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A DEFECT-CORRECTION NODAL FINITE ELEMENT METHOD FOR TIME-DEPENDENT MAXWELL'S EQUATIONS ON POLYGONAL DOMAINS

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Abstract. This paper develops a nodal finite element Crank-Nicolson method of lines to solve the time-dependent Maxwell's equations on polygonal domains with re-entrant corners. Nodal Finite Element methods are used to solve Maxwell's equations with an optimal convergence rate when the domain is convex or has a smooth boundary, but may fail to converge if the domain has a re-entrant corner. The Defect-Correction method presented is based on a decomposition of the solution in terms of Fourier and Bessel's series, an extraction of the singular function and an approximation of the regular part of the solution. Optimal convergence results are recovered using the method in both the energy norm and the L^2-norm.

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1. INTRODUCTION

Electromagnetic waves are of particular importance in applied physics, engineering, and materials science, for example, in radars, antennas, and the detection of cracks in metals. Electromagnetic phenomena are modeled by a particular system of partial differential equations (PDEs) called Maxwell's equations or equations of electromagnetism. Maxwell's equations involve a couple of vector functions $(\mathbf{u}, \mathbf{b})$ called electromagnetic field, where $\mathbf{u}$ is the electric field and $\mathbf{b}$ is the magnetic field.

A general result of numerical methods for PDEs is that the accuracy of approximation depends on the regularity of the unknown solution, see [1, 2, 3, 4]. It is well known that solutions of boundary value problems in domains with corners, cracks, edges and conic vertices may entail singularities, even if the data are smooth. Moreover, the asymptotic behavior of the solution near these geometric singularities has been well studied and the results are now well known, see, for example [5, 6, 7, 8]. However, unlike general elliptic boundary value problems, the solution of Maxwell's equations in domains with geometric

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singularities exhibits some regularity properties that make its approximation particularly difficult.

In the case of time-harmonic Maxwell's equations in a two-dimensional domain Ω with corners and for a given right hand side function in the Hilbert space $L^2(\Omega)^2$, it has been shown that the solution belongs to the Sobolev space $H^m(\Omega)^2$ $(m = 1, 2)$ only if the value of the largest angle is smaller than π/m, see, for example [9, 10, 11, 12]. It follows that the solution belongs to the space $H^1(\Omega)^2$ only if Ω is convex and to the space $H^2(\Omega)^2$ only if the largest angle at the corners is less than $\pi/2$. We would say that the solution of the boundary value problem has geometric singularities if for a given right hand side function in $L^2(\Omega)^2$ the solution belongs only to the Sobolev space $H^s(\Omega)^2$ with $s < 2$. It follows that on non-convex domains, the standard H^1-conforming FEM cannot be employed for the solution of Maxwell's equations.

The work of Costabel [13] in 2002 combined with the bilinear form introduced in [14] has allowed the scientific community to renew the nodal FEM for Maxwell's Equations and some new finite element schemes have been developed in order to recover the convergence rate on non-convex domains. We mention in particular:

- The singular field method (SFM) introduced by Hazard and Lenoir [15], Hazard et al. [16, 17]. This method consists in splitting the solution $\mathbf{u}$ into two parts $\mathbf{u} = \mathbf{u}_r + \gamma_s \mathbf{u}_s$, with a "regular part" $\mathbf{u}_r$ that lies in the usual Sobolev space H^1 and a singular part $\mathbf{u}_s$ that lies in a known finite dimensional space and a constant γ_s to be determined from the finite element linear system. The constant γ_s is known as the coefficient of singularity. With this method one can approximate the solution in non-convex polygonal domains with nodal FEM but the rate of convergence is still lower than the one on smooth domains because the splitting is not optimal.
- The Orthogonal Singular Field Method (OSFM) introduced by Assous et al. [17], where the solution $\mathbf{u}$ is again split into two parts $\mathbf{u} = \mathbf{u}_r + (\mathbf{u}'_r + \gamma_s \mathbf{u}_s)$, a regular part $\mathbf{u}_r$ that lies in a subspace of H^1 and the second part $(\mathbf{u}'_r + \mathbf{u}_s)$ is in a subspace of the space of solutions orthogonal to the regular space. The OSFM and the SFM have the same rate of convergence but the OSFM is more stable due to the omission of the use of cut-off functions present in the SFM for the instabilities due to the use of cut-off functions.
- The "λ-approach" for two-dimensional vector problems developed by Jamelot in 2004 [18], where the splitting $\mathbf{u} = \mathbf{u}_r + \gamma_s \mathbf{u}_s$ is done but γ_s is given by a formula and depends only on the domain and the initial data. This method ameliorates the rate of convergence of the SFM and the OSFM but the rate is still not optimal (of order $\mathcal{O}(h^{2\pi/\omega-1-\varepsilon})$ in the energy norm and $\mathcal{O}(h^{4\pi/\omega-2-\varepsilon})$ in the L^2-norm, ω being the greatest value of the angles of the domain).

- The weighted regularization developed by Costabel and Dauge [13], where the boundary conditions are multiplied by a weight that depends on the distance to the geometric singularities. In this method each singularity requires the construction of a new weight. It is stated in [19] that this method requires the approximation space to contain the gradient of C^1-scalar functions, which excludes low order finite element spaces (C^0-finite element spaces for example). This restriction is removed in the work of Buffa et al. [20] by considering a mixed form of the weighted L^2-stabilization technique on special meshes. This method has also been simplified by Otin in [21, 22] where the method is performed by using a weight equal to zero in the elements near the singularity and equal to one in the other elements.
- The mixed methods with natural boundary conditions developed by Ciarlet Jr. et al.[4]. These methods consist in the dualisation of the equation on the divergence and the relation on the tangential or normal trace of the field with some Lagrange multipliers. Several authors have also contributed to the development of these methods. We highlight the works of Codina et al. [23, 24, 25] where a novel augmented formulation is produced by adding the Laplacian of the Lagrange multiplier multiplied by a mesh dependent stabilizing term to the equation resulting from the dualisation of the divergence equation.
- The L^2-projection methods developed by Duan and al. [26, 27]. In these methods, L^2-projectors are applied to both curl and div formulations and linear continuous finite elements enriched with some higher order bubble functions are employed in order to approximate low regular functions. These methods do not impose information on the geometric singularities of the domain boundary but are still limited to linear continuous elements.
- The interior penalty method [19, 28], where the idea consists of controlling the divergence of the electric field in a Sobolev space with fractional negative exponent. The optimal rate of convergence is recovered with this method.
- The Predictor-Corrector nodal FEM that makes use of the explicit extraction formulas for the coefficients of the singularities of the solution near the corners (see [9, 11, 12, 29, 30]). The optimal convergence rate is recovered by the method. The method can be applied with high order finite element polynomials but faces the problem of logarithmic singularities that may easily happen in the case of Maxwell's equations.

This paper extends the Predictor-Corrector nodal FEM developed for the 2D time-harmonic Maxwell's equations on convex and non-convex polygonal domains with exactly one re-entrant angle Ω centered at the origin $O(0,0)$. The method is based on a Fourier decomposition of the solution and a defect-correction algorithm to recover the optimal convergence known for Maxwell's equations on convex polygonal domains or on domains with a $\mathcal{C}^2$-boundary.

This paper is organized as follows, Section 2 presents the Maxwell's equations on polygonal domains, Section 3 develops a local decomposition of the solution around the singular corner, Section 4 proposes a Defect-Correction FEM with the error estimates in L^2- and the energy norm and Section 5 concludes the paper.

2. THE MODEL PROBLEM

Given a vector function $\mathbf{v} = (v_1, v_2)^T$, define the divergence of $\mathbf{v}$ by

$$\operatorname{div} \mathbf{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}.$$

$$\operatorname{curl} \mathbf{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}.$$

Given a scalar function v, define $\mathbf{curl}\, v$ by

$$\mathbf{curl}\, v = \left(\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1}\right).$$

Define $L^2(\Omega)$ to be the space of classes of measurable and square integrable functions over Ω equipped with the norm $||u||_0 = \left(\int_\Omega |u(x_1,x_2)|^2\, \mathbf{dx}\right)^{1/2}$ that will also denote the norm in the Cartesian product space $L^2(\Omega)^2$, $\mathbf{dx} = dx_1\, dx_2$ being the Lebesgue measure.

$$\begin{aligned}
H_0(\operatorname{curl}\,;\,\Omega) &:= \left\{\mathbf{v} \in L^2(\Omega)^2 :\ \operatorname{curl} \mathbf{v} \in L^2(\Omega) \text{ and } \mathbf{v} \wedge \mathbf{n} = 0 \text{ on } \partial\Omega\right\}\\
H(\operatorname{div}\,;\,\Omega) &:= \left\{\mathbf{v} \in L^2(\Omega)^2 :\ \operatorname{div} \mathbf{v} \in L^2(\Omega)\right\}\\
H(\operatorname{div}^0\,;\,\Omega) &:= \left\{\mathbf{v} \in L^2(\Omega)^2 :\ \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega\right\}\\
H_0(\operatorname{curl},\operatorname{div}\,;\,\Omega) &:= H_0(\operatorname{curl}\,;\,\Omega) \cap H(\operatorname{div}\,;\,\Omega)\\
H^m(\Omega) &:= \left\{v :\ \frac{\partial^{\alpha_1+\alpha_2} v}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \in L^2(\Omega),\ \alpha_1, \alpha_2 \in \mathbb{N},\ \alpha_1 + \alpha_2 \le m\right\}\\
H_N(\Omega) &:= \left\{\mathbf{v} \in H^1(\Omega)^2 : \mathbf{v} \wedge \mathbf{n} = 0 \text{ on } \Gamma\right\}\\
H_0^1(\Omega) &:= \left\{u \in H^1(\Omega) :\ u = 0 \text{ on } \Gamma\right\}
\end{aligned}$$

equipped with the norms

$$||\mathbf{v}||_{\text{curl}} := \left(||\mathbf{v}||_0^2 + ||\text{curl}\,\mathbf{v}||_0^2\right)^{1/2}$$
$$||\mathbf{v}||_{\text{div}} := \left(||\mathbf{v}||_0^2 + ||\text{div}\,\mathbf{v}||_0^2\right)^{1/2}$$
$$||\mathbf{v}||_{\text{cd}} := \left(||\mathbf{v}||_0^2 + ||\text{curl}\,\mathbf{v}||_0^2 + ||\text{div}\,\mathbf{v}||_0^2\right)^{1/2}$$
$$||v||_m := \left(||v||_0^2 + \sum_{\alpha_1,\alpha_2\le m} \left|\left|\frac{\partial^{\alpha_1+\alpha_2} v}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}\right|\right|_0^2\right)^{1/2}.$$
$$||\mathbf{v}||_{H_N(\Omega)} := ||\mathbf{v}||_1 =: ||\mathbf{v}||_{H_0^1(\Omega)}$$

where $||\cdot||_m$ denotes the norm in $H^m(\Omega)$ and also the norm in the cartesian product space $H^m(\Omega)^2$. The quantity defined by

$$|\mathbf{v}|_{\text{cd}} := \left(||\text{curl}\,\mathbf{v}||_0^2 + ||\text{div}\,\mathbf{v}||_0^2\right)^{1/2}$$

is the semi-norm in $H_0(\text{curl,div}\,;\,\Omega)$.

Given an interval $I \subset \mathbb{R}$ and a Hilbert space X, $\mathcal{C}^k(I;X)$ will denote the space of bounded k-continuously differentiable functions u on I of the form $t \mapsto u(\cdot,t) \in X$ equipped with the norm

$$||u||_{\mathcal{C}^k} = \sup_{t\in I, |\alpha|\le k} ||\partial^\alpha u(\cdot,t)||_X.$$

The space $\mathbf{C}^k(I;X)$ will denote the space of functions $\mathbf{u} := (u_1,u_2)$, $u_1,u_2 \in \mathcal{C}^k(I;X)$ equipped with the norm

$$||u||_{\mathbf{C}^k} = (||u_1||_{\mathcal{C}^k}^2 + ||u_2||_{\mathcal{C}^k}^2)^{1/2}.$$

The space $L^2(I;X)$ is the completion of $\mathcal{C}^0(I;X)$ with respect to the norm

$$||u||_{L^2(I;X)} = \left(\int_I ||u(\cdot,t)||_X^2\,dt\right)^{\frac{1}{2}}.$$

The space $\mathbf{L}^2(I;X)$ is the space of functions $\mathbf{u} := (u_1,u_2)$, $u_1,u_2 \in L^2(I;X)$ equipped with the norm

$$||u||_{\mathbf{L}^2(I;X)} = \left(||u_1||_{L^2(I;X)}^2 + ||u_2||_{L^2(I;X)}^2\right)^{1/2}.$$

The space $H^m(I;X)$, $m \in \mathbb{N}$, $m > 0$ is the space of functions $t \mapsto u(\cdot,t) \in X$ such that $\dfrac{\partial^{\alpha_1+\alpha_2} u}{\partial t^{\alpha_1+\alpha_2}} \in L^2(I;\Omega)$ with the norm

$$||u||_{H^m(I;X)} = \left(\int_I \sum_{\alpha_1+\alpha_2\le m} \left|\left|\frac{\partial^{\alpha_1+\alpha_2} u}{\partial t^{\alpha_1+\alpha_2}}\right|\right|_X^2\,dt\right)^{\frac{1}{2}}.$$

The space $\mathbf{H}^m(I;X)$ is the space of functions $\mathbf{u} := (u_1, u_2)$, $u_1, u_2 \in H^m(I;X)$ equipped with the norm

$$||\mathbf{u}||_{\mathbf{H}^m(I;X)} = \left(||u_1||^2_{H^m(I;X)} + ||u_2||^2_{H^m(I;X)}\right)^{1/2}.$$

Given a simply connected polygonal domain $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega$, a function $\mathbf{f} := (f_1, f_2)^T \in \mathbf{H}^1([0,T]; L^2(\Omega)^2)$ such that $\operatorname{div} \mathbf{f} = 0$, two functions $\mathbf{u}^0 := (u_1^0, u_2^0)^T \in H_0(\text{curl,div}; \Omega) \cap H(\text{div}^0; \Omega) \cap H^2(\Omega)^2$ and $\mathbf{u}^1 := (u_1^1, u_2^1)^T \in H(\text{div}^0; \Omega)$, find $\mathbf{u} := (u_1, u_2)^T \in \mathbf{L}^2(0,T; H_0(\text{curl,div}; \Omega))$ such that

$$(1)\qquad \begin{aligned} \kappa^2 \frac{\partial^2 \mathbf{u}}{\partial t^2} + \mathbf{curl}\,\mathrm{curl}\,\mathbf{u} &= \mathbf{f}\text{in} && \Omega \times (0,T) \\ \operatorname{div} \mathbf{u} &= 0\text{in} && \Omega \times (0,T) \\ \mathbf{u} \wedge \mathbf{n} &= 0\text{on} && \partial\Omega \times (0,T) \\ \mathbf{u}(\cdot, 0) &= \mathbf{u}^0(\cdot)\text{in} && \Omega \\ \frac{\partial \mathbf{u}}{\partial t}(\cdot, 0) &= \mathbf{u}^1(\cdot)\text{in} && \Omega \end{aligned}$$

where $\mathbf{n} = (n_1, n_2)$ is the unit outward normal to Ω, $T > 0$ and $\kappa \in \mathbb{R}$.

Using the formula $\mathbf{curl}\,\mathrm{curl}\,\mathbf{u} = -\Delta\mathbf{u} + \nabla(\operatorname{div} \mathbf{u})$ one obtains the system

$$(2)\qquad \begin{aligned} \kappa^2 \frac{\partial^2 \mathbf{u}}{\partial t^2} - \Delta\mathbf{u} &= \mathbf{f}\text{in} && \Omega \times (0,T) \\ \operatorname{div} \mathbf{u} &= 0\text{in} && \Omega \times (0,T) \\ \mathbf{u} \wedge \mathbf{n} &= 0\text{on} && \partial\Omega \times (0,T) \\ \mathbf{u}(\cdot, 0) &= \mathbf{u}^0(\cdot)\text{in} && \Omega \\ \frac{\partial \mathbf{u}}{\partial t}(\cdot, 0) &= \mathbf{u}^1(\cdot)\text{in} && \Omega \end{aligned}$$

Problem (2) is equivalent to an initial boundary value problem useful for the Lagrange nodal finite element investigation due to the continuity of the finite element functions along the tangential and the normal components. That result is given by the following proposition.

PROPOSITION 1. *If κ^2 is not an eigenvalue of the Laplace operator $-\Delta(\cdot)$, Problem (2) is equivalent to the following problem:*
find $\boldsymbol{u} \in L^2(0,T; H_0(curl,div; \Omega))$ such that

$$(3)\qquad \begin{aligned} \kappa^2 \frac{\partial^2 \boldsymbol{u}}{\partial t^2} - \Delta\boldsymbol{u} &= \boldsymbol{f} in && \Omega \times (0,T) \\ div\, \boldsymbol{u} &= 0\, on && \partial\Omega \times (0,T) \\ \boldsymbol{u} \wedge \boldsymbol{n} &= 0\, on && \partial\Omega \times (0,T) \\ \boldsymbol{u}(\cdot, 0) &= \boldsymbol{u}^0(\cdot)\, in && \Omega \\ \frac{\partial \boldsymbol{u}}{\partial t}(\cdot, 0) &= \boldsymbol{u}^1(\cdot)\, in && \Omega \end{aligned}$$

Proof. A solution of Problem (2) is obviously a solution of Problem (3).

Suppose $\mathbf{u}$ is a solution of Problem (3) and let $\varphi = \operatorname{div}\mathbf{u}$, taking the divergence in the first equation of (3), one obtains that $\varphi \in L^2(0,T;H_0^1(\Omega))$ is the unique solution of the Problem

$$
(4)\quad \begin{aligned}
\kappa^2 \frac{\partial^2\varphi}{\partial t^2} - \Delta\varphi &= 0 \text{in} && \Omega\times(0,T)\\
\varphi &= 0 \text{on} && \partial\Omega\times(0,T)\\
\varphi(\cdot,0) &= \operatorname{div}\mathbf{u}^0(\cdot) = 0 \text{in} && \Omega\\
\frac{\partial\varphi}{\partial t}(\cdot,0) &= \operatorname{div}\mathbf{u}^1(\cdot) = 0 \text{in} && \Omega
\end{aligned}
$$

But Problem (4) has the unique null solution provided that κ^2 is not an eigenvalue of the Laplace operator $-\Delta(\cdot)$, then $\varphi = 0$ in $\Omega\times(0,T)$ and $\mathbf{u}$ is also a solution of Problem (2). □

Taking the inner product of the first, fourth and the fifth equation of Problem (3) with $\mathbf{v} \in H_0(\text{curl,div};\Omega)$, the integral over Ω and using integration by parts lead to the variational problem: find $\mathbf{u} \in L^2(0,T;H_0(\text{curl,div};\Omega))$ such that

$$
(5)\quad \begin{aligned}
\kappa^2 \frac{d^2}{dt^2}\int_\Omega \mathbf{u}\cdot\mathbf{v}\,\mathbf{dx} + a(\mathbf{u},\mathbf{v}) &= \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathbf{dx} && \forall\mathbf{v}\in H_0(\text{curl,div};\Omega)\\
\int_\Omega \mathbf{u}(\cdot,0)\cdot\mathbf{v}\,\mathbf{dx} &= \int_\Omega \mathbf{u}^0\cdot\mathbf{v}\,\mathbf{dx} && \forall\mathbf{v}\in H_0(\text{curl,div};\Omega)\\
\int_\Omega \frac{\partial\mathbf{u}}{\partial t}(\cdot,0)\cdot\mathbf{v}\,\mathbf{dx} &= \int_\Omega \mathbf{u}^1\cdot\mathbf{v}\,\mathbf{dx} && \forall\mathbf{v}\in H_0(\text{curl,div};\Omega)
\end{aligned}
$$

where $a(\mathbf{u},\mathbf{v}) := \int_\Omega \operatorname{curl}\mathbf{u}\operatorname{curl}\mathbf{v} + \operatorname{div}\mathbf{u}\operatorname{div}\mathbf{v}\,\mathbf{dx}$. The following results prove the existence and uniqueness of the solution of the variational problem (5). To prove it, one can apply Theorem 8.1, page 287 of [31].

Proposition 2. *If $f \in H^1(0,T;H_0(curl,\ div;\Omega))$, $\boldsymbol{u}^0 \in H_0(curl,div;\Omega)\cap H(div^0;\Omega)\cap H^2(\Omega)^2$ and $\boldsymbol{u}^1 \in H(div^0;\Omega)$, then the variational problem (5) has a unique solution $\boldsymbol{u} \in \boldsymbol{C}^0([0,T];H_0(curl,div;\Omega))$ such that $\dfrac{\partial\boldsymbol{u}}{\partial t}$ belongs to $\boldsymbol{C}^0([0,T];H(div^0;\Omega))$. Furthermore the solution depends continuously on the data.*

3. DECOMPOSITION OF THE SOLUTION

An approximation of the solution of the variational problem (5) involves a particular study of the relation between $H_N(\Omega)$ and $H_0(\text{curl,div};\Omega)$. From [16] page 2033, $H_0(\text{curl,div};\Omega) = H_N(\Omega)\oplus\nabla X_S$ where

$$X_S := \left\{\varphi\in H_0^1(\Omega) : \exists f\in\mathcal{N} :\ -\int_\Omega \nabla\varphi\cdot\nabla\psi\,\mathbf{dx} = \int_\Omega f\psi\,\mathbf{dx}\ \forall\psi\in H_0^1(\Omega)\right\},$$

$\mathcal{N}$ being the orthogonal of $\Delta(H^2(\Omega)\cap H_0^1(\Omega))$ in $L^2(\Omega)$.

This introduction allows the decomposition of the solution into a space regular and a space singular part. The geometric singularity being a local problem (see [8], page 71) it is judicious to study the problem in a circular sector.

For sake of simplicity, it is assumed that the domain Ω has only one corner centered at the origin $O(0,0)$ in the (x_1, x_2)-plane with angle $\omega > \pi/2$, $\omega \neq \pi$. Introduce the polar coordinates (r, θ) with $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $0 \leq \theta \leq \omega$. Let $R_0 > 0$, consider the restriction on a circular sector G_0 with

$$\overline{G_0} = \{(r\cos\theta, r\sin\theta) : \ 0 \leq \theta \leq \omega, \ 0 \leq r \leq R_0\},$$

and the cut-off function

$$\eta(r) := \begin{cases} 1 & \text{if} \quad 0 \leq r < R_0/3 \\ 0 \leq \eta(r) \leq 1 & \text{if} \quad R_0/3 \leq r \leq 2R_0/3 \\ 0 & \text{if} \quad r > 2R_0/3 \end{cases}.$$

Set $\mathbf{u}_\eta := \eta\mathbf{u}$, $\mathbf{u}_\eta$ describes the solution $\mathbf{u}$ around the corner $O(0,0)$ and $\mathbf{u}_\eta$ is the unique solution of the problem: find $\mathbf{u}_\eta \in \mathbf{L}^2(0, T; H_0(\text{curl,div}; G_0))$ such that

$$\kappa^2 \frac{\partial^2 \mathbf{u}_\eta}{\partial t^2} - \Delta\mathbf{u}_\eta = \mathbf{f}_\eta \text{in} \quad G_0 \times (0,T) \tag{6}$$

$$\operatorname{div}\mathbf{u}_\eta = 0 \text{on} \quad \partial G_0 \times (0,T) \tag{7}$$

$$\mathbf{u}_\eta \wedge \mathbf{n} = 0 \text{on} \quad \partial G_0 \times (0,T) \tag{8}$$

$$\mathbf{u}_\eta(\cdot, 0) = \eta\mathbf{u}^0(\cdot) \text{in} \quad G_0 \tag{9}$$

$$\frac{\partial \mathbf{u}_\eta}{\partial t}(\cdot, 0) = \eta\mathbf{u}^1(\cdot) \text{in} \quad G_0 \tag{10}$$

where

$$\mathbf{f}_\eta := \begin{pmatrix} \eta f_1 - u_1\Delta\eta - 2\nabla\eta\cdot\nabla u_1 \\ \eta f_2 - u_2\Delta\eta - 2\nabla\eta\cdot\nabla u_2 \end{pmatrix}.$$

The one-to-one mapping $(x_1, x_2) \mapsto (r, \theta)$ transforms G_0 into a rectangle $\tilde{G}_0 := \{(r, \theta) : 0 \leq r \leq R_0, \ 0 \leq \theta \leq \omega\}$ and we can pass through polar coordinates by setting $\tilde{\mathbf{u}}(r, \theta, t) = \mathbf{u}_\eta(x_1, x_2, t)$ and $\tilde{\mathbf{f}}(r, \theta, t) = \mathbf{f}_\eta(x_1, x_2, t)$ where

$$\begin{pmatrix} u_r \\ u_\theta \end{pmatrix} = \begin{pmatrix} u_{\eta 1}\cos\theta + u_{\eta 2}\sin\theta \\ -u_{\eta 1}\sin\theta + u_{\eta 2}\cos\theta \end{pmatrix}, \ \begin{pmatrix} f_r \\ f_\theta \end{pmatrix} = \begin{pmatrix} f_{\eta 1}\cos\theta + f_{\eta 2}\sin\theta \\ -f_{\eta 1}\sin\theta + f_{\eta 2}\cos\theta \end{pmatrix}$$

$$\begin{pmatrix} u_r^0 \\ u_{0\theta} \end{pmatrix} = \begin{pmatrix} \eta u_1^0\cos\theta + \eta u_2^0\sin\theta \\ -\eta u_1^0\sin\theta + \eta u_2^0\cos\theta \end{pmatrix}, \ \begin{pmatrix} u_r^1 \\ u_{1\theta} \end{pmatrix} = \begin{pmatrix} \eta u_1^1\cos\theta + \eta u_2^1\sin\theta \\ -\eta u_1^1\sin\theta + \eta u_2^1\cos\theta \end{pmatrix},$$

Problem (6) becomes
(11)

$$
\begin{array}{lcll}
\kappa^2 \frac{\partial^2 u_r}{\partial t^2} - \frac{\partial^2 u_r}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} - \frac{1}{r}\frac{\partial u_r}{\partial r} + \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r^2} u_r & = & f_r & \text{in } \tilde{G_0} \times (0,T) \\
\kappa^2 \frac{\partial^2 u_\theta}{\partial t^2} - \frac{\partial^2 u_\theta}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2 u_r \theta}{\partial \theta^2} - \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} + \frac{1}{r^2} u_\theta & = & f_\theta & \text{in } \tilde{G_0} \times (0,T) \\
\frac{\partial u_r}{\partial r} + \frac{1}{r} u_r + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0, \; u_r = 0 & & & \text{if } \theta = 0 \\
\frac{\partial u_r}{\partial r} + \frac{1}{r} u_r + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0, \; u_r = 0 & & & \text{if } \theta = \omega \\
|u_r(0,\theta,t)| < \infty, \; |u_\theta(0,\theta,t)| < \infty & & & 0 \leq t \leq T \\
u_r(R_0,\theta,t) = u_\theta(R_0,\theta,t) = 0 & & 0 < \theta < \omega & 0 \leq t \leq T \\
u_r(\cdot,0) = u_r^0(\cdot), \; u_\theta(\cdot,0) = u_\theta^0(\cdot) & & & \\
\frac{\partial u_r}{\partial t}(\cdot,0) = u_r^1(\cdot), \; \frac{\partial u_\theta}{\partial t}(\cdot,0) = u_\theta^1(\cdot) & & &
\end{array}
$$

The derivatives are interpreted in the sense of distributions. The boundary conditions at $\theta = 0$ and $\theta = \omega$ allow us to consider the following Fourier decompositions for u_r and u_θ,

$$
\begin{aligned}
u_r(r,\theta,t) &= \sum_{k=1}^{\infty} u_{rk}(r,t)\sin\lambda_k\theta, & u_\theta(r,\theta,t) &= \sum_{k=1}^{\infty} u_{\theta k}(r,t)\cos\lambda_k\theta \\
f_r(r,\theta,t) &= \sum_{k=1}^{\infty} f_{rk}(r,t)\sin\lambda_k\theta, & f_\theta(r,\theta,t) &= \sum_{k=1}^{\infty} f_{\theta k}(r,t)\cos\lambda_k\theta \\
u_r^{0,1}(r,\theta) &= \sum_{k=1}^{\infty} u_{rk}^{0,1}(r)\sin\lambda_k\theta, & u_\theta^{0,1}(r,\theta) &= \sum_{k=1}^{\infty} u_{\theta k}^{0,1}(r)\cos\lambda_k\theta.
\end{aligned}
$$

where $\lambda_k = k\pi/\omega$, $k \in \mathbb{N}$, $k > 0$. The system (11) becomes
(12)

$$
\begin{array}{lcll}
\kappa^2 \frac{\partial^2 u_{rk}}{\partial t^2} - \frac{\partial^2 u_{rk}}{\partial r^2} + \frac{\lambda_k}{r^2} u_{rk} - \frac{1}{r}\frac{\partial u_{rk}}{\partial r} - \frac{2\lambda_k}{r^2} u_{\theta k} + \frac{1}{r^2} u_{rk} & = & f_{rk} & \\
\kappa^2 \frac{\partial^2 u_{\theta k}}{\partial t^2} - \frac{\partial^2 u_{\theta k}}{\partial r^2} + \frac{\lambda_k}{r^2} u_{\theta k} - \frac{1}{r}\frac{\partial u_{\theta k}}{\partial r} - \frac{2\lambda_k}{r^2} u_{rk} + \frac{1}{r^2} u_{\theta k} & = & f_{\theta k} & \\
\frac{\partial u_{rk}}{\partial r} + \frac{1}{r} u_{rk} - \frac{\lambda_k}{r} u_{\theta k} = 0, \; u_{rk} = 0 & & & \text{if } \theta = 0 \text{ or } \theta = \omega \\
|u_{rk}(0,t)| < \infty, \; |u_{\theta k}(0,t)| < \infty & & & 0 \leq t \leq T \\
u_{rk}(R_0,t) = u_{\theta k}(R_0,t) = 0 & & & 0 \leq t \leq T
\end{array}
$$

$$u_{rk}(r,0) = \eta(r)u^0_{rk}(r),\ \ u_{\theta k}(r,0) = \eta u^0_{\theta k}(r) \qquad 0 \le r \le R_0$$

$$\frac{\partial u_{rk}}{\partial t}(r,0) = \eta(r)u^1_{rk}(r),\ \ \frac{\partial u_{\theta k}}{\partial t}(r,0) = \eta u^1_{\theta k}(r) \qquad 0 \le r \le R_0.$$

Setting $u_{3k} = u_{rk} + u_{\theta k}$ and $u_{4k} = u_{rk} - u_{\theta k}$ the first and the second equations of Problem (12) become

$$\text{(13)} \qquad \begin{aligned} \kappa^2 \frac{\partial^2 u_{3k}}{\partial t^2} - \frac{\partial^2 u_{3k}}{\partial r^2} + \frac{\lambda_k^2+1}{r^2}u_{3k} - \frac{1}{r}\frac{\partial u_{3k}}{\partial r} - \frac{2\lambda_k}{r^2}u_{3k} &= f_{rk} + f_{\theta k} \\ \kappa^2 \frac{\partial^2 u_{4k}}{\partial t^2} - \frac{\partial^2 u_{4k}}{\partial r^2} + \frac{\lambda_k^2+1}{r^2}u_{4k} - \frac{1}{r}\frac{\partial u_{4k}}{\partial r} - \frac{2\lambda_k}{r^2}u_{4k} &= f_{rk} - f_{\theta k} \end{aligned}$$

The homogeneous equations associated to the equations (13) can be solved by separation of variables by setting $u_{ik}(r,t) = \varphi_{ik}(t)\psi_{ik}(r)$, $i = 3,4$ to obtain the equality $\dfrac{\varphi''_{ik}}{\varphi_{ik}} = \dfrac{1}{\kappa^2\psi_{ik}}\left(\psi''_{ik} - \dfrac{(\lambda_k-1)^2}{r^2}\psi_i + \dfrac{1}{r}\psi'_{ik}\right)$, $i = 3,4$.

Due to the mixed boundary conditions at $t = 0$ from the two last equalities of (12) and the smoothness of the solution $\mathbf{u}$ in time, the operator $-\dfrac{d^2(\cdot)}{dt^2}$ has positive and discrete eigenvalues α_{km^2}, $m \in \mathbb{N}$, $m \ge 1$ arranged in an increasing sequence. One writes $\dfrac{\varphi''_{ik}}{\varphi_{ik}} = \dfrac{1}{\kappa^2\psi_{ik}}\left(\psi''_{ik} - \dfrac{(\lambda_k-1)^2}{r^2}\psi_i + \dfrac{1}{r}\psi'_{ik}\right) = -\alpha^2_{km}$, $i = 3,4$, $m \in \mathbb{N}$, so, the equations $\psi''_{ik} + \dfrac{1}{r}\psi'_{ik} - \left(\dfrac{(\lambda_k-1)^2}{r^2} - \kappa^2\alpha^2_{km}\right)\psi_i = 0$, $i = 3,4$ lead to $\psi_{ik}(r) = \displaystyle\sum_{m=1}^{\infty} C_{1im}J_{\lambda_k-1}(\kappa\alpha_{km}r) + C_{2im}Y_{\lambda_k-1}(\kappa\alpha_{km}r)$, $i = 3,4$ where C_{1im}, C_{2im} are constants and the α_{km}, $m \in \mathbb{N}$, $m \ge 1$ form an increasing sequence of positive numbers such that $J_{\lambda_k-1}(\kappa\alpha_{km}R_0) = Y_{\lambda_k-1}(\kappa\alpha_{km}R_0) = 0$, J_{λ_k-1} and Y_{λ_k-1} are the Bessel functions of the first and second kind respectively.

The boundary conditions $|u_{rk}(0,t)| < \infty$ and $|u_{\theta k}(0,t)| < \infty$ in Problem (12) imply that $C_{2im} = 0$, $i = 3,4$, $m \in \mathbb{N}$, $m \ge 1$. Then

$$\begin{aligned} u_{rk}(r,t) &= \sum_{m=1}^{\infty} u_{rkm}(t)J_{\lambda_k-1}(\kappa\alpha_{km}r), & u_\theta(r,t) &= \sum_{m=1}^{\infty} u_{\theta km}(t)J_{\lambda_k-1}(\kappa\alpha_{km}r) \\ f_{rk}(r,t) &= \sum_{m=1}^{\infty} f_{rkm}(t)J_{\lambda_k-1}(\kappa\alpha_{km}r), & f_{\theta k}(r,t) &= \sum_{m=1}^{\infty} f_{\theta km}(t)J_{\lambda_k-1}(\kappa\alpha_{km}r) \\ u^0_{rk}(r) &= \sum_{m=1}^{\infty} u^0_{rkm}J_{\lambda_k-1}(\kappa\alpha_{km}r), & u^0_{\theta k}(r) &= \sum_{m=1}^{\infty} u^0_{\theta km}J_{\lambda_k-1}(\kappa\alpha_{km}r) \\ u^1_{rk}(r) &= \sum_{m=1}^{\infty} u^1_{rkm}J_{\lambda_k-1}(\kappa\alpha_{km}r), & u^1_{\theta k}(r) &= \sum_{m=1}^{\infty} u^1_{\theta km}J_{\lambda_k-1}(\kappa\alpha_{km}r) \end{aligned}$$

where

$$
\begin{aligned}
f_{rkm}(t) &= \frac{1}{||J_{\lambda_k-1}(\kappa\alpha_{km}\cdot)||_0^2}\int_0^{R_0} f_{rk}(r,t)J_{\lambda_k-1}(\kappa\alpha_{km}r)rdr \\
&= \frac{2}{\omega||J_{\lambda_k-1}(\kappa\alpha_{km}\cdot)||_0^2}\int_0^{\omega}\int_0^{R_0} f_r(r,\theta,t)J_{\lambda_k-1}(\kappa\alpha_{km}r)\sin(\lambda_k\theta)rdrd\theta \\
f_{\theta km}(t) &= \frac{1}{||J_{\lambda_k-1}(\kappa\alpha_{km}\cdot)||_0^2}\int_0^{R_0} f_{\theta k}(r,t)J_{\lambda_k-1}(\kappa\alpha_{km}r)rdr \\
&= \frac{2}{\omega||J_{\lambda_k-1}(\kappa\alpha_{km}\cdot)||_0^2}\int_0^{\omega}\int_0^{R_0} f_\theta(r,\theta,t)J_{\lambda_k-1}(\kappa\alpha_{km}r)\cos(\lambda_k\theta)rdrd\theta \\
u^0_{rkm} &= \frac{1}{||J_{\lambda_k-1}(\kappa\alpha_{km}\cdot)||_0^2}\int_0^{R_0} \eta(r)u^0_{rk}(r)J_{\lambda_k-1}(\kappa\alpha_{km}r)rdr \\
&= \frac{2}{\omega||J_{\lambda_k-1}(\kappa\alpha_{km}\cdot)||_0^2}\int_0^{\omega}\int_0^{R_0} \eta(r)u^0_r(r,\theta)J_{\lambda_k-1}(\kappa\alpha_{km}r)\sin(\lambda_k\theta)rdrd\theta \\
u^1_{rkm} &= \frac{1}{||J_{\lambda_k-1}(\kappa\alpha_{km}\cdot)||_0^2}\int_0^{R_0} \eta(r)u^1_{rk}(r)J_{\lambda_k-1}(\kappa\alpha_{km}r)rdr \\
&= \frac{2}{\omega||J_{\lambda_k-1}(\kappa\alpha_{km}\cdot)||_0^2}\int_0^{\omega}\int_0^{R_0} \eta(r)u^1_r(r,\theta)J_{\lambda_k-1}(\kappa\alpha_{km}r)\sin(\lambda_k\theta)rdrd\theta
\end{aligned}
$$

Replacing u_{rk} and $u_{\theta k}$ by their decompositions in Problem (12) leads to

$$
\begin{aligned}
u_{rkm}(t) &= \left(\frac{1}{\alpha_{km}}u^1_{rkm} + \frac{1}{\kappa^2\alpha_{km}}\int_0^t f_{rkm}(\tau)\cos(\alpha_{km}\tau)d\tau\right)\sin(\alpha_{km}t) \\
&\quad + \left(u^0_{rkm} - \frac{1}{\kappa^2\alpha_{km}}\int_0^t f_{rkm}(\tau)\sin(\alpha_{km}\tau)d\tau\right)\cos(\alpha_{km}t) \\
u_{\theta km}(t) &= \left(\frac{1}{\alpha_{km}}u^1_{\theta km} + \frac{1}{\kappa^2\alpha_{km}}\int_0^t f_{\theta km}(\tau)\cos(\alpha_{km}\tau)d\tau\right)\sin(\alpha_{km}t) \\
&\quad + \left(u^0_{rkm} - \frac{1}{\kappa^2\alpha_{km}}\int_0^t f_{\theta km}(\tau)\sin(\alpha_{km}\tau)d\tau\right)\cos(\alpha_{km}t)
\end{aligned}
. \tag{14}
$$

Then

$$
\begin{aligned}
u_r(r,\theta,t) &= \sum_{k,m=1}^{\infty} u_{rkm}(t)J_{\lambda_k-1}(\kappa\alpha_{km}r)\sin\lambda_k\theta \\
u_\theta(r,\theta,t) &= \sum_{k,m=1}^{\infty} u_{\theta km}(t)J_{\lambda_k-1}(\kappa\alpha_{km}r)\cos\lambda_k\theta
\end{aligned}
$$

where u_{rkm} and $u_{\theta km}$ are defined in (14). Using the fact that

$$
J_{\lambda_k-1}(\kappa\alpha_{km}r) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+\lambda_k)}\left(\frac{\kappa\alpha_{km}r}{2}\right)^{2n+\lambda_k-1},
$$

the solution $\tilde{\mathbf{u}} := (u_r, u_\theta)^T$ has the decomposition

$$
\begin{aligned}
\tilde{\mathbf{u}}(r,\theta,t) &= \sum_{k,m=1}^{\infty}\sum_{n=0}^{\infty}\mathbf{s}_{k,m,n}(r,\theta,t) \\
&= \tilde{\mathbf{w}}(r,\theta,t) + \sum_{k,m=1}^{\infty}\sum_{n=0,2n+\lambda_k-1<2}^{\infty}\tilde{\mathbf{s}}_{k,m,n}(r,\theta,t) \qquad (15)\\
&= \tilde{\mathbf{w}}_M(r,\theta,t) + \sum_{k=1}^{\infty}\sum_{n=0,2n+\lambda_k-1<2}^{\infty}\sum_{m=1}^{M}\tilde{\mathbf{s}}_{k,m,n}(r,\theta,t)
\end{aligned}
$$

where $M \in \mathbb{N}$, $M \geq 1$,

$$\tilde{\mathbf{s}}_{k,m,n}(r,\theta,t) = \frac{(-1)^n}{n!\Gamma(n+\lambda_k)}\left(\frac{\kappa\alpha_{km}r}{2}\right)^{2n+\lambda_k-1}\begin{pmatrix}u_{rkm}(t)\sin\lambda_k\theta\\ u_{\theta km}(t)\cos\lambda_k\theta\end{pmatrix}$$

and $\mathbf{w} \in \mathbf{C}^1(0,T;H^2(\Omega)^2)$.

Remark 3. In cartesian coordinates, $\mathbf{u}_\eta$ has the decomposition

$$\mathbf{u}_\eta(x_1,x_2,t) := \mathbf{w}(x_1,x_2,t) + \sum_{k=1}^{\infty}\sum_{n=0,2n+\lambda_k-1<2}^{\infty}\sum_{m=1}^{M}\mathbf{s}_{k,m,n}(r,\theta,t) \qquad (16)$$

where

$$\mathbf{s}_{k,m,n}(x_1,x_2,t) = \frac{(-1)^n}{n!\Gamma(n+\lambda_k)}\left(\frac{\kappa\alpha_{km}r}{2}\right)^{2n+\lambda_k-1}\begin{pmatrix}u_{rkm}(t)\sin(\lambda_k-1)\theta\\ u_{\theta km}(t)\cos(\lambda_k-1)\theta\end{pmatrix}$$

Lemma 4.

$$\left|\sum_{k,m=1}^{\infty}\sum_{n=0,2n+\lambda_k-1<2}^{\infty}\frac{(-1)^n}{n!\Gamma(n+\lambda_k)}\left(\frac{\kappa\alpha_{km}}{2}\right)^{2n+\lambda_k-1}u_{rkm}(t)\right| < \infty \qquad (17)$$

and

$$\left|\sum_{k,m=1}^{\infty}\sum_{n=0,2n+\lambda_k-1<2}^{\infty}\frac{(-1)^n}{n!\Gamma(n+\lambda_k)}\left(\frac{\kappa\alpha_{km}}{2}\right)^{2n+\lambda_k-1}u_{\theta km}(t)\right| < \infty \qquad (18)$$

Proof.

$$
\begin{aligned}
&\left|\sum_{k,m=1}^{\infty}\sum_{n=0,2n+\lambda_k-1<2}^{\infty}\frac{(-1)^n}{n!\Gamma(n+\lambda_k)}\left(\frac{\kappa\alpha_{km}}{2}\right)^{2n+\lambda_k-1}u_{rkm}(t)\right|\\
\leq\ &\sum_{k,m=1}^{\infty}\sum_{n=0,2n+\lambda_k-1<2}^{\infty}\frac{|\kappa\alpha_{km}|^2}{4n!|\Gamma(n+\lambda_k)|}\left|u^0_{rkm}\cos(\alpha_{km}t)\right.\\
&\left.+\frac{1}{\kappa\alpha_{km}}u^1_{rkm}\sin(\alpha_{km}t)+\frac{1}{\kappa^2\alpha_{km}}\int_0^t f_{rkm}(\tau)\sin(\alpha_{km}\tau)d\tau\right|
\end{aligned}
\qquad (19)
$$

The fact that $rJ_{\lambda_k-1}(\kappa\alpha_{km}r) = \frac{1}{\kappa\alpha_{km}}\left(-\lambda_k J_{\lambda_k}(\kappa\alpha_{km}r) + r\frac{dJ_{\lambda_k}}{dr}(\kappa\alpha_{km}r)\right)$ implies that

$$
\begin{aligned}
&\int_0^{R_0} u_r^0(r)\sin(\lambda_k\theta)J_{\lambda_k-1}(\kappa\alpha_{km}r)dr = -\frac{\lambda_k}{\kappa\alpha_{km}}\int_0^{R_0} u_r^0(r)\sin(\lambda_k\theta)J_{\lambda_k}(\kappa\alpha_{km}r)dr \\
&+\frac{1}{\kappa\alpha_{km}}\int_0^{R_0} u_r^0(r)\sin(\lambda_k\theta)\frac{dJ_{\lambda_k}}{dr}(\kappa\alpha_{km}r)rdr \\
= &-\frac{\lambda_k}{\kappa\alpha_{km}}\int_0^{R_0} u_r^0(r)\sin(\lambda_k\theta)r^{\lambda_k-1}\left(r^{1-\lambda_k}J_{\lambda_k}dr(\kappa\alpha_{km}r)\right)dr \\
&+\left[\frac{r}{\kappa\alpha_{km}}u_r^0(r)\sin(\lambda_k\theta)J_{\lambda_k}(\kappa\alpha_{km}r)\right]_0^{R_0} \\
&-\frac{1}{\kappa\alpha_{km}}\int_0^{R_0}\left(u_r^0(r)+r\frac{du_r^0}{dr}(r)\right)\sin(\lambda_k\theta)J_{\lambda_k}(\kappa\alpha_{km}r)dr \\
= &-\frac{\lambda_k}{\kappa\alpha_{km}}\int_0^{R_0} u_r^0(r)\sin(\lambda_k\theta)r^{\lambda_k-1}\frac{d}{dr}\left(r^{1-\lambda_k}J_{\lambda_k-1}(\kappa\alpha_{km}r)\right)dr \\
&+\frac{1}{\kappa^2\alpha_{km}^2}\int_0^{R_0}\left(u_r^0(r)+r\frac{du_r^0}{dr}\right)\sin(\lambda_k\theta)r^{\lambda_k-1}\left(-\kappa\alpha_{km}r^{1-\lambda_k}J_{\lambda_k}(\kappa\alpha_{km}r)\right)dr \\
= &\frac{1-\kappa\alpha_{km}\lambda_k}{\kappa^2\alpha_{km}}\int_0^{R_0} u_r^0(r)\sin(\lambda_k\theta)r^{\lambda_k-1}\frac{d}{dr}\left(r^{1-\lambda_k}J_{\lambda_k-1}(\kappa\alpha_{km}r)\right)dr \\
&+\frac{1}{\kappa^2\alpha_{km}^2}\int_0^{R_0}\frac{du_r^0}{dr}\sin(\lambda_k\theta)r^{\lambda_k}\frac{d}{dr}\left(r^{1-\lambda_k}J_{\lambda_k-1}(\kappa\alpha_{km}r)\right)dr \\
= &\frac{1}{\kappa^2\alpha_{km}^2}\left[\left((1-\kappa\alpha_{km})u_r^0+r\frac{du_r^0}{dr}\right)J_{\lambda_k-1}(\kappa\alpha_{km}r)\right]_0^{R_0} \\
&-\frac{1}{\kappa^2\alpha_{km}^2}\int_0^{R_0}\left(-\frac{(\kappa\alpha_{km}\lambda_k-1)^2}{r}u_r^0+\frac{du_r^0}{dr}+r\frac{d^2u_r^0}{dr^2}\right)\sin(\lambda_k\theta)J_{\lambda_k-1}(\kappa\alpha_{km})dr \\
= &-\frac{1}{\kappa^2\alpha_{km}^2}\int_0^{R_0}\left(-\frac{(\kappa\alpha_{km}\lambda_k-1)^2}{r^2}u_r^0+\frac{1}{r}\frac{du_r^0}{dr}+\frac{d^2u_r^0}{dr^2}\right)\sin(\lambda_k\theta)J_{\lambda_k-1}(\kappa\alpha_{km})rdr \\
\leq &\; C\frac{||J_{\lambda_k-1}(\kappa\alpha_{km}\cdot)||_0}{|\kappa\alpha_{km}|^2}||\mathbf{u}^0||_2 \text{ if } \mathbf{u}^0\in H^2(G_0)^2
\end{aligned}
$$

One also has

$$
\begin{aligned}
&\int_0^{R_0} u_r^1(r,\theta)\sin(\lambda_k\theta)J_{\lambda_k-1}(\kappa\alpha_{km}r)rdr \\
= &\frac{1}{\kappa\alpha_{km}}\int_0^{R_0} u_r^1(r)\sin(\lambda_k\theta)J_{\lambda_k-1}(\kappa\alpha_{km}r)\kappa\alpha_{km}rdr \\
= &\frac{1}{\kappa\alpha_{km}}\int_0^{\kappa\alpha_{km}R_0} u_r^1\left(\frac{R}{\kappa\alpha_{km}},\theta\right)\sin(\lambda_k\theta)J_{\lambda_k-1}(R)R\,dR \text{ where } R=\kappa\alpha_{km}r
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{|\kappa\alpha_{km}|}\left(\int_0^{\kappa\alpha_{km}R_0}\left|u_r^1\left(\frac{R}{\kappa\alpha_{km}},\theta\right)R^{1/2}\right|^2 dR\int_0^{\kappa\alpha_{km}R_0}\left|J_{\lambda_k-1}(R)R^{1/2}\right|^2 dR\right)^{\frac{1}{2}}\\
&\le C||J_{\lambda_k-1}(\kappa\alpha_{km}\cdot)||_0\,||\mathbf{u}^1||_0.
\end{aligned}
$$

Using the same way one can show that

$$
\int_0^{R_0} f_{rkm}(r)\sin(\lambda_k\theta)J_{\lambda_k-1}(\kappa\alpha_{km}r)dr \le C||J_{\lambda_k-1}(\kappa\alpha_{km}\cdot)||_0||\mathbf{f}||_0.
$$

Hence (19) becomes

$$
\begin{aligned}
&\left|\sum_{k,m=1}^{\infty}\sum_{n=0,2n+\lambda_k-1<2}^{\infty}\frac{(-1)^n}{n!\Gamma(n+\lambda_k)}\left(\frac{\kappa\alpha_{km}}{2}\right)^{2n+\lambda_k-1}u_{rkm}(t)\right|\\
\le\;& C\sum_{k,m=1}^{\infty}\sum_{n=0,2n+\lambda_k-1<2}^{\infty}\left[\frac{\left(||\mathbf{u}^0||_2+\frac{1}{\kappa\alpha_{km}}||\mathbf{u}^1||_0\right)}{2\,\omega\, n!\,|\Gamma(n+\lambda_k)|\,||J_{\lambda_k-1}(\kappa\alpha_{km}\cdot)||_0}\right.\\
&\left.+\frac{1}{|\kappa\alpha_{km}|^2}\int_0^t\sin(\alpha_{km}\tau)d\tau||\mathbf{f}||_0\right]\\
\le\;& \left|\sum_{k,m=1}^{\infty}\sum_{n=0,2n+\lambda_k-1<2}^{\infty}\frac{(-1)^n}{n!\Gamma(n+\lambda_k)}\left(\frac{\kappa\alpha_{km}}{2}\right)^{2n+\lambda_k-1}u_{rkm}(t)\right|\\
\le\;& \sum_{k,m=1}^{\infty}\sum_{n=0,2n+\lambda_k-1<2}^{\infty}\frac{C\left(||\mathbf{u}^0||_2+\frac{1}{|\kappa\alpha_{km}|}||\mathbf{u}^1||_0+\frac{T||\mathbf{f}||_0}{|\kappa\alpha_{km}|^2}\right)}{2\omega n!|\Gamma(n+\lambda_k)|\,||J_{\lambda_k-1}(\kappa\alpha_{km}\cdot)||_0}\\
\le\;& \sum_{k,m=1}^{\infty}\sum_{n=0,2n+\lambda_k-1<2}^{\infty}\frac{C(R_0,T,\mathbf{u}^0,\mathbf{u}^1,\mathbf{f})}{2\,\omega\, n!\,|\Gamma(n+\lambda_k)|\,||J_{\lambda_k-1}(\kappa\alpha_{km}\cdot)||_0}\\
<\;& \infty
\end{aligned}
$$

Similar methods can be used with u_θ^0, u_θ^1 and $f_{\theta km}$ to show (18). □

A truncation of the summation on the integer m can be done and the following error estimate is obtained.

LEMMA 5. *Let $M \in \mathbb{N}$, $M > 1$, and let $\{\alpha_{km}\}_{m\ge1}$ be an increasing sequence of positive numbers such that $J_{\lambda_k-1}(\kappa\alpha_{km}R_0) = 0$ where $\lambda_k = k\pi/\omega$, $R_0 > 0$ is fixed and $\kappa \in \mathbb{R}$ is defined as in (1). Let*

$$
c_r(t) = \sum_{k,m=1}^{\infty}\sum_{n=0,2n+\lambda_k-1<2}^{\infty}\frac{(-1)^n}{n!\Gamma(n+\lambda_k)}\left(\frac{\kappa\alpha_{km}}{2}\right)^{2n+\lambda_k-1}u_{rkm}(t),
$$

$$
c_\theta(t) = \sum_{k,m=1}^{\infty}\sum_{n=0,2n+\lambda_k-1<2}^{\infty}\frac{(-1)^n}{n!\Gamma(n+\lambda_k)}\left(\frac{\kappa\alpha_{km}}{2}\right)^{2n+\lambda_k-1}u_{\theta km}(t),
$$

$$c_{r,M}(t) = \sum_{m=1}^{M} \sum_{k=1}^{\infty} \sum_{n=0, 2n+\lambda_k-1<2}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\lambda_k)} \left(\frac{\kappa\alpha_{km}}{2}\right)^{2n+\lambda_k-1} u_{rkm}(t),$$

$$c_{\theta,M}(t) = \sum_{m=1}^{M} \sum_{k=1}^{\infty} \sum_{n=0, 2n+\lambda_k-1<2}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\lambda_k)} \left(\frac{\kappa\alpha_{km}}{2}\right)^{2n+\lambda_k-1} u_{\theta km}(t).$$

Then

$$|c_r(t) - c_{r,M}(t)| \leq C \max_{k\geq 1, n\geq 0,\, 2n+\lambda_k-1<2} \{|\alpha_{kM}|^{-\lambda_k}\} \tag{20}$$

and

$$|c_\theta(t) - c_{\theta,M}(t)| \leq C \max_{k\geq 1, n\geq 0,\, 2n+\lambda_k-1<2} \{|\alpha_{kM}|^{-\lambda_k}\} \tag{21}$$

Proof.

$$\begin{aligned} |c_r(t) - c_{r,M}(t)| &= \left| \sum_{m=M+1}^{\infty} \sum_{k=1}^{\infty} \sum_{2n+\lambda_k-1<2}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\lambda_k)} \left(\frac{\kappa\alpha_{km}}{2}\right)^{2n+\lambda_k-1} u_{rkm}(t) \right| \\ &\leq \sum_{m=M+1}^{\infty} \sum_{k=1}^{\infty} \sum_{2n+\lambda_k-1<2}^{\infty} \frac{C}{2\omega n! |\Gamma(n+\lambda_k)| \, ||J_{\lambda_k-1}(\kappa\alpha_{km}\cdot)||_0} \end{aligned}$$

But

$$J_{\lambda_k-1}(\kappa\alpha_{km} r) = \frac{(\kappa\alpha_{km} r)^{\lambda_k-1}}{2^{\lambda_k-1}\Gamma(\lambda_k)} \left(1 - \frac{(\kappa\alpha_{km} r)^2}{2(2\lambda_k)} + \frac{(\kappa\alpha_{km} r)^4}{2(4)(2\lambda_k)(2\lambda_k+2)} - \cdots\right)$$

then

$$\begin{aligned} |c_r(t) - c_{r,M}(t)| &\leq C(R_0) \sum_{m=M+1}^{\infty} \sum_{k=1}^{\infty} \sum_{n=0, 2n+\lambda_k-1<2}^{\infty} \frac{2^{\lambda_k-1}\Gamma(\lambda_k)}{2\omega n! |\Gamma(\lambda_k+n)| \, |\alpha_{km}|^{\lambda_k}} \\ &\leq C \max_{k\geq 1, n\geq 0,\, 2n+\lambda_k-1<2} \{|\alpha_{kM}|^{-\lambda_k}\} \end{aligned}$$

The inequality (21) can be deduced using the same manner. □

REMARK 6. $\alpha_{kM} \to \infty$ as $M \to \infty$ but since $\lambda_k > 0$, $|\alpha_{kM}|^{-\lambda_k} \to 0$ as $M \to \infty$.

4. THE DEFECT-CORRECTION FINITE ELEMENT METHOD

4.1. A Semi-discrete Approximation.

This section presents a space discretization of the solution by a Galerkin approximation with H^1-conforming finite element methods. The Galerkin method considers a quasi-uniform and shape-regular triangulation $\mathcal{T}_h$ of Ω

with diameter h, and finds a semi-discrete approximation $\mathbf{u}_h \in \mathbf{L}^2(0,T;V_h)$ of the solution $\mathbf{u} \in \mathbf{L}^2(0,T;H(\text{curl},\text{div};\Omega))$ of Problem (5) satisfying

$$
(22)\qquad
\begin{aligned}
\kappa^2 \frac{d^2}{dt^2}\int_\Omega \mathbf{u}_h\cdot\mathbf{v}_h\,\mathbf{dx} + a(\mathbf{u}_h,\mathbf{v}_h) &= \int_\Omega \mathbf{f}\cdot\mathbf{v}_h\,\mathbf{dx} \quad \forall\mathbf{v}_h\in V_h\\
\int_\Omega \mathbf{u}_h(\cdot,0)\cdot\mathbf{v}_h\,\mathbf{dx} &= \int_\Omega \mathbf{u}^0\cdot\mathbf{v}_h\,\mathbf{dx} \quad \forall\mathbf{v}_h\in V_h\\
\int_\Omega \frac{\partial\mathbf{u}_h}{\partial t}(\cdot,0)\cdot\mathbf{v}_h\,\mathbf{dx} &= \int_\Omega \mathbf{u}^1\cdot\mathbf{v}_h\,\mathbf{dx} \quad \forall\mathbf{v}_h\in V_h
\end{aligned}
$$

where $V_h := \left\{\mathbf{v}_h \in \mathcal{C}(\Omega)^2 : \ \mathbf{v}_{h|K} \in \mathcal{P}^1(K)^2 \text{ and } \mathbf{v}_h\wedge\mathbf{n} = 0 \text{ on } \partial\Omega\right\} \subset H_N(\Omega)$ is a finite dimensional space.

The following result presents the standard error estimates for regular solutions $\mathbf{u} \in L^2(0,T;H^2(\Omega)^2)$ which happens when Ω is convex or has a $\mathcal{C}^2$-boundary. A proof can be adapted from the one in [32], Theorem 4.1, page 575.

Theorem 7. *If Ω is a convex polygonal domain with all its angles less than or equal to $\pi/2$, $\boldsymbol{u}(\cdot,t)$ the solution of Problem (5) and $\boldsymbol{u}_h(\cdot,t)$ the solution of Problem (22), $t\in(0,T)$, if $\boldsymbol{u} \in L^\infty(0,T;H(curl, div;\Omega))$ and $\dfrac{\partial \boldsymbol{u}}{\partial t} \in L^2(0,T;H(curl, div;\Omega))$ and $\dfrac{\partial^k \boldsymbol{u}}{\partial t^k} \in L^2(0,T;L^2(\Omega)^2)$, $k=3,4$, then*

$$
\begin{aligned}
||\boldsymbol{u}(\cdot,t)-\boldsymbol{u}_h(\cdot,t)||_{cd} &\le Ch\\
||\boldsymbol{u}(\cdot,t)-\boldsymbol{u}_h(\cdot,t)||_0 &\le Ch^2.
\end{aligned}
$$

When the domain Ω is convex with an angle greater than $\pi/2$ the solution may converge slowly and if Ω is not convex, the solution may fail to converge. The following algorithm based on the decomposition (16) proposes a way to recover the optimal convergence as in Theorem 7.

Defect-Correction Algorithm 1.

(1) Solve Problem (22) in V_h and obtain the solution $u_h^{(0)}$.

(2) Compute the $\mathbf{s}^{(0)}_{k,m,n}$ using formula (16), replacing $\mathbf{u}$ with $\mathbf{u}_h^{(0)}$, $k,m,n\in\mathbb{N}$, $k\ge 1$, $1\le m\le M$, $M\in\mathbb{N}$, $2n+\lambda_k-1<2$, that is

$$
\mathbf{s}^{(0)}_{k,m,n}(x_1,x_2,t) = \frac{(-1)^n}{n!\Gamma(n+\lambda_k)}\left(\frac{\kappa\alpha_{km}r}{2}\right)^{2n+\lambda_k-1}\begin{pmatrix} u^{(0)}_{rkm}(t)\sin(\lambda_k-1)\theta\\ u^{(0)}_{\theta km}(t)\cos(\lambda_k-1)\theta\end{pmatrix}
$$

where

$$
\begin{aligned}
u^{(0)}_{rkm}(t) &= \left(\frac{1}{\alpha_{km}}u^{(0)1}_{rkm} + \frac{1}{\kappa^2\alpha_{km}}\int_0^t f^{(0)}_{rkm}(\tau)\cos(\alpha_{km}\tau)d\tau\right)\sin(\alpha_{km}t)\\
&\quad + \left(u^{(0)0}_{rkm} - \frac{1}{\kappa^2\alpha_{km}}\int_0^t f^{(0)}_{rkm}(\tau)\sin(\alpha_{km}\tau)d\tau\right)\cos(\alpha_{km}t)\\
u^{(0)}_{\theta km}(t) &= \left(\frac{1}{\alpha_{km}}u^{(0)1}_{\theta km} + \frac{1}{\kappa^2\alpha_{km}}\int_0^t f(0)_{\theta km}(\tau)\cos(\alpha_{km}\tau)d\tau\right)\sin(\alpha_{km}t)\\
&\quad + \left(u^{(0)0}_{rkm} - \frac{1}{\kappa^2\alpha_{km}}\int_0^t f(0)_{\theta km}(\tau)\sin(\alpha_{km}\tau)d\tau\right)\cos(\alpha_{km}t)
\end{aligned}.
$$

$$f^{(0)}_{rkm}(t) = \frac{2}{\omega||J_{\lambda_k-1}(\kappa\alpha_{km}\cdot)||_0^2}\int_0^{\omega}\int_0^{R_0} f^{(0)}_r(r,\theta,t)J_{\lambda_k-1}(\kappa\alpha_{km}r)\sin(\lambda_k\theta)rdrd\theta$$

$$f^{(0)}_{\theta km}(t) = \frac{2}{\omega||J_{\lambda_k-1}(\kappa\alpha_{km}\cdot)||_0^2}\int_0^{\omega}\int_0^{R_0} f^{(0)}_\theta(r,\theta,t)J_{\lambda_k-1}(\kappa\alpha_{km}r)\cos(\lambda_k\theta)rdrd\theta$$

$$\begin{pmatrix} f_r^{(0)} \\ f_\theta^{(0)} \end{pmatrix} = \begin{pmatrix} f_{\eta 1}^{(0)}\cos\theta + f_{\eta 2}^{(0)}\sin\theta \\ -f_{\eta 1}^{(0)}\sin\theta + f_{\eta 2}^{(0)}\cos\theta \end{pmatrix}$$

$$\mathbf{f}_\eta^{(0)} := \begin{pmatrix} f_{\eta 1}^{(0)} \\ f_{\eta 2}^{(0)} \end{pmatrix} = \begin{pmatrix} \eta f_1 - u_1^{(0)}\Delta\eta - 2\nabla\eta\cdot\nabla u_1^{(0)} \\ \eta f_2 - u_2^{(0)}\Delta\eta - 2\nabla\eta\cdot\nabla u_2^{(0)} \end{pmatrix}.$$

$$\begin{pmatrix} u_r^{(0)} \\ u_\theta^{(0)} \end{pmatrix} = \begin{pmatrix} u_{\eta 1}^{(0)}\cos\theta + u_{\eta 2}^{(0)}\sin\theta \\ -u_{\eta 1}^{(0)}\sin\theta + u_{\eta 2}^{(0)}\cos\theta \end{pmatrix}$$

$$\begin{aligned}
u^{(0)0}_{rkm} &= \frac{2}{\omega||J_{\lambda_k-1}(\kappa\alpha_{km}\cdot)||_0^2}\int_0^{\omega}\int_0^{R_0} \eta(r)u_r^0(r,\theta)J_{\lambda_k-1}(\kappa\alpha_{km}r)\sin(\lambda_k\theta)rdrd\theta \\
u^{(0)1}_{rkm} &= \frac{2}{\omega||J_{\lambda_k-1}(\kappa\alpha_{km}\cdot)||_0^2}\int_0^{\omega}\int_0^{R_0} \eta(r)u_r^1(r,\theta)J_{\lambda_k-1}(\kappa\alpha_{km}r)\sin(\lambda_k\theta)rdrd\theta \\
u^{(0)0}_{\theta km} &= \frac{2}{\omega||J_{\lambda_k-1}(\kappa\alpha_{km}\cdot)||_0^2}\int_0^{\omega}\int_0^{R_0} \eta(r)u_\theta^0(r,\theta)J_{\lambda_k-1}(\kappa\alpha_{km}r)\cos(\lambda_k\theta)rdrd\theta \\
u^{(0)1}_{\theta km} &= \frac{2}{\omega||J_{\lambda_k-1}(\kappa\alpha_{km}\cdot)||_0^2}\int_0^{\omega}\int_0^{R_0} \eta(r)u_\theta^1(r,\theta)J_{\lambda_k-1}(\kappa\alpha_{km}r)\cos(\lambda_k\theta)rdrd\theta
\end{aligned}$$

(3) Find $\mathbf{w}^{(1)}_{hM}$ solution of

(23)

$$\kappa^2\frac{d^2}{dt^2}\int_\Omega \mathbf{w}^{(1)}_{hM}\cdot\mathbf{v}_h\,\mathbf{dx} + a(\mathbf{w}^{(1)}_{hM},\mathbf{v}_h) = \int_\Omega\left(\mathbf{f} - \kappa^2\sum_{k,n,m=1}^{M}\frac{\partial^2\mathbf{s}^{(0)}_{k,m,n}}{\partial t^2}\right)\cdot\mathbf{v}_h\,\mathbf{dx} - \sum_{k,m,n} a\left(\mathbf{s}^{(0)}_{k,m,n},\mathbf{v}_h\right)\ \forall\,\mathbf{v}_h\in V_h$$

$$\int_\Omega \mathbf{w}^{(1)}_{hM}(\cdot,0)\cdot\mathbf{v}_h\,\mathbf{dx} = \int_\Omega\left(\mathbf{u}^0 - \sum_{k,n,m=1}^{M}\mathbf{s}^{(0)}_{k,m,n}(\cdot,0)\right)\cdot\mathbf{v}_h\,\mathbf{dx}\ \forall\,\mathbf{v}_h\in V_h$$

$$\int_\Omega \frac{\partial\mathbf{w}^{(1)}_{hM}}{\partial t}(\cdot,0)\cdot\mathbf{v}_h\,\mathbf{dx} = \int_\Omega\left(\mathbf{u}^1 - \sum_{k,n,m=1}^{M}\frac{\partial\mathbf{s}^{(0)}_{k,m,n}}{\partial t}(\cdot,0)\right)\cdot\mathbf{v}_h\,\mathbf{dx}\ \forall\,\mathbf{v}_h\in V_h$$

where $a(\mathbf{u},\mathbf{v}) = \int_\Omega \mathrm{curl}\mathbf{u}\,\mathrm{curl}\mathbf{v} + \mathrm{div}\mathbf{u}\,\mathrm{div}\mathbf{v}\,\mathbf{dx}$.

(4) Compute $\mathbf{u}^{(1)}_{hM} = \mathbf{w}^{(1)}_{hM} + \sum_{k,n,m=1}^{M}\mathbf{s}^{(0)}_{k,m,n}$.

Error Estimates.

From the decomposition (16) and the Defect-correction Algorithm 1, one can derive the following error estimates.

LEMMA 8. *Let $\mathcal{T}_h$ be a quasi-uniform and shape-regular triangulation of Ω and let Π_h be the usual Lagrange interpolation operator over the triangulation $\mathcal{T}_h$. Then for $k,m,n \in \mathbb{N}$, $k \geq 1$, $m \geq 1$, $2n+\lambda_k-1<2$, $\lambda_k = k\pi/\omega$,*

$$||\boldsymbol{s}_{k,m,n}(\cdot,t) - \Pi_h \boldsymbol{s}_{k,m,n}(\cdot,t)||_{cd} \leq Ch^{\lambda_k-1},$$
$$||\boldsymbol{s}_{k,m,n}(\cdot,t) - \Pi_h \boldsymbol{s}_{k,m,n}(\cdot,t)||_0 \leq Ch^{\lambda_k}.$$

Proof. Let us divide the triangulation $\mathcal{T}_h$ of the domain Ω into two parts $\mathcal{M}_h^1 = \{K \in \mathcal{T}_h : (0,0) \in K\}$ and $\mathcal{M}_h^2 = \{K \in \mathcal{T}_h : (0,0) \notin K\}$, one has

$$\begin{aligned}
||\mathbf{s}_{k,m,n}(\cdot,t) - \Pi_h \mathbf{s}_{k,m,n}(\cdot,t)||_0^2 &\leq \sum_{K\in\mathcal{T}_h} ||\mathbf{s}_{k,m,n}(\cdot,t) - \Pi_h \mathbf{s}_{k,m,n}(\cdot,t)||_0^2 \\
&\leq \sum_{K\in\mathcal{M}_h^1} ||\mathbf{s}_{k,m,n}(\cdot,t) - \Pi_h \mathbf{s}_{k,m,n}(\cdot,t)||^2 \\
&\quad + \sum_{K\in\mathcal{M}_h^2} ||\mathbf{s}_{k,m,n}(\cdot,t) - \Pi_h \mathbf{s}_{k,m,n}(\cdot,t)||_0^2 .
\end{aligned}$$

By the Cauchy-Schwartz inequality

$$\sum_{K\in\mathcal{M}_h^1} ||\mathbf{s}_{k,m,n}(\cdot,t) - \Pi_h \mathbf{s}_{k,m,n}(\cdot,t)||_0^2 \leq C \sum_{K\in\mathcal{M}_h^1} ||\mathbf{s}_{k,m,n}(\cdot,t)||_0^2 + ||\Pi_h \mathbf{s}_{k,m,n}(\cdot,t)||_0^2 .$$

$2n+\lambda_k-1<2$ happens when $n=0,1$, then for $K \in \mathcal{M}_h^1$,

$$\begin{aligned}
||\mathbf{s}_{k,m,n}(\cdot,t)||_0^2 &\leq C \int_0^{h_K} (r^{2\lambda_k-1} + r^{2\lambda_k+1}) \left(|u_{rkm}(t)|^2 + |u_{\theta km}(t)|^2\right) dr \\
&\leq C \max\{h_K^{2\lambda_k}, h_K^{2\lambda_k+2}\} \\
&\leq Ch^{2\lambda_k}.
\end{aligned}$$

$\Pi_h \mathbf{s}_{k,m,n}(\cdot,t)$ being a polynomial of degree 1 then for $K \in \mathcal{M}_h^1$,

$$\begin{aligned}
||\Pi_h \mathbf{s}_{k,m,n}(\cdot,t)||_0^2 &\leq C \max\left\{|\mathbf{s}_{k,m,n}(x_1,x_2,t)|^2 ,\ (x_1,x_2) \in K\right\} \text{meas}(K) \\
&\leq Ch^{2\lambda_k-2} h_K^2 \\
&\leq Ch^{2\lambda_k}.
\end{aligned}$$

Hence $\displaystyle\sum_{K\in\mathcal{M}_h^1} ||\mathbf{s}_{k,m,n}(\cdot,t) - \Pi_h \mathbf{s}_{k,m,n}(\cdot,t)||_0^2 \leq Ch^{\lambda_k}$.

For $K \in \mathcal{M}_h^2$, $\mathbf{s}_{k,m,n}(\cdot,t) \in H^2(K)^2$ then

$$\sum_{K\in\mathcal{M}_h^2} ||\mathbf{s}_{k,m,n}(\cdot,t) - \Pi_h \mathbf{s}_{k,m,n}(\cdot,t)||_0^2 \leq Ch^4.$$

Then $||\mathbf{s}_{k,m,n}(\cdot,t)-\Pi_h\mathbf{s}_{k,m,n}(\cdot,t)||_0^2 \le C\max\{h^{2\lambda_k},h^4\}\le Ch^{2\lambda_k}$ and

$$||\mathbf{s}_{k,m,n}(\cdot,t)-\Pi_h\mathbf{s}_{k,m,n}(\cdot,t)||_0 \le Ch^{\lambda_k}.$$

Using the same method as above, one can prove that

$$||\mathbf{s}_{k,m,n}(\cdot,t)-\Pi_h\mathbf{s}_{k,m,n}(\cdot,t)||_{cd} \le Ch^{\lambda_k-1}.$$

□

LEMMA 9. *Let $\mathcal{T}_h$ be a quasi-uniform and shape-regular triangulation of Ω and let Π_h be the usual Lagrange interpolation operator over the triangulation $\mathcal{T}_h$. Then for $M,n,k\in\mathbb{N}$, $m\ge 1$, $k\ge 1$, $n\ge 0$, $2n+\lambda_k-1<2$, $\lambda_k=k\pi/\omega$, there exist $C>0$ such that*

(24)
$$\left|\left|\sum_{m=1}^{M}\left(\boldsymbol{s}_{k,m,n}(\cdot,t)-\Pi_h\boldsymbol{s}_{k,m,n}(\cdot,t)\right)+\sum_{m=M+1}^{\infty}\boldsymbol{s}_{k,m,n}(\cdot,t)\right|\right|_0 \le Ch^{\lambda_k}|\alpha_{kM}|^{-\lambda_k}$$

and

(25)
$$\left|\left|\sum_{m=1}^{M}\left(\boldsymbol{s}_{k,m,n}(\cdot,t)-\Pi_h\boldsymbol{s}_{k,m,n}(\cdot,t)\right)+\sum_{m=M+1}^{\infty}\boldsymbol{s}_{k,m,n}(\cdot,t)\right|\right|_{cd} \le Ch^{\lambda_k-1}|\alpha_{kM}|^{-\lambda_k}$$

Proof. Let us divide the triangulation $\mathcal{T}_h$ of the domain Ω into two parts $\mathcal{M}_h^1=\{K\in\mathcal{T}_h:(0,0)\in K\}$ and $\mathcal{M}_h^2=\{K\in\mathcal{T}_h:(0,0)\notin K\}$.

$$\begin{aligned}&\left|\left|\sum_{m=1}^{M}\left(\mathbf{s}_{k,m,n}(\cdot,t)-\Pi_h\mathbf{s}_{k,m,n}(\cdot,t)\right)+\sum_{m=M+1}^{\infty}\mathbf{s}_{k,m,n}(\cdot,t)\right|\right|_0^2\\
\le\ &\sum_{K\in\mathcal{M}_h^1}\left|\left|\sum_{m=1}^{M}\left(\mathbf{s}_{k,m,n}(\cdot,t)-\Pi_h\mathbf{s}_{k,m,n}(\cdot,t)\right)+\sum_{m=M+1}^{\infty}\mathbf{s}_{k,m,n}(\cdot,t)\right|\right|^2\\
&+\sum_{K\in\mathcal{M}_h^2}\left|\left|\sum_{m=1}^{M}\left(\mathbf{s}_{k,m,n}(\cdot,t)-\Pi_h\mathbf{s}_{k,m,n}(\cdot,t)\right)+\sum_{m=M+1}^{\infty}\mathbf{s}_{k,m,n}(\cdot,t)\right|\right|_0^2.\end{aligned}$$

For $K\in\mathcal{M}_h^1$, one can use the inequalities (20), (21) and (8) to obtain

$$\begin{aligned}&\left|\left|\sum_{m=1}^{M}\left(\mathbf{s}_{k,m,n}(\cdot,t)-\Pi_h\mathbf{s}_{k,m,n}(\cdot,t)\right)+\sum_{m=M+1}^{\infty}\mathbf{s}_{k,m,n}(\cdot,t)\right|\right|_0\\
\le\ &\sum_{m=1}^{M}\left|\left|\frac{(-1)^n}{n!\Gamma(n+\lambda_k)}\left(\frac{\kappa\alpha_{km}r}{2}\right)^{2n+\lambda_k-1}\begin{pmatrix}(u_{rkm}(t)-u_{rkm}^{(0)}(t))\sin\lambda_k\theta\\(u_{\theta km}(t)-u_{\theta km}^{(0)}(t))\cos\lambda_k\theta\end{pmatrix}\right|\right|_0\\
&+Ch_K^{\lambda_k}|\alpha_{kM}|^{-\lambda_k}\end{aligned}$$

$$\leq C\left(h_K^{\lambda_k}\max_{m=1,\cdots,M}|\alpha_{km}|^{-\lambda_k}+h_K^{\lambda_k}|\alpha_{kM}|^{-\lambda_k}\right)$$
$$\leq Ch^{\lambda_k}|\alpha_{kM}|^{-\lambda_k}.$$

For $K\in\mathcal{M}_h^2$,
$$\left\|\sum_{m=1}^{M}(\mathbf{s}_{k,m,n}(\cdot,t)-\Pi_h\mathbf{s}_{k,m,n}(\cdot,t))+\sum_{m=M+1}^{\infty}\mathbf{s}_{k,m,n}(\cdot,t)\right\|_0\leq Ch^2|\alpha_{kM}|^{-\lambda_k}.$$

Then
$$\left\|\sum_{m=1}^{M}(\mathbf{s}_{k,m,n}(\cdot,t)-\Pi_h\mathbf{s}_{k,m,n}(\cdot,t))+\sum_{m=M+1}^{\infty}\mathbf{s}_{k,m,n}(\cdot,t)\right\|_0^2\leq Ch^{\lambda_k}|\alpha_{kM}|^{-\lambda_k}.$$

This shows (24).

The inequality (25) can be shown using the same way as previous. □

THEOREM 10. *Let $\mathcal{T}_h$ be a quasi-uniform and shape-regular triangulation of Ω, let $M\in\mathbb{N}$, $M\geq 1$, let $\boldsymbol{u}\in\boldsymbol{L}^2(0,T;H_0(curl,div;\Omega))$ be the solution of Problem (5), and $\boldsymbol{u}_{hM}^{(1)}$ be the solution of Problem (22) using the Defect-Correction Algorithm 1, then for any $t\in(0,T)$,*

$$\left\|\boldsymbol{u}(\cdot,t)-\boldsymbol{u}_{hM}^{(1)}(\cdot,t)\right\|_{cd}\leq C\max_{2n+\lambda_k-1<2}\{h,h^{\lambda_k-1}|\alpha_{kM}|^{-\lambda_k}\},\tag{26}$$

$$\left\|\boldsymbol{u}(\cdot,t)-\boldsymbol{u}_{hM}^{(1)}(\cdot,t)\right\|_{0}\leq C\max_{2n+\lambda_k-1<2}\{h^2,h^{\lambda_k-1}|\alpha_{kM}|^{-\lambda_k}\}.\tag{27}$$

Proof.

$$\begin{aligned}\mathbf{u}-\mathbf{u}_{hM}^{(1)}&=\mathbf{w}-\sum_{k,n,m=1}^{\infty}\mathbf{s}_{k,m,n}-\mathbf{w}_{hM}^{(1)}-\sum_{k,n,m=1}^{M}\mathbf{s}_{k,m,n}^{(0)}\\&=\mathbf{w}-\mathbf{w}_{hM}^{(1)}+\sum_{k,n,m=1}^{M}\left[\mathbf{s}_{k,m,n}-\Pi_h\mathbf{s}_{k,m,n}+(\Pi_h\mathbf{s}_{k,m,n}-\mathbf{s}_{k,m,n}^{(0)})\right]\\&\quad+\sum_{k,n,m=M+1}^{\infty}\mathbf{s}_{k,m,n}\end{aligned}$$

then
$$\begin{aligned}\left\|\mathbf{u}-\mathbf{u}_{hM}^{(1)}\right\|_{cd}&\leq\|\mathbf{w}-\mathbf{w}_{hM}^{(1)}\|_{cd}\\&\quad+\sum_{k,n,m=1}^{M}\|\mathbf{s}_{k,m,n}-\Pi_h\mathbf{s}_{k,m,n}\|_{cd}+\|\Pi_h\mathbf{s}_{k,m,n}-\mathbf{s}_{k,m,n}^{(0)}\|_{cd}\\&\quad+\sum_{k,n,m=M+1}^{\infty}\|\mathbf{s}_{k,m,n}-\Pi_h\mathbf{s}_{k,m,n}\|_{cd}+\|\Pi_h\mathbf{s}_{k,m,n}-\mathbf{s}_{k,m,n}^{(0)}\|_{cd}\\&\leq\|\mathbf{w}-\mathbf{w}_{hM}^{(1)}\|_{cd}+Ch^{\lambda_k-1}\left[\left(\int_0^t\|\mathbf{u}-\mathbf{u}_h^{(0)}\|_0^2d\tau\right)^{1/2}+|\alpha_{kM}|^{-\lambda_k}\right]\end{aligned}$$

$$
\begin{aligned}
&\leq C\left(||\mathbf{w}-\mathbf{w}_{hM}^{(1)}||_{cd}+h^{\lambda_k-1}||\mathbf{u}-\mathbf{u}_{hM}^{(0)}||_0+h^{\lambda_k-1}|\alpha_{kM}|^{-\lambda_k}\right)\\
&\leq C\Big(||\mathbf{w}-\mathbf{w}_{hM}^{(1)}||_{cd}+h^{\lambda_k-1}||\mathbf{w}-\mathbf{w}_{hM}^{(0)}||_{cd}\\
&\quad +h^{\lambda_k-1}\sum_{m=1}^{M}||\mathbf{s}_{k,m,n}-\Pi_h\mathbf{s}_{k,m,n}||_0+h^{\lambda_k-1}\sum_{m=M+1}^{\infty}||\mathbf{s}_{k,m,n}||_0\\
&\quad + h^{\lambda_k-1}|\alpha_{kM}|^{-\lambda_k}\Big)\\
&\leq C\Big(||\mathbf{w}-\mathbf{w}_{hM}^{(1)}||_{cd}+h^{\lambda_k-1}||\mathbf{w}-\mathbf{w}_{hM}^{(0)}||_{cd}\\
&\quad +h^{\lambda_k-1}(2+h^{\lambda_k})|\alpha_{kM}|^{-\lambda_k}\Big)
\end{aligned}
$$

(28)

$$
\leq C\left(||\mathbf{w}-\mathbf{w}_{hM}^{(1)}||_{cd}+h^{\lambda_k-1}||\mathbf{w}-\mathbf{w}_{hM}^{(0)}||_{cd}+h^{\lambda_k-1}|\alpha_{kM}|^{-\lambda_k}\right)
$$

The next step consists to find an appropriate bound of $||\mathbf{w}-\mathbf{w}_{hM}^{(1)}||_{cd}$.

Set $\Pi_h w = w_h$, $\mathbf{s}=\sum_{k,m,n}\mathbf{s}_{k,m,n}$ and $\mathbf{s}^{(0)}=\sum_{k,n,m=1}^{M}\mathbf{s}_{k,m,n}^{(0)}$, one has

(29)

$$
\kappa^2\frac{d^2}{dt^2}\int_\Omega \mathbf{w}_h\cdot\mathbf{v}_h\,\mathbf{dx}+a(\mathbf{w}_h,\mathbf{v}_h)=\int_\Omega\left(\mathbf{f}-\kappa^2\frac{\partial^2\mathbf{s}}{\partial t^2}\right)\cdot\mathbf{v}_h\,\mathbf{dx}-a\left(\mathbf{s}\,,\mathbf{v}_h\right)\ \forall\,\mathbf{v}_h\in V_h
$$

Taking the difference of (29) with the first equation of (22) leads to

$$
\begin{aligned}
\kappa^2\frac{d^2}{dt^2}\int_\Omega(\mathbf{w}_h-\mathbf{w}_{hM}^{(1)})\cdot\mathbf{v}_h\,\mathbf{dx}+a(\mathbf{w}_h-\mathbf{w}_{hM}^{(1)},\mathbf{v}_h) &= -\int_\Omega\kappa^2\frac{\partial^2(\mathbf{s}-\mathbf{s}^{(0)})}{\partial t^2}\cdot\mathbf{v}_h\,\mathbf{dx}\\
&\quad -a\left(\mathbf{s}-\mathbf{s}^{(0)}\,,\mathbf{v}_h\right)
\end{aligned}
$$

for any $\mathbf{v}_h\in V_h$. Then for $\mathbf{v}_h(\cdot)=\dfrac{\partial^2(\mathbf{w}_h-\mathbf{w}_{hM}^{(1)})}{\partial t^2}(\cdot,t)\in V_h$, integration by parts gives

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left|\left|\kappa^2\frac{\partial(\mathbf{w}_h-\mathbf{w}_{hM}^{(1)})}{\partial t}\right|\right|_0^2+\frac{1}{2}\frac{d}{dt}\left(a(\mathbf{w}_h-\mathbf{w}_{hM}^{(1)},\mathbf{w}_h-\mathbf{w}_{hM}^{(1)})\right)=\\
&\int_\Omega\kappa^2\frac{\partial(\mathbf{s}-\mathbf{s}^{(0)})}{\partial t}\cdot\frac{\partial(\mathbf{w}_h-\mathbf{w}_{hM}^{(1)})}{\partial t}\,\mathbf{dx}-a\left(\mathbf{s}-\mathbf{s}^{(0)}\,,\frac{\partial(\mathbf{w}_h-\mathbf{w}_{hM}^{(1)})}{\partial t}\right).
\end{aligned}
$$

Taking the integral between 0 and t and the Inequality (25) give

$$\begin{aligned}
&\left|\left|\kappa^2\frac{\partial(\mathbf{w}_h-\mathbf{w}_{hM}^{(1)})}{\partial t}\right|\right|_0^2+a(\mathbf{w}_h-\mathbf{w}_{hM}^{(1)},\mathbf{w}_h-\mathbf{w}_{hM}^{(1)})\\
= \;& 2\left[\int_0^t\int_\Omega\kappa^2\frac{\partial(\mathbf{s}-\mathbf{s}^{(0)})}{\partial t}\cdot\frac{\partial(\mathbf{w}_h-\mathbf{w}_{hM}^{(1)})}{\partial t}\,\mathbf{dx}\,d\tau-\int_0^t a\left(\mathbf{s}-\mathbf{s}^{(0)},\frac{\partial(\mathbf{w}_h-\mathbf{w}_{hM}^{(1)})}{\partial t}\right)d\tau\right]\\
= \;& 2\left[\int_0^t\int_\Omega\kappa^2\frac{\partial(\mathbf{s}-\mathbf{s}^{(0)})}{\partial t}\cdot\frac{\partial(\mathbf{w}_h-\mathbf{w}_{hM}^{(1)})}{\partial t}\,\mathbf{dx}\,d\tau-a(\mathbf{s}-\mathbf{s}^{(0)},\mathbf{w}_h-\mathbf{w}_{hM}^{(1)})\right.\\
&+\int_0^t a\left(\mathbf{s}-\mathbf{s}^{(0)},\frac{\partial(\mathbf{w}_h-\mathbf{w}_{hM}^{(1)})}{\partial t}\right)d\tau-\frac{\kappa^2}{2}\int_\Omega\frac{\partial(\mathbf{s}-\mathbf{s}^{(0)})}{\partial t}\cdot\frac{\partial(\mathbf{s}-\mathbf{s}^{(0)})}{\partial t}\,\mathbf{dx}\\
&\left.-\frac{1}{2}a(\mathbf{s}-\mathbf{s}^{(0)},\mathbf{s}-\mathbf{s}^{(0)})-\int_0^t a\left(\mathbf{s}-\mathbf{s}^{(0)},\frac{\partial(\mathbf{w}_h-\mathbf{w}_{hM}^{(1)})}{\partial t}\right)d\tau\right]\\
\leq \;& 2\left|\left|\frac{\partial(\mathbf{s}-\mathbf{s}^{(0)})}{\partial t}\right|\right|_0\left|\left|\kappa^2\frac{\partial(\mathbf{w}_h-\mathbf{w}_{hM}^{(1)})}{\partial t}\right|\right|_0+2||\mathbf{s}-\mathbf{s}^{(0)}||_{cd}|\mathbf{w}_h-\mathbf{w}_{hM}^{(1)}|_{cd}\\
&+\left|\left|\frac{\partial(\mathbf{s}-\mathbf{s}^{(0)})}{\partial t}\right|\right|_0\left|\left|\kappa^2\frac{\partial(\mathbf{s}-\mathbf{s}^{(0)})}{\partial t}\right|\right|_0+|\mathbf{s}-\mathbf{s}^{(0)}|_{cd}^2\\
\leq \;& C\left[h^{\lambda_k-1}|\alpha_{kM}|^{-\lambda_k}\left(\left|\left|\kappa^2\frac{\partial(\mathbf{w}_h-\mathbf{w}_{hM}^{(1)})}{\partial t}\right|\right|_0^2+|\mathbf{w}_h-\mathbf{w}_{hM}^{(1)}|_0^2\right)+h^{2(\lambda_k-1)}|\alpha_{kM}|^{-2\lambda_k}\right].
\end{aligned}$$

Since $a(\mathbf{w}_h-\mathbf{w}_{hM}^{(1)},\mathbf{w}_h-\mathbf{w}_{hM}^{(1)})=|\mathbf{w}_h-\mathbf{w}_{hM}^{(1)}|_{cd}^2$ one deduces that

$$\begin{aligned}
\left(\left|\left|\kappa^2\frac{\partial(\mathbf{w}_h-\mathbf{w}_{hM}^{(1)})}{\partial t}\right|\right|_0^2+|\mathbf{w}_h-\mathbf{w}_{hM}^{(1)}|_{cd}^2\right)^{1/2} &\leq \left(\frac{h^{2\lambda_k-2}|\alpha_{kM}|^{-2\lambda_k}}{1-Ch^{\lambda_k-1}|\alpha_{kM}|^{-\lambda_k}}\right)^{1/2}\\
&\leq Ch^{\lambda_k-1}|\alpha_{kM}|^{-\lambda_k}
\end{aligned}$$

The semi-norm $|\cdot|_{cd}$ being equivalent to the norm $||\cdot||_{cd}$ in $H_0(\text{curl,div}\,;\,\Omega)$ (see [18]), one has
(30)

$$||\mathbf{w}_h-\mathbf{w}_{hM}^{(1)}||_{cd}\leq\left(\left|\left|\kappa^2\frac{\partial(\mathbf{w}_h-\mathbf{w}_{hM}^{(1)})}{\partial t}\right|\right|_0^2+|\mathbf{w}_h-\mathbf{w}_{hM}^{(1)}|_{cd}^2\right)^{1/2}\leq Ch^{\lambda_k-1}|\alpha_{kM}|^{-\lambda_k}.$$

Using (30) in (28) leads to

$$\begin{aligned}
\left|\left|\mathbf{u}-\mathbf{u}_{hM}^{(1)}\right|\right|_{cd} &\leq C\left[||\mathbf{w}-\mathbf{w}_{hM}^{(1)}||_{cd}+\left|\left|\sum_{k,n,m=1}^{M}(\mathbf{s}_{k,m,n}-\Pi_h\mathbf{s}_{k,m,n})\right.\right.\right.\\
&\qquad\left.\left.\left.+\sum_{k,n,m=M+1}^{\infty}\mathbf{s}_{k,m,n}\right|\right|_{cd}\right]\\
&\leq C(||\mathbf{w}-\mathbf{w}_{hM}^{(1)}||_{cd}+h^{\lambda_k-1}|\alpha_{kM}^{-\lambda_k}|)\\
&\leq C(||\mathbf{w}-\mathbf{w}_h||_{cd}+||\mathbf{w}_h-\mathbf{w}_{hM}^{(1)}||_{cd}+h^{\lambda_k-1}|\alpha_{kM}^{-\lambda_k}|)
\end{aligned}$$

$$\leq C(h + h^{\lambda_k - 1}|\alpha_{kM}^{-\lambda_k}|)$$
$$\leq C \max_{2n+\lambda_k-1<2}\{h, h^{\lambda_k-1}|\alpha_{kM}|^{-\lambda_k}\}$$

Hence, inequality (26) is proved.

Following the same steps used to establish Inequality (28) replacing the norm $||\cdot||_{cd}$ by the norm $||\cdot||_0$ one has

$$\begin{aligned}
\left|\left|\mathbf{u} - \mathbf{u}_{hM}^{(1)}\right|\right|_0 &\leq C\left[||\mathbf{w} - \mathbf{w}_{hM}^{(1)}||_0 + \left|\left|\sum_{k,n,m=1}^{M}(\mathbf{s}_{k,m,n} - \Pi_h\mathbf{s}_{k,m,n})\right.\right.\right.\\
&\qquad \left.\left.\left. + \sum_{k,n,m=M+1}^{\infty}\mathbf{s}_{k,m,n}\right|\right|_0\right]\\
&\leq C(||\mathbf{w} - \mathbf{w}_{hM}^{(1)}||_0 + h^{\lambda_k-1}|\alpha_{kM}^{-\lambda_k}|)\\
&\leq C(||\mathbf{w} - \mathbf{w}_h||_0 + ||\mathbf{w}_h - \mathbf{w}_{hM}^{(1)}||_0 + h^{\lambda_k-1}|\alpha_{kM}^{-\lambda_k}|)\\
&\leq C(h^2 + h^{\lambda_k-1}|\alpha_{kM}^{-\lambda_k}|)\\
&\leq C\max_{2n+\lambda_k-1<2}\{h^2, h^{\lambda_k-1}|\alpha_{kM}|^{-\lambda_k}\}
\end{aligned}$$

and (27) is proved. □

Remark 11. If $M \in \mathbb{N}$ is chosen so that $h^{\lambda_k-1}|\alpha_{kM}|^{-\lambda_k} \leq h^2$, the error estimates (26) and (27) are optimal.

4.2. A Crank-Nicolson full discretization.

Considering a quasi-uniform and shape-regular triangulation $\mathcal{T}_h$ of the polygonal domain Ω and a Galerkin solution $\mathbf{u}_h$ of the variational problem (22), let $\{\phi_{jh}\}_{1\leq j\leq N}$ be a basis of the space V_h, one has $\mathbf{u}_h(\cdot,t) = \sum_{j=1}^{N} u_{jh}(t)\phi_{jh}(\cdot) = U_h^T(t)\Phi_h(\cdot)$ where $U_h(t) = (u_{jh}(t))_{1\leq j\leq N}^T$ and $\Phi_h = (\phi_{jh})_{1\leq j\leq N}$, satisfies the variational system

$$\begin{aligned}
\kappa^2 M_h\frac{d^2U_h}{dt^2} + A_hU_h &= F_h(t) \qquad t\in(0,T)\\
M_hU_h(0) &= \left(\int_\Omega u^0\cdot\phi_{lh}\,\mathbf{dx}\right)^T_{1\leq l\leq N}\\
M_h\frac{dU_h}{dt}(0) &= \left(\int_\Omega u^1\cdot\phi_{lh}\,\mathbf{dx}\right)^T_{1\leq l\leq N}
\end{aligned}$$

where $M_h = \left(\int_\Omega \phi_{jh}\cdot\phi_{lh}\,\mathbf{dx}\right)_{1\leq j,l\leq N}$, $A_h = (a(\phi_{jh},\phi_{lh}))_{1\leq j,l\leq N}$ and
$F_h(t) = \left(\int_\Omega \mathbf{f}(\cdot,t)\cdot\phi_{lh}\,\mathbf{dx}\right)^T_{1\leq l\leq N}$.

The full discretization consists to subdivide the interval $[0,T]$ into L subintervals $[t_{j-1},t_j]$ of size $\tau = t_j - t_{j-1} = \dfrac{T}{L}$, $j = 1,\cdots,L$ such that $[0,T] =$

$\bigcup_{j=1}^{L}[t_{j-1}, t_j]$, $0 = t_0 < t_1 < \cdots < t_L = T$. The σ-schemes $(0 \le \sigma \le 1)$ consists of approximating $U_h(t_l) = (u_{jh}(t_l))^T_{1\le j\le N}$, $0 \le l \le L$ such that

(31)

$$\begin{aligned}
\frac{\kappa^2}{\tau^2}M_h(U_h(t_{l+1}) - 2U_h(t_l) + U_h(t_{l-1})) \\
+\sigma A_h U_h(t_{l+1}) + (1-2\sigma)A_h U_h(t_l) \\
+\sigma A_h U_h(t_{l-1}) &= \sigma F_h(t_{l+1}) + (1-2\sigma)F_h(t_l) + \sigma F_h(t_{l-1}) \\
M_h U_h(0) &= \left(\int_\Omega u^0 \cdot \phi_{lh}\,\mathbf{dx}\right)^T_{1\le l\le N} \\
M_h \frac{U_h}{dt}(0) &= \left(\int_\Omega u^1 \cdot \phi_{lh}\,\mathbf{dx}\right)^T_{1\le l\le N}
\end{aligned}$$

So if we set $u_{h\tau}(\cdot, t)$ be an interpolation of the solution $(U_h(t_l))_{1\le l\le L}$, that is, $u_{h\tau}(\cdot, t_l) = U_h(t_l)\Phi(\cdot)$ with a Crank-Nicolson scheme $(\sigma = 1/2)$, one has the following error estimates from Theorem 7

$$\begin{aligned}
||\mathbf{u}(\cdot,t_l) - \mathbf{u}_{h\tau}(\cdot,t_l)||_{cd} \le C(h+\tau^2) \\
||\mathbf{u}(\cdot,t_l) - \mathbf{u}_{h\tau}(\cdot,t_l)||_0 \le C(h^2+\tau^2)
\end{aligned}$$

for $l = 1, \cdots, L$ if Ω is a convex polygonal domain and if $\mathbf{u}$ belongs to $\mathbf{L}^\infty(0,T;H(\mathrm{curl},\mathrm{div};\Omega))$, $\dfrac{\partial \mathbf{u}}{\partial t}$ is in $\mathbf{L}^2(0,T;H(\mathrm{curl},\mathrm{div};\Omega))$ and $\dfrac{\partial^k \mathbf{u}}{\partial t^k}$ belongs to $\mathbf{L}^2(0,T;L^2(\Omega)^2)$, $k = 3, 4$.

The following algorithm is developed to recover the convergence rate of the Crank-Nicolson scheme when the domain Ω is not convex.

Defect-Correction Algorithm 2.

(1) Solve Problem (31) and obtain the solution $u^{(0)}_{h\tau}$.
(2) Compute the $\mathbf{s}^{(0)}_{k,m,n}$ using formula (16), replacing $\mathbf{u}$ with $\mathbf{u}^{(0)}_{h\tau}$, $k, m, n \in \mathbb{N}$, $k \ge 1$, $1 \le m \le M$, $M \in \mathbb{N}$, $M \ge 1$, $2n + \lambda_k - 1 < 2$.
(3) Find $\mathbf{w}^{(1)}_{h\tau M}$ solution of

(32)

$$\begin{aligned}
\kappa^2\frac{d^2}{dt^2}\int_\Omega \mathbf{w}^{(1)}_{h\tau M}\cdot\mathbf{v}_h\,\mathbf{dx} + a(\mathbf{w}^{(1)}_{h\tau M}, \mathbf{v}_h) &= \int_\Omega\left(\mathbf{f} - \kappa^2\sum_{k,m,n}\frac{\partial^2 \mathbf{s}^{(0)}_{k,m,n}}{\partial t^2}\right)\cdot\mathbf{v}_h\,\mathbf{dx} \\
&\quad - \sum_{k,m,n} a\left(\mathbf{s}^{(0)}_{k,m,n}, \mathbf{v}_h\right)\ \forall\,\mathbf{v}_h \in V_h
\end{aligned}$$

$$\begin{aligned}
\int_\Omega \mathbf{w}^{(1)}_{h\tau M}(\cdot,0)\cdot\mathbf{v}_h\,\mathbf{dx} &= \int_\Omega\left(\mathbf{u}^0 - \sum_{k,m,n}\mathbf{s}^{(0)}_{k,m,n}(\cdot,0)\right)\cdot\mathbf{v}_h\,\mathbf{dx}\ \forall\,\mathbf{v}_h \in V_h \\
\int_\Omega \frac{\partial \mathbf{w}^{(1)}_{h\tau M}}{\partial t}(\cdot,0)\cdot\mathbf{v}_h\,\mathbf{dx} &= \int_\Omega\left(\mathbf{u}^1 - \sum_{k,m,n}\frac{\partial \mathbf{s}^{(0)}_{k,m,n}}{\partial t}(\cdot,0)\right)\cdot\mathbf{v}_h\,\mathbf{dx}\ \forall\,\mathbf{v}_h \in V_h
\end{aligned}$$

(4) Compute $\mathbf{u}^{(1)}_{h\tau M} = \mathbf{w}^{(1)}_{h\tau M} + \sum_{k,m,n} \mathbf{s}^{(0)}_{k,m,n}$.

Error Estimates of the Crank-Nicolson FEM.

THEOREM 12. *Let $\mathcal{T}_h$ be a quasi-uniform and shape-regular triangulation of Ω, let $M \in \mathbb{N}$, $M \geq 1$, let $\boldsymbol{u} \in \boldsymbol{L}^2(0,T;H_0(curl,div;\Omega))$ be the solution of Problem (5), and $\boldsymbol{u}^{(1)}_{h\tau M}$ be the interpolation of the solution of Problem (31) using the Defect-Correction Algorithm 2 in a subdivision of the interval $[0,T]$ in sub-interval $[t_{l-1},t_l]$ of size $\tau = \frac{T}{L}$, $l = 1,\cdots,L$ such that $[0,T] = \bigcup_{l=1}^{L}[t_{l-1},t_l]$, $0 = t_0 < t_1 < \cdots < t_L = T$. Then for any $1 \leq l \leq L$,*

(33)
$$\left|\left|\boldsymbol{u}(\cdot,t_l) - \boldsymbol{u}^{(1)}_{h\tau M}(\cdot,t_l)\right|\right|_{cd} \leq C\left(\max_{k\geq 1, n\geq 0,\, 2n+\lambda_k-1<2}\{h, h^{\lambda_k-1}|\alpha_{kM}|^{-\lambda_k}\} + \tau^2\right),$$

(34)
$$\left|\left|\boldsymbol{u}(\cdot,t_l) - \boldsymbol{u}^{(1)}_{h\tau M}(\cdot,t_l)\right|\right|_{0} \leq C\left(\max_{k\geq 1, n\geq 0,\, 2n+\lambda_k-1<2}\{h^2, h^{\lambda_k-1}|\alpha_{kM}|^{-\lambda_k}\} + \tau^2\right).$$

Proof.

$$\begin{aligned}\left|\left|\mathbf{u}(\cdot,t_l) - \mathbf{u}^{(1)}_{h\tau M}(\cdot,t_l)\right|\right|_{cd} &\leq \left|\left|\mathbf{u}(\cdot,t_l) - \mathbf{u}^{(1)}_{hM}(\cdot,t_l)\right|\right|_{cd} + \left|\left|\mathbf{u}^{(1)}_{hM}(\cdot,t_l) - \mathbf{u}^{(1)}_{h\tau M}(\cdot,t_l)\right|\right|_{cd} \\ &\leq C\left(\max_{2n+\lambda_k-1}\{h, h^{\lambda_k-1}|\alpha_{kM}|^{-\lambda_k}\}\right) \\ &\quad + \left|\left|\mathbf{u}_h(\cdot,t_l) - \mathbf{u}^{(1)}_{h\tau M}(\cdot,t_l)\right|\right|_{cd}.\end{aligned}$$

But $\mathbf{u}^{(1)}_{hM}(\cdot,t_l) = \mathbf{w}^{(1)}_{hM}(\cdot,t_l) + \sum_{k,n,m=1}^{M} \mathbf{s}^{(0)}_{k,m,n}$ and $\mathbf{u}^{(1)}_{h\tau M}(\cdot,t_l) = \mathbf{w}^{(1)}_{h\tau M}(\cdot,t_l) + \sum_{k,n,m=1}^{M} \mathbf{s}^{(0)}_{k,m,n,\tau}$ where $\mathbf{s}^{(0)}_{k,m,n,\tau}$ is the interpolation of the $\mathbf{s}^{(0)}_{k,m,n}(\cdot,t_l)$, $0 \leq l \leq L$. Then

$$\begin{aligned}\left|\left|\mathbf{u}^{(1)}_{hM}(\cdot,t_l) - \mathbf{u}^{(1)}_{h\tau M}(\cdot,t_l)\right|\right|_{cd} &\leq \left|\left|\mathbf{w}^{(1)}_{hM}(\cdot,t_l) - \mathbf{w}^{(1)}_{h\tau M}(\cdot,t_l)\right|\right|_{cd} \\ &\quad + \sum_{k,n,m=1}^{M} ||\mathbf{s}^{(0)}_{k,m,n} - \mathbf{s}^{(0)}_{k,m,n,\tau}||_{cd} \\ &\leq \left|\left|\mathbf{w}^{(1)}_{hM}(\cdot,t_l) - \mathbf{w}^{(1)}_{h\tau M}(\cdot,t_l)\right|\right|_{cd} \\ &\quad + C\left(\max_{k,n,\, 2n+\lambda_k-1<2}\{h^{\lambda_k-1}|\alpha_{kM}|^{-\lambda_k}\} + \tau^2\right).\end{aligned}$$

Also, using (30)

$$\begin{aligned}
\left|\left|\mathbf{w}^{(1)}_{hM}(\cdot,t_l)-\mathbf{w}^{(1)}_{h\tau M}(\cdot,t_l)\right|\right|_{cd} &\leq \left|\left|\mathbf{w}^{(1)}_{hM}(\cdot,t_l)-\mathbf{w}_{h}(\cdot,t_l)\right|\right|_{cd}+\left|\left|\mathbf{w}_{h}(\cdot,t_l)-\mathbf{w}^{(1)}_{h\tau M}(\cdot,t_l)\right|\right|_{cd}\\
&\leq \left|\left|\mathbf{w}^{(1)}_{hM}(\cdot,t_l)-\mathbf{w}_{h}(\cdot,t_l)\right|\right|_{cd}+\left|\left|\mathbf{w}_{h}(\cdot,t_l)-\mathbf{w}^{(1)}_{h\tau}(\cdot,t_l)\right|\right|_{cd}\\
&\quad+\left|\left|\mathbf{w}_{h\tau}(\cdot,t_l)-\mathbf{w}^{(1)}_{h\tau M}(\cdot,t_l)\right|\right|_{cd}\\
&\leq C\Bigg(\max_{k,n,2n+\lambda_k-1<2}\{h^{\lambda_k-1}|\alpha_{kM}|^{-\lambda_k}\}+h+\tau^2\\
&\quad+\left|\left|\mathbf{w}_{h\tau}(\cdot,t_l)-\mathbf{w}^{(1)}_{h\tau M}(\cdot,t_l)\right|\right|_{cd}\Bigg).
\end{aligned}$$

But one can use a similar method as the one in Theorem 10 to show that

$$\left|\left|\mathbf{w}_{h\tau}(\cdot,t_l)-\mathbf{w}^{(1)}_{h\tau M}(\cdot,t_l)\right|\right|_{cd}\leq C(\max_{2n+\lambda_k-1<2}\{h^{\lambda_k-1}|\alpha_{kM}|^{-\lambda_k}\}+\tau^2).$$

Hence

$$\left|\left|\mathbf{u}(\cdot,t_l)-\mathbf{u}^{(1)}_{h\tau M}(\cdot,t_l)\right|\right|_{cd}\leq C\left(\max_{k\geq1,n\geq0,\,2n+\lambda_k-1<2}\{h,h^{\lambda_k-1}|\alpha_{kM}|^{-\lambda_k}\}+\tau^2\right).$$

One can use the same method as above to show (34). □

Remark 13. One can remark that if M is chosen such that

$$\max_{k\geq1,n\geq0,\,2n+\lambda_k-1<2}\{h^{\lambda_k-1}|\alpha_{kM}|^{-\lambda_k}\}\leq h^2,$$

then the optimal convergence of a linear Crank-Nicolson FEM is obtained, that is

$$\begin{aligned}
\left|\left|\mathbf{u}(\cdot,t_l)-\mathbf{u}^{(1)}_{h\tau M}(\cdot,t_l)\right|\right|_{cd}&\leq C(h+\tau^2)\\
\left|\left|\mathbf{u}(\cdot,t_l)-\mathbf{u}^{(1)}_{h\tau M}(\cdot,t_l)\right|\right|_{0}&\leq C(h^2+\tau^2)
\end{aligned}$$

5. CONCLUSION

This paper presents a finite element method to solve the time-dependent Maxwell's equations on non-convex polygonal domains. The nodal FEM is coupled with a Fourier decomposition that extracts the singular behavior of the solution and recovers the optimal linear convergence. The extension to polygonal domains with multiple re-entrant corners can be handled by the method through the use of local cut-off functions near the corners and adding the different singular parts to the final solution. However, the method assumes that the solution is sufficiently regular in time, and the initial data are also sufficiently smooth, so time-singularities and singularities due to the smoothness of the initial data or the change of boundary conditions are not handled by the method.

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