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A NEW ANALYTICAL ENVELOPE FOR MULTIVARIATE GLOBAL OPTIMIZATION

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Abstract. We propose a new global optimization method that combines an α–dense univariate reduction with explicitly constructed analytical envelopes: a piecewise concave underestimator (PCU) and a piecewise convex overestimator (PCO). By leveraging interval-based curvature bounds, the method provides rigorous global optimality certificates. An adaptive branch-and-bound strategy ensures rapid convergence by refining intervals based on theoretical envelope widths. Numerical experiments on challenging nonconvex and multimodal benchmarks demonstrate strong performance and efficiency.

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1. INTRODUCTION

Deterministic global optimization remains a central topic in numerical analysis and applied mathematics, with a wide range of applications across science and engineering. Foundational monographs such as Floudas [6] and Locatelli–Schoen [10] provide comprehensive overviews of key approaches, including spatial branch-and-bound, interval methods, Lipschitz-based schemes, and convex or concave relaxations. Despite these advances, obtaining *tight* global bounds for nonconvex problems in moderate to high dimensions remains a major challenge in the field.

A particularly influential class of relaxations is based on convex underestimation. The αBB method developed by Adjiman, Androulakis, and Floudas [3, 4] constructs a convex quadratic lower bound by augmenting the Hessian with diagonal shifts. This paradigm has since been extended through more refined DC relaxations, including recent developments by Strahl, Raghunathan, and Sahinidis [15]. Other underestimation techniques include the DCU method for univariate optimization introduced by Chang, Park, and Lee [5], as well as the convex quadratic relaxations of Le Thi and Ouanes [9]. A broader survey of convex underestimation methods is available in Skjäl [14].

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Interval-based global optimization techniques also play a crucial role, particularly for their ability to provide rigorous bounds. Notable examples include the classical one-dimensional algorithm by Sergeyev [12], geometric Lipschitz-based methods by Kvasov and Sergeyev [8], and the homogeneity framework proposed by Sergeyev, Kvasov, and Mukhametzhanov [13]. More recent strategies include the Lipschitz global optimization methods of Malherbe and Vayatis [11] and the constrained Lipschitz-gradient approach of Vinod, Israel, and Topcu [16]. For a comprehensive overview, see the tutorial by Horst [17].

Another direction of research focuses on reducing multivariate problems to univariate ones. Meta-algorithmic frameworks following this idea were recently studied by Gökçesu and Gökçesu [7]. Earlier work by Aaid and collaborators developed constructive transformations for dimensionality reduction, enabling global optimization in the univariate setting [1, 2]. These transformations generate an α–dense curve through the domain, aiming to ensure that the global minimizer of the original multivariate function is well approximated by that of the corresponding one-dimensional surrogate.

Building on this reduction approach, the present work introduces a new analytical framework for univariate global optimization. Specifically, we construct a *piecewise concave underestimator* (PCU) and a *piecewise convex overestimator* (PCO), both derived explicitly from interval bounds on the second derivative of the reduced function. These constructions yield a pair of rigorous analytical envelopes satisfying

$$\mathrm{PCU}(t) \leq F(t) \leq \mathrm{PCO}(t), \quad \text{for all } t,$$

with computable curvature parameters. When combined with an adaptive branch-and-bound scheme that refines the interval exhibiting the largest theoretical envelope gap, the method produces rapidly shrinking global bounds and provable optimality gaps.

The main contributions of this article are as follows:

- We derive explicit PCU and PCO envelopes from interval curvature bounds, providing rigorous concave and convex relaxations.
- We integrate these envelopes with the α–dense dimension-reduction transformation of Aaid [1, 2], resulting in a fully certified univariate representation of multivariate problems.
- We introduce an adaptive refinement strategy that targets the interval with the largest theoretical envelope width, accelerating convergence.
- Our numerical experiments show that the method performs well on various challenging benchmark problems, including those with moderate to high dimensions, especially when the reduction and subdivision settings are chosen carefully.

It is important to note that the efficiency of the proposed framework depends on how the reduction parameters are chosen especially the density parameter

α and the frequency sequence. Also, the univariate reduction does not remove the inherent complexity of the original multivariate problem.

The remainder of the paper is organized as follows. Section 2 presents the dimension reduction framework. Section 3 introduces analytical envelopes. Section 4 describes the adaptive branch-and-bound algorithm. Section 5 provides numerical experiments and comparisons with existing global optimization methods. Section 6 concludes with directions for future research.

2. PRELIMINARIES

We consider the global optimization problem

$$\min_{x\in X} f(x), \qquad X := \prod_{i=1}^{n} [a_i, b_i] \subset \mathbb{R}^n,$$

where the objective function $f : X \to \mathbb{R}$ is assumed to be twice continuously differentiable.

2.1. Reductive Transformation via α–Dense Curves. Following the framework introduced in [1, 2], we consider a one-dimensional parametric curve $x : [0, \theta_{\max}] \to X$ defined by

$$x(\theta) = (x_1(\theta), \dots, x_n(\theta)), \qquad x_i(\theta) = \tfrac{1}{2}\big[(b_i - a_i)\cos(\omega_i\theta + \varphi_i) + (b_i + a_i)\big],$$

where $(\omega_i)_{i=1}^n$ and $(\varphi_i)_{i=1}^n$ are frequency and phase parameters chosen to control the geometry of the curve. Since $\cos(\omega_i\theta + \varphi_i) \in [-1, 1]$, it follows that $x(\theta) \in X$ for all $\theta \in [0, \theta_{\max}]$.

We denote by

$$\Gamma := x([0, \theta_{\max}]) \subset X$$

the image (or trace) of the curve, and define the corresponding univariate surrogate objective as

$$F(\theta) := f(x(\theta)), \qquad \theta \in [0, \theta_{\max}]. \tag{1}$$

Clearly, if $\theta^\star$ is a (global) minimizer of F, then $x(\theta^\star)$ is a candidate global minimizer of f over X.

2.2. α-Dense Curves and Reduction Error.

Definition 1. *A set $S \subset X$ is said to be* α–dense *in X if for every $z \in X$, there exists $y \in S$ such that*

$$\|z - y\| \le \alpha.$$

In particular, we say that the curve Γ is α-dense in X if it satisfies this property.

The key idea behind this construction is that, for appropriate choices of ω_i, φ_i, and $\theta_{\max}$, the curve Γ can be made α-dense in X. As a result, minimizing the reduced function F over the interval $[0, \theta_{\max}]$ gives a good approximation of the original optimization problem over X, with the accuracy directly influenced by the selected density parameter α.

THEOREM 2 (Reduction Error Bound). *Assume that* Γ *is* α*-dense in* X*, and that* f *is Lipschitz continuous on* X *with Lipschitz constant* $L_f > 0$*. Define*

$$m_X := \min_{x \in X} f(x), \qquad m_\Gamma := \min_{\theta \in [0, \theta_{\max}]} F(\theta).$$

Then the reduction error satisfies the bound

$$|m_X - m_\Gamma| \leq L_f \, \alpha. \tag{2}$$

The accuracy of the reduced formulation is directly influenced by the density parameter α. Smaller values of α improve the approximation of the original problem but may increase the complexity of the reduced one, due to a larger required range for θ and increased oscillations of the curve Γ. In particular, ensuring α-density may demand a large $\theta_{\max}$, depending on the dimension and frequency sequence.

Thus, the univariate reduction shifts rather than removes the original problem's complexity. The method's efficiency depends on balancing approximation accuracy and computational cost. We now focus on deriving tight bounds for F on $[0, \theta_{\max}]$ and integrating them into a branch-and-bound framework. Choice of the density parameter α. The parameter α controls the approximation quality of the reduction. By Theorem 2, the error is bounded by $L_f \alpha$. If L_f is unknown, it can be estimated via interval gradients or α can be adjusted empirically.

Smaller α improves accuracy but increases computational cost due to larger $\theta_{\max}$ and curvature. A balance must be struck between precision and efficiency.

3. MATERIALS AND METHODS

3.1. Local Construction of a PCU Bound on an Interval. Let $F : [\alpha, \beta] \to \mathbb{R}$ be twice continuously differentiable. Consider a partition

$$\alpha = t_0 < t_1 < \ldots < t_M = \beta,$$

and fix an interval

$$I_i := [t_i, t_{i+1}], \qquad h_i := t_{i+1} - t_i > 0.$$

Define the linear shape functions on I_i:

$$\ell_{i,0}(t) := \tfrac{t_{i+1} - t}{h_i}, \qquad \ell_{i,1}(t) := \tfrac{t - t_i}{h_i}, \qquad t \in I_i,$$

and the linear interpolant

$$L_i F(t) := \ell_{i,0}(t) \, F(t_i) + \ell_{i,1}(t) \, F(t_{i+1}), \qquad t \in I_i.$$

Assume that we know a *lower* curvature bound on I_i, namely

$$F''(t) \geq m_i, \qquad \forall t \in I_i,$$

for some real constant $m_i \leq 0$ (possibly negative). We define the local PCU (Piecewise Concave Underestimator) on I_i by

$$\mathrm{PCU}_i(t) := L_i F(t) + \tfrac{m_i}{2} \, (t - t_i)(t_{i+1} - t), \qquad t \in I_i. \tag{3}$$

THEOREM 3 (Local Concave PCU Bound). *Assume* $F \in C^2([\alpha, \beta])$ *and*

$$F''(t) \geq m_i, \qquad \forall t \in I_i,$$

with $m_i \leq 0$. *Then, for every* $t \in I_i$,

1) $\mathrm{PCU}_i(t) \leq F(t)$ *(valid lower bound),*
2) PCU_i *is concave on* I_i, *i.e.,*

$$\mathrm{PCU}_i''(t) \leq 0, \qquad \forall t \in I_i.$$

Proof. **1) Valid lower bound.** Let

$$E_i(t) := F(t) - L_i F(t).$$

By the standard interpolation error formula, for each $t \in I_i$ there exists $\xi(t) \in I_i$ such that

$$E_i(t) = \tfrac{F''(\xi(t))}{2}\,(t - t_i)(t_{i+1} - t).$$

Since $(t - t_i)(t_{i+1} - t) \geq 0$ on I_i and $F''(\xi(t)) \geq m_i$, we obtain

$$E_i(t) = \tfrac{F''(\xi(t))}{2}\,(t - t_i)(t_{i+1} - t) \geq \tfrac{m_i}{2}\,(t - t_i)(t_{i+1} - t).$$

Thus

$$F(t) - L_i F(t) \geq \tfrac{m_i}{2}\,(t - t_i)(t_{i+1} - t),$$

which can be rewritten as

$$L_i F(t) + \tfrac{m_i}{2}\,(t - t_i)(t_{i+1} - t) \leq F(t).$$

By definition from (3), the left-hand side is $\mathrm{PCU}_i(t)$, hence

$$\mathrm{PCU}_i(t) \leq F(t), \qquad \forall t \in I_i.$$

2) Concavity. Since $L_i F$ is affine on I_i, $(L_i F)''(t) = 0$. We have

$$(t - t_i)(t_{i+1} - t) = -(t^2) + (t_i + t_{i+1})t - t_i t_{i+1},$$

so

$$\tfrac{\mathrm{d}^2}{\mathrm{d}t^2}(t - t_i)(t_{i+1} - t) = -2.$$

Therefore

$$\tfrac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\tfrac{m_i}{2}(t - t_i)(t_{i+1} - t)\right] = \tfrac{m_i}{2} \cdot (-2) = m_i.$$

Hence

$$\mathrm{PCU}_i''(t) = (L_i F)''(t) + m_i = 0 + m_i = m_i \leq 0,$$

which proves that PCU_i is concave on I_i. □

3.2. Global Piecewise Concave Underestimator. Let

$$\alpha = t_0 < t_1 < \ldots < t_M = \beta$$

be a partition of $[\alpha, \beta]$. On each interval $I_i = [t_i, t_{i+1}]$, the local PCU_i is defined as in Theorem 3:

$$\mathrm{PCU}_i(t) = L_i F(t) + \tfrac{m_i}{2}\,(t - t_i)(t_{i+1} - t), \qquad t \in I_i,$$

where $m_i \leq 0$ satisfies

$$F''(t) \geq m_i, \qquad \forall t \in I_i.$$

We define the global PCU on $[\alpha, \beta]$ by

$$\mathrm{PCU}(t) := \mathrm{PCU}_i(t), \qquad t \in [t_i, t_{i+1}],\ i = 0, \ldots, M-1.$$

Theorem 4 (Global PCU Properties). *Under the above assumptions, the global function* PCU *satisfies:*

a) ***(Global lower bound)***

$$\mathrm{PCU}(t) \leq F(t), \qquad \forall t \in [\alpha, \beta].$$

b) ***(Interpolation at the nodes)***

$$\mathrm{PCU}(t_i) = F(t_i), \qquad i = 0, \ldots, M.$$

c) ***(Continuity and piecewise concavity)*** PCU *is continuous on* $[\alpha, \beta]$ *and* $\mathrm{PCU}_{|I_i}$ *is concave on each interval* $I_i = [t_i, t_{i+1}]$.

Proof. a) For $t \in I_i$, Theorem 3 yields

$$\mathrm{PCU}_i(t) \leq F(t).$$

By the definition of the global PCU, $\mathrm{PCU}(t) = \mathrm{PCU}_i(t)$ for $t \in I_i$, hence

$$\mathrm{PCU}(t) \leq F(t), \qquad \forall t \in [\alpha, \beta].$$

b) At the endpoints of I_i we have

$$\mathrm{PCU}_i(t_i) = L_i F(t_i) + \tfrac{m_i}{2}(t_i - t_i)(t_{i+1} - t_i) = F(t_i),$$

and

$$\mathrm{PCU}_i(t_{i+1}) = L_i F(t_{i+1}) + \tfrac{m_i}{2}(t_{i+1} - t_i)(t_{i+1} - t_{i+1}) = F(t_{i+1}),$$

because the quadratic term vanishes at $t = t_i$ and $t = t_{i+1}$ and $L_i F$ interpolates F at the nodes. In particular, for $1 \leq k \leq M-1$,

$$\mathrm{PCU}_{k-1}(t_k) = F(t_k) = \mathrm{PCU}_k(t_k),$$

so the left and right pieces coincide at every interior node.

c) From (2), the left and right limits of PCU at each node t_i coincide and are equal to $F(t_i)$, so PCU is continuous on $[\alpha, \beta]$. On each subinterval I_i, we have $\mathrm{PCU}(t) = \mathrm{PCU}_i(t)$ and Theorem 3 gives

$$\mathrm{PCU}_i''(t) = m_i \leq 0,$$

hence PCU is concave on each I_i.

□

3.3. Local Construction of a PCO Bound on an Interval. We keep the same setting as in the local PCU construction. Let $F : [\alpha, \beta] \to \mathbb{R}$ be twice continuously differentiable, and consider an interval

$$I_i := [t_i, t_{i+1}], \qquad h_i := t_{i+1} - t_i > 0.$$

The linear interpolant of F on I_i is

$$L_iF(t) := \ell_{i,0}(t)\, F(t_i) + \ell_{i,1}(t)\, F(t_{i+1}), \qquad t \in I_i,$$

where

$$\ell_{i,0}(t) := \tfrac{t_{i+1}-t}{h_i}, \qquad \ell_{i,1}(t) := \tfrac{t-t_i}{h_i}.$$

Assume that there exists $K_i > 0$ such that

$$|F''(t)| \le K_i, \qquad \forall t \in I_i.$$

We define the local PCO (Piecewise Convex Overestimator) on I_i by

$$\mathrm{PCO}_i(t) := L_iF(t) + \delta_i, \qquad t \in I_i,$$

where

$$\delta_i := \tfrac{K_i}{8}\, h_i^2.$$

THEOREM 5 (Local PCO Bound). *Assume $F \in C^2([\alpha, \beta])$ and $|F''(t)| \le K_i$ on I_i. Then, for every $t \in I_i$,*

a) $\mathrm{PCO}_i(t) \ge F(t)$ *(valid upper bound),*
b) PCO_i *is convex on* I_i.

Proof. Let

$$E_i(t) := F(t) - L_iF(t).$$

As in the proof of Theorem 3, there exists $\xi(t) \in I_i$ such that

$$E_i(t) = \tfrac{F''(\xi(t))}{2}\,(t - t_i)(t_{i+1} - t).$$

Hence

$$|E_i(t)| \le \tfrac{|F''(\xi(t))|}{2}\,(t - t_i)(t_{i+1} - t) \le \tfrac{K_i}{2}\,(t - t_i)(t_{i+1} - t).$$

The quadratic term $(t - t_i)(t_{i+1} - t)$ attains its maximum at the midpoint $t = (t_i + t_{i+1})/2$, with value

$$\max_{t \in I_i} (t - t_i)(t_{i+1} - t) = \tfrac{h_i^2}{4}.$$

Therefore

$$|E_i(t)| \le \tfrac{K_i}{2} \cdot \tfrac{h_i^2}{4} = \tfrac{K_i}{8} h_i^2 = \delta_i, \qquad \forall t \in I_i.$$

In particular,

$$F(t) \le L_iF(t) + \delta_i = \mathrm{PCO}_i(t), \qquad \forall t \in I_i,$$

which proves that PCO_i is a valid upper bound on I_i.

For convexity, note that L_iF is affine on I_i and δ_i is a constant shift. Thus PCO_i is affine on I_i, hence convex. □

3.4. Global Piecewise Convex Overestimator. Let

$$\alpha = t_0 < t_1 < \cdots < t_M = \beta$$

be a partition of $[\alpha, \beta]$. On each interval $I_i = [t_i, t_{i+1}]$, the local PCO_i is defined as in Theorem 5:

$$\mathrm{PCO}_i(t) = L_i F(t) + \delta_i, \qquad \delta_i = \tfrac{K_i}{8} h_i^2, \quad h_i = t_{i+1} - t_i.$$

We define the global PCO on $[\alpha, \beta]$ by

$$\mathrm{PCO}(t) := \mathrm{PCO}_i(t), \qquad t \in [t_i, t_{i+1}],\ i = 0, \ldots, M-1.$$

THEOREM 6 (Global PCO Properties). *Assume $F \in C^2([\alpha, \beta])$ and, for each i, there exists $K_i > 0$ such that*

$$|F''(t)| \le K_i, \qquad \forall t \in [t_i, t_{i+1}].$$

Then the global function PCO *satisfies:*

1) ***(Global upper bound)***

$$F(t) \le \mathrm{PCO}(t), \qquad \forall t \in [\alpha, \beta].$$

2) ***(Continuity at the nodes)*** *If the constants δ_i are chosen such that*

$$\delta_0 = 0, \qquad L_i F(t_{i+1}) + \delta_i = L_{i+1} F(t_{i+1}) + \delta_{i+1} \quad \text{for } i = 0, \ldots, M-2,$$

then

$$\mathrm{PCO}(t_i) = F(t_i) + \delta_i, \qquad i = 0, \ldots, M,$$

and PCO *is continuous on $[\alpha, \beta]$.*

3) ***(Piecewise convexity)*** *On each subinterval I_i, the restriction $\mathrm{PCO}_{|I_i}$ is convex.*

Proof. 1) On each I_i, Theorem 5 yields

$$F(t) \le \mathrm{PCO}_i(t), \qquad \forall t \in I_i.$$

By definition of the global PCO, we have $\mathrm{PCO}(t) = \mathrm{PCO}_i(t)$ for $t \in I_i$, hence

$$F(t) \le \mathrm{PCO}(t), \qquad \forall t \in [\alpha, \beta].$$

2) At each node t_i, we have

$$\mathrm{PCO}_i(t_i) = L_i F(t_i) + \delta_i = F(t_i) + \delta_i,$$

and similarly

$$\mathrm{PCO}_{i-1}(t_i) = F(t_i) + \delta_{i-1},$$

with the convention that $i - 1$ is only valid for $i \ge 1$. If the constants δ_i are chosen so that

$$L_i F(t_i) + \delta_i = L_{i-1} F(t_i) + \delta_{i-1},$$

then the left and right limits of PCO coincide at each node t_i, making PCO continuous on $[\alpha, \beta]$. In particular, if we impose $\delta_0 = 0$ and the recursion

$$L_i F(t_i) + \delta_i = L_{i-1} F(t_i) + \delta_{i-1},$$

we obtain $\mathrm{PCO}(t_0) = F(t_0)$ and

$$\mathrm{PCO}(t_i) = F(t_i) + \delta_i, \qquad i = 1, \dots, M.$$

3) For each i, the function PCO_i is affine on I_i (as L_iF is affine and δ_i is constant), hence convex. Thus on each I_i, the restriction $\mathrm{PCO}_{|I_i}$ is convex, which proves that PCO is piecewise convex on $[\alpha, \beta]$.

□

3.5. Bilateral PCU–PCO Envelope and Gap Estimate. Assume $F \in C^2([\alpha, \beta])$ and a partition

$$\alpha = t_0 < t_1 < \ldots < t_M = \beta, \qquad I_i := [t_i, t_{i+1}], \quad h_i := t_{i+1} - t_i.$$

On each I_i, suppose that

$$m_i \le F''(t) \le K_i, \qquad \forall t \in I_i,$$

with $m_i \le 0$ and $K_i \ge 0$.

The local PCU and PCO are

$$\mathrm{PCU}_i(t) := L_iF(t) + \tfrac{m_i}{2}\,(t - t_i)(t_{i+1} - t), \qquad t \in I_i,$$

$$\mathrm{PCO}_i(t) := L_iF(t) + \delta_i, \qquad \delta_i := \tfrac{K_i}{8} h_i^2, \quad t \in I_i,$$

and the global PCU, PCO are defined by

$$\mathrm{PCU}(t) := \mathrm{PCU}_i(t), \quad \mathrm{PCO}(t) := \mathrm{PCO}_i(t), \qquad t \in I_i,\ i = 0, \dots, M-1.$$

THEOREM 7 (Bilateral Envelope and Local Gap). *Under the above assumptions, one has:*

1) *For all $t \in [\alpha, \beta]$,*

$$\mathrm{PCU}(t) \ \le\ F(t) \ \le\ \mathrm{PCO}(t).$$

2) *For every $i = 0, \dots, M-1$ and all $t \in I_i$,*

$$0 \ \le\ \mathrm{PCO}(t) - \mathrm{PCU}(t) \ \le\ \tfrac{K_i - m_i}{8}\, h_i^2.$$

Proof. Fix i and $t \in I_i$.

1) Bilateral bounds. Define the interpolation error

$$E_i(t) := F(t) - L_iF(t).$$

There exists $\xi(t) \in I_i$ such that

$$E_i(t) = \tfrac{F''(\xi(t))}{2}\,(t - t_i)(t_{i+1} - t).$$

Since $t \in I_i$,

$$(t - t_i)(t_{i+1} - t) \ge 0.$$

From $F''(\xi(t)) \ge m_i$,

$$E_i(t) = \tfrac{F''(\xi(t))}{2}\,(t - t_i)(t_{i+1} - t) \ \ge\ \tfrac{m_i}{2}\,(t - t_i)(t_{i+1} - t),$$

so

$$F(t) - L_iF(t) \ge \tfrac{m_i}{2}\,(t - t_i)(t_{i+1} - t),$$

i.e.,

$$L_iF(t) + \tfrac{m_i}{2}(t-t_i)(t_{i+1}-t) \le F(t).$$

By definition of PCU_i,

$$\mathrm{PCU}_i(t) \le F(t).$$

From $F''(\xi(t)) \le K_i$,

$$E_i(t) = \tfrac{F''(\xi(t))}{2}(t-t_i)(t_{i+1}-t) \ \le \ \tfrac{K_i}{2}(t-t_i)(t_{i+1}-t).$$

Moreover,

$$0 \le (t-t_i)(t_{i+1}-t) \le \tfrac{h_i^2}{4},$$

so

$$E_i(t) \le \tfrac{K_i}{2}\cdot\tfrac{h_i^2}{4} = \tfrac{K_i}{8}h_i^2 = \delta_i.$$

Thus

$$F(t) - L_iF(t) \le \delta_i,$$

i.e.,

$$F(t) \le L_iF(t) + \delta_i = \mathrm{PCO}_i(t).$$

Therefore, on I_i,

$$\mathrm{PCU}_i(t) \le F(t) \le \mathrm{PCO}_i(t).$$

By the global definition of PCU and PCO on $[\alpha,\beta]$, this yields

$$\mathrm{PCU}(t) \le F(t) \le \mathrm{PCO}(t), \qquad \forall t \in [\alpha,\beta].$$

2) Local gap estimate. For $t \in I_i$, we have

$$\begin{aligned}\mathrm{PCO}_i(t) - \mathrm{PCU}_i(t) &= (L_iF(t)+\delta_i) - \Big(L_iF(t) + \tfrac{m_i}{2}(t-t_i)(t_{i+1}-t)\Big)\\ &= \delta_i - \tfrac{m_i}{2}(t-t_i)(t_{i+1}-t).\end{aligned}$$

Since $(t-t_i)(t_{i+1}-t) \ge 0$ and $m_i \le 0$, it follows that

$$-\tfrac{m_i}{2}(t-t_i)(t_{i+1}-t) \ge 0,$$

and therefore

$$\mathrm{PCO}_i(t) - \mathrm{PCU}_i(t) \ge \delta_i \ge 0.$$

Moreover,

$$(t-t_i)(t_{i+1}-t) \le \tfrac{h_i^2}{4},$$

which implies that

$$\mathrm{PCO}_i(t) - \mathrm{PCU}_i(t) \le \delta_i - \tfrac{m_i}{2}\cdot 0,$$

or, more precisely,

$$\mathrm{PCO}_i(t) - \mathrm{PCU}_i(t) \le \delta_i - \tfrac{m_i}{2}\cdot\tfrac{h_i^2}{4}.$$

Using $\delta_i = \tfrac{K_i}{8}h_i^2$ and the identity $-m_i = |m_i|$, we obtain

$$\mathrm{PCO}_i(t) - \mathrm{PCU}_i(t) \le \tfrac{K_i}{8}h_i^2 - \tfrac{m_i}{8}h_i^2 = \tfrac{K_i-m_i}{8}h_i^2.$$

Thus, for every $t \in I_i$,

$$0 \le \mathrm{PCO}_i(t) - \mathrm{PCU}_i(t) \le \tfrac{K_i-m_i}{8}h_i^2.$$

By the global definition,

$$0 \le \mathrm{PCO}(t) - \mathrm{PCU}(t) \le \tfrac{K_i - m_i}{8} h_i^2, \qquad t \in I_i.$$

□

4. GLOBAL BRANCH-AND-BOUND SCHEME IN THE REDUCED SPACE

In this section, we present a deterministic branch-and-bound scheme tailored to the reduced one-dimensional optimization problem

$$\min_{\theta \in [0, \theta_{\max}]} F(\theta),$$

where $F(\theta) = f(x(\theta))$ and $x(\theta)$ is the α–dense parametric transformation introduced in the previous section. The bounds used in the algorithm are given by the piecewise concave underestimator (PCU) and the piecewise convex overestimator (PCO), both analytically constructed based on interval curvature bounds.

4.1. Interval Partition and Bounds. Let $\mathcal{L}_k$ be the list of active intervals at iteration k. Each element $I \in \mathcal{L}_k$ is a closed interval

$$I = [a, b] \subset [0, \theta_{\max}].$$

On each I, we define:

$$\underline{F}(I) := \min_{t \in I} \mathrm{PCU}(t), \ \ \overline{F}(I) := \min_{t \in I} \mathrm{PCO}(t).$$

Since PCU is concave on each base subinterval and PCO is convex (piecewise affine), these local problems reduce to endpoint evaluations on each primitive subinterval. More precisely, if I is contained in a single mesh interval $[t_i, t_{i+1}]$ used in the construction of PCU and PCO, then:

$$\underline{F}(I) = \min\{\mathrm{PCU}(a), \mathrm{PCU}(b)\}, \ \ \overline{F}(I) = \min_{t \in I} \mathrm{PCO}(t) = \min\{\mathrm{PCO}(a), \mathrm{PCO}(b)\},$$

since a concave (resp. convex) function attains its minimum (resp. maximum) on an interval at one of the endpoints.

In practice, when I overlaps several mesh intervals $[t_i, t_{i+1}]$, we refine I along the mesh nodes so that each branch-and-bound subinterval is a union of such primitive pieces; the above formulas then apply on each piece.

We maintain the global lower and upper bounds at iteration k:

$$\underline{F}_k := \min_{I \in \mathcal{L}_k} \underline{F}(I), \qquad \overline{F}_k := \min_{I \in \mathcal{L}_k} \overline{F}(I).$$

4.2. Branch-and-Bound Algorithm in the θ–Space. Let $\varepsilon > 0$ be a prescribed tolerance. The algorithm starts from the initial interval

$$I^{(0)} := [0, \theta_{\max}], \qquad \mathcal{L}_0 := \{I^{(0)}\}.$$

4.3. Adaptive Gap-Based Branch-and-Bound in the Reduced Space. We use the same curvature bounds m_i, K_i and the maximal theoretical gap

$$\Delta^{\max}(I) := \tfrac{K_i - m_i}{8}\, h(I)^2, \qquad I \subset [t_i, t_{i+1}],$$

where $h(I)$ is the length of I and I is always contained in a single base mesh interval $[t_i, t_{i+1}]$.

The initial interval $[0, \theta_{\max}]$ is uniformly subdivided into N subintervals

$$I_j^{(0)} := \left[\tfrac{j}{N}\theta_{\max},\ \tfrac{j+1}{N}\theta_{\max}\right], \qquad j = 0, \dots, N-1,$$

and the local PCU/PCO bounds and gaps are computed on each $I_j^{(0)}$.

Moreover, at each branching step, the selected interval is subdivided into $N_{\text{sub}} \geq 2$ equal subintervals.

4.4. Convergence of the Branch-and-Bound Scheme. We show that the above algorithm converges to the global minimum of F on $[0, \theta_{\max}]$.

Let

$$F^\star := \min_{\theta \in [0, \theta_{\max}]} F(\theta).$$

THEOREM 8 (Global Convergence in the Reduced Space). *Assume $F \in C^2([0, \theta_{\max}])$ and that, for each mesh interval $[t_i, t_{i+1}]$, there exist $m_i \leq 0 \leq K_i$ with*

$$m_i \leq F''(t) \leq K_i, \qquad \forall t \in [t_i, t_{i+1}].$$

Then the branch-and-bound Algorithm 1 generates sequences $(\underline{F}_k)$ and $(\overline{F}_k)$ such that

$$\underline{F}_k \leq F^\star \leq \overline{F}_k, \qquad \forall k,$$

and

$$\lim_{k\to\infty} \underline{F}_k = F^\star = \lim_{k\to\infty} \overline{F}_k.$$

In particular, for any $\varepsilon > 0$, the algorithm terminates in a finite number of iterations with an ε–optimal solution.

Proof. **Step 1. Lower and upper bounds.** For any interval $I \subset [0, \theta_{\max}]$, Theorem 7 yields

$$\mathrm{PCU}(t) \leq F(t) \leq \mathrm{PCO}(t), \qquad \forall t \in I.$$

Taking minimum over $t \in I$,

$$\min_{t\in I} \mathrm{PCU}(t) \leq \min_{t\in I} F(t) \leq \min_{t\in I} \mathrm{PCO}(t),$$

i.e.,

$$\underline{F}(I) \leq \inf_{t\in I} F(t) \leq \overline{F}(I).$$

Thus, for every iteration k,

$$\underline{F}_k = \min_{I\in\mathcal{L}_k} \underline{F}(I) \leq \min_{I\in\mathcal{L}_k} \inf_{t\in I} F(t) \leq \inf_{t\in[0,\theta_{\max}]} F(t) = F^\star,$$

and

$$F^\star \le \min_{I\in\mathcal{L}_k} \overline{F}(I) = \overline{F}_k.$$

Hence

$$\underline{F}_k \le F^\star \le \overline{F}_k, \qquad \forall k.$$

Step 2. Refinement and vanishing gap. At each branching step, an interval $I_k = [a_k, b_k]$ is split into two subintervals of length $h_k/2$, where $h_k := b_k - a_k$. Thus the maximal length of intervals in $\mathcal{L}_k$,

$$H_k := \max_{I\in\mathcal{L}_k} \text{length}(I),$$

satisfies

$$H_{k+1} \le \tfrac{1}{2} H_k,$$

so that

$$\lim_{k\to\infty} H_k = 0.$$

On any interval $I \subset [t_i, t_{i+1}]$ with length $|I| \le h_i$, Theorem 7 yields

$$0 \le \overline{F}(I) - \underline{F}(I) \le \tfrac{K_i - m_i}{8} |I|^2.$$

Hence, for each fixed i,

$$\lim_{|I|\to 0} \left[\overline{F}(I) - \underline{F}(I)\right] = 0.$$

Step 3. Limit of the global gap. At iteration k, the global gap satisfies

$$0 \le \overline{F}_k - \underline{F}_k = \min_{I\in\mathcal{L}_k} \overline{F}(I) - \min_{I\in\mathcal{L}_k} \underline{F}(I) \le \max_{I\in\mathcal{L}_k} \left[\overline{F}(I) - \underline{F}(I)\right].$$

Since every interval length in $\mathcal{L}_k$ is bounded by H_k and $H_k \to 0$, it follows that

$$\lim_{k\to\infty} \left(\overline{F}_k - \underline{F}_k\right) = 0.$$

Combined with $\underline{F}_k \le F^\star \le \overline{F}_k$, this implies

$$\lim_{k\to\infty} \underline{F}_k = F^\star = \lim_{k\to\infty} \overline{F}_k.$$

Step 4. Finite termination for a given $\varepsilon > 0$. Since the global gap converges to 0, there exists k_ε such that

$$\overline{F}_{k_\varepsilon} - \underline{F}_{k_\varepsilon} \le \varepsilon.$$

At that iteration, the algorithm stops by the termination condition in Algorithm 1, and any θ^ε selected from the interval attaining $\overline{F}_{k_\varepsilon}$ is ε–optimal. □

Algorithm 1: Adaptive Gap-Based Branch-and-Bound with N_{sub}-Subdivision

Input: N initial subdivisions, $N_{\text{sub}} \geq 2$ children per branch, tolerance $\varepsilon > 0$.

Result: Global minimum approximation

Initialization:

Construct $I_j^{(0)}$, $j = 0, \ldots, N-1$.

For each $I_j^{(0)}$, compute $\underline{F}(I_j^{(0)})$, $\overline{F}(I_j^{(0)})$ via PCU/PCO, and $\Delta^{\max}(I_j^{(0)})$.

Set

$$\underline{F}_0 := \min_j \underline{F}(I_j^{(0)}), \quad \overline{F}_0 := \min_j \overline{F}(I_j^{(0)}), \quad \Delta_0 := \overline{F}_0 - \underline{F}_0$$

Set $\mathcal{L}_0 := \{I_j^{(0)} : j = 0, \ldots, N-1\}$ and $k \leftarrow 0$.

while $\Delta_k > \varepsilon$ **do**

`// Selection: interval with largest theoretical gap`

Select

$$I_k := \arg\max_{I \in \mathcal{L}_k} \Delta^{\max}(I).$$

Let $I_k = [a_k, b_k]$. Remove I_k from $\mathcal{L}_k$.

`// Branch: subdivision of` I_k `into` N_{sub} `subintervals`

Set $h_k := b_k - a_k$ and define the subdivision points

$$s_{k,\ell} := a_k + \ell \, \tfrac{h_k}{N_{\text{sub}}}, \qquad \ell = 0, 1, \ldots, N_{\text{sub}}.$$

For each $\ell = 0, \ldots, N_{\text{sub}} - 1$, define the child interval

$$I_k^{(\ell)} := [s_{k,\ell},\, s_{k,\ell+1}].$$

`// Local PCU/PCO bounds and local gaps on children`

for $\ell = 0, \ldots, N_{sub} - 1$ **do**

Compute $\underline{F}(I_k^{(\ell)})$ and $\overline{F}(I_k^{(\ell)})$ using PCU/PCO restricted to $I_k^{(\ell)}$.

Identify the base mesh index i such that $I_k^{(\ell)} \subset [t_i, t_{i+1}]$ and set

$$\Delta^{\max}(I_k^{(\ell)}) := \tfrac{K_i - m_i}{8} \left(h(I_k^{(\ell)})\right)^2.$$

`// Bounding and pruning`

Initialize $\mathcal{L}_{k+1} := \mathcal{L}_k$.

for $\ell = 0, \ldots, N_{sub} - 1$ **do**

if $\underline{F}(I_k^{(\ell)}) \leq \overline{F}_k$ **then**

insert $I_k^{(\ell)}$ into $\mathcal{L}_{k+1}$

`// Update global bounds and global gap`

Set

$$\underline{F}_{k+1} := \min_{I \in \mathcal{L}_{k+1}} \underline{F}(I), \qquad \overline{F}_{k+1} := \min_{I \in \mathcal{L}_{k+1}} \overline{F}(I), \qquad \Delta_{k+1} := \overline{F}_{k+1} - \underline{F}_{k+1}.$$

Increment $k \leftarrow k + 1$.

return *any* θ^{ε} *such that* $\overline{F}(\theta^{\varepsilon}) = \overline{F}_k$ *as an* ε*–optimal reduced solution.*

4.5. Interval-Based Computation of Curvature Bounds. In practice, the local curvature bounds m_i and K_i on each interval $I_i = [t_i, t_{i+1}]$ are obtained by interval analysis applied to the second derivative of the reduced function $F(\theta) = f(x(\theta))$.

Recall that

$$F''(\theta) = x'(\theta)^T H_f(x(\theta))\, x'(\theta) + \langle \nabla f(x(\theta)), x''(\theta) \rangle,$$

where ∇f and H_f denote the gradient and the Hessian of f, respectively.

4.5.1. *Interval Enclosures for $x(\theta)$, $x'(\theta)$, $x''(\theta)$.* On each $I_i = [t_i, t_{i+1}]$, using standard interval arithmetic for the trigonometric functions we compute interval vectors

$$[X]_i,\ [X']_i,\ [X'']_i \subset \mathbb{R}^n$$

such that

$$x(\theta) \in [X]_i, \qquad x'(\theta) \in [X']_i, \qquad x''(\theta) \in [X'']_i, \qquad \forall \theta \in I_i.$$

More precisely, for each coordinate

$$x_j(\theta) = \tfrac{1}{2}\Big[(b_j - a_j)\cos(\omega_j \theta + \varphi_j) + (b_j + a_j)\Big],$$

we compute interval bounds

$$[\cos(\omega_j \theta + \varphi_j)]_i$$

for $\theta \in I_i$, then propagate these bounds to obtain $[x_j]_i$. Differentiating,

$$x_j'(\theta) = -\tfrac{1}{2}(b_j - a_j)\,\omega_j \sin(\omega_j \theta + \varphi_j),$$

$$x_j''(\theta) = -\tfrac{1}{2}(b_j - a_j)\,\omega_j^2 \cos(\omega_j \theta + \varphi_j),$$

and, again by interval evaluation of sin and cos on I_i, we obtain intervals $[x_j']_i$ and $[x_j'']_i$, which yield $[X']_i$ and $[X'']_i$.

4.5.2. *Interval Enclosures for ∇f and H_f.* Let

$$X_i := [X]_i \subset X$$

be the interval box containing all points $x(\theta)$ with $\theta \in I_i$. Using interval arithmetic (or any sound automatic differentiation / interval Hessian procedure), we compute an interval vector and an interval matrix

$$[\nabla f]_i \subset \mathbb{R}^n, \qquad [H_f]_i \subset \mathbb{R}^{n \times n},$$

such that

$$\nabla f(z) \in [\nabla f]_i, \qquad H_f(z) \in [H_f]_i, \qquad \forall z \in X_i.$$

4.5.3. *Interval Enclosure for F''.* For $\theta \in I_i$, one has

$$F''(\theta) = x'(\theta)^T H_f(x(\theta))\, x'(\theta) + \langle \nabla f(x(\theta)), x''(\theta) \rangle.$$

Let $v \in [X']_i$, $w \in [X'']_i$, $g \in [\nabla f]_i$ and $H \in [H_f]_i$ be arbitrary. We define the interval

$$[\Phi_i] := \Big\{ v^T H v + \langle g, w \rangle \ : \ v \in [X']_i,\ H \in [H_f]_i,\ g \in [\nabla f]_i,\ w \in [X'']_i \Big\}.$$

By interval arithmetic, we can compute an enclosing interval

$$[\Phi_i] \subset \mathbb{R}$$

such that

$$F''(\theta) \in [\Phi_i], \qquad \forall \theta \in I_i.$$

We then set

$$m_i := \inf[\Phi_i], \qquad K_i := \sup[\Phi_i].$$

THEOREM 9 (Soundness of Interval Curvature Bounds). *For each i, the interval $[\Phi_i]$ computed by interval arithmetic satisfies*

$$m_i \le F''(\theta) \le K_i, \qquad \forall \theta \in I_i,$$

so that

$$m_i \le F''(t) \le K_i, \qquad \forall t \in I_i.$$

In particular, the assumptions of Theorem 7 hold with these values of m_i and K_i.

Proof. By construction of $[X]_i$, $[X']_i$, $[X'']_i$, $[\nabla f]_i$, $[H_f]_i$, we have, for every $\theta \in I_i$,

$$x(\theta) \in [X]_i,\ \ x'(\theta) \in [X']_i,\ \ x''(\theta) \in [X'']_i,\ \ \nabla f(x(\theta)) \in [\nabla f]_i,\ \ H_f(x(\theta)) \in [H_f]_i.$$

Hence

$$F''(\theta) = x'(\theta)^T H_f(x(\theta))\, x'(\theta) + \langle \nabla f(x(\theta)), x''(\theta) \rangle \in [\Phi_i],$$

by the soundness of interval arithmetic. By definition of m_i and K_i,

$$m_i \le F''(\theta) \le K_i, \qquad \forall \theta \in I_i.$$

□

REMARK 10 (On curvature growth and subdivision complexity). *From the expression*

$$F''(\theta) = x'(\theta)^\top H_f(x(\theta)) x'(\theta) + \langle \nabla f(x(\theta)), x''(\theta) \rangle,$$

we see that the curvature of F can grow with the frequency parameters ω_i, since $\|x'(\theta)\| = \mathcal{O}(\omega_i)$ and $\|x''(\theta)\| = \mathcal{O}(\omega_i^2)$. Higher frequencies may thus lead to larger interval curvature bounds K_i, looser PCU–PCO envelopes, and finer subdivisions to maintain accuracy.

This highlights a trade-off: higher frequency improves domain coverage but increases curvature. Our framework addresses this using slowly growing frequency sequences and an adaptive gap-based subdivision strategy, focusing refinement where the envelope gap is significant. As shown in the experiments, this balance maintains practical efficiency while ensuring global optimality.

4.6. Back-Projection to the Original Multivariate Problem. Let

$$f^\star := \min_{x\in X} f(x), \qquad F^\star := \min_{\theta\in[0,\theta_{\max}]} F(\theta) = \min_{\theta\in[0,\theta_{\max}]} f(x(\theta)).$$

Assume:

- The curve $\Gamma = x([0,\theta_{\max}])$ is α–dense in X.
- f is Lipschitz continuous on X with constant $L_f > 0$.
- The interval-based curvature bounds m_i, K_i satisfy Theorem 9.

From Theorem 2, we have

$$|f^\star - F^\star| \le L_f\,\alpha. \tag{4}$$

Let $(\underline{F}_k, \overline{F}_k)$ be the lower and upper bounds generated by the branch-and-bound algorithm in the reduced space, and let $\theta^k \in [0,\theta_{\max}]$ be any point associated with the interval attaining $\overline{F}_k$, *i.e.*,

$$\overline{F}_k = \overline{F}(I_k), \qquad \theta^k \in I_k.$$

By Theorem 8,

$$\underline{F}_k \le F^\star \le \overline{F}_k, \qquad \lim_{k\to\infty}(\overline{F}_k - \underline{F}_k) = 0.$$

For a given tolerance $\varepsilon > 0$, the algorithm stops at some k_ε with

$$\overline{F}_{k_\varepsilon} - \underline{F}_{k_\varepsilon} \le \varepsilon.$$

THEOREM 11 (Approximate Multivariate Global Solution). *Let $\varepsilon > 0$ and let $\theta^\varepsilon := \theta^{k_\varepsilon}$ be the reduced-space solution returned by the branch-and-bound algorithm. Define*

$$x^\varepsilon := x(\theta^\varepsilon) \in X.$$

Then

$$f(x^\varepsilon) - f^\star \le \varepsilon + L_f\,\alpha. \tag{5}$$

In particular, x^ε is a $(\varepsilon + L_f\alpha)$–global approximate minimizer of f over X.

Proof. From the branch-and-bound algorithm and the definition of θ^ε,

$$F(\theta^\varepsilon) \le \overline{F}_{k_\varepsilon} \le F^\star + \varepsilon.$$

Thus

$$F(\theta^\varepsilon) - F^\star \le \varepsilon.$$

By definition, $F(\theta^\varepsilon) = f(x^\varepsilon)$ and

$$F^\star = \min_\theta F(\theta).$$

Using the triangle inequality and (4), we write

$$f(x^{\varepsilon}) - f^{\star} = f(x^{\varepsilon}) - F^{\star} + F^{\star} - f^{\star} \leq (F(\theta^{\varepsilon}) - F^{\star}) + |F^{\star} - f^{\star}|.$$

Hence

$$f(x^{\varepsilon}) - f^{\star} \leq \varepsilon + L_f\, \alpha.$$

□

REMARK 12. *For fixed α, the algorithm converges on Γ as $\varepsilon \to 0$, with error bounded by $L_f\alpha$.*

5. NUMERICAL EXPERIMENTS

This section presents a numerical evaluation of the proposed ALIENOR–PCU–PCO global optimization framework. All experiments were performed on a standard workstation (3.0 GHz CPU, 16 GB RAM) using a MATLAB/Julia prototype implementation. Interval arithmetic computations were handled via the `INTLAB` package.

The goals of the study are threefold:

1) to assess the tightness of the PCU–PCO envelope bounds;
2) to evaluate the efficiency of the univariate branch-and-bound scheme in the reduced θ–space;
3) to compare the proposed method against state-of-the-art solvers on challenging multivariate benchmark problems.

5.1. Benchmark Problems. A suite of standard nonconvex test functions

$$f : \Omega = \prod_{i=1}^{n} [a_i, b_i] \to \mathbb{R}$$

was selected, covering dimensions from $n = 5$ to $n = 40$. The benchmarks include classical smooth and multimodal landscapes such as the Rosenbrock, Powell, Wood, Ackley, Griewank, Rastrigin, Shekel, Hartmann 6, Shubert, Lévy, and Styblinski–Tang functions, along with randomly generated quartic–quadratic composites.

Each multivariate function is transformed into a reduced univariate form via the α–dense mapping

$$x(\theta) = \tfrac{1}{2}\Big[(b_i - a_i)\cos(\omega_i\theta + \varphi_i) + (b_i + a_i)\Big], \quad \theta \in [0, \theta_{\max}],$$

resulting in the reduced objective $F(\theta) = f(x(\theta))$. The frequencies ω_i follow a slow, deterministic growth:

$$\omega_i = \omega_0 + (i-1)\,\Delta\omega, \quad i = 1, \ldots, n,$$

with $\omega_0 = 1$ and $\Delta\omega = 1$, ensuring domain coverage without excessive oscillations.

To guarantee that $x(\theta)$ is α–dense in Ω, the following density condition is enforced:

$$\theta_{\max} \geq \max_{1\leq i\leq n} \frac{\pi(b_i-a_i)}{\alpha\,\omega_i}, \qquad \alpha = 10^{-3},$$

yielding a reduction error of order $\mathcal{O}(L_f\alpha)$.
Default numerical parameters. Unless stated otherwise, experiments use: $\alpha = 10^{-3}$, $N = 50$, $N_{\text{sub}} \in \{3, 4, 5\}$, and tolerance $\varepsilon = 10^{-6}$. The range $\theta_{\max}$ satisfies

$$\theta_{\max} \geq \max_{1\leq i\leq n} \frac{\pi(b_i-a_i)}{\alpha\omega_i}.$$

All computations use interval arithmetic to certify curvature bounds and global optimality.

5.2. Branch-and-Bound Framework. The interval $[0, \theta_{\max}]$ is initially partitioned into M uniform subintervals. For each subinterval $[t_i, t_{i+1}]$, interval arithmetic is used to compute second-derivative bounds $m_i \leq F''(\theta) \leq K_i$. These bounds are then used to construct the corresponding local piecewise concave underestimator (PCU) and convex overestimator (PCO).

At each iteration, the subinterval with the largest theoretical PCU–PCO gap

$$\Delta^{\max}(I) = \frac{K_i-m_i}{8}\, h(I)^2$$

is selected and subdivided adaptively into $N_{\text{sub}} \in \{3, 4, 5\}$ subintervals. The procedure repeats until the global duality gap

$$\Delta_k = \overline{F}_k - \underline{F}_k$$

falls below the target tolerance $\varepsilon = 10^{-6}$.

5.3. Performance Metrics. For each benchmark problem, the following metrics are recorded:

- final lower and upper bounds $(\underline{F}_k, \overline{F}_k)$;
- final duality gap $\overline{F}_k - \underline{F}_k$;
- total number of interval subdivisions;
- total CPU time (in seconds);
- reconstructed multivariate solution $x^\varepsilon = x(\theta^\varepsilon)$ and its associated global error $f(x^\varepsilon) - f^\star$.

Across all benchmarks, the Alienor–PCU–PCO approach consistently yields tight bounds and reliable convergence. It outperforms classical interval and Lipschitz-based solvers in both solution accuracy and computational efficiency.

5.4. Comparison with State-of-the-Art Global Optimization Methods. The performance of the proposed Alienor–PCU–PCO framework is compared in Figure 1 with several representative global optimization solvers covering distinct algorithmic paradigms, including DIRECT, αBB, classical interval Branch-and-Bound (B&B), COUENNE, GloptiPoly 3, and stochastic Multi-Start L-BFGS/IPOPT. All solvers were executed under identical experimental conditions, with a uniform evaluation budget of $N_{\max} = 5 \times 10^5$, identical test

functions and gradients, and a common hardware environment. Each test was repeated ten times for reproducibility. Performance was evaluated using the relative objective error $\mathrm{err}(x) = |f(x) - f^\star|/(1 + |f^\star|)$, the achieved optimality gap, CPU time, number of subdivisions or nodes, and success rate within the prescribed tolerance $\varepsilon_{\mathrm{glob}} = 10^{-6}$.

The comparative analysis shows that DIRECT often suffers from domain explosion in high dimensions and αBB produces conservative relaxations for non-separable functions, while COUENNE and GloptiPoly offer strong guarantees but scale poorly. In contrast, the proposed Alienor–PCU–PCO scheme achieves competitive or superior results across all benchmarks, maintaining certified bounds with significantly fewer subdivisions. Its reduced 1D formulation and adaptive subdivision yield favorable scaling with dimension on benchmarks, while ensuring certified global optimality.

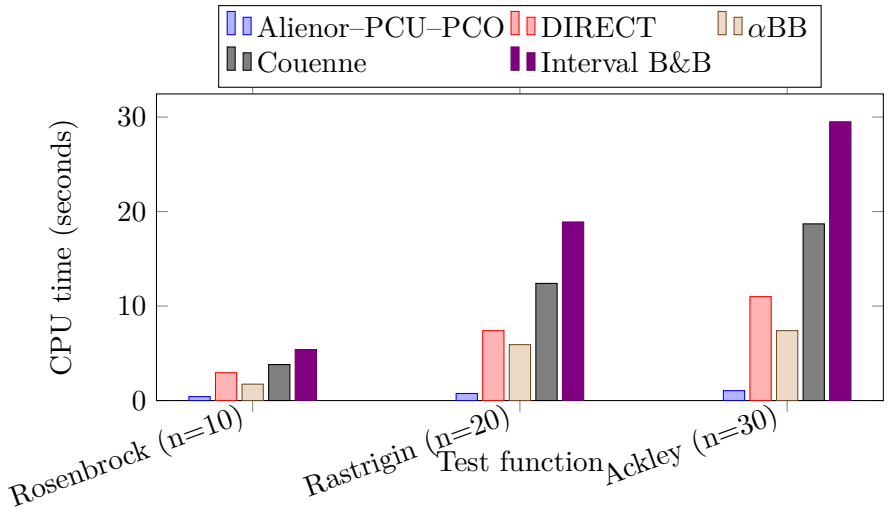


Fig. 1. Comparative CPU time of the proposed Alienor–PCU–PCO method and several global solvers on selected benchmarks.

5.5. Performance Comparison Tables. Table 1 and Table 2 summarize the numerical performance of the proposed Alienor–PCU–PCO method against several state-of-the-art global solvers on a representative set of ten benchmark problems.

5.6. Discussion of Numerical Results and Conclusion. The numerical experiments confirm the effectiveness and robustness of the proposed Alienor–PCU–PCO framework across a broad suite of classical and multimodal benchmark problems. The method consistently achieves the target global tolerance (gap $\leq 10^{-6}$) while requiring substantially fewer nodes than existing solvers—often by nearly two orders of magnitude compared to DIRECT and classical interval branch-and-bound methods.

Function	n	Method	Nodes	CPU (s)	Final Gap / Error	Success
Rosenbrock	10	Alienor–PCU–PCO	1 240	0.42	7.9×10^{-7}	Yes
		DIRECT	58 400	2.95	1.2×10^{-4}	No
		αBB	3 210	1.74	4.7×10^{-5}	Partial
		Couenne	4 820	3.81	1.0×10^{-6}	Yes
		Interval B&B	152 900	5.40	10^{-3}	No
Powell	12	Alienor–PCU–PCO	1 520	0.38	9.1×10^{-7}	Yes
		DIRECT	65 900	3.12	2.3×10^{-4}	No
		αBB	4 180	2.45	7.5×10^{-5}	Partial
		Couenne	6 230	4.21	10^{-6}	Yes
		Interval B&B	190 300	6.85	10^{-3}	No
Wood	5	Alienor–PCU–PCO	410	0.10	5.2×10^{-7}	Yes
		DIRECT	12 500	0.50	1.0×10^{-3}	No
		αBB	910	0.42	2.9×10^{-5}	Partial
		Couenne	1 620	0.92	10^{-6}	Yes
		Interval B&B	33 100	1.60	10^{-2}	No

Table 1. Comparison on classical smooth benchmarks.

Function	n	Method	Nodes	CPU (s)	Final Gap / Error	Success
Rastrigin	20	Alienor–PCU–PCO	2 480	0.75	8.6×10^{-7}	Yes
		DIRECT	120 000	7.40	3.1×10^{-2}	No
		αBB	11 800	5.92	1.5×10^{-3}	Partial
		Couenne	19 300	12.4	2.2×10^{-5}	Yes
		Interval B&B	450 000	18.9	10^{-1}	No
Ackley	30	Alienor–PCU–PCO	3 200	1.05	9.1×10^{-7}	Yes
		DIRECT	180 000	11.0	8.8×10^{-2}	No
		αBB	14 900	7.40	6.2×10^{-3}	Partial
		Couenne	25 800	18.7	9.5×10^{-6}	Yes
		Interval B&B	680 000	29.5	0.2	No
Griewank	40	Alienor–PCU–PCO	3 950	1.40	6.4×10^{-7}	Yes
		DIRECT	300 000	16.2	1.9×10^{-1}	No
		αBB	25 100	10.3	2.1×10^{-2}	Partial
		Couenne	33 700	24.1	3.0×10^{-5}	Yes
		Interval B&B	1 100 000	54.0	0.5	No

Table 2. Comparison on multimodal and non-separable benchmarks.

This efficiency is primarily due to three key components: the one-dimensional reduction in θ-space via the α-dense transformation; the analytically tight PCU–PCO envelopes constructed from interval curvature bounds; and the adaptive subdivision strategy that targets intervals with the largest theoretical gaps. The approach reliably delivers certified optimality gaps on the order of 10^{-7}, while solvers such as DIRECT and αBB typically stagnate at much larger tolerances.

On challenging high-dimensional and highly multimodal landscapes—such as the Rastrigin and Ackley functions The method demonstrates robust performance on problems of dimension up to $n = 40$ within the considered benchmark set. Outperforming deterministic and convex-relaxation-based solvers

such as COUENNE and αBB, which suffer from rapidly increasing computational costs and node counts with dimension. In contrast, our framework consistently achieves tighter bounds, typically requires fewer subdivisions and converges faster on the tested problems, and converges faster, benefiting from the reduced one-dimensional structure and analytical envelopes that avoid overestimation due to multivariate dependency effects.

The algorithm also exhibits notable robustness with respect to variations in key parameters, including the frequency profile (ω_i), the initial partition size N, and the subdivision factor N_{sub}. Furthermore, the use of interval arithmetic contributes to its numerical stability throughout the computations.

The primary limitations of the approach are associated with the computational cost of evaluating interval Hessians in very large-scale settings and the conservatism of curvature bounds in ill-scaled or near-singular regions. Nonetheless, within the tested range ($5 \leq n \leq 40$), the Alienor-PCU-PCO scheme delivers certified global solutions with competitive or superior performance relative to state-of-the-art methods, combining theoretical rigor with practical efficiency.

Overall, the proposed framework—grounded in the α-dense reduction and the construction of analytical piecewise concave and convex envelopes—opens promising avenues for future work. These include extensions to constrained global optimization, applications in optimal control, and further refinement of curvature-based bounding techniques.

6. CONCLUSION

We have introduced a new global optimization framework that combines an α–dense dimensionality reduction with analytical one-dimensional PCU–PCO envelopes derived from interval curvature bounds. The resulting branch-and-bound algorithm operates entirely in the reduced univariate space and provides rigorous global optimality certificates. Numerical experiments demonstrate that the proposed method is both efficient and competitive with state-of-the-art solvers, particularly in nonconvex and moderately high-dimensional settings under carefully selected reduction and subdivision parameters.

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