0\| + K\beta^* e^{-\mu\tau_V} \int_{\tau_v}^{\iota-\tau_v} e^{-\beta_V(\iota-\tau_v-\sigma)}\|V(\sigma,\cdot)\|d\sigma$$
$$+ K\beta^* e^{-\mu\tau_V} \int_{\tau_v}^{\iota-\tau_v} e^{-\beta_V(\iota-\tau_v-\sigma)}\|I(\sigma,\cdot)\|d\sigma.$$

Thus,

$$\begin{aligned}
&\|V(\iota,\cdot)\| \leq \\
&\leq e^{-\beta_V \iota}\|V_0\| + K\beta^* e^{-\mu\tau_V} \int_{\tau_v}^{\iota-\tau_v} e^{-\beta_V(\iota-\tau_v-\sigma)}\|V(\sigma,\cdot)\|d\sigma \\
&\quad + K\beta^* e^{-\mu\tau_V} \int_{\tau_v}^{\iota-\tau_v} e^{-\beta_V(\iota-\tau_v-\sigma)} \left\{ e^{-\lambda\sigma}\|I_0\| + \|\beta_1\|_0^\sigma e^{-\lambda(\sigma-s)}\|V(s,\cdot)\|ds \right\} d\sigma \\
&\leq e^{-\beta_V \iota}\|V_0\| + K\beta^* e^{-\mu\tau_V} \int_{\iota_v}^{\iota-\tau_v} e^{-\beta_V(\tau-\tau_v-\sigma)}\|V(\sigma,\cdot)\|d\sigma \\
&\quad + K\beta^* e^{-\mu\tau_V}\|I_0\| \int_{-\tau_v}^{\iota-\tau_v} e^{-\beta_V(\iota-\tau_v-\sigma)} e^{-\lambda\sigma} d\sigma \\
&\quad + \|\beta_1\| K\beta^* e^{-\mu\tau_V} \int_{-\tau_v}^{\iota-\tau_v} e^{-\beta_V(\iota-\tau_v-\sigma)} \int_{-\tau_v}^{\sigma} e^{-\lambda(\sigma-s)}\|V(s,\cdot)\|ds d\sigma \\
&\leq e^{-\beta_V \iota}\|V_0\| + K\beta^* e^{-\mu\tau_V} \int_{-\tau_v}^{\iota-\tau_v} e^{-\beta_V(\tau-\tau_v-\sigma)}\|V(\sigma,\cdot)\|d\sigma \\
&\quad + K\beta^* e^{-\mu\tau_V}\|I_0\| \int_{-\tau_v}^{\iota-\tau_v} e^{-\lambda\sigma} d\sigma \\
&\quad + \|\beta_1\| M\beta^* e^{-\mu\tau_V} \int_{-\tau_v}^{\iota-\tau_v} e^{-\beta_V(\iota-\tau_v-\sigma)} \int_{-\tau_v}^{\sigma} e^{-\lambda(\sigma-s)}\|V(s,.)\|ds d\sigma.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\|V(\iota,\cdot)\| \leq \\
&\leq e^{-\beta_V \iota}\|V_0\| + K\beta^* e^{-\mu\tau_V} \int_{-\tau_v}^{\iota-\tau_v} e^{-\beta_V(\tau-\tau_v-\sigma)}\|V(\sigma,\cdot)\|d\sigma \\
&\quad + K\beta^* e^{-\mu\tau_V}\|I_0\| \tfrac{e^{\lambda\tau_v} - e^{-\lambda(\iota-\tau_v)}}{\lambda} \\
&\quad + \|\beta_1\| K\beta^* e^{-\mu\tau_V} e^{-\beta_V(\iota-\tau_v)} \int_{-\tau_v}^{\iota-\tau_v} e^{\lambda s}\|V(s,\cdot)\| \int_{s}^{\iota-\tau_v} e^{(\beta_V-\lambda)\sigma} d\sigma ds \\
&\leq e^{-\beta_V \tau}\|V_0\| + K\beta^* e^{-\mu\tau_V} \int_{-\tau_v}^{\iota-\tau_v} e^{-\beta_V(\tau-\tau_v-\sigma)}\|V(\sigma,\cdot)\|d\sigma
\end{aligned}$$

$$+ K\beta^* e^{-\mu\tau_V}\|I_0\|\tfrac{e^{\lambda\tau_v} - e^{-\lambda(\iota-\tau_v)}}{\lambda}$$

$$+ \|\beta_1\| K\beta^* e^{-\mu\tau_V}\tfrac{e^{-\varpi(\tau-\tau_v)}}{\lambda-\beta_V}\int\limits_{-\tau_v}^{\iota-\tau_v} e^{\varpi s}\|V(s,\cdot)\|ds$$

$$\leq \|V_0\| + K\beta^* e^{-\mu\tau_V}\int\limits_{-\tau_v}^{\iota-\tau_v} e^{-\varpi(\tau-\tau_v-\sigma)}\|V(\sigma,\cdot)\|d\sigma$$

$$+ K\beta^* e^{-\mu\tau_V}\|I_0\|\tfrac{e^{\lambda\tau_v}}{\lambda}$$

$$+ \tfrac{\|\beta_1\| K\beta^* e^{-\mu\tau_V}}{\lambda-\beta_V}\int\limits_{-\tau_v}^{\iota-\tau_v} e^{-\varpi(\iota-\tau_v-s)}\|V(s,\cdot)\|ds$$

$$\leq C_1 + C_2\int\limits_{-\tau_v}^{\iota-\tau_v}\|V(s,\cdot)\|ds,$$

with

$$\varpi = \min\left\{\beta_V^-, \tfrac{\lambda}{2}\right\}, C_1 = \|V_0\| + K\beta^* e^{-\mu\tau_V}\|I_0\|\tfrac{e^{\lambda\tau_V}}{\lambda},$$

$$C_2 = \max\left\{K\beta^* e^{-\mu\tau_V}, \tfrac{\|\beta_1\| K\beta^* e^{-\mu\tau_V}}{\lambda-\beta_V}\right\}.$$

Gronwall's inequality implies

$$\|V\| \leq C_1 e^{C_2\iota},\ \iota \geq 0.$$

The last equation of (2) implies

$$R(\iota,.) = e^{-\beta_R\iota}R_0 + \int\limits_0^{\iota}\zeta(x)e^{-\beta_R(\iota-\sigma)}I(\sigma-\tau_R,x)R(\iota-\tau_R,x)d\sigma.$$

Therefore, $\sigma^{**} = \sigma - \tau_R$

$$\|R(\iota,.)\| \leq e^{-\beta_R\iota}\|R_0\| + \zeta^*\int\limits_{-\tau_R}^{\iota-\tau_R} e^{-\beta_R(\iota-\sigma)}\|I(\sigma,\cdot)\|\|R(\sigma,\cdot)\|d\sigma.$$

It follows

$$\|R(\iota,\cdot)\| \leq C_4 e^{C_5\iota},\ \iota \geq 0, \tag{7}$$

with

$$\|I(\iota,\cdot)\| \leq \|I_0\| + \tfrac{C_1\|\beta_1\|e^{C_2\iota}}{\lambda} = C_3,$$

$$\begin{aligned}\|I(\iota,\cdot)\| &\leq \|I_0\|e^{-\lambda\iota} + \|\beta_1\|\int_0^{\iota} e^{-\lambda(\iota-\sigma)}\|V(\sigma,\cdot)\|d\sigma\\ &\leq \|I_0\|e^{-\lambda\iota} + C_1\|\beta_1\|\int_0^{\iota} e^{-\lambda(\iota-\sigma)}e^{C_2\iota}d\sigma\\ &\leq \|I_0\| + \tfrac{C_1\|\beta_1\|e^{C_2\iota}}{\lambda} = C_3.\end{aligned}$$

Therefore, the solution is globally defined, that is, $\tau_0 = +\infty$. □

3. COMPACTNESS

We now examine the solution map's compactness. The semiflow produced by system (2)–(3) is denoted by $\{\Xi(\iota)\}_{\iota\geq 0}$. $\Xi(\iota)$ is not compact. we may demonstrate the asymptotic compactness of forward orbits in order to demonstrate that $\Xi(\iota)$ has a global compact attractor. Thus, using Kuratowski measure κ , which is defined as follows,

$$\kappa(\mathbf{B}) = \inf\{r\colon \text{has a finite cover of diameter} < r\}.$$

Then

$$\mathbf{B} \text{ precompact} \Leftrightarrow \kappa(P) = 0.$$

To achieve this goal, we apply several lemmas.

Lemma 4. *The semiflow $\{\Xi(\iota)\}_{\iota\geq 0}$ is point dissipative.*

Proof. We let $H(\iota,\eta) = S(\iota,\eta) + V(\iota+\tau_v,\eta)$; hence,

$$\begin{aligned}\tfrac{\partial H(\iota,\eta)}{\partial\iota} &= \lambda_1(\eta) - (1-e^{-\mu\tau_V})\beta(\eta)S(\iota,\eta)I(\iota,\eta)\\ &\quad - (1-e^{-\mu\tau_V})\alpha(\eta)S(\iota,\eta)V(\iota,\eta) - \beta_S(\eta)S(\iota,\eta) - \beta_V(\eta)V(\iota+\tau_v,\eta)\\ &\leq \lambda_1(\eta) - \beta_S(\eta)S(\iota,\eta) - \beta_V(\eta)V(\iota+\tau_v,\eta)\\ &\leq \lambda_1(\eta) - \omega H(\iota,\eta)\end{aligned}$$

with $\omega = \min\{\beta_S^-,\beta_V^-\}$. Therefore, we get $H(\iota,\eta) \leq g_1(\eta)$, $\iota\in\mathbb{R}^+$, with $g_1(\eta) = H(0,\eta) + \frac{\lambda_1(\eta)}{\omega}$. Therefore, the existence of ξ_1 independent of the initial conditions verifying

$$\lim_{\iota\to\infty}\sup\|H(\iota,\eta)\| \leq \xi_1.$$

On Ω,we integrate the third equation of (2).

$$\begin{aligned}\tfrac{\partial}{\partial\iota}\int_\Omega I(\iota,\eta)d\eta =& K_I\int_\Omega\int_\Omega J(\eta-\epsilon)(I(\iota,\epsilon)-I(\iota,\eta))d\epsilon d\eta\\ &+\int_\Omega \beta_1(\eta)V(\iota,\eta)d\eta - \int_\Omega\beta_I(\eta)I(\iota,\eta)d\eta\\ &-\int_\Omega\gamma_1(\eta)I(\iota,\eta)R(\iota,\eta)d\eta,\\ \leq& \int_\Omega\beta_1(\eta)V(\iota,\eta)d\eta - \beta_I^-\int_\Omega I(\iota,\eta)d\eta,\end{aligned}$$

which gives

$$\tfrac{\partial}{\partial\iota}\int_\Omega I(\iota,\eta)d\eta + \beta_I^-\int_\Omega I(\iota,\eta)d\eta \leq \int_\Omega\beta_1(\eta)V(\iota,\eta)d\eta.$$

Let us multiply by $e^{\beta_I^-\iota}$,

$$e^{\beta_I^-\iota}\tfrac{\partial}{\partial\iota}\int_\Omega I(\iota,\eta)d\eta + \beta_I^- e^{\beta_I^-\iota}\int_\Omega I(\iota,\eta)d\eta \;\leq\; e^{\beta_I^-\iota}\int_\Omega\beta_1(\eta)V(\iota,\eta)d\eta,$$

$$\tfrac{d}{d\iota}e^{\beta_I^-\iota}\int_\Omega I(\iota,\eta)d\eta \quad\le\quad e^{\beta_I^-\iota}\int_\Omega \beta_1(\eta)V(\iota,\eta)d\eta,$$

Integrating by parts, we find

$$\begin{aligned}\int_0^\iota \tfrac{\partial}{\partial s}e^{\beta_I^- s}\int_\Omega I(s,\eta)d\eta ds &\le \int_0^\iota e^{\beta_I^- s}\int_\Omega \beta_1(\eta)V(s,\eta)d\eta ds,\\ e^{\beta_I^-\iota}\int_\Omega I(\iota,\eta)d\eta+\int_\Omega I(0,\eta)d\eta &\le \int_0^\iota e^{\beta_I^- s}\int_\Omega \beta_1(\eta)V(s,\eta)d\eta ds,\\ \int_\Omega I(\iota,\eta)d\eta &\le e^{-\beta_I^-\iota}\int_\Omega I(0,\eta)d\eta\\ &\quad+\tfrac{1}{\beta_I^-}(1-e^{-\beta_I^-\iota})\int_\Omega \beta_1(\eta)V(s,\eta)d\eta := D_1.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\tfrac{\partial}{\partial\iota}I(\iota,\eta)d\eta \quad&=\quad K_I\int_\Omega J(\eta-\epsilon)(I(\iota,\epsilon)-I(\iota,\eta))d\epsilon+\beta_1(\eta)V(\iota,\eta)-\beta_I(\eta)I(\iota,\eta)\\ &\le\quad K_I(\sup_{\eta\in\Omega}J)\int_\Omega I(\iota,\epsilon)d\epsilon+\beta_1(\eta)V(\iota,\eta)-\beta_I(\eta)I(\iota,\eta).\end{aligned}$$

By applying this inequality, the third equation in (2) suggests

$$\tfrac{dI(\iota,\cdot)}{d\iota}\le K_I(\sup_{\eta\in\Omega}J)D_1+\beta_1^+\xi_1-(K_I+\beta_I^-)I(\iota,\cdot).$$

Then,

$$I(\iota,\cdot)\le I_0(\iota,\cdot)e^{-(K_I+\beta_I^-)\iota}+\frac{K_I(\sup_{\eta\in\Omega}J)D_1+\beta_1^+\xi_1}{K_I+\beta_I^-}(1-e^{-(K_I+\beta_I^-)\iota}).$$

After that, $\exists\xi_2$ independent of the initial condition satisfying

$$\limsup_{\iota\to\infty}\|I(\iota,\eta)\|\le\xi_2.$$

We integrate the final equation of (2) on Ω

$$\begin{aligned}\tfrac{\partial}{\partial\iota}\int_\Omega R(\iota,\eta)dx =&K_R\int_\Omega\int_\Omega J(\eta-\epsilon)(R(\iota,\epsilon)-R(\iota,\eta))d\epsilon d\eta\\ &+\int_\Omega \zeta(\eta)V(\iota-\tau_R,\eta)I(\iota-\tau_R,\eta)d\eta-\int_\Omega\beta_R(\eta)R(\iota,\eta)d\eta\\ \le&\int_\Omega \zeta(\eta)R(\iota-\tau_R,\eta)I(\iota-\tau_R,\eta)d\eta-\beta_R^-\int_\Omega R(\iota,\eta)d\eta.\end{aligned}$$

By similar reasoning, there is a ξ_3 independent of the initial condition satisfying

$$\limsup_{\iota\to\infty}\|R(\iota,\eta)\|\le\xi_3.$$

Therefore, the semiflow $\Xi(\iota)$ is point dissipative. □

We now demonstrate that $\Xi(\iota)$ is a $\kappa -$ contraction; that is, for all $\iota > 0$, there exists a continuous function $0 \leq k(\iota) < 1$, and a bounded set θ, such that $\{\Xi(\sigma)\theta, 0 \leq \sigma \leq \iota\}$ is bounded and $\kappa(\Xi(\sigma)\theta) \leq k(\iota)\kappa(\theta)$. First, we demonstrate the asymptotic compactness of S and V.

LEMMA 5. *The forward orbit* $\delta^+(\Psi) : \mathbb{U}(\iota, \cdot, \mathbb{U}_0), \iota \geq 0$ *for* (2)–(3) *is asymptotically compact for all* $\Psi \in X^+$, *in that for all sequences* $\iota_k \to \infty$. *Consequently,* $\exists \iota_{ki}$ *to satisfy* $\mathbb{U}(\iota_{ki}, \cdot; \mathbb{U}_0)$ *converges in* X^+ *as* $i \to \infty$.

Proof. Let us assume that $\mathbb{U}_0 \in X^+$, there exists $\eta > 0$ verifying

$$| S(\iota, \eta; \mathbb{U}_0) | \leq \varpi, \ | I(\iota, \eta; \mathbb{U}_0) | \leq \varpi, \ for \ all \ \eta \in \bar{\Omega}, \ \iota \geq 0.$$

It is sufficient to demonstrate that $\mathbb{U}(\iota, \eta; \mathbb{U}_0)_{n \geq 1}$ is equicontinuous in $\eta \in \bar{\Omega}$, $\iota > 0$ using the Arzelà–Ascoli theorem. In other words, for all $\rho > 0$, there exists $\varepsilon > 0$ guaranteeing $|h(\iota, \eta_1) - h(\iota, \eta_2)| < \varepsilon^2$ for all $\eta_1, \eta_2 \in \bar{\Omega}$ as long as $|\eta_1 - \eta_2| < \rho$. It is evident that $h(\iota, \eta) = \lambda(\eta) + \beta(\eta)S(\iota, \eta)I(\iota, \eta)e^{-\mu\tau_V} + \alpha(\eta)S(\iota, \eta)V(\iota, \eta)e^{-\mu\tau_V}$ is uniformly continuous in $\eta \in \bar{\Omega}$ uniformly for $\iota \geq 0$. Clearly,

$$\begin{aligned}
&\tfrac{\partial}{\partial\iota} [S(\iota, \eta_1) - S(\iota, \eta_2)]^2 = \\
=&2(S(\iota, \eta_1) - S(\iota, \eta_2))\Big[h(\iota, \eta_1) - h(\iota, \eta_2) - \beta_S(\eta_1)(S(\iota, \eta_1) \\
&\quad -S(\iota, \eta_2)) - S(\iota, \eta_2)(\beta_S(\eta_1) - \beta_S(\eta_2))\big] \\
\leq&4\eta(h(\iota, \eta_1) - h(\iota, \eta_2)) - \beta_S(\eta_1)(S(\iota, \eta_1) - S(\iota, \eta_2))2 \\
&\quad + | S(\iota, \eta_2) \cdot (\beta_S(\eta_1) - \beta_S(\eta_2)) | \\
\leq&(4\varpi\varepsilon^2 + 2\varepsilon) - \beta_S^-(\eta_1)(S(\iota, \eta_1) - S(\iota, \eta_2))^2,
\end{aligned}$$

for all $\iota > -\tau_V$ and $\eta_1, \eta_2 \in \bar{\Omega}$ verifying $|\eta_1 - \eta_2| < \rho$, we have

$$[S(\iota, \eta_1) - S(\iota, \eta_2)]^2 \leq e^{-\beta_S^- \iota} \left[[S(0, \eta_1) - S(0, \eta_2)]^2\right] + (4\varpi\varepsilon^2 + 2\varepsilon) \int_0^\iota e^{-2\beta_S^-(\iota - \sigma)} d\sigma.$$

Then

$$[S(\iota, \eta_1) - S(\iota, \eta_2)]^2 \leq [S(0, \eta_1) - S(0, \eta_2)]^2 + (4\varpi\varepsilon^2 + 2\varepsilon) \int_0^\iota e^{-2\beta_S^-(\iota - \sigma)} d\sigma.$$

Given that $\eta \in \bar{\Omega}$, the initial condition's uniform continuity suggests that $| S_0(0, \eta_1) - S_0(0, \eta_2) | < \frac{\varepsilon}{2}$ for all $\eta_1, \eta_2 \in \bar{\Omega}$ verifying that $|\eta_1 - \eta_2| < \rho_1$. Therefore, for any $|\eta_1 - \eta_2| < min\{\rho, \rho_1\}$, with $\eta_1, \eta_2 \in \bar{\Omega}$, we have

$$[S(\iota, \eta_1) - S(\iota, \eta_2)]^2 \leq \tfrac{\varepsilon^2}{4} + \tfrac{(4\varpi\varepsilon^2 + 2\varepsilon)}{2\beta_S^-}.$$

Since S is equicontinuous, we can infer that ε is arbitrary. We demonstrate that I, V and R are equicontinuous by using a similar logic. $\square$

LEMMA 6. *The semiflow* $\Xi(\iota)$ *is a* κ*-contraction.*

Proof. Clearly, $I(\iota,\eta,\mathbb{U}_0)$ and $R(\iota,\eta,\mathbb{U}_0)$ satisfy

$$\begin{aligned}
\tfrac{\partial I}{\partial \iota}(\iota,\eta) &= K_I \int_\Omega J(\eta-\epsilon)(I(\iota,\epsilon)-I(\iota,\eta))d\epsilon - \beta_I(\eta)I(\iota,\eta),\\
\tfrac{\partial R}{\partial \iota}(\iota,\eta) &= K_R \int_\Omega J(\eta-\epsilon)(R(\iota,\epsilon)-R(\iota,\eta))d\epsilon - \beta_R(\eta)R(\iota,\eta),\\
I(s,\eta) &= I_0(s,\eta), R(s,\eta) = R_0(s,\eta),\ s\in[-\tau,0],\ \eta\in\bar{\Omega}.
\end{aligned}$$

Therefore,

$$I(\iota,\cdot) = T_1(\iota)I_0 + \int_0^\iota T_1(\iota-\sigma)(\beta_1(\cdot)V(\sigma,\cdot,\mathbb{U}_0) - \gamma_1(\cdot)I(\sigma,\cdot,\mathbb{U}_0)R(\sigma,\cdot,\mathbb{U}_0))d\sigma,$$

and

$$R(\iota,\cdot) = T_2(\iota)R_0 + \zeta(\cdot)\int_0^\iota T_2(v-\sigma)V(\sigma-\tau_R,\cdot,\mathbb{U}_0)I(\sigma-\tau_R,\cdot,\mathbb{U}_0)d\sigma,$$

according to Lemma 2.1 in [10], the operators L_V and L_R have strictly positive eigenfunctions, which correspond to strictly negative eigenvalues μ_0 and μ_1, respectively. These eigenvalues are defined by

$$\mu_0 = -\inf_{\substack{\Psi_0\in L^2(\Omega)\\ \|\Psi_0\|_{L^2(\Omega)}=1}} \left\{\int_\Omega\int_\Omega J(\eta-\varepsilon)\big(\Psi_0(\varepsilon)-\Psi_0(\eta)\big)^2\,d\varepsilon\,d\eta + \int_\Omega \beta_I(\eta)\Psi_0^2(\eta)\,d\eta\right\},$$

and

$$\mu_1 = -\inf_{\substack{\Psi_1\in L^2(\Omega)\\ \|\Psi_1\|_{L^2(\Omega)}=1}} \left\{\int_\Omega\int_\Omega J(\eta-\varepsilon)\big(\Psi_1(\varepsilon)-\Psi_1(\eta)\big)^2\,d\varepsilon\,d\eta + \int_\Omega \beta_R(\eta)\Psi_1^2(\eta)\,d\eta\right\}.$$

Now, we decompose the semiflow into $\Phi(\iota) = \Phi_1(\iota) + \Phi_2(\iota)$, where

$$\Phi_1(\iota)U_0 = (0,0,T_1(\iota)I_0,T_2(\iota)R_0),\ \iota\geq 0$$

and

$$\begin{aligned}
\Phi_2(\iota)\mathbb{U}_0 = \Big(&S(\cdot,\cdot,\mathbb{U}_0), V(\cdot,\cdot,\mathbb{U}_0),\\
&\int_0^\iota T_1(\iota-\sigma)(\beta_1(\cdot)V(\sigma,\cdot,\mathbb{U}_0)-\gamma_1(\cdot)I(\sigma,\cdot,\mathbb{U}_0)R(\sigma,\cdot,\mathbb{U}_0))d\sigma,\\
&\zeta(\cdot)\int_0^\iota T_2(\tau-\sigma)V(\sigma-\tau_R,\cdot,\mathbb{U}_0)I(\sigma-\tau_R,\cdot,\mathbb{U}_0)d\sigma\Big),\ \iota\geq 0
\end{aligned}$$

We infer that S, V, I, and R are asymptotically compact from Lemma 5. Consequently, Φ_2 is compact. Thus, we have $\kappa(\Phi_2(\iota)\mathbf{A}) = 0$ for a bounded set $\mathbf{A}\subset X^+$ and $\iota\geq 0$. Additionally,

$$\|\Phi_1(\iota)\mathbb{U}_0\| \leq \|e^{L_V\iota}\mathbb{U}_0 + e^{L_R\iota}\mathbb{U}_0\|$$

$$\begin{aligned} &\leq e^{\mu_0 \iota}\|\mathbb{U}_0\| + e^{\mu_1 \iota}\|\mathbb{U}_0\| \\ &\leq e^{\mu^* \iota}\|\mathbb{U}_0\|, \ \iota > 0, \end{aligned}$$

with $\mu^* = \max\{\mu_0, \mu_1\}$ Hence,

$$\|\Phi_1(\iota)\| \leq e^{\mu^* \iota}.$$

Hence, for $\iota > 0$

$$\kappa(\Phi(\iota)\mathbf{A}) \leq \kappa(\Phi_1(\iota)\mathbf{A}) + \kappa(\Phi_2(\iota)\mathbf{A}) \leq e^{\mu^* \iota}\kappa(\mathbf{A}).$$

The conclusion that follows is that $\Xi(\iota)$ is a κ- contraction. □

We infer from Lemma 4 that the semiflow $\Xi(\iota)$ is point dissipative. Moreover, Lemma 6 shows that $\Xi(\iota)$ is a κ-contraction. By Lemma 2.3.5 in [11], every κ-contraction semiflow is asymptotically smooth. Furthermore, Theorem 2.6 in [12] implies that $\Xi(\iota)$ possesses a global compact attractor that attracts all bounded subsets of X^+. We denote this attractor by $\mathbb{C}$.

4. BASIC REPRODUCTION NUMBER AND STEADY-STATE

A steady-state $(S(\eta), V(\eta), I(\eta), R(\eta))$ of (2) is the solution of the following system

(8)

$$\begin{aligned} 0 =&\lambda_1(\eta) - \beta_S(\eta)S(\eta) - \beta(\eta)S(\eta)I(\eta) - \alpha(\eta)S(\eta)V(\eta), \\ 0 =&\beta(\eta)S(\eta)I(\eta)e^{-\mu\tau_V} + \alpha(\eta)S(\eta)V(\eta)e^{-\mu\tau_V} - \beta_V(\eta)V(\eta), \\ 0 =&K_I \int_\Omega J(\eta-\epsilon)(I(\epsilon) - I(\eta))d\epsilon + \beta_1(\eta)V(\eta) - \beta_I(\eta)I(\eta) - \gamma_1(\eta)I(\eta)R(\eta), \\ 0 =&K_R \int_\Omega J(\eta-\epsilon)(R(\epsilon) - R(\eta))d\epsilon + \zeta(\eta)I(\eta)R(\eta) - \beta_R(\eta)R(\eta), \quad \eta \in \bar{\Omega} \end{aligned}$$

Clearly, (2) has a unique VFSS $H_0 = (\frac{\lambda_1(\eta)}{\beta_S(\eta)}, 0, 0, 0), \eta \in \bar{\Omega}$. We linearize (2) in order to get the basic reproduction number $\Re_0$ corresponding to (2) at H_0, we have

(9)

$$\begin{aligned} \frac{\partial V}{\partial \iota} =&\beta(\eta)\frac{\lambda_1(\eta)}{\beta_S(\eta)}I(\iota-\tau_v,\eta)e^{-\mu\tau_V} + \alpha(\eta)\frac{\lambda_1(\eta)}{\beta_S(\eta)}V(\iota-\tau_v,\eta)e^{-\mu\tau_V} - \beta_V(\eta)V(\iota,\eta), \\ \frac{\partial I}{\partial \iota} =&K_I \int_\Omega J(\eta-\epsilon)(I(\iota,\epsilon) - I(\iota,\eta))d\epsilon + \beta_1(\eta)V(\iota,\eta) \\ &- \beta_I(\eta)I(\iota,\eta) - \gamma_1(\eta)I(\iota,\eta)R(\iota,\eta), \quad \iota > 0, \eta \in \bar{\Omega}. \end{aligned}$$

We only take into consideration the subsystem because S and R are absent from the middle two equations

$$\begin{cases} \frac{\partial}{\partial \iota}\begin{pmatrix} V \\ I \end{pmatrix} = \mathbf{A}\begin{pmatrix} V \\ I \end{pmatrix}, & \eta \in \Omega, \ \iota \geq 0, \\ \frac{\partial I}{\partial \upsilon} = 0, & \eta \in \partial\Omega, \ \iota \geq 0, \end{cases}$$

we define the following linear operators $\mathcal{B}, \mathcal{F} : \mathbb{Z} \to \mathbb{Z}(\mathbb{Z} := C(\Omega, \mathbb{R}^2))$,

$$\mathcal{B} = \begin{pmatrix} -\beta_V(\eta) & 0 \\ \beta_1(\eta) & A_I \end{pmatrix}, \ \mathcal{F} = \begin{pmatrix} \beta(\eta)\frac{\lambda_1(\eta)}{\beta_S(\eta)}e^{-\mu\tau_V} & \alpha(\eta)\frac{\lambda_1(\eta)}{\beta_S(\eta)}e^{-\mu\tau_V} \\ 0 & 0 \end{pmatrix}, \tag{10}$$

and the linear operator $\mathbf{A} := \mathcal{B} + \mathcal{F}$

We designate $\mathbf{T}_{\mathcal{B}}(\iota)$ as the semigroup that $\mathcal{B}$ generates. Therefore, we can define $\Re_0$ as the spectral radius of $\mathcal{K}$, which is

$$\Re_0 = \mathfrak{r}(\mathcal{K}).$$

Here, the next generation operator, $\mathcal{K} := -\mathcal{F}\mathcal{B}^{-1}$, can be written as

$$\mathcal{K}(\Theta)(\eta) = \mathcal{F}(\eta)\int_0^\infty \mathbf{T}_B(\iota)(\Theta)(\eta)d\iota, \Theta \in C(\bar{\Omega}, \mathbb{R}^2), \ \eta \in \bar{\Omega}.$$

To derive a variational expression for the basic reproduction number $\Re_0$, we obtain the following result.

Lemma 7. *$\Re_0 = \mathfrak{r}(-\mathcal{F}\mathcal{B}^{-1}) = r(d_V + d_I)$ with $d_V = \beta\frac{\lambda_1}{\beta_S\beta_V}e^{-\mu\tau_V}$ and $d_I = -\alpha\beta_1 e^{-\mu\tau_V}\frac{\lambda_1(x)}{\beta_S\beta_V}A_I^{-1}$.*

Proof. Let $\phi = \mathcal{F}\Psi$ and $\Psi = -\mathcal{B}^{-1}\Theta$. Clearly,

$$\begin{cases} \beta_V\Psi_1 = \Theta_1 \\ \beta_I\Psi_1 - A_I\Psi_2 = \Theta_2. \end{cases}$$

Solving this system, we get

$$\begin{aligned} \Psi_1 &= \tfrac{\Theta_1}{\beta_V}, \\ \Psi_2 &= A_I^{-1}(\beta_I\Psi_1 - \Theta_2). \end{aligned}$$

Consequently,

$$\begin{aligned} \phi_1 &= d_1\Psi_1 + d_2\Psi_2, \\ \phi_2 &= 0, \end{aligned}$$

with $d_1 = \beta\frac{\lambda_1}{\beta_S\beta_V}e^{-\mu\tau_V} + \alpha e^{-\mu\tau_V}\frac{\lambda_1\beta_1}{\beta_S\beta_V}A_I^{-1}$, and $d_2 = -A_I^{-1}\alpha\frac{\lambda_1}{\beta_S}e^{-\mu\tau_V}$. Hence, we can write

$$-\mathcal{F}\mathcal{B}^{-1}\begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix} = \begin{pmatrix} d_1\Psi_1 + d_2\Psi_2 \\ 0 \end{pmatrix}.$$

Iteratively, we get

$$(-\mathcal{F}\mathcal{B}^{-1})^n\begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix} = \begin{pmatrix} d_1^n\Psi_1 + d_1^{n-1}d_2\Psi_2 \\ 0 \end{pmatrix}.$$

Hence $\|d_I^n\| \le \|(-\mathcal{F}\mathcal{B}^{-1})^n \le \|d_1^{n-1}\|(\|d_1\| + \|d_2\|)$. By applying the squeeze theorem and *Gelfand's formula*, we obtain $r(-\mathcal{F}\mathcal{B}^{-1}) = \mathfrak{r}(d_1)$. Since $d_1 = d_V + d_I$, the proof is achieved. From Lemma 7, we get

$$\Re_0 = \mathfrak{r}(d_V + d_I), \tag{11}$$

where $d_V = \frac{\beta\lambda_1}{\beta_S\beta_V}$ is the cell-to-cell transmission's next generation operator, and $d_I = \frac{\alpha\lambda_1\beta_1}{\beta_S\beta_V}A_I^{-1}$ is the next generation operator for virus-to-cell transmission. The basic reproduction number for cell-to-cell transmission only, when virus-to-cell transmission is not present, is

$$\Re_0^V = \mathfrak{r}(d_V) = \max_{\eta\in\bar{\Omega}} \frac{\beta(\eta)\lambda_1(\eta)e^{-\mu\tau_V}}{\beta_S(\eta)\beta_V(\eta)}. \tag{12}$$

Furthermore, the basic reproduction number can also be stated by the following variational formula in the situation of virus-to-cell only:

$$\Re_0^I = \mathfrak{r}(d_I) = \sup_{\Theta\in L^2,\Theta\neq 0} \frac{\int_\Omega \frac{\alpha(\eta)\lambda_1(\eta)\beta(\eta)e^{-\mu\tau_V}}{\beta_S(\eta)\beta_V(\eta)}\Theta^2(\eta)d\eta}{K_I\int_\Omega\int_\Omega \frac{1}{2}J(\eta-\epsilon)(\Theta(\epsilon)-\Theta(\eta))^2 d\epsilon d\eta + \int_\Omega \beta_I(\eta)\Theta^2(\eta)d\eta}. \tag{13}$$

Clearly, $\Re_0 \leq \Re_0^V + \Re_0^I$. The equality holds if $\frac{\beta(\eta)\lambda_1(\eta)}{\beta_S(\eta)\beta_V(\eta)}$ is space independent.
The basic reproduction number $\Re_1$ for the antibody response is determined by

$$\Re_1 = \sup_{\Theta\in L^2,\Theta\neq 0} \frac{\int_\Omega \zeta(\eta)I^{**}(\eta)\Theta^2(\eta)d\eta}{K_R\int_\Omega\int_\Omega \frac{1}{2}J(\eta-\epsilon)(\Theta(\epsilon)-\Theta(\eta))^2 d\epsilon d\eta + \int_\Omega \beta_R(\eta)\Theta^2(\eta)d\eta}.$$

□

It is evident that $\Re_1$ is increasing in $\zeta(\eta)$ and $I^{**}(\eta)$, and decreasing in β_R and K_R. In particular, we obtain the spatially homogeneous model, where model (2)–(3) degenerates.

$$\Re_1 = \frac{\zeta(\eta)I^{**}(\eta)}{\beta_R(\eta)}.$$

Lemma 8. *$\Re_0 - 1$ and $\delta(\mathbf{A})$ have the same sign.*

5. EXTINCTION SCENARIO OF VIRUS

Here, we show that for $\Re_0 < 1$, the VFSS is globally stable. First, we use the following theorem to show that this steady state is locally stable:

Theorem 9. *H_0 is locally asymptotically stable if $\Re_0 < 1$, and unstable if $\Re_0 > 1$.*

Proof. To begin with, we notice that $\mathbf{A}_{\tau_V}$, the generator of the linearized system (9), and $\mathbf{A}$, the generator of the corresponding linear system without delay ($\tau_V = 0$), are not identical. Moreover, the principal eigenvalue of the problem is $S(\mathbf{A}_{\tau_V})$.

$$\begin{aligned} \beta(\eta)\frac{\lambda_1(\eta)}{\beta_S(\eta)}\Theta_3(\eta)e^{-\lambda\tau_V}e^{-\mu\tau_V} + \alpha(\eta)\frac{\lambda_1(\eta)}{\beta_S(\eta)}\Theta_2(\eta)e^{-\lambda\tau_V}e^{-\mu\tau_V} - \beta_V(\eta)\Theta_2(\eta) \\ = \lambda\Theta_2(\eta), \\ K_I\int_\Omega J(\eta-\epsilon)(\Theta_3(\iota,\epsilon)-\Theta_3(\iota,\eta))d\epsilon + \beta_1(\eta)\Theta_2(\eta) - \beta_I(\eta)\Theta_3(\eta) \\ = \lambda\Theta_3(\eta), \end{aligned} \tag{14}$$

with $\mathbf{A}_{\tau_V}$ as the generator of the semigroup $P_{\tau_V}(\iota)$ related to the delayed linear system (9), for all $\eta \in \bar{\Omega}$. The principal eigenvalue of the problem is $\delta(\mathbf{A})$.

$$(15)\quad \beta(\eta)\tfrac{\lambda_1(\eta)}{\beta_S(\eta)}\Theta_3(\eta)e^{-\mu\tau_V} + \alpha(\eta)\tfrac{\lambda_1(\eta)}{\beta_S(\eta)}\Theta_2(\eta)e^{-\mu\tau_V} - \beta_V(\eta)\phi_2(\eta) = \lambda\Theta_2(\eta),$$

$$K_I\int_\Omega J(\eta-\epsilon)(\Theta_3(\iota,\epsilon) - \Theta_3(\iota,\eta))d\epsilon + \beta_1(\eta)\Theta_2(\eta) - \beta_I(\eta)\Theta_3(\eta) = \lambda\Theta_3(\eta).$$

We determine that $\delta(A)$ has the same sign as $S(A_{\tau_V})$ by applying Theorem 9.2.1 in [14] to the eigenvalue problem associated with the first equation of (14) and (15). As a result, there is a sign difference between the exponential growth rate associated with the semi-group linked to the delayed system (14), known as $\omega(P_{\tau_V})$), and the one associated with $P(\iota)$, the semi-group linked to the generator $\mathbf{A}$. The exponential growth bound of $P(\iota)$ is now found and is defined as

$$\omega(\mathbb{T}) = \max\{\delta(\mathbf{A}), \omega_0(\mathbb{T})\},$$

utilizing $e^{\omega_0(\mathbb{T})} := \kappa(P(\iota))$ where κ represents the non-compactness measure. Finding the sign of $\omega(P_{\tau_V})$ is necessary to demonstrate the local stability of H_0. If $\omega(\mathbb{T}) < 0$, then H_0 is locally asymptotically stable; if $\omega(\mathbb{T}) > 0$, then H_0 is unstable. Since $\omega(\mathbb{T})$ and $\omega(P_{\tau_V})$ have the same sign, it is possible to infer the local asymptotic stability of VFSS by using Theorem 2.1 in [16] to determine the sign of $\omega(\mathbb{T})$. Lemma 4 gives us $\omega_0 \le \mu_0 < 0$. As a result, H_0stability (or instability) is determined by $\delta(A)$. Lemma 8 gives us that $\Re_0 - 1$ and $\delta(\mathbf{A})$ have the same sign. Consequently, if $R_0 < 1$, then H_0 is locally asymptotically stable since $\delta(\mathbf{A}) < 0$ and $\omega(\mathbb{T}) < 0$. Nevertheless, if $\Re_1 \le 1 < \Re_0$, $\delta(\mathbf{A}) > 0$. Consequently, $\omega(\mathbb{T}) = \delta(\mathbf{A}) > 0$, and H_0 becomes unstable. □

The following lemma is used to construct test functionals.

Lemma 10. *Given the assumption that $\Re_0 < 1$, then there exists a $\rho > 0$ that satisfies $\rho < 1$. It may be written as*

$$(16)\qquad \rho = \sup_{\Theta\in L^2,\Theta\neq 0}\Bigg\{\frac{1}{K_I\int_\Omega\int_\Omega \frac{1}{2}J(\eta-\epsilon)(\Theta(\eta)-\Theta(\epsilon))\Theta(\eta)d\epsilon d\eta + \int_\Omega\beta_I(\eta)\Theta^2(\eta)d\eta}\times$$
$$\times\int_\Omega \frac{\alpha(\eta)\lambda_1(\eta)\beta_I(\eta)e^{-\mu\tau_V}}{\beta_S(\eta)\beta_V(\eta) - \beta_* e^{-\mu\tau_V}\lambda(\eta)}\Theta^2(\eta)d\eta\Bigg\}.$$

Additionally, the following system is also satisfied by the positive functions $\Theta^0 = (\Theta_1^0(\eta), \Theta_2^0(\eta))^T$.

(17)

$$0 = -\left(\beta_V(\eta) - \alpha(\chi)\tfrac{\lambda_1(\eta)}{\beta_S(\eta)}e^{-\mu\tau_V}\right)\Theta_1^0(\eta) + \beta_1(\eta)\Theta_2^0(\eta), \quad \eta\in\bar{\Omega},$$

$$0 = \tfrac{1}{\rho}\beta(\eta)e^{-\mu\tau_V}\tfrac{\lambda_1(\eta)}{\beta_S(\eta)}\Theta_1^0(\eta) + K_I\int_\Omega J(-\epsilon)(\Theta_2^0(\iota,\epsilon) - \Theta_2^0(\iota,\eta))d\epsilon - \beta_I(\eta)\Theta_2^0(\eta),$$

$$\eta\in\bar{\Omega}.$$

Proof. Notice that for $\Re_0 < 1$, we obtain that $\Re(\eta) := \frac{\beta(\eta)\lambda_1(\eta)}{\beta_S(\eta)\beta_V(\eta)} \leq 1$. As a consequence, we use a different decomposition from ($\mathbf{A}$:=$\mathsf{B}+\mathsf{F}$) of the operator $\mathbf{A}$ as

(18)

$$\mathbf{A} := \begin{pmatrix} \beta_V(\eta) - \alpha(\eta)\frac{\lambda_1(\eta)}{\beta_S(\eta)}e^{-\mu\tau_V} & 0 \\ \beta_1(\eta) & A_I \end{pmatrix} + \begin{pmatrix} 0 & \beta(\eta)e^{-\mu\tau_V}\frac{\lambda_1(\eta)}{\beta_S(\eta)} \\ 0 & 0 \end{pmatrix} := \mathsf{B}+\mathsf{F}$$

We assume that the semigroup generated by B is $\mathbb{T}_{\mathsf{B}}(\iota)$. Clearly, $S(\mathsf{B}) < 0$. We assume that ρ is the spectral radius of $\mathbf{K}$, which is

$$\rho = \mathfrak{r}(\mathbf{K})$$

Using $\mathbf{K} := -\mathsf{F}\mathsf{B}^{-1}$ and applying Theorem 3.5 in [13], we conclude that if $\mathcal{R}_0 < 1$, then $\delta(\mathbf{A}) < 0$ and, consequently, $\rho < 1$. We then consider an eigenvalue ϱ of $\mathbf{A}$ that solves the following problem

$$\varrho\Theta(\eta) = A[\Theta](\eta), \eta \in \bar{\Omega}, \Theta = (\Theta_1(\eta), \Theta_2(\eta))^T \in C(\Omega, \mathbb{R}^2).$$

Next, we claim that the problem

$$\mathsf{B}[\Theta](\eta) = \varrho\mathsf{F}[\Theta](\eta), \eta \in \bar{\Omega}, \tag{19}$$

has a unique principal eigenpair (η^*, Θ^0) that fulfills

$$\rho = \mathfrak{r}(-\mathsf{F}\mathsf{B}^{-1}) = \mathfrak{r}(-\mathsf{B}^{-1}\mathsf{F}) = \tfrac{1}{\varrho^*}.$$

We assume that $\mathbb{T}_{\mathsf{B}}(\iota)$ is the semigroup related to the generator B in order to demonstrate this assertion. Theorem 3.5 in [13] suggests that since B is a positive resolvent and $T_{\mathsf{B}}(\iota)$ is a positive semigroup

$$(\gamma I - \mathsf{B})^{-1}\Theta = \int_0^\infty e^{-\epsilon\iota}\, T_{\mathsf{B}}(\iota)d\iota, \Theta \in \mathfrak{D}(\mathsf{B}).$$

Assume that $\gamma = 0$. Next,

$$-\mathsf{B}^{-1}\Theta = \int_0^\infty T_B(\iota)d\iota, \Theta \in \mathfrak{D}(\mathsf{B}).$$

Additionally, we get

$$\mathsf{F}(-\mathsf{B}^{-1})\Theta = \mathbf{K}[\Theta](\eta).$$

Now, we set $\Theta = -\mathsf{B}\bar{\Theta}$, then we get

$$\mathsf{F}\bar{\Theta} = -\mathbf{K}(\eta)\mathsf{B}\bar{\Theta}.$$

Therefore,

$$-\mathsf{B}\bar{\Theta} = \tfrac{\mathsf{F}\bar{\Theta}}{\mathbf{K}(\eta)}, \eta \in \bar{\Omega}.$$

Given that (19) has a unique positive eigenvalue ϱ^*, then $\varrho^* = \frac{1}{\rho}$ with a positive eigenfunction. In (18), we obtain ϱ by substituting ϱ^* with $\Theta^* = (\Theta_1^*, \Theta_2^*), \eta \in \bar{\Omega}$ is the corresponding positive eigenfunction , we have

(20)

$$0 = -(\beta_V(\eta) - \alpha(\eta)\tfrac{\lambda_1(\eta)}{\beta_S(\eta)}e^{-\mu\tau_V})\Theta_1^*(\eta) + \beta_1(\eta)\Theta_2^*(\eta), \eta \in \bar{\Omega},$$

$$0 = \varrho^*\beta(\eta)e^{-\mu\tau_V}\tfrac{\lambda_1(\eta)}{\beta_S(\eta)}\Theta_1^*(\eta) + K_I\int_\Omega J(\eta-\epsilon)(\Theta_2^*(\iota,\epsilon) - \Theta_2^*(\iota,\eta))d\epsilon - \beta_I(\eta)\Theta_2^*(\eta),$$

$\eta \in \bar{\Omega}$.

The first equation in (20) suggests that $\Re(\eta) < 1$

$$\Theta_1^*(\eta) = \tfrac{\beta_1(\eta)\beta_S(\eta)}{\beta_V(\eta)\beta_S(\eta) - \alpha(\eta)\lambda_1(\eta)e^{-\mu\tau_V}}\Theta_2^*(\eta), \quad \eta \in \bar{\Omega}.$$

Therefore, we enter it in the second equation in (20) to get

$$\begin{aligned} 0 =& \varrho^*\beta(\eta)e^{-\mu\tau_V}\tfrac{\lambda_1(\eta)}{\beta_S(\eta)} \cdot \tfrac{\beta_1(\eta)\beta_S(\eta)}{\beta_V(\eta)\beta_S(\eta) - \alpha(\eta)\lambda_1(\eta)e^{-\mu\tau_V}}\Theta_2^*(\eta) \\ &+ K_I\int_\Omega J(\eta-\epsilon)(\Theta_2^*(\iota,\epsilon) - \Theta_2^*(\iota,\eta))d\epsilon - \beta_I(\eta)\Theta_2^*(\eta), \\ &\eta \in \bar{\Omega}, \end{aligned} \tag{21}$$

hence,

$$\begin{aligned} &K_I\int_\Omega J(\eta-\epsilon)(\Theta_2^*(\iota,\epsilon) - \Theta_2^*(\iota,\eta))d\epsilon - \beta_I(\eta)\Theta_2^*(v) \\ &+ \varrho^*\tfrac{\beta(\eta)e^{-\mu\tau_V}\lambda_1(\eta)\beta_1(\eta)}{\beta_V(\eta)\beta_S(\eta) - \alpha(\eta)\lambda_1(\eta)e^{-\mu\tau_V}}\Theta_2^*(\eta) = 0. \end{aligned}$$

By considering the decomposition (18) and taking into account that $\varrho^* = \frac{1}{\rho}$, we deduce that

$$\begin{aligned} \rho = \sup_{\Theta \in L^2, \Theta \neq 0} \Bigg\{ & \tfrac{1}{K_I\int_\Omega\int_\Omega \frac{1}{2}J(\eta-\epsilon)(\Theta(\eta)-\Theta(\epsilon))\Theta(\eta)d\epsilon d\eta + \int_\Omega \beta_I(\eta)\Theta^2(\eta)d\eta} \times \\ & \times \int_\Omega \tfrac{\alpha(\eta)\lambda_1(\eta)\beta_1(\eta)e^{-\mu\tau_V}}{\beta_S(\eta)\beta_V(\eta) - \beta_* e^{-\mu\tau_V}\lambda_1(\eta)}\Theta^2(\eta)d\eta \Bigg\}. \end{aligned} \tag{22}$$

Consider that $\Theta^0 = (\Theta_1^0(\eta), \Theta_2^0(\eta))^T$ is the eigenfunction associated with the eigenvalue ρ. Consequently, ρ and Θ^0 satisfy the following system

$$\begin{aligned} &-(\beta_V(\eta) - \alpha(\eta)\tfrac{\lambda_1(\eta)}{\beta_S(\eta)}e^{-\mu\tau_V})\Theta_1^0(\eta) + \beta_1(\eta)\Theta_2^0(\eta) = 0, \eta \in \bar{\Omega} \\ &\tfrac{1}{\rho}\beta(\eta)e^{-\mu\tau_V}\tfrac{\lambda_1(\eta)}{\beta_S(\eta)}\Theta_1^0(\eta) + K_I\int_\Omega J(\eta-\epsilon)(\Theta_2^0(\iota,\epsilon) \\ &\qquad - \Theta_2^0(\iota,\eta))d\epsilon - \beta_I(\eta)\Theta_2^0(\eta) = 0, \eta \in \bar{\Omega} \end{aligned} \tag{23}$$

□

Remark 11. *System (24) is used to establish the global stability of the VFSS.*

The following theorem presents the principal result of this section.

Theorem 12. *Let $\Re_0$ be defined by (22) and (23). Then H_0 is globally stable in $\mathbb{C}$ if $\Re_0 < 1$.*

Proof. It has been proved that $\limsup_{\iota\to\infty} S(\iota,\eta) \leqslant \frac{\lambda_1(\eta)}{\beta_S(\eta)}$.Therefore, there exists $\theta > 0$ sufficiently small such that $S(\iota,\eta) \leq \frac{\lambda_1(\eta)}{\beta_S(\eta)} + \theta$ for ι sufficiently large. Furthermore, given the fact that ρ is defined by (22), if $\Re_0 < 1$, then for ρ_θ small enough there exist positive functions $(\Theta_1^\theta(\eta), \Theta_2^\theta(\eta))$ satisfying the following system:

$$\begin{aligned}0 =& -(\beta_V(\eta) - \alpha(\eta)e^{-\mu\tau_V}(\tfrac{\lambda_1(\eta)}{\beta_S(v)} + \theta))\Theta_1^\theta(\eta) + \beta_1(\eta)\Theta_2^\theta(\eta),\\ 0 =& \tfrac{1}{\rho_\theta}\beta(\eta)e^{-\mu\tau_V}(\tfrac{\lambda_1(\eta)}{\beta_S(\eta)} + \theta)\Theta_1^\theta(\eta)\\ &+ K_I\int_\Omega J(\eta-\epsilon)(\Theta_2^\theta(\epsilon) - \Theta_2^\theta(\eta))d\epsilon - \beta_I(\eta)\Theta_2^\theta(\eta), \quad \eta \in \bar{\Omega}.\end{aligned} \tag{24}$$

We then define the following Lyapunov function

$$E(\iota) = \vartheta(\iota) + \vartheta_{\tau_V}(\iota), \tag{25}$$

with

$$\vartheta(\iota) = \int_\Omega \Theta_1^\theta(\eta)V(\iota,\eta) + \Theta_2^\theta(\eta)V(\iota,\eta)d\eta,$$

and

$$\begin{aligned}\vartheta_{\tau_V}(\iota) =& e^{-\mu\tau_V}\int_\Omega \Theta_1^\theta(\eta)\beta(\eta)(\tfrac{\lambda_1(\eta)}{\beta_S(\eta)} + \theta)\int_0^{\tau_V} I(\tau - s,\eta)ds\\ &+ \Theta_1^\theta(\eta)\alpha(\eta)(\tfrac{\lambda_1(\eta)}{\beta_S(\eta)} + \theta)\int_0^{\tau_V} V(\iota - s,\eta)dsd\eta,\end{aligned}$$

where system (24) defines $\Theta_1^\theta(\eta), \Theta_2^\theta(\eta)$. The derivative of $\vartheta(\iota)$ is

$$\begin{aligned}&\vartheta'(\iota) =\\ &= \int_\Omega \Theta_1^\theta(\eta)\tfrac{\partial V(\iota,\eta)}{\partial\tau} + \Theta_2^\theta(\eta)\tfrac{\partial V(\iota,\eta)}{\partial\iota}d\eta\\ &= \int_\Omega \Big\{\Theta_1^\theta(\eta)\left[\beta(\eta)S(\iota-\tau_v,\eta)I(\iota-\tau_v,\eta)e^{-\mu\tau_V}\right.\\ &\quad \left.+\alpha(\eta)S(\iota-\tau_v,\eta)V(\iota-\tau_v,\eta)e^{-\mu\tau_V} - \beta_V(\eta)V(\iota,\eta)\right]\Big\}d\eta\\ &\quad +\int_\Omega\Big\{\Theta_2^\theta(\eta)\Big[K_I\int_\Omega J(\eta-\epsilon)(I(\iota,\epsilon)-I(\iota,\eta))d\epsilon+\beta_1(\eta)V(\iota,\eta)-\beta_I(\eta)I(\iota,\eta)\Big]\Big\}d\eta.\end{aligned}$$

Using the inequality $S(\iota,\eta) \leq \frac{\lambda_1(\eta)}{\beta_S(\eta)} + \theta$, we get

$$\vartheta'(\iota) \leq \int_\Omega \Big\{\Theta_1^\theta(\eta)\Big[\beta(\eta)(\tfrac{\lambda_1(\eta)}{\beta_S(\eta)} + \theta)I(\iota-\tau_v,\eta)e^{-\mu\tau_V}$$

$$
\begin{aligned}
&+ \alpha(\eta)(\tfrac{\lambda_1(\eta)}{\beta_S(\eta)} + \theta)V(\iota - \tau_v, \eta)e^{-\mu\tau_V} - \beta_V(\eta)V(\iota,\eta)\Big]\Big\}d\eta \\
&+ \int_\Omega \Big\{\Theta_2^\theta(\eta)\Big[K_I \int_\Omega J(\eta-\epsilon)I(\iota,\epsilon)d\epsilon + \beta_1(\eta)V(\iota,\eta) \\
&- \beta_I(\eta)I(\iota,\eta) - (\gamma_1(\eta) + K_I J_1(\eta))I(\iota,\eta)\Big]\Big\}d\eta,
\end{aligned}
$$

with $J_1(\eta) = \int_\Omega J(\eta-\epsilon)d\epsilon$. Using (24), we get

$$
\begin{aligned}
&\vartheta'(\iota) \leq \\
&\leq \int_\Omega \Big\{\Theta_1^\theta(\eta)[\beta(\eta)(\tfrac{\lambda_1(\eta)}{\beta_S(\eta)} + \theta)I(\iota-\tau_v,\eta)e^{-\mu\tau_V} \\
&+ \alpha(\eta)(\tfrac{\lambda_1(\eta)}{\beta_S(\eta)} + \theta)V(\iota-\tau_v,\eta)e^{-\mu\tau_V}]\Big\}d\eta \\
&+ \int_\Omega \Big\{\Theta_2^\theta(\eta)[K_I \int_\Omega J(\eta-\epsilon)I(\iota,\epsilon)d\epsilon + \beta_1(\eta)V(\iota,\eta)]\Big\}\, d\eta \\
&- \int_\Omega V(\iota,\eta)\Big\{\alpha(\eta)\tfrac{\lambda_1(\eta)}{\beta_S(\eta)}e^{-\mu\tau_V}\phi_1^\theta(\eta) + \beta_1(\eta)\Theta_2^\theta(\eta)\Big\}\, d\eta \\
&- \int_\Omega I(\iota,\eta)\Big\{\tfrac{1}{\rho_\theta}\beta(\eta)e^{-\mu\tau_V}(\tfrac{\lambda_1(\eta)}{\beta_S(\eta)} + \theta)\Theta_1^\theta(\eta) + K_I \int_\Omega J(\eta-\epsilon)\Theta_2^\theta(\epsilon)d\epsilon\Big\}\, d\eta.
\end{aligned}
$$

Using the assumptions on J, we have

$$
\int_\Omega\int_\Omega J(\eta-\epsilon)\Theta_2^\theta(\epsilon)I(\iota,\eta)d\epsilon d\eta = \int_\Omega\int_\Omega J(\eta-\epsilon)\Theta_2^\theta I(\iota,\epsilon)d\eta d\epsilon.
$$

Therefore, some simplification gives

$$
\begin{aligned}
\vartheta'(\iota) \quad \leq \quad & \int_\Omega \Theta_1^\theta(\eta)\beta(\eta)(\tfrac{\lambda_1(\eta)}{\beta_S(\eta)} + \theta)e^{-\mu\tau_V}[\tfrac{-I(\iota,\eta)}{\rho_\theta} + I(\iota-\tau_v,\eta)]d\eta \\
&+ \int_\Omega \Theta_1^\theta(\eta)\alpha(\eta)(\tfrac{\lambda_1(\eta)}{\beta_S(\eta)} + \theta)e^{-\mu\tau_V}[V(\iota-\tau_v,\eta) - V(\iota,\eta)].
\end{aligned}
$$

$\vartheta'_{\tau_V}(\iota)$ is given by

$$
\begin{aligned}
\vartheta'_{\tau_V}(\tau) =& e^{-\mu\tau_V}\int_\Omega \Theta_1^\theta(\eta)\beta(\eta)(\tfrac{\lambda_1(\eta)}{\beta_S(\eta)} + \theta)\tfrac{\partial}{\partial\iota}\int_0^{\tau_V} I(\iota-s,\eta)ds \\
&+ \Theta_1^\theta(\eta)\alpha(\eta)(\tfrac{\lambda_1(\eta)}{\beta_S(\eta)} + \theta)\tfrac{\partial}{\partial\iota}\int_0^{\tau_V} V(\iota-s,\eta)dsd\eta \\
=& e^{-\mu\tau_V}\int_\Omega \Theta_1^\theta(\eta)\beta(\eta)(\tfrac{\lambda_1(\eta)}{\beta_S(\eta)} + \theta)\int_0^{\tau_V} \tfrac{\partial}{\partial t}I(\iota-s,\eta)ds \\
&+ \Theta_1^\theta(\eta)\alpha(\eta)(\tfrac{\lambda_1(\eta)}{\beta_S(\eta)} + \theta)\int_0^{\tau_V} \tfrac{\partial}{\partial\iota}V(\iota-s,\eta)dsd\eta \\
=& e^{-\mu\tau_V}\int_\Omega \Theta_1^\theta(\eta)\beta(\eta)(\tfrac{\lambda_1(\eta)}{\beta_S(\eta)} + \theta)\int_0^{\tau_V} (-\tfrac{\partial}{\partial s})I(\iota-s,\eta)ds \\
&+ \Theta_1^\theta(\eta)\alpha(\eta)(\tfrac{\lambda_1(\eta)}{\beta_S(\eta)} + \theta)\int_0^{\tau_V} (-\tfrac{\partial}{\partial s})V(\iota-s,\eta)dsd\eta
\end{aligned}
$$

$$
\begin{aligned}
&= -\left\{ e^{-\mu\tau_V} \int_\Omega \Theta_1^\theta(\eta)\beta(\eta)(\tfrac{\lambda_1(\eta)}{\beta_S(\eta)} + \theta) I(\iota - s, \eta) \mid_0^{\tau_V} \right. \\
&\quad \left. + \Theta_1^\theta(\eta)\alpha(\eta)(\tfrac{\lambda_1(\eta)}{\beta_S(\eta)} + \theta) V(\iota - s, \eta) \mid_0^{\tau_V} d\eta \right\} \\
&= e^{-\mu\tau_V} \int_\Omega \Theta_1^\theta(\eta)\beta(\eta)(\tfrac{\lambda_1(\eta)}{\beta_S(\eta)} + \theta)(I(\iota, \eta) - I(\iota - \tau_V, \eta)) \\
&\quad + \Theta_1^\theta(\eta)\alpha(\eta)(\tfrac{\lambda_1(\eta)}{\beta_S(\eta)} + \theta)(V(\iota, \eta) - V(\iota - \tau_V, \eta)) d\eta
\end{aligned}
$$

. By summing $\vartheta'(\iota)$ and $\vartheta'_{\tau_V}(\iota)$, we obtain

$$
E'(\iota) \leq e^{-\mu\tau_V} \int_\Omega \Theta_1^\theta(\eta)\beta(\eta) I(\iota, \eta)(\tfrac{\lambda_1(\eta)}{\beta_S(\eta)} + \theta)(1 - \tfrac{1}{\rho_\theta}) d\eta \leq 0.
$$

Equality holds for $I(\iota, \eta) = 0$. Given this, the third equation of (2) implies that $V(\iota, \eta) = 0$. Therefore, $(I(\iota, \eta), V(\iota, \eta)) \to (0, 0)$ uniformly for $\eta \in \Omega$ as $\iota \longrightarrow \infty$.
We further obtain that $S(\iota, \eta) \to \frac{\lambda_1(\eta)}{\beta_S(\eta)}$ uniformly for $\eta \in \Omega$ as $\iota \to \infty$. To sum up, the theorem is proved. □

6. UNIFORM PERSISTENCE

The following spaces should be defined:

$$
\begin{aligned}
\xi_0 &= \left\{ \mathbb{U} \in X^+ : V(\cdot, \eta) > 0, I(\cdot, \eta) > 0 \ , \eta \in \bar{\Omega} \right\}; \\
\partial\xi_0 &= \left\{ \mathbb{U} \in X^+ : V(\cdot, \eta) \equiv 0, I(\cdot, \eta) \equiv 0 \ , \eta \in \bar{\Omega} \right\}; \\
M_\partial &= \left\{ \mathbb{U}_0 \in \partial\xi_0 : \Phi(\iota, \mathbb{U}_0) \in \partial\xi_0 \ for \ \iota \geq 0 \right\}.
\end{aligned}
$$

Then, we obtain the following results.

Theorem 13. *Assume that $\Re_1 \leq 1 < \Re_0$. Then the system (2)–(3) exhibits strong uniform persistence. Additionally, in ξ_0, it possesses at least one positive steady state that remains uniformly persistent.*

Proof. To establish the uniform persistence of the system, it is sufficient to verify that all the assumptions of Theorem 3 in [18] are satisfied. □

The following claims provide the detailed proof:

Claim 14. *H_0 is globally stable for semiflow $\{\Xi(\iota)\}_{\iota \geq 0}$ restricted to $\partial\xi_0$. For $\mathbb{U}_0 \in \partial\xi_0$, (2) becomes*

$$
\left\{
\begin{array}{l}
\frac{dS}{d\iota}(\iota, \eta) = \lambda_1(\eta) - \beta_S(\eta) S(\iota, \eta), \\
V(\cdot, \eta) = 0, I(\cdot, \eta) = 0,
\end{array}
\right. \qquad \iota > 0, \eta \in \bar{\Omega}. \tag{26}
$$

Therefore, $S(\iota, \eta)$ goes to $\frac{\lambda_1(\eta)}{\beta_S(\eta)}$ for all $\eta \in \bar{\Omega}$. Hence, $\omega(\mathbb{U}_0) = \{(\frac{\lambda_1(\eta)}{\beta_S(\eta)}, 0, 0, 0)\}$, where $\omega(\phi)$ is the ω-limit set.

CLAIM 15. *For any* $\mathbb{U}_0 \in \xi_0$*, we have* $\mathbb{U} > 0$ *for all* $\eta \in \bar{\Omega}$ *, and* $\iota > 0$*. We consider the problem*

$$\frac{\partial i(\iota,\eta)}{\partial \iota} = K_I \int_\Omega J(\eta-\epsilon)(i(\iota,\epsilon)-i(\iota,\eta))d\epsilon - \beta_I(\eta)i(\iota,\eta),\ \iota > 0, \eta \in \bar{\Omega}.$$

$$i(0,\eta) = I_0(\eta), \eta \in \bar{\Omega}.$$

Comparison principle implies that $I(\iota,\eta) \geq i(\iota,\eta)$*. Now, it is sufficient to show that* $i(\iota,\eta) > 0$ *for all* $\iota > 0$*,* $\eta \in \bar{\Omega}$*. This proof follows the approach of Proposition 2.2 in* [19]*. We define the operator* $\Upsilon : \mathbb{C}(\bar{\Omega}) \to \mathbb{C}(\bar{\Omega})$

$$\Upsilon i(\iota,\eta) = K_I \int_\Omega J(\eta-y)(I(\iota,y)-I(\iota,\eta))dy, \eta \in \bar{\Omega}.$$

Clearly, Υ *is continuous, and* $\{e^{\Upsilon\iota}\}_{\iota\geq 0}$ *forms a uniformly continuous, positive semigroup on* $\mathbb{C}(\overline{\Omega})$ *generated by* Υ*. , where for* $\iota \geq 0$*, we get* $e^{\Upsilon\iota} = \sum_{n=0}^{\infty} \frac{\iota^n \Upsilon^n}{n!}$*. Further,*

$$e^{\Upsilon\tau}(I_0(s,\eta)) = \sum_{n=0}^{\infty} \frac{\tau^n \Upsilon^n (I_0(s,\eta))}{n!}, \iota > 0, s \in [-\tau, 0].$$

Since $i(\tau,\eta) \neq 0$*, we have* $\Upsilon^{n+1}(I_0(s,\eta)) = J(\eta-y)\Upsilon^n I_0(s,\upsilon)d\upsilon, \eta \in \bar{\Omega}$*,* $s \in [-\tau,0]$*, and* $n \geq 1$*. The iteration process implies the existence of* n_0 *such that* $\Upsilon^{n+1}(I(0,\eta)) > 0, \eta \in \bar{\Omega}$ *and* $n \geq n_0$*. Hence,*$\Upsilon^n(I_0(s,\eta)) > 0, \eta \in \bar{\Omega}, s \in [-\tau,0]$*. Consequently,*

$$\begin{aligned} i(\iota,\eta) &= e^{-(K_I-\beta_I(\eta))} e^{\Upsilon\tau}(I_0(s,\eta)) + \int_0^\tau e^{-(K_I-\beta_I(\eta))(\tau-\sigma)}\ e^{\Upsilon(\tau-\sigma)} i(\iota,\eta)d\sigma, \\ &\geq e^{-(K_I-\beta_I(\eta))} e^{\Upsilon\tau}(I_0(s,\eta)) > 0, \end{aligned}$$

for $\iota > 0, \eta \in \bar{\Omega}$*. The second equation in* (2) *yields* $\mathbb{U} > 0$ *from this inequality. The value* ξ_0 *is therefore positively invariant.*

CLAIM 16. H_0 *is a uniform weak repeller for* ξ_0 *for* $\Re_1 \leq 1 < \Re_0$*. In order to prove this assertion, we demonstrate that there exists a sufficiently small* $\theta_1 > 0$ *such that the semiflow* $\{\Xi(\iota)\}_{\iota\geq 0}$ *satisfies*

$$\limsup_{\iota\to\infty} \|\Xi(\iota,\mathbb{U}_0) - H_0\| \geq \theta_1.$$

We use contradiction arguments to demonstrate this result. Suppose that $\limsup_{\iota\to\infty} \|\boldsymbol{\Xi}(\iota;\mathbb{U}_0) - H_0\| \leq \theta_1$*. Hence, there exists* ι_1 *such that* $\mathbb{U}(\iota;\mathbb{U}_0)$ *satisfies*

$$\frac{\lambda_1(\eta)}{\beta_S(\eta)} - \theta_1 \leq S(\iota,\eta) \leq \frac{\lambda_1(\eta)}{\beta_S(\eta)} + \theta_1, 0 \leq V(\iota;\eta) \leq \theta_1, \iota \geq \iota_1,\ \eta \in \bar{\Omega}.$$

Since $\Re_1 \leq 1 < \Re_0$, we have $S_\theta(A) > 0$. Then, there exists ε sufficiently small satisfying

$$\chi\phi(\eta) = \left(\beta_V(\eta) - \alpha(\eta)(\tfrac{\lambda_1(\eta)}{\beta_S(\eta)} + \theta_1)\beta(\eta)(\tfrac{\lambda_1(\eta)}{\beta_S(\eta)} + \theta_1)\beta_1(\eta)A_I\right)\Theta(\eta), \Theta(\eta) = (\phi^1(\eta), \Theta^2(\eta))^T, \ \eta \in \Omega.$$

has a principal eigenpair (χ^p, Θ^p) with $\chi^p > 0$ and $\Theta^p = (\Theta_1^p, \Theta_2^p)$. Hence, for $\iota > \iota_1$,

$$\begin{aligned}\frac{\partial V(\iota,\eta)}{\partial\iota} =&\beta(\eta)(\tfrac{\lambda_1(\eta)}{\beta_S(\eta)} + \theta_1)I(\iota - \tau_v, \eta)e^{-\mu\tau_V}\\ &+ \alpha(\eta)(\tfrac{\lambda_1(\eta)}{\beta_S(\eta)} + \theta_1)V(\iota - \tau_v, \eta)e^{-\mu\tau_V} - \beta_V(\eta)V(\iota,\eta),\\ \frac{\partial I}{\partial\iota}(\iota,\eta) =&K_I\int_\Omega J(\eta - y)(I(\iota,y) - I(\iota,\eta))dy + \beta_1(\eta)V(\iota,\eta) - \beta_I(\eta)I(\iota,\eta),\\ \iota >&\iota_1, \quad \eta \in \bar{\Omega}.\end{aligned}$$

By comparison principle, we obtain

$$(V(\eta), I(\eta))^T = y^P e^{y^P(\iota - \iota_1)})\Theta^P(\eta).$$

There is a contradiction since $y^P > 0$.

Claim 17. *System (2) is strongly uniformly persistent in ξ_0 for $\Re_1 \leq 1 < \Re_0$. For the orbit $\varkappa^+(\mathbb{U}_0) := \mathbb{U}_{\iota\geq 0}\{\Xi(\iota)\}$, let $\varrho_1(\mathbb{U}_0)$ be the ϱ - limit set. Since $\Phi(\iota;\xi_0) \subseteq \xi_0$ and $\Xi(\iota;\partial\xi_0) \subseteq \partial\xi_0$, then $M_\partial = \partial\xi_0$ and $\varrho(\mathbb{U}_0) = H_0$ for all $\mathbb{U}_0 \in \partial\xi_0$. Hence, H_0 is isolated in X. Since $\varrho(\mathbb{U}_0) = H_0$ for all $\mathbb{U}_0 \in \partial\xi_0$, then there is no cycle in $\partial\xi_0$ from H_0 to H_0. We obtain $W^s(H_0) \cap \xi_0 = \varnothing$, with $W^s(E_0) = \{\mathbb{U}_0 \in X : \lim_{\iota\to\infty} \Xi(\iota, \mathbb{U}_0) = H_0\}$, based on the outcomes of Claim 16. Since all the assumptions of Theorem 3 in [18] are satisfied, system (2) is strongly uniformly persistent for $\mathcal{R}_1 \leq 1 < \mathcal{R}_0$. Consequently, the assertion follows. Moreover, the endemic equilibrium $E^* = (S^*, V^*, I^*)$ satisfies the following steady-state system:*

(27)

$$\begin{aligned}0 =&\lambda_1(\eta) - \beta_S(\eta)S^*(\iota,\eta) - \beta(\eta)S^*(\iota,\eta)I^*(\iota,\eta) - \alpha(\eta)S^*(\iota,\eta)V^*(\iota,\eta)\\ 0 =&\beta(\eta)S^*(\eta)I^*(\eta)e^{-\mu\tau_V} + \alpha(\eta)S^*(\eta)V^*(\eta)e^{-\mu\tau_V} - \beta_V(\eta)V^*(\eta) \quad \eta \in \bar{\Omega}.\\ 0 =&K_I\int_\Omega J(\eta - \epsilon)(I^*(\epsilon) - I^*(\eta))d\epsilon + \beta_1(\eta)V^*(\eta) - \beta_I(\eta)I^*(\eta).\end{aligned}$$

7. GLOBAL BEHAVIOR ANALYSIS

Theorem 18. *If $\Re_1 \leq 1 < \Re_0$, then besides H_0, there is also a unique immunity-free infected steady state $H_1 = (S^*, V^*, I^*, 0)$.*

Proof. From (27), we see that system (27) has a unique endemic equilibrium $H_1 = (S^*, V^*, I^*, 0)$. Let us construct the following Lyapunov function,

$$E_*(\iota) = \varrho_1(\iota) + \varrho_{1\tau_V},$$

with

$$\varrho_1(\iota) = \int_\Omega \zeta_1(\eta)\Big\{ e^{-\mu\tau_V} P^*(\eta) H\left(\tfrac{S(\iota,\eta)}{S^*(\eta)}\right) + V^*(\eta) H\left(\tfrac{V(\iota,\eta)}{V^*(\eta)}\right)$$
$$+ \zeta_2(\eta) I^*(\eta) H\left(\tfrac{I(\iota,\eta)}{I^*(\eta)}\right)\Big\} d\eta,$$

and

$$\varrho_{1\tau_V} = \int_\Omega e^{-\mu\tau_V} \zeta_1(\eta) \int_0^{\tau_V} \beta(\eta) S^*(\eta) I^*(\eta) H\left(\tfrac{S(\iota-\vartheta)I(\iota-\vartheta)}{S^*I^*}\right)$$
$$+ \alpha(\eta) S^*(\eta) V^*(\eta) H\left(\tfrac{S(\iota-\vartheta)V(\iota-\vartheta)}{S^*V^*}\right) d\vartheta d\eta,$$

where $H(t) = t - 1 - \ln(t)$, for $t > 0$, $\zeta_1(\eta) = 2e^{-\mu\tau_V} \frac{\beta_1(\eta)V^*(\eta)}{\beta(\eta)S^*(\eta)}, \zeta_2(x) = 2e^{-\mu\tau_V} \frac{\beta(\eta)S^*(\eta)I^*(\eta)}{\beta_1(\eta)V^*(\eta)}$. For simplicity, we use the notations

$S = S(\iota,\eta),\ I = I(\iota,\eta), V = V(\iota,\eta),\ S_{\tau_V} = S(\iota-\tau_V,\eta),\ I_{\tau_V} = I(\iota-\tau_V,\eta),$ $V_{\tau_V} = V(\iota-\tau_V,\eta),\ S^* = S^*(\eta),\ I^* = I^*(\eta),\ V^* = V^*(\eta).$

First, we compute $\varrho_1'(\iota)$, which is

$$\varrho_1'(\iota) =$$
$$= \int_\Omega \zeta_1(\eta)\left\{ e^{-\mu\tau_V}\left(1-\tfrac{S^*}{S}\right)\tfrac{\partial S}{\partial\iota} + \left(1-\tfrac{V^*}{V}\right)\tfrac{\partial V}{\partial\iota} + \zeta_2(\eta)\left(1-\tfrac{I^*}{I}\right)\tfrac{\partial I}{\partial\iota}\right\} d\eta$$
$$= \int_\Omega \zeta_1(\eta)\Big\{ e^{-\mu\tau_V}\left(1-\tfrac{S^*}{S}\right)[\lambda_1 - \beta_S S - \beta S I - \alpha S V]$$
$$+ \left(1-\tfrac{V^*}{V}\right)[\beta S_{\tau_V} I_{\tau_V} e^{-\mu\tau_V} + \alpha S_{\tau_V} V_{\tau_V} e^{-\mu\tau_V} - \beta_V V]$$
$$+ \zeta_2(\eta)\left(1-\tfrac{I^*}{I}\right)\Big[K_I \int_\Omega J(\eta-\epsilon) I(\iota,\epsilon) d\epsilon + \beta_1(\eta) V(\eta) - (\beta_I(\eta)+K_I) I(\eta)\Big]\Big\} d\eta.$$

$H_1 = (S^*, V^*, I^*, 0)$ satisfies

$$\begin{cases} \lambda_1(\eta) = \beta_S(\eta)S^* + \beta(\eta)S^*I^* + \alpha(\eta)S^*V^*, & \\ \beta_V(\eta)V^* = \beta(\eta)S^*I^*e^{-\mu\tau_V} + \alpha(\eta)S^*V^*e^{-\mu\tau_V}, & \eta \in \bar{\Omega}. \\ (\beta_I(\eta)+K_I)I^* = K_I \int_\Omega J(\eta-\epsilon)I^*(\epsilon)d\epsilon + \beta_1(\eta)V^*, & \end{cases}$$

Thus, after some simplification, $\varrho_1'(\iota)$ becomes

$$\varrho_1'(\iota) =$$
$$= \int_\Omega \zeta_1(\eta)\Big\{ e^{-\mu\tau_V}\beta_S(\eta)S^*\left(2 - \tfrac{S^*}{S} - \tfrac{S}{S^*}\right)$$
$$+ e^{-\mu\tau_V}\beta(\eta)S^*I^*\left[2 - \tfrac{S^*}{S} - \tfrac{SI}{S^*I^*} + \tfrac{I}{I^*} - \tfrac{V^*}{V} - \tfrac{S_{\tau_V}I_{\tau_V}}{S^*I^*} - \tfrac{S_{\tau_V}I_{\tau_V}V^*}{S^*I^*V}\right]$$
$$+ e^{-\mu\tau_V}\alpha(\eta)S^*V^*\left[2 - \tfrac{S^*}{S} - \tfrac{SV}{S^*V^*} + \tfrac{S_{\tau_V}V_{\tau_V}}{S^*V^*} - \tfrac{S_{\tau_V}V_{\tau_V}}{S^*V}\right]$$
$$+ K_I\zeta_2(\eta)\left[\left(1-\tfrac{I^*}{I}\right)K_I\int_\Omega J(\eta-\epsilon)I(\iota,\epsilon)d\epsilon + \left(1-\tfrac{I}{I^*}\right)\int_\Omega J(\eta-\epsilon)I^*(\epsilon)d\epsilon\right]$$

$$+ \zeta_2(\eta)\beta_1(\eta)V^* \left(1 + \tfrac{V}{V^*} - \tfrac{I}{I^*} - \tfrac{V^*I}{VI^*}\right) \Big\} d\eta.$$

Notice that $\zeta_2(\eta)\beta_1(\eta)V^* = \beta(\eta)S^*I^*e^{-\mu\tau_V}$ and $\zeta_1(\eta)\zeta_2(\eta) = 2I^*$. Then, we obtain

$$\begin{aligned}
\varrho_1'(\iota) =& \int_\Omega \zeta_1(\eta)\Big\{ e^{-\mu\tau_V}\beta_S(\eta)S^*\left(2 - \tfrac{S^*}{S} - \tfrac{S}{S^*}\right) \\
&+ e^{-\mu\tau_V}\beta(\eta)S^*I^*\left[3 - \tfrac{S^*}{S} - \tfrac{SI}{S^*I^*} + \tfrac{S_{\tau_V}I_{\tau_V}}{S^*I^*} - \tfrac{S_{\tau_V}I_{\tau_V}V^*}{S^*I^*V} - \tfrac{V^*I}{VI^*}\right] \\
&+ e^{-\mu\tau_V}\alpha(\eta)S^*V^*\left[2 - \tfrac{S^*}{S} - \tfrac{SV}{S^*V^*} + \tfrac{S_{\tau_V}V_{\tau_V}}{S^*V^*} - \tfrac{S_{\tau_V}V_{\tau_V}}{S^*V}\right]\Big\} d\eta \\
&+ 2K_I \int_\Omega I^* \Big\{ \left(1 - \tfrac{I^*}{I}\right)\int_\Omega J(\eta-\epsilon)I^*(\epsilon)d\epsilon \\
&+ \left(1 - \tfrac{I}{I^*}\right)\int_\Omega J(\eta-\epsilon)I(\iota,\epsilon)d\epsilon\Big\} d\eta \\
=& \int_\Omega \zeta_1(\eta)\Big\{ e^{-\mu\tau_V}\beta_S(\eta)S^*\left(2 - \tfrac{S^*}{S} - \tfrac{S}{S^*}\right) \\
&+ e^{-\mu\tau_V}\beta(\eta)S^*I^*\left[3 - \tfrac{S^*}{S} - \tfrac{SI}{S^*I^*} + \tfrac{S_{\tau_V}I_{\tau_V}}{S^*I^*} - \tfrac{S_{\tau_V}I_{\tau_V}V^*}{S^*I^*V} - \tfrac{V^*I}{VI^*}\right] \\
&+ e^{-\mu\tau_V}\alpha(\eta)S^*V^*\left[2 - \tfrac{S^*}{S} - \tfrac{SV}{S^*V^*} - \tfrac{S_{\tau_V}V_{\tau_V}}{S^*V^*} - \tfrac{S_{\tau_V}V_{\tau_V}}{S^*V}\right]\Big\} d\eta \\
&+ K_I \int_\Omega\int_\Omega J(\eta-\epsilon)I^*(\epsilon)I^*(\eta)\Big\{2 - \tfrac{I^*(\eta)}{I(\iota,\epsilon)}\tfrac{I(\iota,\epsilon)}{I^*(\epsilon)} - \tfrac{I^*(\epsilon)}{I(\iota,\epsilon)}\tfrac{I(\iota,\epsilon)}{I^*(\eta)}\Big\} d\eta d\epsilon.
\end{aligned}$$

Alternatively, $\varrho_{1\tau_V}'$ is computed as follows:

$$\begin{aligned}
\varrho_{1\tau_V}' =& \tfrac{\partial}{\partial\iota}\int_\Omega e^{-\mu\tau_V}\zeta_1(\eta)\int_0^{\tau_V} \beta(\eta)S^*I^*H\left(\tfrac{S_\vartheta I_\vartheta}{S^*I^*}\right) + \alpha(\eta)S^*V^*H\left(\tfrac{S_\vartheta V_\vartheta}{S^*V^*}\right) d\vartheta d\eta \\
=& \int_\Omega e^{-\mu\tau_V}\zeta_1(\eta)\Big\{\beta(\eta)S^*I^*\left[H\left(\tfrac{SI}{S^*I^*}\right) - H\left(\tfrac{S_{\tau_V}I_{\tau_V}}{S^*I^*}\right)\right] \\
&+ \alpha(\eta)S^*V^*\left[H\left(\tfrac{SV}{S^*V^*}\right) - H\left(\tfrac{S_{\tau_V}V_{\tau_V}}{S^*V^*}\right)\right]\Big\} d\eta.
\end{aligned}$$

Adding $\varrho_1'(\iota)$ and $\varrho_{1\tau_V}'$, we find

$$\begin{aligned}
E_*'(\iota) =& \int_\Omega \zeta_1(\eta)\Big\{ e^{-\mu\tau_V}\beta_S(\eta)S^*\left(2 - \tfrac{S^*}{S} - \tfrac{S}{S^*}\right) \\
&+ e^{-\mu\tau_V}\beta(\eta)S^*I^*\left[3 - \tfrac{S^*}{S} - \tfrac{SI}{S^*I^*} - \tfrac{S_{\tau_V}I_{\tau_V}}{S^*I^*} - \tfrac{S_{\tau_V}I_{\tau_V}V^*}{S^*I^*V} - \tfrac{V^*I}{VI^*}\right] \\
&+ e^{-\mu\tau_V}\alpha(\eta)S^*V^*\left[2 - \tfrac{S^*}{S} - \tfrac{SV}{S^*V^*} - \tfrac{S_{\tau_V}V_{\tau_V}}{S^*V^*} - \tfrac{S_{\tau_V}V_{\tau_V}}{S^*V}\right]\Big\} d\eta
\end{aligned}$$

$$
\begin{aligned}
&+ K_I \int_\Omega \int_\Omega J(\eta-\epsilon) I^*(\epsilon) I^*(\eta) \left\{2 - \tfrac{I^*(\eta)}{I(\iota,\epsilon)} \tfrac{I(\iota,\epsilon)}{I^*(\epsilon)} - \tfrac{I^*(\epsilon)}{I(\iota,\epsilon)} \tfrac{I(\iota,\epsilon)}{I^*(\eta)}\right\} d\eta d\epsilon \\
&+ \int_\Omega e^{-\mu\tau_V} \zeta_1(\eta) \Big\{ \beta(\eta) S^* I^* \left[H\left(\tfrac{SI}{S^*I^*}\right) - H\left(\tfrac{S_{\tau_V} I_{\tau_V}}{S^*I^*}\right)\right] \\
&+ \alpha(\eta) S^* V^* \left[H\left(\tfrac{SV}{S^*V^*}\right) - H\left(\tfrac{S_{\tau_V} V_{\tau_V}}{S^*V^*}\right)\right] \Big\} d\eta \\
= & \int_\Omega e^{-\mu\tau_V} \zeta_1(\eta) \Big\{ \beta_S(\eta) S^* \left(2 - \tfrac{S^*}{S} - \tfrac{S}{S^*}\right) \\
&+ \beta(\eta) S^* I^* \left[-H\left(\tfrac{S^*}{S}\right) - H\left(\tfrac{S_{\tau_V} I_{\tau_V} V^*}{S^* I^* V}\right) - H\left(\tfrac{I^* V}{I V^*}\right)\right] \\
&+ \alpha(\eta) S^* V^* \left[-H\left(\tfrac{S^*}{S}\right) - H\left(\tfrac{S_{\tau_V} V_{\tau_V}}{S^* V}\right)\right] \Big\} d\eta \\
&+ K_I \int_\Omega \int_\Omega J(\eta-\epsilon) I^*(\epsilon) I^*(\eta) \left\{2 - \tfrac{I^*(\eta)}{I(\iota,\epsilon)} \tfrac{I(\iota,\epsilon)}{I^*(\epsilon)} - \tfrac{I^*(\epsilon)}{I(\iota,\epsilon)} \tfrac{I(\iota,\epsilon)}{I^*(\eta)}\right\} d\eta d\epsilon.
\end{aligned}
$$

Thus, $E'_*(\iota) \leq 0$, and equality holds when $S = S^*$, $V = V^*$, $I = I^*$. As a result, the immunity-free infected steady state H_1 is globally asymptotically stable. □

8. GRAPHICAL REPRESENTATION

Here, we validate our theoretical results by numerical simulation. We set $\Omega = (-1, 1)$ and

$$
J(\eta) = \begin{cases} 2.2523 e^{\frac{1}{\eta^2-1}}, & -1 < \eta < 1, \\ 0, & \text{otherwise}. \end{cases}
$$

This ensures that

$$
\int_{\mathbb{R}} J(\eta)\, d\eta \approx 1,
$$

and that assumptions (P0) and (P1) are fulfilled.

We consider the following set of parameters:

$$
\begin{array}{llll}
\mu = 10^{-1}, & \beta_S(\eta) = 15 \cdot 10^{-1}, & \beta_V(\eta) = 12 \cdot 10^{-1}, & \beta_I(\eta) = 13 \cdot 10^{-1}, \\
\lambda_1(\eta) = 29 \cdot 10^{-1}, & \tau_V = 98 \cdot 10^{-1}, & \beta(\eta) = \bar{\beta}, & \alpha(\eta) = 1, \\
K_I = 5 \cdot 10^{-2}, & \zeta = 98 \cdot 10^{-1}. & &
\end{array}
$$

$$
\begin{aligned}
S_0(\iota, \eta) &= 3 \cdot 10^{-1} + 5 \cdot 10^{-2} \cos(\eta), \\
V_0(\iota, \eta) &= 41 \cdot 10^{-2} + 10^{-2} \cos(\eta) + 10^{-3} \cos(\iota), \\
I_0(\iota, \eta) &= 71 \cdot 10^{-2} + 10^{-1} \cos(\eta) + 10^{-2} \cos(3\iota).
\end{aligned}
$$

First, we take $\bar{\beta} = 47 \cdot 10^{-2}$. Then $\mathcal{R}_0 = 0.7614 < 1$, and the infection-free steady state H_0 is globally asymptotically stable; see Figure 8.2.

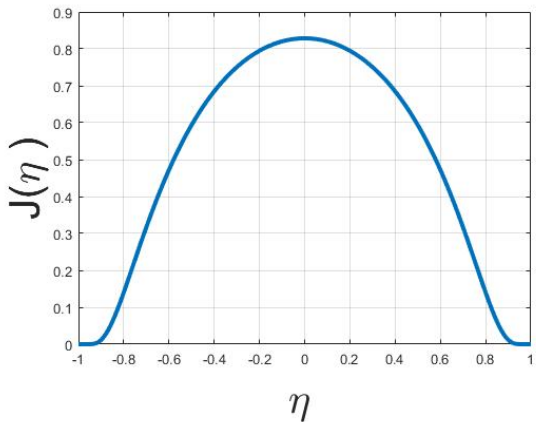

Fig. 8.1. Kernel function J.

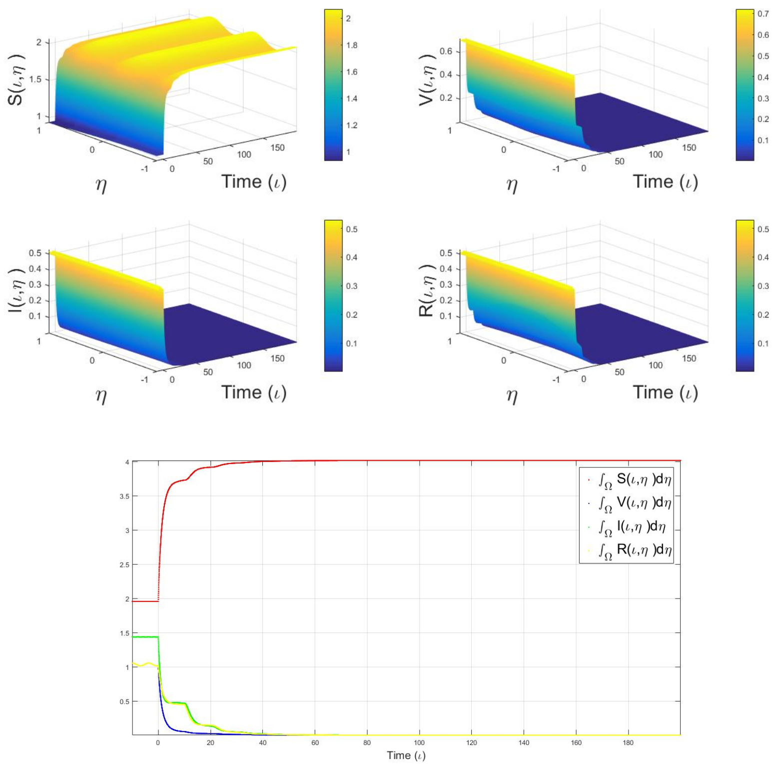

Fig. 8.2. When $\mathcal{R}_0 = 0.7614 < 1$, the infection-free steady state H_0 is globally asymptotically stable.

Next, the immunity-free infected steady state H_1 is globally asymptotically stable for $\bar{\beta} = 23 \cdot 10^{-2}$ and $\beta_R(\eta) = 187 \cdot 10^{-1}$, where $\mathcal{R}_0 = 3.5874 > 1$ and $\mathcal{R}_1 = 0.3956 < 1$; see Figure 8.3.

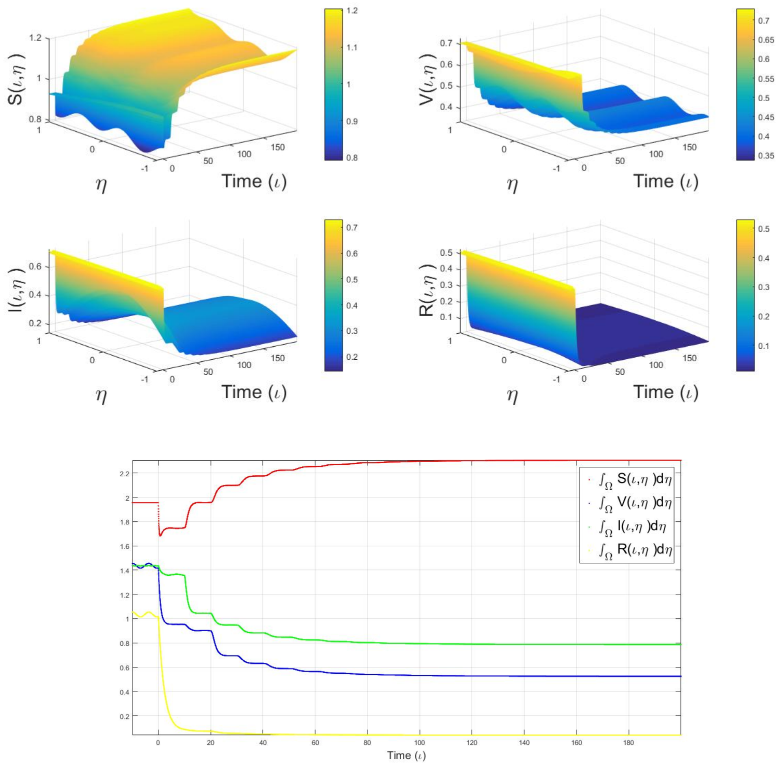

Fig. 8.3. When $\mathcal{R}_0 = 3.5874 > 1$ and $\mathcal{R}_1 = 0.3956 < 1$, the immunity-free infected steady state H_1 is globally asymptotically stable.

Finally, for $\beta_R(\eta) = 53 \cdot 10^{-1}$ and $\mathcal{R}_1 = 2.1894 > 1$, the infected-immune steady state H_2 is globally asymptotically stable; see Figure 8.4.

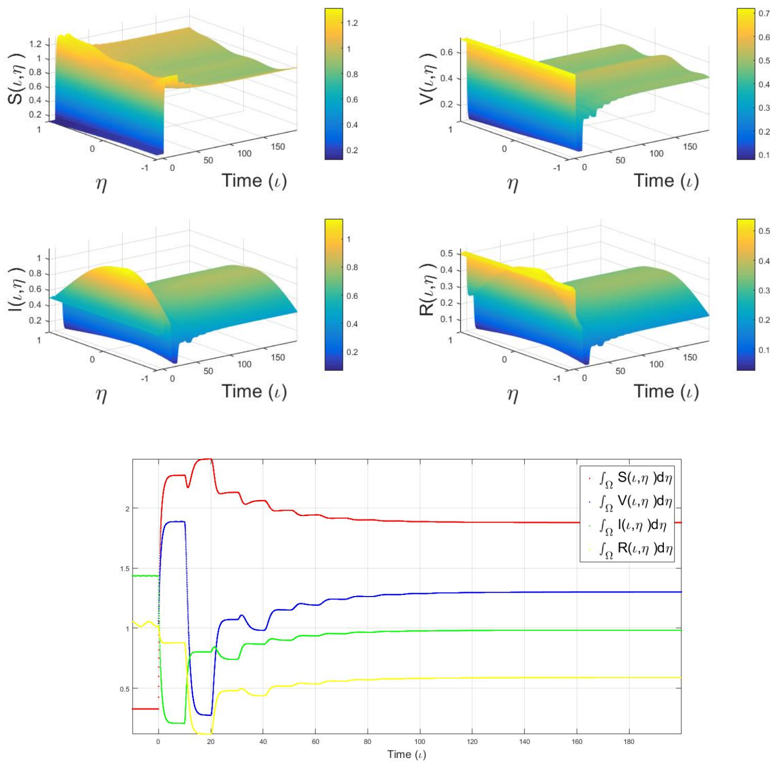

Fig. 8.4. When $\mathcal{R}_1 = 2.1894 > 1$, the infected-immune steady state H_2 is globally asymptotically stable.

9. CONCLUSIONS

This paper presents and examines a model of nonlocal dispersal viral infection that includes humoral immunity, intracellular delay, and transmissions from virus to cell as well as cell to cell.

We have proved that the global dynamics of system (2)–(3) is determined by the basic reproduction number of infection $\Re_0$ and the basic reproduction number of immunity $\Re_1$. By analyzing the characteristic equations and constructing Lyapunov functionals, we have obtained the following conclusions: if $\Re_0 < 1$, then the infection-free steady state H_0 is globally asymptotically stable; if $\Re_1 \leqslant 1 < \Re_0$, then the immunity-free infected steady state H_1 is globally asymptotically stable. We note that constructing a Lyapunov functional for a delayed nonlocal diffusive problem is not straightforward.

Additionally, using a set of assertions that confirm every assumption of Theorem 3 in [18], we demonstrated the system's uniform persistence. As a result, the semiflow $\Xi(\iota)$ is uniformly persistent, and the system admits at least one positive steady state. Finally, we supported our findings using graphical representations.

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