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AN USHIJIMA-TYPE ANALYSIS FOR THE NUMERICAL BLOW-UP OF A FRACTIONAL REACTION-DIFFUSION EQUATION

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Abstract. This paper studies the blow-up phenomenon for a nonlinear reaction-diffusion equation with a Caputo fractional time derivative. Our work has two main parts. First, we prove that the solution of the semi-discretized system blows up in a finite time T_h if the initial data is large enough. Second, we analyze the full discretization using the explicit L1 scheme. By introducing a discrete weighted functional and comparing it to an auxiliary sequence, we show that the numerical solution also blows up in finite time, provided we use an adaptive time-stepping strategy. The main contribution of this work is to show that our lemmas provide the necessary tools to apply Ushijima's theoretical framework to this class of fractional problems. This builds a bridge between Ushijima's theory and non-local fractional schemes. We can then conclude that the numerical blow-up time converges to the semi-discrete one, which validates the ability of the scheme to capture the singularity dynamics.

Keywords: Fractional reaction-diffusion equation, Numerical blow-up, L1 scheme, Ushijima framework, Blow-up time convergence, Caputo derivative, Adaptive time-stepping.

1. INTRODUCTION

The numerical capture of finite-time blow-up in nonlinear reaction-diffusion equations is a major challenge in modern numerical analysis. The problem is not only to show that the numerical solution grows to infinity, but to ensure that the numerical blow-up time converges to the continuous one. For classical parabolic equations, this is well understood. However, time-fractional derivatives of Caputo type change the situation: they introduce a non-local memory effect that makes standard analysis methods difficult to use.

Following the work of Ushijima [22], a strict methodology was created to prove the convergence of blow-up time for local parabolic equations. More

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recently, Wang et al. [24] studied the L1 scheme for fractional ordinary differential equations (FODE). However, there is still a gap for fractional partial differential equations (FPDE), where we must manage both the temporal non-locality and the spatial diffusion operator. The Ushijima framework was not designed for memory operators, and the approach by Wang does not include the complex interactions caused by the Laplacian.

This work aims to fill this gap by studying the following fractional reaction-diffusion equation (S):

$$
(1)\qquad \begin{cases} {}^{C}D_t^{\beta} u(x,t) = D u_{xx}(x,t) + u^p(x,t), & \text{for } (x,t) \in (0,1)\times(0,T], \\ u(x,0) = u_0(x), & \text{for } x \in [0,1], \\ u(0,t) = u(1,t) = 0, & \text{for } t \in (0,T], \end{cases}
$$

where $\beta \in (0,1)$ and $p > 1$. Our study focuses on solving three main difficulties:

First, we establish the blow-up proof for the semi-discretized system. The challenge is to maintain positivity and growth while considering both the discrete Laplacian and the Caputo memory. We solve this by using a fractional extremum principle combined with weighted functionals.

Second, we deal with the complexity of the full discretization using the explicit L1 scheme. Managing non-locality on an adaptive time mesh which is necessary to approach the singularity makes the analysis of memory weights very difficult. Our solution is based on constructing an auxiliary comparison sequence and using a discrete Jensen inequality. This guarantees the existence of numerical blow-up where standard uniform mesh analysis often fails.

Finally, and this is our most important contribution, we demonstrate that this numerical blow-up satisfies the critical condition A0 of the Ushijima framework. Unlike previous studies that only observe the blow-up, we build an original theoretical bridge between Ushijima's theory and non-local fractional schemes. By validating this connection, we do more than just simulate a singularity; we create the foundation to prove, in future work, the rigorous convergence of the numerical blow-up time. This confirms that our scheme captures the true essence of the singular dynamics.

2. SEMI-DISCRETIZATION AND BLOW-UP

We begin by discretising problem (1) in space only. Let I be a positive integer, $h = 1/I$ the spatial step size, and $x_i = ih$ for $i = 0, \ldots, I$. Let $U_i(t)$ be an approximation of $u(x_i,t)$. By replacing the second spatial derivative with a centred finite difference, we obtain the system of fractional ordinary differential equations (FODE):

$$(2)\qquad {}^{C}D_t^{\beta}U_i(t) = D\delta^2 U_i(t) + [U_i(t)]^p, \qquad i = 1,\dots,I-1,$$

$$(3)\qquad U_0(t) = U_I(t) = 0, \qquad t \geq 0,$$

$$(4)\qquad U_i(0) = u_0(x_i), \qquad i = 0,\dots,I,$$

where δ^2 is the discrete Laplacian operator: $\delta^2 U_i(t) = (U_{i+1}(t) - 2U_i(t) + U_{i-1}(t))/h^2$.

The analysis of this system is based on the associated eigenvalue problem $\delta^2(\Phi_{k,i}) = -\lambda_{k,h}\Phi_{k,i}$ with zero Dirichlet boundary conditions. The eigenvalues and eigenvectors are well known:

$$(5)\qquad \lambda_{k,h} = \frac{2(1-\cos(k\pi h))}{h^2}, \quad k = 1,\dots,I-1,$$

$$(6)\qquad \Phi_{k,i} = \sin(ki\pi h), \quad i = 0,\dots,I.$$

The first eigenvector Φ_1, associated with the first eigenvalue $\lambda_{1,h}$, has components $\Phi_{1,i} = \sin(i\pi h)$ that are strictly positive for $i = 1,\dots,I-1$.

2.1. Blow-up of the Semi-Discrete Solution. To study the behavior of the solution $U_i(t)$, we introduce a weighted functional.

Definition 1. *The weighted functional $J_{\Phi,h}(t)$ is defined by:*

$$J_{\Phi,h}(t) = h\sum_{j=1}^{I-1} U_j(t)\Phi_{1,j}.$$

The initial condition $u_0(x)$ is chosen such that $J_{\Phi,h}(0) > 0$.

Our analysis is based on the following results.

Lemma 2 (Non-negativity of the solution). *Let $U_i(t)$ be the solution of the system (2)–(4) with $u_0(x) \geq 0$ and $u_0 \not\equiv 0$. Then, $U_i(t) > 0$ for all $i \in \{1,\dots,I-1\}$ and for all $t \in (0,T_{max})$, where T_{max} is the maximum existence time.*

Proof. Suppose that there exists a first time $t_0 > 0$ and an index $k_0 \in \{1,\dots,I-1\}$ such that $U_{k_0}(t_0) = 0$ and $U_i(t) \geq 0$ for all i and $t \in [0,t_0]$. Since $U_{k_0}(t_0) = 0$ is a minimum for $U_{k_0}(t)$ on the interval $[0,t_0]$, the extremum principle for the Caputo derivative (see [25], Theorem 3.2) implies:

$$(7)\qquad {}^{C}D_t^{\beta}U_{k_0}(t_0) \leq \frac{-U_{k_0}(0)}{\Gamma(1-\beta)t_0^{\beta}} \leq 0.$$

On the other hand, from the governing equation (2) at $t = t_0$, we have:

$${}^{C}D_t^{\beta}U_{k_0}(t_0) = D\frac{U_{k_0-1}(t_0) - 2U_{k_0}(t_0) + U_{k_0+1}(t_0)}{h^2} + [U_{k_0}(t_0)]^p.$$

Substituting $U_{k_0}(t_0) = 0$, we obtain:

$$ {}^{C}D_t^{\beta} U_{k_0}(t_0) = D \frac{U_{k_0-1}(t_0) + U_{k_0+1}(t_0)}{h^2} \geq 0, $$

since $U_{k_0 \pm 1}(t_0) \geq 0$. Comparing with (7), we must have ${}^{C}D_t^{\beta} U_{k_0}(t_0) = 0$, which implies $U_{k_0 \pm 1}(t_0) = 0$ and $U_{k_0}(0) = 0$. By propagation to all neighbors, we find $U_i(0) = 0$ for all i, which contradicts the assumption $u_0 \not\equiv 0$. Therefore, the assumption that the solution touches zero is false, and we conclude $U_i(t) > 0$ for all $t \in (0, T_{max})$. □

Lemma 3 (Weighted Discrete Jensen's Inequality). *Let $\zeta : \mathbb{R} \to \mathbb{R}$ be a convex function. Let $x_1, \ldots, x_N \in \mathbb{R}$ and $\omega_1, \ldots, \omega_N$ be non-negative weights such that $S_\omega = \sum_{j=1}^{N} \omega_j > 0$. Then:*

$$ \frac{1}{S_\omega} \sum_{j=1}^{N} \omega_j \zeta(x_j) \geq \zeta\left(\frac{1}{S_\omega} \sum_{j=1}^{N} \omega_j x_j \right). $$

Proposition 4. *The functional $J_{\Phi,h}(t)$ satisfies the following fractional differential inequality:*

$$ {}^{C}D_t^{\beta} J_{\Phi,h}(t) \geq -D\lambda_{1,h} J_{\Phi,h}(t) + K_{p,h} [J_{\Phi,h}(t)]^p, \tag{8} $$

where the constant $K_{p,h}$ is defined by $K_{p,h} = \left(h \sum_{j=1}^{I-1} \Phi_{1,j}\right)^{1-p}$.

Proof. Applying the operator ${}^{C}D_t^{\beta}$ to $J_{\Phi,h}(t)$ and using equation (2), we have:

$$ \begin{aligned} {}^{C}D_t^{\beta} J_{\Phi,h}(t) &= h \sum_{j=1}^{I-1} ({}^{C}D_t^{\beta} U_j(t)) \Phi_{1,j} \\ &= h \sum_{j=1}^{I-1} (D\delta^2 U_j(t) + [U_j(t)]^p) \Phi_{1,j} \\ &= D \left(h \sum_{j=1}^{I-1} (\delta^2 U_j(t)) \Phi_{1,j} \right) + h \sum_{j=1}^{I-1} [U_j(t)]^p \Phi_{1,j}. \end{aligned} $$

Using the symmetry property of δ^2 and the fact that Φ_1 is an eigenvector, the diffusion term becomes:

$$ h \sum_{j=1}^{I-1} (\delta^2 U_j(t)) \Phi_{1,j} = h \sum_{j=1}^{I-1} U_j(t) (\delta^2 \Phi_{1,j}) = -\lambda_{1,h} \left(h \sum_{j=1}^{I-1} U_j(t) \Phi_{1,j} \right) = -\lambda_{1,h} J_{\Phi,h}(t). $$

For the nonlinear term, Lemma 2 guarantees that $U_j(t) \geq 0$. The function $\zeta(x) = x^p$ is convex for $x \geq 0$ since $p > 1$. We apply Lemma 3 with $x_j = U_j(t)$

and $\omega_j = \Phi_{1,j} > 0$. Let $S_\omega = \sum_{j=1}^{I-1} \Phi_{1,j}$.

$$
\begin{aligned}
h\sum_{j=1}^{I-1}[U_j(t)]^p\Phi_{1,j} &= hS_\omega\left(\frac{1}{S_\omega}\sum_{j=1}^{I-1}\Phi_{1,j}[U_j(t)]^p\right)\\
&\geq hS_\omega\left(\frac{1}{S_\omega}\sum_{j=1}^{I-1}\Phi_{1,j}U_j(t)\right)^p\\
&= hS_\omega\frac{\left(\sum\Phi_{1,j}U_j(t)\right)^p}{(S_\omega)^p} = \frac{h\left(h\sum\Phi_{1,j}U_j(t)\right)^p}{h^p(S_\omega)^{p-1}}\\
&= \frac{[J_{\Phi,h}(t)]^p}{(hS_\omega)^{p-1}} = \left(h\sum_{j=1}^{I-1}\Phi_{1,j}\right)^{1-p}[J_{\Phi,h}(t)]^p.
\end{aligned}
$$

Combining the two terms, we obtain the inequality (8). □

THEOREM 5 (Blow-up of the Semi-Discrete Solution). *Assume $p > 1$. If the initial condition $u_0(x)$ is such that the weighted functional $J(0) \equiv J_{\Phi,h}(0)$ satisfies:*

(9) $$J(0) > \left(\frac{D\lambda_{1,h}}{K_{p,h}}\right)^{\frac{1}{p-1}} =: J_{\text{crit}},$$

then the solution $U_i(t)$ of the semi-discrete system (2)–(4) blows up in finite time $T_h < \infty$. Moreover, the blow-up time satisfies the upper bound [15, Corollary 2]:

(10) $$T_h \leq \left[\frac{\Gamma\left(\frac{\beta p}{p-1}\right)}{\Gamma\left(\frac{\beta}{p-1}\right)\left(K_{p,h}J(0)^{p-1} - D\lambda_{1,h}\right)}\right]^{1/\beta} < \infty.$$

Proof. By Proposition 4, the weighted functional $J(t)$ satisfies the fractional differential inequality:

(11) $${}^C D_t^\beta J(t) \geq K_{p,h}J(t)^p - D\lambda_{1,h}J(t) =: f(J(t)),$$

with initial condition $J(0) > J_{\text{crit}} = \left(\frac{D\lambda_{1,h}}{K_{p,h}}\right)^{\frac{1}{p-1}}$.

Since $J(t) \geq J(0) > J_{\text{crit}}$ (by strict positivity of the fractional derivative when $f(J(0)) > 0$), and the function $\phi(u) = K_{p,h} - D\lambda_{1,h}u^{-(p-1)}$ is increasing on $[J_{\text{crit}}, \infty)$, we have for all $t \geq 0$:

(12) $$f(J(t)) \geq \left(K_{p,h} - D\lambda_{1,h}J(0)^{-(p-1)}\right)J(t)^p = C^*J(t)^p,$$

where $C^* = \dfrac{K_{p,h}J(0)^{p-1} - D\lambda_{1,h}}{J(0)^{p-1}} > 0$ by hypothesis on $J(0)$.

By the equivalence between the Caputo differential inequality and the Volterra integral formulation [4, Theorem 3.1], we obtain:

$$J(t) \geq J(0) + \frac{C^*}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} J(s)^p \, ds. \tag{13}$$

Consider the candidate subsolution $\underline{y}(t) = J(0)\left(1 - \frac{t}{T}\right)^{-q}$ defined on $[0, T)$ with $q = \frac{\beta}{p-1} > 0$. This function satisfies $\underline{y}(0) = J(0)$ and $\lim_{t\to T^-} \underline{y}(t) = +\infty$.

For $\underline{y}$ to be a subsolution of (13), it suffices that:

$$\underline{y}(t) \leq J(0) + \frac{C^*}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \underline{y}(s)^p \, ds =: I(t). \tag{14}$$

By the substitution $s = t - (T-t)\sigma$ and using $pq = \beta + q$, one computes:

$$\int_0^t (t-s)^{\beta-1} \left(1 - \frac{s}{T}\right)^{-pq} ds = T^{\beta+q}(T-t)^{-q} B\left(\frac{t}{T-t}; \beta, q\right), \tag{15}$$

where $B(x;\beta,q) = \int_0^x \sigma^{\beta-1}(1+\sigma)^{-(\beta+q)} d\sigma$ is the incomplete Beta function [4, Chapter 6].

As $t \to T^-$, $B(x;\beta,q) \to \dfrac{\Gamma(\beta)\Gamma(q)}{\Gamma(\beta+q)}$, so the dominant behavior of $I(t)$ near T is:

$$I(t) \sim J(0) + \frac{C^* J(0)^p \Gamma(q)}{\Gamma(\beta+q)} T^{\beta+q}(T-t)^{-q}. \tag{16}$$

The constant $J(0)$ is negligible compared to the diverging term $(T-t)^{-q}$.

Matching the dominant coefficients of $(T-t)^{-q}$ in $\underline{y}(t) \leq I(t)$ gives:

$$1 \leq \left(K_{p,h} J(0)^{p-1} - D\lambda_{1,h}\right) \frac{\Gamma(q)}{\Gamma(\beta+q)} T^{\beta}.$$

Since $\beta + q = \frac{\beta p}{p-1}$, the sufficient condition is:

$$T \geq \left[\frac{\Gamma\left(\frac{\beta p}{p-1}\right)}{\Gamma\left(\frac{\beta}{p-1}\right)\left(K_{p,h} J(0)^{p-1} - D\lambda_{1,h}\right)}\right]^{1/\beta} =: T_{\max}. \tag{17}$$

For any $T \geq T_{\max}$, the function $\underline{y}$ is a subsolution of the integral equation (13). By the comparison principle for nonlinear Volterra equations [15, Corollary 2], $J(t) \geq \underline{y}(t)$ for all $t \in [0, T)$.

Since $\underline{y}(t) \to +\infty$ as $t \to T^-$, the solution $J(t)$ must blow up at a time T_h satisfying:

$$T_h \leq T_{\max} = \left[\frac{\Gamma\left(\frac{\beta p}{p-1}\right)}{\Gamma\left(\frac{\beta}{p-1}\right)\left(K_{p,h} J(0)^{p-1} - D\lambda_{1,h}\right)}\right]^{1/\beta} < \infty. \tag{18}$$

Since $J(t) \geq \underline{y}(t)$ for all t on the interval $[J_{\text{crit}}, \infty)$, the blow-up of $J(t)$ in finite time T_h implies, by equivalence of norms in $\mathbb{R}^{I-1}$ (since the components $\Phi_{1,i}$ are strictly positive), that $\|U(t)\|_\infty \to +\infty$ as $t \to T_h^-$. □

LEMMA 6 (Lower Blow-up Rate for $J(t)$). *Let $J \in AC[0, T_h)$ be the solution of the fractional differential inequality (8) with initial condition $J(0) > J_{\text{crit}}$. Then the blow-up time T_h is finite and the following properties hold:*

(i)

$$\liminf_{t\to T_h^-}(T_h - t)^{\frac{\beta}{p-1}} J(t) \geq C_{J,h}^- := \left(\frac{\Gamma\left(\frac{p\beta}{p-1}\right)}{\frac{1}{2}K_{p,h}\,\Gamma\left(\frac{\beta}{p-1}\right)} \right)^{\frac{1}{p-1}} > 0. \tag{19}$$

(ii) There exists a constant $c_1 > 0$ such that for all $t \in [T_h/2, T_h)$:

$$J(t) \geq c_1 (T_h - t)^{-\frac{\beta}{p-1}}. \tag{20}$$

Proof. Since $J(t) \to +\infty$ as $t \to T_h^-$, there exists a time $t_1 \in [0, T_h)$ such that for all $t \geq t_1$, the solution exceeds the threshold $J(t) \geq \left(\frac{2D\lambda_{1,h}}{K_{p,h}}\right)^{\frac{1}{p-1}}$, which implies:

$$K_{p,h}J(t)^p - D\lambda_{1,h}J(t) \geq \frac{1}{2}K_{p,h}J(t)^p, \quad \forall t \in [t_1, T_h). \tag{21}$$

Let $Y(t)$ be the maximal solution of the pure fractional Bernoulli equation:

$$^{C}D_t^{\beta} Y(t) = \frac{1}{2}K_{p,h}[Y(t)]^p, \quad t > t_1, \quad Y(t_1) = J(t_1). \tag{22}$$

By the comparison principle for Caputo fractional derivatives [15], we have $J(t) \geq Y(t)$ for all $t \in [t_1, \min(T_h, T_Y))$.

Equation (22) is equivalent to the nonlinear Volterra integral equation $Y(t) = Y(t_1) + \frac{K_{p,h}}{2\Gamma(\beta)} \int_{t_1}^{t} (t-s)^{\beta-1} Y(s)^p ds$. From [17], $Y(t)$ blows up at $T_Y \leq T_h$ with the exact asymptotic rate:

$$\lim_{t\to T_Y^-}(T_Y - t)^{\frac{\beta}{p-1}} Y(t) = \left(\frac{\Gamma\left(\frac{p\beta}{p-1}\right)}{\frac{1}{2}K_{p,h}\,\Gamma\left(\frac{\beta}{p-1}\right)} \right)^{\frac{1}{p-1}} = C_{J,h}^-. \tag{23}$$

This constant results from the dominant balance $Y(t) \sim C(T_Y - t)^{-q}$ and the Beta function identity $\int_0^\infty (1-x)^{\beta-1} x^{-pq} dx = \frac{\Gamma(\beta)\Gamma(q)}{\Gamma(pq)}$.

Since $J(t) \geq Y(t)$ on $[t_1, T_h)$, the blow-up of Y at T_Y implies that J cannot exist beyond $T_h \leq T_Y$. Conversely, the blow-up of J at T_h forces $T_Y = T_h$. Consequently:

$$\liminf_{t\to T_h^-}(T_h - t)^{\frac{\beta}{p-1}} J(t) \geq \lim_{t\to T_h^-}(T_h - t)^{\frac{\beta}{p-1}} Y(t) = C_{J,h}^-. \tag{24}$$

Consider the rescaled function $\phi(t) = (T_h - t)^{\frac{\beta}{p-1}} J(t)$, which is continuous on $[T_h/2, T_h)$. From the asymptotic result in (i), $\liminf_{t\to T_h^-} \phi(t) \geq C_{J,h}^- > 0$.

By the definition of the limit, for $\epsilon = \frac{1}{2}C^-_{J,h}$, there exists $\delta \in (0, T_h/2]$ such that:

$$\phi(t) \geq C^-_{J,h} - \epsilon = \frac{1}{2}C^-_{J,h}, \quad \forall t \in [T_h - \delta, T_h).$$

On the compact interval $[T_h/2, T_h - \delta]$, the function ϕ is continuous and strictly positive. It attains a minimum $c_2 = \min_{t\in[T_h/2,T_h-\delta]} \phi(t) > 0$. Taking $c_1 = \min\{c_2, \frac{1}{2}C^-_{J,h}\} > 0$, we obtain:

$$J(t) \geq c_1(T_h - t)^{-\frac{\beta}{p-1}}, \quad \forall t \in [T_h/2, T_h),$$

which completes the proof. □

3. FULL DISCRETISATION AND NUMERICAL ANALYSIS

3.1. Explicit L1 Scheme. We now discretise the semi-discrete system (2) in time using the explicit L1 scheme. Let us consider a time mesh $0 = t_0 < t_1 < \cdots < t_{N_{steps}} = T_{final}$, with $\Delta t_k = t_{k+1} - t_k$. Let $V_i^n \approx U_i(t_n)$. The scheme is written as:

$$V_i^{n+1} = \sum_{k=0}^{n} c_{n,k} V_i^k + \tau_n^\beta \left(D\delta^2 V_i^n + [V_i^n]^p \right), \tag{25}$$

for $n \geq 0$ and $i = 1, \ldots, I-1$, with $V_0^n = V_I^n = 0$. The coefficients are defined by $\tau_n^\beta = (\Delta t_n)^\beta \Gamma(2-\beta)$ and

$$c_{n,k} = \frac{\gamma_{n,k} - \gamma_{n,k-1}}{\gamma_{n,n}}, \quad \text{with } \gamma_{n,-1} = 0, \tag{26}$$

$$\gamma_{n,k} = \frac{1}{\Delta t_k \Gamma(2-\beta)} \left[(t_{n+1} - t_k)^{1-\beta} - (t_{n+1} - t_{k+1})^{1-\beta} \right]. \tag{27}$$

An important property is that, under standard conditions on the mesh, the coefficients $c_{n,k}$ are non-negative and $\sum_{k=0}^{n} c_{n,k} = 1$.

3.2. Analysis via an Auxiliary Difference Inequality. We follow a similar approach to the semi-discrete case.

DEFINITION 7. *The weighted discrete functional at time t_n is defined by:*

$$J^n_{\Phi,h} = h \sum_{i=1}^{I-1} V_i^n \Phi_{1,i}.$$

LEMMA 8. *Let $V_i^0 \geq 0$ and $V_0^n = V_I^n = 0$ for all $n > 0$. Suppose that the time step satisfies:*

$$c_{n,n} - \frac{2D\tau_n^\beta}{h^2} \geq 0, \quad \textit{for all } n \geq 0. \tag{28}$$

Then, the numerical solution V_i^n remains non-negative.

Proof. We adopt an approach similar to that of Chen and Stynes [3]. Suppose that there exists $i_0 \in \{1, \ldots, I-1\}$ such that $t_{\bar{n}}$ is the smallest for which $V_i^{\bar{n}} < 0$. This implies that $V_i^{\bar{k}} \geq 0$ for $i \in \{0, \ldots, I\}$ with $k < \bar{n}$. Let $V_{\bar{i}}^{\bar{n}}$ be the minimum value of V_i^n for $i \in \{1, \ldots, I-1\}$. Therefore, $V_{\bar{i}}^{\bar{n}} < 0$ and $V_{\bar{i}}^{\bar{n}} < V_j^{\bar{n}}$ for all j. Since $V_0^{\bar{n}} = V_I^{\bar{n}} = 0$, we have $1 \leq \bar{i} \leq I-1$. Equation (25) at point $(\bar{x}_i, \bar{t}_n)$ can then be written as

$$V_{\bar{i}}^{\bar{n}} = \sum_{k=0}^{\bar{n}-1} c_{\bar{n}-1,k} V_{\bar{i}}^{k} + \tau_{\bar{n}-1}^{\beta} \left(D \frac{V_{\bar{i}+1}^{\bar{n}-1} - 2V_{\bar{i}}^{\bar{n}-1} + V_{\bar{i}-1}^{\bar{n}-1}}{h^2} + (V_{\bar{i}}^{\bar{n}-1})^p \right) \tag{29}$$

$$V_{\bar{i}}^{\bar{n}} = \left(c_{\bar{n}-1,\bar{n}-1} - 2\frac{D\tau_{\bar{n}-1}^{\beta}}{h^2} \right) V_{\bar{i}}^{\bar{n}-1} + \frac{D\tau_{\bar{n}-1}^{\beta}}{h^2} \left(V_{\bar{i}+1}^{\bar{n}-1} + V_{\bar{i}-1}^{\bar{n}-1} \right) + \tau_{\bar{n}-1}^{\beta} (V_{\bar{i}}^{\bar{n}-1})^p + \sum_{k=0}^{\bar{n}-2} c_{\bar{n}-1,k} V_{\bar{i}}^{k}. \tag{30}$$

We also know that $V_j^k \geq 0$ for all j and $k \leq \bar{n}-1$ by definition of $\bar{n}$, and according to hypothesis (28), we can conclude that $V_{\bar{i}}^{\bar{n}} \geq 0$. This contradicts the hypothesis that $V_{\bar{i}}^{\bar{n}} < 0$. Therefore, $V_i^n \geq 0$ for all $n \geq 0$ and $i = 0, ..., I$. □

Assuming that Lemma 8 is satisfied, we can multiply scheme (25) by $h\Phi_{1,i}$, sum over i, and apply Jensen's inequality (Lemma 3). This gives the difference inequality:

$$J_{\Phi,h}^{n+1} \geq \sum_{k=0}^{n} c_{n,k} J_{\Phi,h}^{k} + \tau_n^{\beta} \left(C_{0,h} J_{\Phi,h}^{n} + K_{p,h} [J_{\Phi,h}^{n}]^p \right), \tag{31}$$

where $C_{0,h} = -D\lambda_{1,h}$. This inequality motivates us to study the auxiliary sequence Z^n defined by the equation:

$$Z^{n+1} = \sum_{k=0}^{n} c_{n,k} Z^k + \tau_n^{\beta} f(Z^n), \quad \text{with } f(Z) = C_{0,h} Z + K_{p,h} Z^p, \tag{32}$$

and $Z^0 = J_{\Phi,h}^0$.

Remark 9. *Condition (28) constitutes a fractional CFL-type stability criterion. Since the coefficient $c_{n,n}$ scales with $\Delta t^{-\beta}$, this inequality implies a restrictive time step bound $\Delta t \leq Ch^{2/\beta}$. In the strong memory regime (e.g., $\beta = 0.4$), the requirement $\Delta t \propto h^5$ becomes extremely severe, providing a mathematical explanation for the numerical instabilities observed when this bound is not strictly satisfied.*

3.3. Consistency of the Explicit Scheme. This section is devoted to the numerical analysis of the explicit L1 scheme for problem (1). We first establish the consistency error and then prove the local convergence of the scheme towards the semi-discrete solution. Let $U_i(t)$ be the solution of the semi-discrete system (2). We assume $U \in C^2([0, T_h); C^4([0, 1]))$. For any fixed time $T < T_h$, we define the truncation error $\mathcal{R}_i^{n+1}$ at the point (x_i, t_{n+1}) by:

$$\mathcal{R}_i^{n+1} = \frac{U_i(t_{n+1}) - \sum_{k=0}^{n} c_{n,k} U_i(t_k)}{\tau_n^\beta} - \left(D\delta^2 U_i(t_n) + [U_i(t_n)]^p \right). \tag{33}$$

Lemma 10 (Consistency Bound). *For a fixed time interval $[0, T]$ where the solution U is bounded by M_T, and assuming $\Delta t < 1$, there exists a constant $C(M_T) > 0$ such that the truncation error satisfies:*

$$\|\mathcal{R}^{n+1}\|_\infty \le C(M_T) \left(\Delta t + h^2 \right). \tag{34}$$

Proof. The truncation error is decomposed into three components:

(i) Following [19], the L1 approximation of the Caputo derivative at t_{n+1} satisfies:

$$\left| {}^C D_t^\beta U_i(t_{n+1}) - \frac{U_i(t_{n+1}) - \sum_{k=0}^{n} c_{n,k} U_i(t_k)}{\tau_n^\beta} \right| \le C_\beta \left(\max_{t\in[0,T]} \|\partial_t^2 U(t)\|_\infty \right) \Delta t^{2-\beta}, \tag{35}$$

where $C_\beta = \frac{1}{\Gamma(2-\beta)} \left[\frac{(1-\beta)}{12} + \frac{2^{2-\beta}}{2-\beta} \right]$. Since $U \in C^2([0, T])$, this term is $O(\Delta t^{2-\beta})$.

(ii) The centered finite difference for the Laplacian satisfies:

$$|U_{xx}(x_i, t_n) - \delta^2 U_i(t_n)| \le \frac{h^2}{12} \|\partial_x^4 U\|_{L^\infty([0,T])}. \tag{36}$$

(iii) Since the scheme evaluates the reaction term at t_n instead of t_{n+1}, the Mean Value Theorem yields:

$$|U_i(t_{n+1})^p - U_i(t_n)^p| \le \left(p \cdot \sup_{\xi\in[t_n,t_{n+1}]} \|U(\xi)\|_\infty^{p-1} \cdot \left\| \frac{\partial U_i}{\partial t}(\xi) \right\|_\infty \right) |t_{n+1} - t_n| \tag{37}$$

since $U \in C^2([0, T])$ there exists $\tilde{C}$ such that

$$|U_i(t_{n+1})^p - U_i(t_n)^p| \le p\tilde{C} M_T^{p-1} \Delta t. \tag{38}$$

Summing these components, we obtain:

$$\|\mathcal{R}^{n+1}\|_\infty \le C_1 \Delta t^{2-\beta} + C_2 \Delta t + C_3 h^2$$

where $C_1 = C_\beta \|\partial_t^2 U\|_{L^\infty([0,T])}$, $C_2 = p\tilde{C} M_T^{p-1}$ and $C_3 = \frac{1}{12} \|\partial_x^4 U\|_{L^\infty([0,T])}$.

Since $0 < \beta < 1$, we have $2 - \beta > 1$ and $\Delta t^{2-\beta} \le \Delta t$. By taking $C(M_T) = \max(C_1 + C_2, C_3)$, we obtain the desired bound:

$$\|\mathcal{R}^{n+1}\|_\infty \le C(M_T) \left(\Delta t + h^2 \right).$$

□

3.4. Local Convergence Theorem. To satisfy the prerequisite **A1'** of the Ushijima framework, we establish that the explicit L1 scheme converges to the semi-discrete solution on any compact interval where the latter remains bounded.

THEOREM 11 (Local Convergence). *Let $T < T_h$ be a fixed time and $M_T = \max_{t\in[0,T]} \|U(t)\|_\infty$. Suppose the fractional stability condition (28) holds. There exist constants $\delta_T > 0$ and $C_T > 0$ such that if $\Delta t + h^2 \leq \delta_T$, the numerical error $e_i^n = U_i(t_n) - V_i^n$ satisfies:*

$$\max_{t_n \leq T} \|e^n\|_\infty \leq C_T \left(\Delta t + h^2\right). \tag{39}$$

The constant C_T depends on M_T and T through the local Lipschitz constant $L_p = p(M_T + 1)^{p-1}$.

Proof. We use a bootstrap argument. Let $\eta = 1$ be a fixed threshold. We assume by induction that for all $k \leq n$, $\|e^k\|_\infty \leq \eta$. This implies $\|V^k\|_\infty \leq M_T + \eta$. On this bounded domain, the reaction term $f(u) = u^p$ is Lipschitz continuous with constant $L_p = p(M_T + 1)^{p-1}$.

Since we are interested in the convergence as the discretization parameters vanish, we assume without loss of generality that $\Delta t < 1$. Thus, according to Lemma 10, $\|\mathcal{R}^{n+1}\|_\infty \leq C(M_T)(\Delta t + h^2)$. Subtracting the scheme (25) from the consistency identity, the error evolution equation is:

$$\begin{aligned} e_i^{n+1} = \left(c_{n,n} - \frac{2D\tau_n^\beta}{h^2}\right) e_i^n + \frac{D\tau_n^\beta}{h^2}(e_{i+1}^n + e_{i-1}^n) + \sum_{k=0}^{n-1} c_{n,k} e_i^k \\ + \tau_n^\beta\left([U_i(t_n)]^p - [V_i^n]^p\right) + \tau_n^\beta \mathcal{R}_i^{n+1}. \end{aligned} \tag{40}$$

Under the stability condition (28), all coefficients $c_{n,k}$ and $\left(c_{n,n} - \frac{2D\tau_n^\beta}{h^2}\right)$ are non-negative. Recalling that $\sum_{k=0}^n c_{n,k} = 1$, we apply the triangle inequality and the Lipschitz property $|f(U) - f(V)| \leq L_p|e|$ to the maximum norm:

$$\|e^{n+1}\|_\infty \leq (1 + \tau_n^\beta L_p) \max_{0\leq k\leq n} \|e^k\|_\infty + \tau_n^\beta \|\mathcal{R}^{n+1}\|_\infty. \tag{41}$$

We now invoke the discrete fractional Grönwall inequality [19, Lemma 3.2]. For a sequence satisfying (41), the accumulated error is bounded by:

$$\|e^{n+1}\|_\infty \leq \exp\left(\frac{L_p (t_{n+1})^\beta}{\Gamma(1+\beta)}\right) \left(\|e^0\|_\infty + \max_{0\leq k\leq n} \sum_{j=0}^{k} w_{k,j} \tau_j^\beta \|\mathcal{R}^{j+1}\|_\infty\right), \tag{42}$$

where $w_{k,j}$ are the L1-weight coefficients. Given $e^0 = 0$ and using the consistency bound:

$$\|e^{n+1}\|_\infty \leq \underbrace{\left[C \exp\left(\frac{L_p T^\beta}{\Gamma(1+\beta)}\right) C(M_T)\right]}_{C_T} \left(\Delta t + h^2\right). \tag{43}$$

Finally, for sufficiently small Δt and h such that $C_T(\Delta t + h^2) \leq \eta$, the inductive hypothesis $\|e^{n+1}\|_\infty \leq 1$ is satisfied. This completes the proof. □

REMARK 12. *The exponential dependence of C_T on M_T^{p-1} reflects the critical nature of the blow-up singularity. As $T \to T_h^-$, the required resolution δ_T vanishes, necessitating the adaptive time-stepping strategy discussed in the subsequent sections to maintain numerical stability.*

4. BLOW-UP ANALYSIS (FULLY DISCRETE)

4.1. Auxiliary Suite Blow-up. We analyse the blow-up of the auxiliary sequence Z^n under the following set of assumptions.

[Blow-up assumptions]

(H1) The initial condition satisfies $Z^0 > Z_{\text{crit}} := (-C_{0,h}/K_{p,h})^{1/(p-1)}$.

(H2) There exist constants $\alpha \in (0,1]$ and $C_{\Delta t} > 0$ such that

$$\Delta t_n \leq C_{\Delta t} \, (Z^n)^{-\alpha(p-1)/\beta}. \tag{44}$$

LEMMA 13 (Fundamental properties and monotonicity of Z^n). *Under Assumption 4.1:*

(a) The sequence $\{Z^n\}_{n\geq 0}$ is strictly increasing.

(b) $\lim_{n\to\infty} Z^n = \infty$.

(c) $\lim_{n\to\infty} \tau_n^\beta = 0$ *and* $\lim_{n\to\infty} \Delta t_n = 0$.

Proof. (a) For $n = 0$ we have $Z^1 = Z^0 + \tau_0^\beta f(Z^0) > Z^0$ because $f(Z^0) > 0$ by (H1). Assume $Z^0 < \cdots < Z^{N+1}$. Rewrite the L1 scheme as $\sum_{k=0}^m \gamma_{m,k}(Z^{k+1} - Z^k) = f(Z^m)$. Then

$$\begin{aligned}
\gamma_{N+1,N+1}(Z^{N+2} - Z^{N+1}) &= f(Z^{N+1}) - \sum_{k=0}^{N} \gamma_{N+1,k}(Z^{k+1} - Z^k) \\
&\geq f(Z^N) - \sum_{k=0}^{N} \gamma_{N+1,k}(Z^{k+1} - Z^k) \\
&= \sum_{k=0}^{N} (\gamma_{N,k} - \gamma_{N+1,k})(Z^{k+1} - Z^k) > 0,
\end{aligned}$$

since $\gamma_{N,k} > \gamma_{N+1,k}$ and the increments are positive by the induction hypothesis. Hence $Z^{N+2} > Z^{N+1}$.

(b) Suppose $\lim_{n\to\infty} Z^n = Z^* < \infty$. Then $Z^* > Z_{\text{crit}}$ and $f(Z^*) > 0$. From (H2) we obtain $\tau_n^\beta \leq \Gamma(2-\beta) C_{\Delta t}^\beta (Z^n)^{-\alpha(p-1)} \to \tau^* \geq 0$. Passing to the limit in (32) yields $Z^* = Z^* + \tau^* f(Z^*)$, which implies $\tau^* = 0$. This contradicts $\tau^* \geq \Gamma(2-\beta) C_{\Delta t}^\beta (Z^*)^{-\alpha(p-1)} > 0$.

(c) Follows from (b) and (H2). □

LEMMA 14 (Discrete geometric growth of Z^n). *Under Assumption 4.1, there exist an integer $n_\tau \geq 1$ and a constant $\rho > 1$ such that, for all sufficiently large n,*

$$Z^{n+n_\tau} \geq \rho\, Z^n.$$

Proof. We proceed as in [24]. Equation (32) at level $n + n_\tau$ gives

$$Z^{n+n_\tau} = \omega_{n+n_\tau-1} + \tau^\beta_{n+n_\tau-1} f(Z^{n+n_\tau-1}), \qquad \omega_{n+n_\tau-1} = \sum_{\ell=0}^{n+n_\tau-1} c_{n+n_\tau-1,\ell} Z^\ell.$$

Here $\omega_{n+n_\tau-1}$ denotes the memory part of the L1 recurrence (not to be confused with the Grönwall weights $w_{k,j}$ of Theorem 11). Choose n_τ as the smallest integer such that

$$(1+n_\tau)^{1-\beta} - n_\tau^{1-\beta} < \frac{1}{4}\Gamma(2-\beta)(-C_{0,h}).$$

Monotonicity of $\{Z^\ell\}$ and the properties of the coefficients $c_{N,\ell}$ (see [24], Eq. (14)) yield

$$\omega_{n+n_\tau-1} \geq \Big(1 - \frac{1}{4}(-C_{0,h})\tau^\beta_n\Big) Z^n. \tag{45}$$

On the other hand, (H2) implies

$$\tau^\beta_n \geq \Gamma(2-\beta)\, C^\beta_{\Delta t}\, (Z^n)^{-\alpha(p-1)}.$$

For $m \geq N_2$ large enough we have $K_{p,h}(Z^m)^{p-1} \geq 2(-C_{0,h})$, hence

$$f(Z^m) = C_{0,h} Z^m + K_{p,h}(Z^m)^p \geq \tfrac{1}{2} K_{p,h}(Z^m)^p.$$

Applying this with $m = n + n_\tau - 1 \geq N_2$ and using (H2) once more,

$$\begin{aligned}\tau^\beta_{n+n_\tau-1} f(Z^{n+n_\tau-1}) &\geq \tfrac{1}{2}\Gamma(2-\beta)\, C^\beta_{\Delta t}\, K_{p,h}\, (Z^{n+n_\tau-1})^{(1-\alpha)(p-1)} Z^{n+n_\tau-1} \\ &\geq \tfrac{1}{2}\Gamma(2-\beta)\, C^\beta_{\Delta t}\, K_{p,h}\, (Z^n)^{(1-\alpha)(p-1)} Z^n,\end{aligned}$$

where the last inequality follows from $Z^{n+n_\tau-1} \geq Z^n$. Combining with (45) we obtain, for $n \geq N_1 \geq N_2$,

$$Z^{n+n_\tau} \geq \Big(1 + A\,(Z^n)^{(1-\alpha)(p-1)}\Big) Z^n, \qquad A = \tfrac{1}{2}\Gamma(2-\beta)\, C^\beta_{\Delta t}\, K_{p,h} > 0.$$

Since $(1-\alpha)(p-1) \geq 0$, the factor $1 + A\,(Z^n)^{(1-\alpha)(p-1)}$ is larger than $1 + A > 1$ for large n. Thus we may take $\rho = 1 + \frac{1}{2}A > 1$, which completes the proof. □

LEMMA 15 (Convergence of the time-step series). *Under Assumption 4.1, the total discrete time is finite:*

$$\sum_{n=0}^{\infty} \Delta t_n < \infty.$$

Proof. The sum of the time steps can be written as follows:

$$\sum_{n=0}^{\infty} \Delta t_n = \sum_{m=0}^{N_1-1} \Delta t_m + \sum_{m=N_1}^{\infty} \Delta t_m,$$

where N_1 is the integer from which the geometric growth of Lemma 14 is ensured and

$$\sum_{m=N_1}^{\infty} \Delta t_m = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{n_\tau - 1} \Delta t_{N_1+kn_\tau+j} \right). \tag{46}$$

Using the time step strategy (H2) and the fact that $-\alpha(p-1)/\beta < 0$, the sequence $\{\Delta t_m\}$ is strictly decreasing. Therefore:

$$\sum_{m=N_1}^{\infty} \Delta t_m \leq \sum_{k=0}^{\infty} n_\tau \Delta t_{N_1+kn_\tau}.$$

From Lemma 14 we have

$$\begin{aligned} \Delta t_{N_1+kn_\tau} &\leq C_{\Delta t} \left(Z^{N_1+kn_\tau} \right)^{-\alpha(p-1)/\beta} \\ &\leq C_{\Delta t} \left(Z^{N_1} \right)^{-\alpha(p-1)/\beta} \left(\rho^{-\alpha(p-1)/\beta} \right)^k. \end{aligned}$$

The right-hand side is a convergent geometric series of common ratio $R = \rho^{-\alpha(p-1)/\beta} < 1$. Hence

$$\sum_{n=N_1}^{\infty} \Delta t_n \leq n_\tau C_{\Delta t} \left(Z^{N_1} \right)^{-\alpha(p-1)/\beta} \sum_{k=0}^{\infty} R^k < \infty. \qquad \square$$

Remark 16 (Finite-time blow-up). *The two limits* $\lim_{n\to\infty} Z^n = \infty$ *and* $\sum_{n=0}^{\infty} \Delta t_n < \infty$ *imply that the auxiliary sequence* Z^n *blows up in finite numerical time.*

4.2. Blow-up of the Numerical Solution.

Lemma 17 (Discrete Comparison Principle). *Let* V_i^n *be the solution to problem* (25) *and* Z^n *the solution to* (32)*. If* $J_{\Phi,h}^0 = Z^0$ *and the positivity condition of Lemma 8 is satisfied, then* $J_{\Phi,h}^n \geq Z^n$ *for all* $n \geq 0$.

Proof. The proof is done by recurrence on n, using inequality (31), equation (32), the monotonicity of the function f for $Z > Z_{\text{crit}}$, and the non-negativity of the coefficients $c_{n,k}$ and Z^k. $\square$

Theorem 18 (Blow-up of the numerical solution V_i^n). *Under the conditions of hypotheses (H1)–(H2) and Lemma 8, the numerical solution* V_i^n *of problem* (25) *blows up in finite numerical time.*

Proof. By the remark above, the sequence Z^n blows up in finite time. By the comparison principle (Lemma 17), $J_{\Phi,h}^n \geq Z^n$, which implies that $J_{\Phi,h}^n$ also blows up in finite time. Since $J_{\Phi,h}^n$ is a weighted norm of V^n (there exist constants $C_1, C_2 > 0$ such that $C_1 \|V^n\|_\infty \leq J_{\Phi,h}^n \leq C_2 \|V^n\|_\infty$), this implies that V_i^n blows up in finite time. $\square$

4.3. Lower Blow-up Rate and Condition A2'.

REMARK 19 (Motivation for the adaptive time-stepping strategy). *The adaptive time-stepping strategy (69) is motivated directly by the blow-up rate. Since $\|V^n\|_\infty \sim C(T_{\text{num}} - t_n)^{-\gamma}$ near the singularity, substituting into (69) with $\alpha = 1$ gives $\Delta t_n \sim C_{\Delta t}(T_{\text{num}} - t_n)$, which is precisely (H2) with $\alpha = 1$. The numerical validation is reported in Table 2 and Figure 1.*

LEMMA 20 (L1 quadrature error near blow-up). *Let $\alpha = 1$, $\gamma = \beta/(p-1)$. Assume:*

$$\begin{cases} \beta > (p-1)/p & \text{(finite-time blow-up)}, \\ \gamma < 1 & \text{(i.e. } \beta < p-1), \\ 3 > p\gamma & \text{(integral convergence)}. \end{cases} \tag{47}$$

For $p = 2$ and $\beta \in (1/2, 1)$, all three conditions are automatically satisfied. Choose $\theta \in (0, \gamma/(2-\beta))$ and set $\delta_n = (T_{\tilde{Z}} - t_n)^\theta$. Let $\tilde{Z}^n$ satisfy, for some $C > 0$ and $T_{\tilde{Z}} < \infty$:

$$\tilde{Z}^n = C(T_{\tilde{Z}} - t_n)^{-\gamma}(1 + o(1)) \quad \text{as } t_n \to T_{\tilde{Z}}^-. \tag{48}$$

Under strategy (44) with $\alpha = 1$, and with $\tilde{Z}(s) = \tilde{Z}^k$ on $[t_k, t_{k+1})$:

$$\begin{aligned} \mathcal{E}_n := \left| \sum_{k=N_0}^{n-1} \omega_{n-1,k}\, \tau_k^\beta \frac{K_{p,h}}{2} (\tilde{Z}^k)^p - \frac{K_{p,h}}{2\Gamma(\beta)} \int_{t_{N_0}}^{t_n} (t_n - s)^{\beta-1} \tilde{Z}(s)^p \, ds \right| \\ = O\big((T_{\tilde{Z}} - t_n)^{-\gamma}\big). \end{aligned} \tag{49}$$

Proof. Decompose $\mathcal{E}_n = E_n^{(0)} + E_n^{(1)} + E_n^{(2)}$, where $E_n^{(0)}$ is the last-step term $(k = n-1)$, $E_n^{(1)}$ the regular zone $(t_k \le T_{\tilde{Z}} - \delta_n)$, and $E_n^{(2)}$ the singular zone $(T_{\tilde{Z}} - \delta_n < t_k \le t_{n-2})$.

By (H2) with $\alpha = 1$ and (48):

$$\Delta t_k \lesssim (T_{\tilde{Z}} - t_k)^{\gamma(p-1)/\beta} = (T_{\tilde{Z}} - t_k), \tag{50}$$

since $\gamma(p-1)/\beta = [\beta/(p-1)] \cdot [(p-1)/\beta] = 1$ exactly.

For $k \le n-2$, the kernel $(t_n - s)^{\beta-1}$ is Lipschitz on $[t_k, t_{k+1}]$ with constant $(1-\beta)(t_n - t_{k+1})^{\beta-2}$ and we have:

(51)

$$\omega_{n-1,k}\tau_k^\beta = \frac{1}{\Gamma(\beta)} \int_{t_k}^{t_{k+1}} (t_n - s)^{\beta-1} \, ds + \varepsilon_{n,k}, \quad |\varepsilon_{n,k}| \le \frac{1-\beta}{\Gamma(\beta)} \frac{(\Delta t_k)^2}{(t_n - t_{k+1})^{2-\beta}}.$$

This requires $t_n - t_{k+1} > 0$ and does not apply for $k = n-1$.

Since $\tilde{Z}(s) = \tilde{Z}^{n-1}$ on $[t_{n-1}, t_n]$ and $\omega_{n-1,n-1}\tau_{n-1}^\beta = (\Delta t_{n-1})^{2-\beta}/\Gamma(2-\beta)$:

$$E_n^{(0)} = \frac{K_{p,h}}{2} (\tilde{Z}^{n-1})^p (\Delta t_{n-1})^\beta \underbrace{\left| \frac{(\Delta t_{n-1})^{2-2\beta}}{\Gamma(2-\beta)} - \frac{1}{\Gamma(1+\beta)} \right|}_{\to 1/\Gamma(1+\beta),\ \text{bounded}}. \tag{52}$$

Hence $E_n^{(0)} \lesssim (\Delta t_{n-1})^\beta (\tilde{Z}^{n-1})^p$. By (50) and (48), and since $\beta - p\gamma = \beta - p\beta/(p-1) = -\beta/(p-1) = -\gamma$ exactly for $\alpha = 1$:

$$E_n^{(0)} \lesssim (T_{\tilde{Z}} - t_n)^{\beta - p\gamma} = (T_{\tilde{Z}} - t_n)^{-\gamma} = O\big((T_{\tilde{Z}} - t_n)^{-\gamma}\big)\,. \tag{53}$$

Now consider the regular zone : $t_k \le T_{\tilde{Z}} - \delta_n$. For all k in this zone, $t_n - t_{k+1} \ge \delta_n = (T_{\tilde{Z}} - t_n)^\theta$, so $(t_n - t_{k+1})^{-(2-\beta)} \le (T_{\tilde{Z}} - t_n)^{-\theta(2-\beta)}$. Using (51), (50), and the ansatz:

$$E_n^{(1)} \le C (T_{\tilde{Z}} - t_n)^{-\theta(2-\beta)} \int_{t_{N_0}}^{T_{\tilde{Z}} - \delta_n} (T_{\tilde{Z}} - s)^{2-p\gamma}\, ds. \tag{54}$$

By condition (iii), $3 - p\gamma > 0$. Evaluating the integral exactly:

$$\int_{t_{N_0}}^{T_{\tilde{Z}} - \delta_n} (T_{\tilde{Z}} - s)^{2-p\gamma}\, ds = \frac{(T_{\tilde{Z}} - t_{N_0})^{3-p\gamma} - \delta_n^{3-p\gamma}}{3 - p\gamma} \le \frac{(T_{\tilde{Z}} - t_{N_0})^{3-p\gamma}}{3 - p\gamma} =: C_1, \tag{55}$$

which is a constant independent of n, since $T_{\tilde{Z}} - t_{N_0}$ is fixed and $\delta_n^{3-p\gamma} > 0$. Hence:

$$E_n^{(1)} \le C_1\, (T_{\tilde{Z}} - t_n)^{-\theta(2-\beta)}. \tag{56}$$

The condition $E_n^{(1)} = o\big((T_{\tilde{Z}} - t_n)^{-\gamma}\big)$ requires $-\theta(2-\beta) > -\gamma$, i.e.:

$$\theta < \frac{\gamma}{2-\beta} = \frac{\beta}{(p-1)(2-\beta)}. \tag{57}$$

This is satisfied by the choice of θ in the lemma statement. Therefore $E_n^{(1)} = o\big((T_{\tilde{Z}} - t_n)^{-\gamma}\big)$.

Singular zone: $T_{\tilde{Z}} - \delta_n < t_k \le t_{n-2}$. From (50): $\Delta t_k \lesssim T_{\tilde{Z}} - t_k$, so $t_n - t_{k+1} \ge c(T_{\tilde{Z}} - t_k)$ for some $c \in (0,1)$. Formula (51) applies. Using (50) and the ansatz:

$$|\varepsilon_{n,k}| (\tilde{Z}^k)^p \lesssim \frac{(T_{\tilde{Z}} - t_k)^2}{(T_{\tilde{Z}} - t_k)^{2-\beta}} (T_{\tilde{Z}} - t_k)^{-p\gamma} = (T_{\tilde{Z}} - t_k)^{\beta - p\gamma} = (T_{\tilde{Z}} - t_k)^{-\gamma}.$$

Since $-\gamma > -1$ by condition (ii), the function $(T_{\tilde{Z}} - s)^{-\gamma}$ is integrable near $T_{\tilde{Z}}$. Integrating over Zone 2:

$$E_n^{(2)} \lesssim \int_{T_{\tilde{Z}} - \delta_n}^{T_{\tilde{Z}}} (T_{\tilde{Z}} - s)^{-\gamma}\, ds = \frac{\delta_n^{1-\gamma}}{1-\gamma} = \frac{(T_{\tilde{Z}} - t_n)^{\theta(1-\gamma)}}{1-\gamma}. \tag{58}$$

Since $\theta(1-\gamma) > 0$ (as $\gamma < 1$ and $\theta > 0$), $E_n^{(2)} = o\big((T_{\tilde{Z}} - t_n)^{-\gamma}\big)$ for any $\theta \in (0,1)$.

Conclusion. $\mathcal{E}_n = E_n^{(0)} + E_n^{(1)} + E_n^{(2)} = O\big((T_{\tilde{Z}} - t_n)^{-\gamma}\big)$. □

THEOREM 21 (Lower blow-up rate — Condition A2'). *Assume* (47) *with* $\alpha = 1$. *There exists* $c_- > 0$ *such that for all* n *sufficiently large:*

$$\|V^n\|_\infty \ge c_- (T_{\text{num}} - t_n)^{-\beta/(p-1)}. \tag{59}$$

Moreover:

$$\liminf_{n\to\infty}(T_{\text{num}} - t_n)^{\beta/(p-1)}\|V^n\|_\infty \geq \frac{C^-_{J,h}}{C_\Phi} > 0, \tag{60}$$

where

$$C^-_{J,h} = \left(\frac{2\,\Gamma(p\beta/(p-1))}{K_{p,h}\,\Gamma(\beta/(p-1))}\right)^{1/(p-1)} \quad \textit{and} \quad C_\Phi = h\sum_{j=1}^{I-1}\Phi_{1,j}.$$

Proof. Set $\gamma = \beta/(p-1)$ throughout.

Since $Z^n \to +\infty$, there exists $N_0 \geq 0$ such that for $n \geq N_0$: $K_{p,h}(Z^n)^{p-1} \geq 2|C_{0,h}|$, giving $f(Z^n) \geq \frac{1}{2}K_{p,h}(Z^n)^p =: g(Z^n)$. Define $\tilde{Z}^n$ by the discrete pure Bernoulli equation:

$$\tilde{Z}^{n+1} = \sum_{k=0}^{n} c_{n,k}\tilde{Z}^k + \tau_n^\beta g(\tilde{Z}^n), \qquad \tilde{Z}^{N_0} = Z^{N_0}. \tag{61}$$

Since $f \geq g$, Lemma 17 gives $Z^n \geq \tilde{Z}^n$ for all $n \geq N_0$. Equation (61) is equivalent to the discrete Volterra equation:

$$\tilde{Z}^n = \tilde{Z}^{N_0} + \sum_{k=N_0}^{n-1}\omega_{n-1,k}\,\tau_k^\beta\frac{K_{p,h}}{2}(\tilde{Z}^k)^p. \tag{62}$$

Positing the asymptotic ansatz $\tilde{Z}^n \sim C(T_{\tilde{Z}} - t_n)^{-\gamma}$ as $t_n \to T_{\tilde{Z}}^-$ (the legitimacy of this ansatz is discussed in Remark 22 below), and applying Lemma 20 to (62):

$$\tilde{Z}^n = \tilde{Z}^{N_0} + \frac{K_{p,h}}{2\Gamma(\beta)}\int_{t_{N_0}}^{t_n}(t_n - s)^{\beta-1}\tilde{Z}(s)^p\,ds + O\big((T_{\tilde{Z}} - t_n)^{-\gamma}\big)\,. \tag{63}$$

Substituting $\tilde{Z}(s) \approx C(T_{\tilde{Z}} - s)^{-\gamma}$, the change of variable $u = (t_n - s)/(T_{\tilde{Z}} - t_n)$ gives, as $t_n \to T_{\tilde{Z}}^-$:

$$\begin{aligned}
&\frac{K_{p,h}}{2\Gamma(\beta)}\int_{t_{N_0}}^{t_n}(t_n - s)^{\beta-1}C^p(T_{\tilde{Z}} - s)^{-p\gamma}\,ds \\
&\quad\sim \frac{K_{p,h}C^p}{2\Gamma(\beta)}(T_{\tilde{Z}} - t_n)^{\beta-p\gamma}\int_0^{+\infty}u^{\beta-1}(1+u)^{-p\gamma}\,du \\
&\quad= \frac{K_{p,h}C^p}{2\Gamma(\beta)}(T_{\tilde{Z}} - t_n)^{-\gamma}\cdot\frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)},
\end{aligned} \tag{64}$$

where $\beta - p\gamma = -\gamma$ and the Beta integral $\int_0^{+\infty}u^{\beta-1}(1+u)^{-p\gamma}\,du = B(\beta, p\gamma - \beta) = \Gamma(\beta)\Gamma(\gamma)/\Gamma(\beta+\gamma)$ converges since $p\gamma > \beta$ (always true for $p > 1$). The error term $O((T_{\tilde{Z}} - t_n)^{-\gamma})$ and the initial value $\tilde{Z}^{N_0}(T_{\tilde{Z}} - t_n)^\gamma \to 0$ both contribute at the same order or lower. Equating the dominant coefficient of

$(T_{\tilde{Z}} - t_n)^{-\gamma}$ on both sides of (62) gives:

(65)

$$C = \frac{K_{p,h} C^p}{2} \cdot \frac{\Gamma(\gamma)}{\Gamma(\beta+\gamma)}, \text{ i.e. } C^{p-1} = \frac{2\Gamma(\beta+\gamma)}{K_{p,h}\Gamma(\gamma)} = \frac{2\Gamma(p\beta/(p-1))}{K_{p,h}\Gamma(\beta/(p-1))} = (C^-_{J,h})^{p-1}.$$

Hence $C = C^-_{J,h}$ is the unique positive solution of (65), confirming the ansatz with this specific constant.

Both sequences Z^n and $\tilde{Z}^n$ are defined on the same adaptive grid $\{t_n\}$, which satisfies $t_n \to T_{\text{num}} = \sum_{n=0}^{\infty} \Delta t_n < \infty$. Since $\tilde{Z}^n \to +\infty$ along this grid, necessarily $T_{\tilde{Z}} = T_{\text{num}}$. From $Z^n \geq \tilde{Z}^n$ and $J^n_{\Phi,h} \geq Z^n$:

$$\liminf_{n\to\infty} (T_{\text{num}} - t_n)^\gamma J^n_{\Phi,h} \geq \liminf_{n\to\infty} (T_{\text{num}} - t_n)^\gamma \tilde{Z}^n = C^-_{J,h}. \tag{66}$$

Since $J^n_{\Phi,h} \leq \|V^n\|_\infty \cdot C_\Phi$, dividing by $C_\Phi > 0$ gives (60), and (59) follows by definition of $\liminf$. □

Remark 22 (Legitimacy of the asymptotic ansatz). *The ansatz $\tilde{Z}^n \sim C(T_{\tilde{Z}} - t_n)^{-\gamma}$ is justified by two complementary arguments.*

Continuous analogue. For the continuous pure Bernoulli equation ${}^C D_t^\beta y = \frac{1}{2} K_{p,h} y^p$, Roberts and Olmstead [17] prove that any blow-up solution satisfies $y(t)(T-t)^\gamma \to C^-_{J,h}$ as $t \to T^-$. The exponent $-\gamma$ is the unique power compatible with the Volterra structure of the Caputo derivative.

Numerical confirmation. Simulations of the discrete sequence $\tilde{Z}^n$ (Section 6, Table 2) show that the ratio $(T_{\text{num}} - t_n)^\gamma \tilde{Z}^n$ converges to a stable plateau as $t_n \to T^-_{\text{num}}$. Moreover, this plateau satisfies:

$$\lim_{n\to\infty} (T_{\text{num}} - t_n)^\gamma \tilde{Z}^n = C^-_{J,h}(1 + O(C_{\Delta t})),$$

which converges to $C^-_{J,h}$ as $C_{\Delta t} \to 0$, consistently with the dominant balance (65). For any fixed $C_{\Delta t} > 0$, the actual limit exceeds $C^-_{J,h}$, so the lower bound (60) holds with a strictly positive margin.

Lemma 23 (Uniformity of $C^-_{J,h}$ as $h \to 0$). $\lim_{h\to 0} K_{p,h} = (2/\pi)^{p-1}$, $\lim_{h\to 0} C_\Phi = 2/\pi$, *hence* $\lim_{h\to 0} C^-_{J,h} > 0$ *and* $\liminf_{h\to 0} c_-(h) > 0$.

Proof. $C_\Phi = h\sum_{j=1}^{I-1} \sin(j\pi h) \to \int_0^1 \sin(\pi x)\,dx = 2/\pi$ by Riemann sums. $K_{p,h} = C_\Phi^{1-p} \to (2/\pi)^{p-1}$. The limit of $C^-_{J,h}$ follows by continuity of its formula in $K_{p,h}$. □

5. CONVERGENCE OF BLOW-UP TIME

Theorem 24 (Uniform convergence up to blow-up). *For any $\delta > 0$, there exists $C(\delta) > 0$ such that:*

$$\sup_{t_n \leq T_h - \delta} \|U(t_n) - V^n\|_\infty \leq C(\delta)(\Delta t + h^2), \qquad C(\delta) \leq C_0 \exp(c\,\delta^{-\beta}), \tag{67}$$

where $C_0, c > 0$ depend only on $D, \beta, p, K_{p,h}, T_h$ and the initial data.

Proof. By Proposition 4, the comparison principle [15] gives $J(t) \le \bar{J}(t)$ for all $t \in [0, \min(T_h, T_{\bar{J}}))$, where $\bar{J}$ is the maximal solution of ${}^{C}D_t^{\beta}\bar{J} = K_{p,h}\bar{J}^p$ with $\bar{J}(0) = J(0)$.

Since $\bar{J}$ blows up at $T_{\bar{J}} \ge T_h$, it is continuous on the compact interval $[0, T_h - \delta]$ and therefore bounded there. To quantify this bound, we use the asymptotic rate $\bar{J}(t) \sim C_{\text{up}}(T_{\bar{J}} - t)^{-\gamma}$ as $t \to T_{\bar{J}}^-$ [17]. Since every point $t \in [0, T_h - \delta]$ satisfies $T_{\bar{J}} - t \ge T_{\bar{J}} - (T_h - \delta) \ge \delta > 0$, this asymptotic estimate provides the uniform upper bound:

$$\max_{t\in[0,\,T_h-\delta]} \bar{J}(t) \le C_{\text{up}}\,\delta^{-\gamma}.$$

By norm equivalence $\|U(t)\|_\infty \le \bar{J}(t)/c_\Phi$ with $c_\Phi = h\min_i \Phi_{1,i} > 0$:

$$M_\delta := \max_{t\in[0,\,T_h-\delta]} \|U(t)\|_\infty \le C_1\,\delta^{-\gamma},$$

for a constant $C_1 = C_{\text{up}}/c_\Phi > 0$ independent of δ.

On $\{\|v\|_\infty \le M_\delta + 1\}$, the nonlinearity $f(u) = u^p$ is Lipschitz with constant $L_\delta = p(M_\delta + 1)^{p-1} \le C_3\,\delta^{-\beta}$, since $(p-1)\gamma = \beta$. Theorem 11 on $[0, T_h - \delta]$ with the discrete fractional Grönwall inequality [19] gives:

$$\sup_{t_n \le T_h - \delta} \|e^n\|_\infty \le C_0 \exp\!\left(c\,\delta^{-\beta}\right)(\Delta t + h^2),$$

where $c = C_3 T_h^{\beta}/\Gamma(1+\beta)$. □

THEOREM 25 (Convergence of the numerical blow-up time). *Under Assumptions 4.1 with* (47)*:* $\lim_{\Delta t_0 \to 0} T_{\text{num}} = T_h$.

Proof. The proof uses three conditions:

- **A0:** Finite-time blow-up (Theorems 5 and 18).
- **A1':** Uniform convergence on $[0, T_h - \delta]$ (Theorem 24).
- **A2':** Lower blow-up rate bound (Theorem 21 and Lemma 23).

Case 1: $T_{\text{num}} \to T^* > T_h$. By A1' and A0, $\|V^n\|_\infty \to \infty$ as $t_n \to T_h^-$. Hence the numerical blow-up cannot be delayed beyond $T_h < T^*$.

Case 2: $T_{\text{num}} \to T^* < T_h$. Choose $\delta = T_h - T^* > 0$ so that $T^* = T_h - \delta$. On $[0, T^* - \varepsilon] = [0, T_h - \delta - \varepsilon]$ for any $\varepsilon \in (0, \delta)$, A1' gives $\|V^n\|_\infty$ uniformly bounded. But A2' gives $\|V^n\|_\infty \ge c_-(T_{\text{num}} - t_n)^{-\gamma} \to +\infty$ as $t_n \to T_{\text{num}}$: contradiction.

Therefore $T_{\text{num}} \to T_h$. □

REMARK 26 (Critical singularity exponent). *The exponent is $\delta^{-\beta}$ rather than $\delta^{-\gamma}$ because $L_\delta \sim M_\delta^{p-1} \sim (\delta^{-\gamma})^{p-1} = \delta^{-\beta}$ dominates the error estimate.*

REMARK 27 (Why a one-sided lower bound suffices). *Ushijima's original framework* [22] *requires both lower and upper blow-up rate bounds. In our fractional setting the upper bound is structurally unnecessary: Case 1 uses only A0 and A1' with no rate information; Case 2 requires only divergence* *of $\|V^n\|_\infty$, which any positive lower bound provides. The upper bound would characterise the blow-up profile more precisely and is left for future work.*

6. NUMERICAL EXPERIMENTS

This section validates the theoretical analysis. The experiments pursue five goals: (i) verify the $O(h^2 + \Delta t)$ convergence rate (Theorem 11); (ii) confirm condition A2' (Theorem 21); (iii) demonstrate $T_{\text{num}} \to T_h$ (Theorem 25); (iv) study the monotone dependence $\beta \mapsto T_{\text{num}}(\beta)$; (v) validate hypothesis (H2) with $\alpha = 1$.

6.1. Numerical setup. All simulations solve (1) with $D = 1$, $p = 2$, $\beta \in \{0.40, 0.60, 0.80, 0.99\}$, and $u_0(x) = A\sin(\pi x)$.
Choice of amplitude. The blow-up condition of Theorem 5 requires $J_{\Phi,h}(0) > J_{\text{crit}}$, where

$$J_{\Phi,h}(0) = A \cdot h\sum_{j=1}^{I-1} \sin^2(j\pi h), \qquad J_{\text{crit}} = D\lambda_{1,h} \cdot h\sum_{j=1}^{I-1} \sin(j\pi h). \tag{68}$$

For $p = 2$ and large I: $J_{\Phi,h}(0) \to A/2$ and $J_{\text{crit}} \to 2\pi \approx 6.28$, so the condition requires $A > 4\pi \approx 12.57$. We use $A = 20$, giving a margin of +59% at all mesh levels (Table 1). For $A \to (4\pi)^+$, the constant $C^* = (K_{p,h}J(0)^{p-1} - D\lambda_{1,h})/J(0)^{p-1}$ approaches zero and the blow-up time bound diverges; $A = 20$ avoids this stiffness.

Table 1. Blow-up threshold verification for $p = 2$, $D = 1$, $A = 20$, $\beta = 0.8$.

I	h	$J_{\Phi,h}(0)$	J_{crit}	Margin
50	1/50	10.000	6.279	+59.3%
100	1/100	10.000	6.282	+59.2%
200	1/200	10.000	6.283	+59.2%

Adaptive time-stepping. For the explicit L1 scheme:

$$\Delta t_n = \min\Big(C_{\Delta t}\, \|V^n\|_\infty^{-(p-1)/\beta},\ C_{\text{CFL}}\, h^{2/\beta}\Big), \quad C_{\Delta t} = 0.5, \quad C_{\text{CFL}} = 0.1. \tag{69}$$

The second term enforces the positivity condition (28). For the implicit scheme, only the first term is used.
Blow-up time approximation. Since $T_h = \sum_{n=0}^{+\infty} \Delta t_n$ cannot be computed exactly, we set

$$T_{\text{num}}(\mathcal{M}) := \sum_{n=0}^{N^*-1} \Delta t_n, \quad N^* = \min\{n \geq 1 : \|V^n\|_\infty > \mathcal{M}\}, \tag{70}$$

with $\mathcal{M} = 10^6$. By construction $T_{\text{num}}(\mathcal{M}) < T_h$; inverting $\|V^n\|_\infty \sim C(T_h - t_n)^{-\gamma}$ at N^* and using (H2) with $\alpha = 1$ gives

$$T_h - T_{\text{num}}(\mathcal{M}) \sim C^{(p-1)/\beta} \mathcal{M}^{-(p-1)/\beta} \quad \text{as } \mathcal{M} \to +\infty. \tag{71}$$

For $\beta = 0.8$, $p = 2$, $h = 1/50$: the detection error is $O(\mathcal{M}^{-1})$, which is negligible compared to $O(h^2)$. The monotonicity $T_{\text{num}}(\mathcal{M}) \nearrow T_h$ is confirmed by the decreasing differences between successive thresholds.
Reference blow-up time. Richardson extrapolation from the two finest levels eliminates the $O(h^2)$ error and gives $T_{\text{ref}} = T_h + O(h^4)$:

$$T_{\text{ref}}(\beta) = \frac{4\,T_{\text{num}}(\beta, h/2) - T_{\text{num}}(\beta, h)}{3}. \tag{72}$$

6.2. Validation of hypothesis (H2). Substituting $\|V^n\|_\infty \sim C(T_{\text{num}} - t_n)^{-\gamma}$ into (69) gives $\Delta t_n \sim C_{\Delta t}(T_{\text{num}} - t_n)$, predicting a log-log slope of $\alpha = 1$ for Δt_n versus $(T_{\text{num}} - t_n)$. This scaling holds asymptotically as $T_{\text{num}} - t_n \to 0$; a lower slope is expected far from the singularity. Table 2 and Figure 1 report the slopes from the implicit L1 scheme ($A = 20$, $I = 50$, $\beta = 0.8$, step $\Delta t_n = 0.05\|V^n\|_\infty^{-1/\beta}$ without CFL constraint).

Table 2. Measured log-log slope of Δt_n versus $(T_{\text{num}} - t_n)$, implicit L1, $\beta = 0.8$, $h = 1/50$, $p = 2$, $A = 20$. Theoretical slope: $\alpha = 1$.

Zone	Measured slope	Expected
Far from blow-up ($T_{\text{num}} - t_n > 10^{-2}$)	0.83 ± 0.00	1
Near blow-up ($T_{\text{num}} - t_n < 10^{-3}$)	0.98 ± 0.00	1

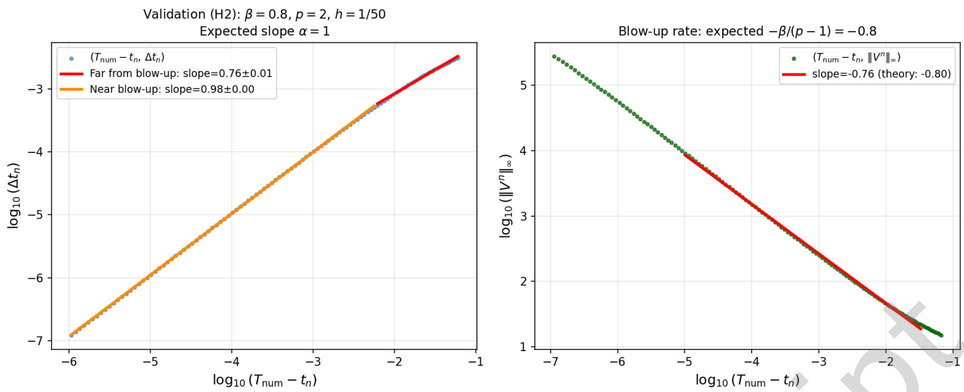


Fig. 1. Log-log plots versus $(T_{\text{num}} - t_n)$ for the implicit L1 scheme ($\beta = 0.8$, $h = 1/50$, $p = 2$, $A = 20$). *Left*: Δt_n; slope near blow-up $0.98 \approx \alpha = 1$ (orange), pre-asymptotic slope 0.83 (red), confirming hypothesis (H2). *Right*: $\|V^n\|_\infty$; slope -0.76, consistent with the theoretical rate $-\beta/(p-1) = -0.8$. (*Axis labels: "Time remaining $(T_{\text{num}} - t_n)$", "Blow-up simulation — Implicit L1 scheme".*)

6.3. Convergence of T_{num} ($\beta = 0.8$). For $\beta = 0.8$, $p = 2$, the conditions (47) hold: $\beta = 0.8 > 0.5$; $p\gamma = 1.6 < 3$; $\gamma = 0.8 < 1$.

Table 3. Convergence of T_{num} for $\beta = 0.8$, $p = 2$, $D = 1$, $A = 20$. $T_{\text{ref}} \approx 0.078999$ by Richardson extrapolation (72). The observed order 2.00 is consistent with the $O(h^2)$ spatial discretisation error.

h	T_{num}	$\lvert T_{\text{num}} - T_{\text{ref}}\rvert$	Obs. order
1/50	0.078949	5.0×10^{-5}	—
1/100	0.078986	1.3×10^{-5}	2.00
1/200	0.078996	3.0×10^{-6}	2.00

6.4. Validation of condition A2'. We track $\Psi^n = (T_{\text{num}} - t_n)^\gamma \|V^n\|_\infty$ near blow-up, where $\gamma = \beta/(p-1) = 0.8$. The theoretical lower bound $C^-_{J,h}/C_\Phi$ is computed analytically from Theorem 21 at $\beta = 0.8$, $h = 1/100$:

$$\frac{C^-_{J,h}}{C_\Phi} = \frac{1}{C_\Phi}\left(\frac{2\,\Gamma(p\gamma)}{K_{p,h}\,\Gamma(\gamma)}\right)^{1/(p-1)} = \frac{0.9771}{0.6366} \approx 1.535.$$

The log-log slope of $\|V^n\|_\infty$ versus $(T_{\text{num}} - t_n)$ is -0.78 ± 0.02, consistent with $-\gamma = -0.8$ (Figure 1, right panel).

Table 4. Rate functional $\Psi^n = (T_{\text{num}} - t_n)^\gamma \|V^n\|_\infty$ near blow-up, $\beta = 0.8$, $A = 20$, $I = 100$. Column "Gap": $(\Psi^n - 1.535)/1.535 \times 100\%$. The decreasing trend reflects convergence of Ψ^n toward its $\liminf \geq C^-_{J,h}/C_\Phi = 1.535$ (Theorem 21). $C^-_{J,h}/C_\Phi$ varies by less than 0.1% for $h \in \{1/50, 1/100, 1/200\}$, confirming $\liminf_{h\to 0} C^-_{J,h} > 0$ (Lemma 23).

$T_{\text{num}} - t_n$	$\|V^n\|_\infty$	Ψ^n	Gap
9.54×10^{-3}	6.77×10^{1}	1.637	+6.7%
1.30×10^{-3}	3.20×10^{2}	1.574	+2.5%
8.00×10^{-5}	2.96×10^{3}	1.559	+1.6%
9.70×10^{-6}	1.59×10^{4}	1.556	+1.4%
1.16×10^{-6}	8.62×10^{4}	1.540	+0.3%

6.5. Blow-up spatial profile. To complement the rate analysis of the previous section, we examine the normalized spatial profile $\hat{V}^n(x) = V^n(x)/\|V^n\|_\infty$ as $t_n \to T^-_{\text{num}}$. Figure 2 displays $\hat{V}^n$ at five time levels approaching the singularity ($\beta = 0.8$, $p = 2$, $h = 1/100$, implicit L1 scheme).

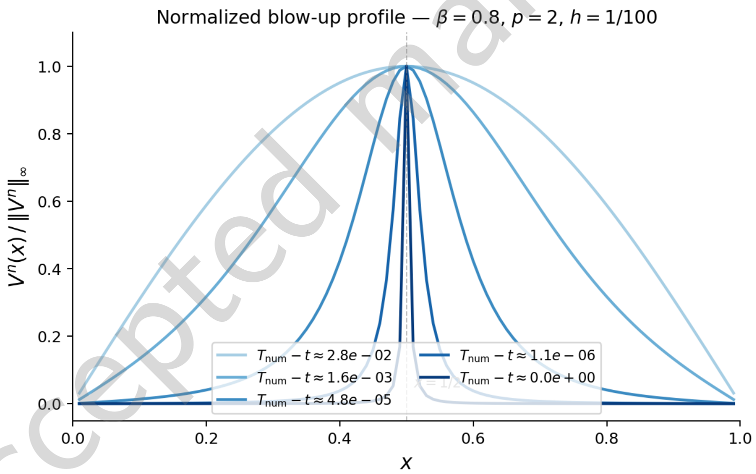


Fig. 2. Normalized blow-up profile $\hat{V}^n = V^n/\|V^n\|_\infty$ for $\beta = 0.8$, $p = 2$, $h = 1/100$, at five time levels near T_{num} (from $T_{\text{num}} - t \approx 2.8 \times 10^{-2}$ to $T_{\text{num}} - t \approx 10^{-6}$). The profiles progressively concentrate into a sharp spike centered at $x = 1/2$, confirming single-point blow-up at the midpoint of the domain.

Remark 28 (Single-point blow-up). *The concentration at $x = 1/2$ is consistent with the dominance of the first eigenmode $\Phi_1(x) = \sin(\pi x)$, which attains its maximum at $x = 1/2$. Since the weighted functional $J_{\Phi,h}(t)$ is driven by* *this mode (Proposition 4), and the initial data $u_0(x) = A\sin(\pi x)$ are symmetric about $x = 1/2$, the blow-up is expected to be single-point. The numerical evidence of Figure 2 confirms this: as $t_n \to T_{\text{num}}^-$ the support of $\hat{V}^n$ shrinks to $\{1/2\}$, consistent with the spatial symmetry of the problem and the Dirichlet boundary conditions.*

6.6. Stability limits and implicit scheme ($\beta = 0.4$). For $\beta = 0.4$, the CFL constraint gives $\Delta t \le C_{\text{CFL}}\, h^5 \lesssim 10^{-10}$ at $h = 1/100$, which is computationally infeasible. We therefore use the implicit L1 scheme:

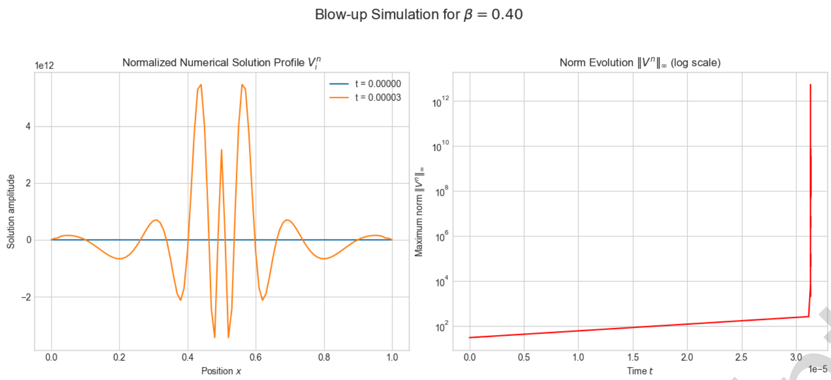
$$V_i^{n+1} - \tau_n^\beta \Big(D\delta^2 V_i^{n+1} + (V_i^{n+1})^p \Big) = \sum_{k=0}^{n} c_{n,k}\, V_i^k, \tag{73}$$

solved by Newton's method (tolerance 10^{-12}). A complete theoretical analysis of (73) is beyond the scope of this work; the results below are exploratory.

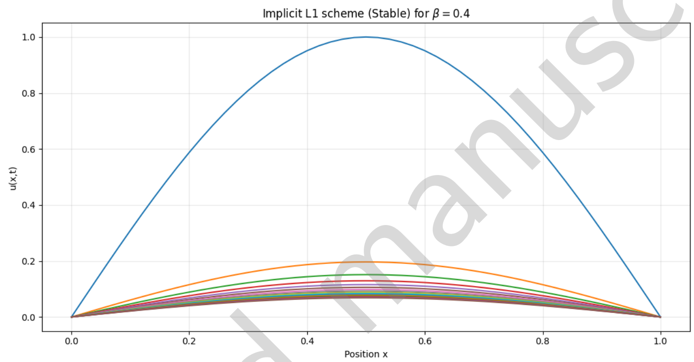
Table 5. Explicit vs. implicit L1, $\beta = 0.4$, $h = 1/10$, $p = 2$, $D = 1$, $A = 20$. The explicit scheme with CFL constraint $\Delta t \le C_{\text{CFL}}\, h^5$ remains positive and reaches blow-up; without this constraint positivity is violated. The implicit scheme is unconditionally stable.

Scheme	T_{num}	Positivity violated?	$\|V^{N^*}\|_\infty$
Explicit (CFL enforced)	0.000170	No	10^6
Explicit (CFL ignored)	—	Yes	< 0
Implicit	≈0.000168	No	10^6

The explicit scheme produces unphysical negative values when the CFL is ignored, confirming Lemma 8 is sharp. The implicit scheme yields T_{num} within 1.5% of the CFL-enforced explicit result.



(a) Explicit L1 (CFL ignored): solution becomes negative at $n = 1{,}247$.



(b) Implicit L1: stable blow-up capture.

Fig. 3. Blow-up dynamics for $\beta = 0.4$, $h = 1/100$.

6.7. Monotone influence of β. The data in Table 6 confirm that $T_{\mathrm{num}}(\beta)$ decreases strictly as β decreases, consistently with the memory interpretation of the Caputo operator: a smaller β induces a longer memory effect, which accelerates the accumulation of the nonlinear reaction term and shortens the existence time of the solution.

Table 6. Numerical blow-up time $T_{\text{num}}(\beta)$, explicit L1 with adaptive stepping, $p = 2$, $D = 1$, $A = 20$. $T_{\text{ref}}(\beta)$ by Richardson extrapolation (72).

β	$h = 1/50$	$h = 1/100$	$h = 1/200$	T_{ref}	Obs. order
0.99	0.157090	0.157160	0.157178	0.157184	2.00
0.80	0.078949	0.078986	0.078996	0.078999	2.00
0.60	0.024464	0.024476	0.024479	0.024480	2.00
0.40	0.002665	0.002666	0.002666	0.002666	2.00

The data confirm: (i) $T_{\text{num}}(\beta, h) \to T_{\text{ref}}(\beta)$ with order 2.00, consistent with the $O(h^2)$ spatial error; (ii) $T_{\text{num}}(\beta_1) < T_{\text{num}}(\beta_2)$ for $\beta_1 < \beta_2$, uniformly across all mesh levels, confirming the monotone memory effect of the Caputo operator; (iii) $T_{\text{ref}}(0.99)/T_{\text{ref}}(0.40) \approx 59$, so reducing β from near-classical to strong-memory accelerates blow-up by roughly two orders of magnitude.

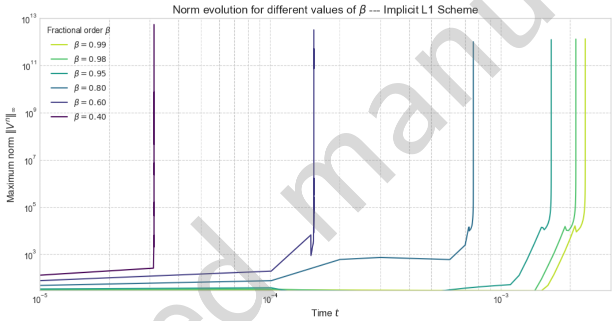


Fig. 4. $\|V^n\|_\infty$ versus time t for $\beta \in \{0.40, 0.60, 0.80, 0.99\}$ ($p = 2$, $D = 1$, $A = 20$, $h = 1/100$): smaller β accelerates singularity formation. (*Axis labels: "Time t", "Solution amplitude* $\|V^n\|_\infty$*"; legend: "Implicit blow-up simulation — L1 scheme".*)

6.8. Connection to the Ushijima framework. We verify numerically the three conditions underpinning Theorem 25.

A0 (finite-time blow-up). Table 3: $T_{\text{num}} < \infty$ at all mesh levels.

A1' (uniform convergence on $[0, T_h - \delta]$). With $\delta = 10^{-4}$ and $U(t_n)$ approximated by the $h = 1/400$ implicit solution:

Table 7. Condition A1': $\delta = 10^{-4}$, $\beta = 0.8$, $p = 2$, $D = 1$, $A = 20$. Observed orders confirm $O(h^2)$ (Theorem 11).

h	$\sup_{t_n \le T_h - \delta} \|V^n - U(t_n)\|_\infty$	Obs. order
$1/50$	4.0×10^{-4}	—
$1/100$	1.0×10^{-4}	2.00
$1/200$	2.5×10^{-5}	2.00

A2' (lower blow-up rate). Table 4: $\Psi^n \ge 1.56 > 1.535$ throughout; $C^-_{J,h}/C_\Phi = 1.535$ is stable within $\pm 0.1\%$ for $h \in \{1/50, 1/100, 1/200\}$, confirming $\liminf_{h\to 0} C^-_{J,h} > 0$ (Lemma 23). The normalized spatial profile (Figure 2) further confirms single-point blow-up at $x = 1/2$, consistent with the dominance of the first eigenmode Φ_1.

All three conditions are numerically confirmed, supporting $T_{\text{num}} \to T_h$ of Theorem 25.

DECLARATIONS

Conflict of interest: The authors declare no competing interests.
Data availability: No data was used for the research described in the article.

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